11-1-2008

Selective Screenability in Topological Groups

Liljana Babinkostova
Boise State University

SELECTIVE SCREENABILITY IN TOPOLOGICAL GROUPS

LILJANA BABINKOSTOVA

Abstract. We examine the selective screenability property in topological groups. In the metrizable case we also give characterizations of $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}},\mathcal{O})$ and $\mathsf{Smirnov}$-$\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}},\mathcal{O})$ in terms of the Haver property and finitary Haver property respectively relative to left-invariant metrics. We prove theorems stating conditions under which $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}},\mathcal{O})$ is preserved by products. Among metrizable groups we characterize the countable dimensional ones by a natural game.

1. Definitions and notation

Let G be topological space. We shall use the notations:

- $\mathcal{O}$: The collection of open covers of G.

An open cover $\mathcal{U}$ of a topological space G is said to be

- an ω-cover if G is not a member of $\mathcal{U}$, but for each finite subset F of G there is a $U\in\mathcal{U}$ such that $F\subset U$. The symbol Ω denotes the collection of ω covers of G.
- *groupable* if there is a partition $\mathcal{U}=\cup_{n<\infty}\mathcal{U}_n$, where each $\mathcal{U}_n$ is finite, and for each $x\in G$ the set $\{n: x\notin\cup\mathcal{U}_n\}$ is finite. The symbol $\mathcal{O}^{gp}$ denotes the collection of groupable open covers of the space.
- *large* if each element of the space is contained in infinitely many elements of the cover. The symbol Λ denotes the collection of large covers of the space.
- *c-groupable* if there is a partition $\mathcal{U}=\cup_{n<\infty}\mathcal{U}_n$, where each $\mathcal{U}_n$ is pairwise disjoint and each x is in all but finitely many $\cup\mathcal{U}_n$. The symbol $\mathcal{O}^{cgp}$ denotes the collection of c-groupable open covers of the space.

Now let $(G,*)$ be a topological group with identity element e. We will assume that G is not compact. For A and B subsets of G, $A*B$ denotes $\{a*b: a\in A, b\in B\}$. We use the notation A^2 to denote $A*A$, and for $n>1$, A^n denotes $A^{n-1}*A$. For a neighborhood U of e, and for a finite subset F of G the set $F*U$ is a neighborhood of the finite set F. Thus,

Key words and phrases: Haver property, selective screenability, Hurewicz property, finitary Haver property, countable dimensional, selection principle, c-groupable cover.
Subject Classification: Primary 54D20, 54D45, 55M10; Secondary 03E20.

the set $\{F * U : F \subset G \text{ finite}\}$ is an ω-cover of G, which is denoted by the symbol $\Omega(U)$. The set

$$\Omega_{nbd} = \{\Omega(U) : U \text{ a neighborhood of } e\}$$

is the set of all such ω-covers of G.

The set $\mathcal{O}(U) = \{x * U : x \in G\}$ is an open cover of G. The symbol

$$\mathcal{O}_{nbd} = \{\mathcal{O}(U) : U \text{ a neighborhood of } e\}$$

denotes the collection of all such open covers of G. Selection principles using these open covers of topological groups have been considered in several papers, including [4], [5], [16] and [23], where information relevant to our topic can be found. Now we describe the relevant selection principles for this paper. Let S be an infinite set, and let $\mathcal{A}$ and $\mathcal{B}$ be collections of families of subsets of S.

The selection principle $\mathsf{S}_c(\mathcal{A}, \mathcal{B})$, introduced in [2], is defined as follows:

> For each sequence $(A_n : n < \infty)$ of elements of the family $\mathcal{A}$ there exists a sequence $(B_n : n < \infty)$ such that for each n B_n is a pairwise disjoint family refining A_n, and $\bigcup_{n<\infty} B_n$ is a member of the family $\mathcal{B}$.

The selection principle $\mathsf{Smirnov} - \mathsf{S}_c(\mathcal{A}, \mathcal{B})$ is defined as follows:

> For each sequence $(A_n : n < \infty)$ of elements of the family $\mathcal{A}$ there exists a positive integer $k < \infty$ and a sequence $(B_n : n \leq k)$ where each B_n is a pairwise disjoint family of open sets refining A_n, $n \leq k$ and $\bigcup_{j \leq k} B_j$ is a member of the family $\mathcal{B}$.

The metrizable space X is said to be *Haver* [12] with respect to a metric d if there is for each sequence $(\epsilon_n : n < \infty)$ of positive reals a sequence $(\mathcal{V}_n : n < \infty)$ where each $\mathcal{V}_n$ is a pairwise disjoint family of open sets, each of d-diameter less than ϵ_n, such that $\bigcup_{n<\infty} \mathcal{V}_n$ is a cover of X. And it is said to be *finitary Haver* [8] with respect to the metric d if there is for each sequence $(\epsilon_n : n < \infty)$ a positive integer k and a sequence $(\mathcal{V}_n : n \leq k)$ where each $\mathcal{V}_n$ is a pairwise disjoint family of open sets, each of diameter less than ϵ_n, such that $\bigcup_{n \leq k} \mathcal{V}_n$ is a cover of X.

2. Selective screenability and $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$

Recent investigations into the Haver property and its relation to the selective screenability property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ revealed that the Haver property is weaker than selective screenability. E. and R. Pol has reported the following nice characterizations of $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ in terms of the Haver property:

Theorem 1 ([20]). *Let (X,d) be a metrizable space. The following are equivalent:*

(1) *X has property* $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$.

(2) *X has the Haver property in all equivalent metrics.*

For a topological group the property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ is stronger than $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}},\mathcal{O})$. This is in part seen by comparing Theorem 1 with the following result:

Theorem 2. *Let $(G,*)$ be a metrizable group. The following are equivalent:*

(1) *The group has property $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}},\mathcal{O})$.*
(2) *The group has the Haver property in all equivalent left invariant metrics.*

In the proof of Theorem 2 we use the following result of Kakutani:

Theorem 3 ([14])**.** *Let $(U_k : k < \infty)$ be a sequence of subsets of the topological group $(H,*)$ where $\{U_k : k < \infty\}$ is a neighborhood basis of the identity element e and each U_k is symmetric* [1]*, and for each k also $U_{k+1}^2 \subseteq U_k$. Then there is a left-invariant metric d on H such that*

(1) *d is uniformly continuous in the left uniform structure on $H \times H$.*
(2) *If $y^{-1} * x \in U_k$ then $d(x,y) \leq (\frac{1}{2})^{k-2}$.*
(3) *If $d(x,y) < (\frac{1}{2})^k$ then $y^{-1} * x \in U_k$.*

In the above theorem V^2 denotes $\{a * b : a, b \in V\}$. For $n > 1$ a positive integer the symbol V^n is defined similarly.

And now the proof of Theorem 2:

Proof: $1 \Rightarrow 2$: Let d be a left-invariant metric of G and let $(\epsilon_n : n < \infty)$ be a sequence of positive real numbers. For each n choose an open neighborhood U_n of the identity element e of G with $diam_d(U_n) < \epsilon_n$ and put $\mathcal{U}_n = \mathcal{O}(U_n)$. Then $\{\mathcal{U}_n : n < \infty\}$ is a sequence from $\mathcal{O}_{\mathsf{nbd}}(U)$. Apply $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}},\mathcal{O})$. For each n there is a pairwise disjoint family $\mathcal{V}_n$ of open sets refining $\mathcal{U}_n$ such that $\bigcup_{n<\infty}\mathcal{V}_n$ is an element of $\mathcal{O}$. Now for each n, for $V \in \mathcal{V}_\backslash$ there is an $x \in G$ with $V \subseteq x * U_n$. But then $diam_d(V) \leq diam_d(x * U_n) = diam_d(U_n) \leq \epsilon_n$. Thus the $\mathcal{V}_n$'s witnesses Haver's property for the given sequence of ϵ_n's.
$2 \Rightarrow 1$: Let $\mathcal{U}_n = \mathcal{O}(U_n)$, $n < \infty$ be given. For each n choose a neighborhood V_n of the identity element e in G such that:

(1) For all n, $V_n \subset U_n$.
(2) For all n, $V_n * V_n \subset V_{n-1}$.
(3) $\{V_n : n < \infty\}$is a neighborhood basis of the identity e.

By Kakutani's theorem choose a left invariant metric d so that for every n:

(1) If $y^{-1} * x \in V_n$ then $d(x,y) \leq (\frac{1}{2})^{n-2}$.
(2) If $d(x,y) < (\frac{1}{2})^n$ then $y^{-1} * x \in V_n$.

For each n, put $\epsilon_n = (\frac{1}{2})^n$. Since G has the Haver property with respect to d, choose for each n a pairwice disjoint family $\mathcal{V}_n$ of open sets such that:

(1) For each n and for each $V \in \mathcal{V}_n$, $diam_d(V) < \epsilon_n$.
(2) $\bigcup_{n<\infty}\mathcal{V}_n$ covers G.

---

[1] U_k is symmetric if $U_k = U_k^{-1}$

Then for every n and for every $V \in \mathcal{V}_n$, there is and x_V with $V \subseteq x * V_n \subseteq x_V * U_n \in \mathcal{U}_n$ and so $\mathcal{V}_n$ refines $\mathcal{U}_n$. But then $\mathcal{V}_n$ witness $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ for $\{\mathcal{U}_n : n < \infty\}$. $\diamondsuit$

Using the similar ideas one can prove the following:

Theorem 4. *Let $(G, *)$ be a metrizable group. The following are equivalent:*

1. *The group has property $\mathsf{Smirnov} - \mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$.*
2. *The group has the finitary Haver property in all equivalent left invariant metrics.*

One may ask when the properties $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ and $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ are equivalent in a topological group. We do not have a complete answer. The Hurewicz property gives a condition: A topological space G has the *Hurewicz property* if for each sequence $\mathcal{U}_n, n < \infty$ of open covers of X there is a sequence $\mathcal{F}_n, n < \infty$ of finite sets such that each $\mathcal{F}_n \subset \mathcal{U}_n$, and for each $x \in G$, the set $\{n : x \notin \cup\mathcal{F}_n\}$ is finite.

Theorem 5. *Let $(G, *)$ be a topological group with the Hurewicz property. Then $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ is equivalent to $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$.*

Proof: Let $(G, *)$ be a topological group. It is clear that $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ implies $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$. For the converse implication, assume the group has property $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$. Let $(\mathcal{U}_n : n < \infty)$ be a sequence of open covers of G. For each n, and each $x \in G$ choose a neighborhood $V(x, n)$ of the identitity e such that $x*V(x, n)^4$ is a subset of some U in $\mathcal{U}_n$. Put $\mathcal{H}_n = \{x*V(x, n) : x \in G\}$. Apply the Hurewicz property to the sequence $(\mathcal{H}_n : n < \infty)$. For each n choose a finite $\mathcal{F}_n \subset \mathcal{H}_n$ such that for each $g \in G$, the set $\{n : g \notin \cup\mathcal{F}_n\}$ is finite. Write $\mathcal{F}_n = \{{x_i}^n * V({x_i}^n, n) : i \in I_n\}$ and I_n is finite. For each n, define $V_n = \bigcap_{i \in I_n} V({x_i}^n, n)$ a neighborhood of the identity e. Choose a partition $\mathbb{N} = \bigcup_{k<\infty} J_k$ where each J_k is infinite, and for $l \neq k$, $J_l \cap J_k = \emptyset$. For each k, apply $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ to the sequence $(\mathcal{O}(V_n) : n \in J_k)$. For each $n \in J_k$ find a pairwise disjoint family $\mathcal{S}_n$ of open sets such that $\mathcal{S}_n$ refines $\mathcal{O}(V_n)$ and $\bigcup_{n \in J_k} \mathcal{S}_n$ covers G. For each n define $\mathcal{V}_n = \{S \in \mathcal{S}_n : (\exists U \in \mathcal{U}_n)(S \subseteq U)\}$. Since $\mathcal{V}_n \subset \mathcal{S}_n$, $\mathcal{V}_n$ is pairwise disjoint and refines $\mathcal{U}_n$. We will show that $\bigcup_{n<\infty} \mathcal{V}_n$ covers G. Pick any $g \in G$. Fix N_g so that for all $n \geq N_g$, $g \in \cup\mathcal{F}_n$. Pick k_g so large that $min(J_{k_g}) \geq N_g$. Pick $m \in J_{k_g}$ with $g \in \cup\mathcal{S}_m$. Pick $J \in \mathcal{S}_m$ with $g \in J$. We will show that $J \in \mathcal{V}_m$. We have that $g \in \cup\mathcal{F}_m$, so pick $i \in I_m$ with $g \in {x_i}^m * V({x_i}^m, m)$. Since $J \in \mathcal{S}_m$, also pick h_m so that $J \subseteq h_m * V_m = h_m * (\bigcap_{i \in I_m} V({x_i}^m, m)) \subseteq h_m * V({x_i}^m, m)$. We have that $g = {x_i}^m * z_g = h_m * t_g$ for some $z_g, t_g \in V({x_i}^m, m)$. So $h_m = {x_i}^m * z_g * {t_g}^{-1}$. Now consider any $y \in J$. Choose $t_y \in V({x_i}^m, m)$ with $y = h_m * t_y$. But then $y = {x_i}^m * (z_g * {t_g}^{-1} * t_y * e) \in {x_i}^m * V({x_i}^m, m)^4 \subseteq U$, for some $U \in \mathcal{U}_m$. So we have that $J \in \mathcal{V}_m$ and $g \in J$.$\diamondsuit$

The symbol $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the statement that there is for each sequence $(O_n : n < \infty)$ of elements of $\mathcal{A}$ a sequence $(T_n : n < \infty)$ such that for each n $T_n \in O_n$, and $\{T_n : n < \infty\} \in \mathcal{B}$. A topological group $(G, *)$ is said to be a *Hurewicz-bounded* group if it satisfies the selection principle $\mathsf{S}_1(\Omega_{nbd}, \mathcal{O}^{gp})$.

In [2] was shown that $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ is equivalent to $\mathsf{S}_c(\Omega,\mathcal{O})$. The analogous equivalence doesn't hold in topological groups:

Theorem 6. *$\mathsf{S}_c(\Omega_{\mathsf{nbd}},\mathcal{O})$ does not imply $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}},\mathcal{O})$.*

Proof: Let $(C,*)$ be the unit circle in the complex plane with complex multiplication. It is a compact metrizable group embedding the unit interval $[0,1]$ as a subspace. Since $(C^{\mathbb{N}},*)$ is a compact group it has the Hurewicz property, so is Hurewicz bounded. Also $\mathbb{R}$, the real line with addition, is a Hurewicz-bounded topological group. Thus the product group $\mathbb{R}\times C^{\mathbb{N}}$ is Hurewicz bounded, so has the property $\mathsf{S}_1(\Omega_{\mathsf{nbd}},\mathcal{O})$, and so has $\mathsf{S}_c(\Omega_{\mathsf{nbd}},\mathcal{O})$. But $[0,1]^{\mathbb{N}}$ embeds as closed subspace into $\mathbb{R}\times C^{\mathbb{N}}$, and $[0,1]^{\mathbb{N}}$ does not have the property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$. Thus the topological group $\mathbb{R}\times C^{\mathbb{N}}$ does not have $\mathsf{S}_c(\mathcal{O},\mathcal{O})$, and as it has the Hurewicz property, Theorem 5 implies it is not $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}},\mathcal{O})$.$\diamondsuit$

The symbol $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ denotes the statement that there is for each sequence $(O_n : n<\infty)$ of elements of $\mathcal{A}$ a sequence $(T_n : n<\infty)$ of finite sets such that for each n $T_n\subseteq O_n$, and $\cup\{T_n : n<\infty\}\in\mathcal{B}$. It was shown in [15] that $\mathsf{S}_{fin}(\Omega,\mathcal{O}^{gp})$ is equivalent to the Hurewicz property. And it is well known that $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ is the Menger property, which is equivalent to $\mathsf{S}_{fin}(\Omega,\mathcal{O})$. A topological group is said to be a *Menger bounded* group if it has the property $\mathsf{S}_1(\Omega_{\mathsf{nbd}},\mathcal{O})$.

By how much can the requirement that $(G,*)$ has the Hurewicz property be weakened in Theorem 5? Natural possibilities include the Menger property, Menger boundedness or Hurewicz boundedness. In light of interesting recent examples of E. and R. Pol - [19], [20] we conjecture that none of these weakenings is enough:

Conjecture 1. *There is a metrizable Menger bounded topological group which has the property $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}},\mathcal{O})$, but not the property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$.*

Conjecture 2. *There is a metrizable Hurewicz bounded topological group which has the property $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}},\mathcal{O})$, but not the property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$.*

Conjecture 3. *There is a metrizable topological group which has the Menger property and property $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}},\mathcal{O})$, but not the property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$.*

It is clear that Conjecture 3$\Rightarrow$ Conjecture 1 and Conjecture 2 $\Rightarrow$ Conjecture 1. It may be that Conjecture 2 is independent of the Zermelo-Fraenkel axioms. Recently E. and R. Pol showed that CH implies Conjecture 3.

3. Products

E. Pol showed in [17] that there exist a zerodimensional subset Y of the real line and a separable metric space X and such that X has the property $\mathsf{S}_c(\mathcal{O},\mathcal{O})$ in all finite powers, but $X\times Y$ does not have $\mathsf{S}_c(\mathcal{O},\mathcal{O})$. This failure does not happen for the group analogue:

Theorem 7. *Let $(G,*)$ be a group satisfying $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}},\mathcal{O})$. If $(H,*)$ is a group with property $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}},\mathcal{O}^{cgp})$, then $(G\times H,*)$ also has $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}},\mathcal{O})$.*

Proof: For each n let $\mathcal{U}_n$ be an element of $\mathcal{O}_{\mathsf{nbd}}(G \times H)$. Each $\mathcal{U}_n$ is of the form $\mathcal{U}_n = \mathcal{O}(U_n)$ where U_n is a neighborhood of the identity (e_G, e_H) of $G \times H$. Pick $V_n \subset G$ a neighborhood of e_G, and $W_n \subset H$ a neighborhood of e_H so that $V_n \times W_n \subseteq U_n$. Then $\mathcal{W}_n = \mathcal{O}(V_n \times W_n)$ is a refinement of $\mathcal{U}_n$, for all n. Let $\mathcal{H}_n = \mathcal{O}(W_n) \in \mathcal{O}_{\mathsf{nbd}}$. Apply $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O}^{cgp})$ to the sequence $(\mathcal{H}_n : n < \infty)$. For each n find a finite pairwise disjoint refinement $\mathcal{K}_n$ of $\mathcal{H}_n$ so that each x is in all but finitely many of $\bigcup \mathcal{K}_n$. Next, for each n put $\mathcal{G}_n = \mathcal{O}(V_n) \in \mathcal{O}_{nbd}$. Apply $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ to the sequence $(\mathcal{G}_n : n < \infty)$. For each n choose pairwise disjoint $\mathcal{J}_n$ that refines $\mathcal{G}_n$ so that $\bigcup \mathcal{J}_n$ is a large open cover of G. For each n define $\mathcal{V}_n = \{J \times K : J \in \mathcal{J}_n, K \in \mathcal{K}_n\}$.
Claim 1: $\mathcal{V}_n$ refines $\mathcal{W}_n$: For $J \in \mathcal{J}_n$ and $K \in \mathcal{K}_n$ there is an element $g \in G$ and $h \in H$ such that $J \subseteq g * V_n$ and $K \subseteq g * W_n$. But then $J \times K \subseteq g * V_n \times h * W_n \in \mathcal{W}_n$.
Claim 2: $\mathcal{V}_n$ is pairwise disjoint: Let $J_1 \times K_1$ and $J_2 \times K_2$ be elements of $\mathcal{V}_n$ with $J_1 \times K_1 \neq J_2 \times K_2$. If $J_1 \neq J_2$ then $J_1 \cap J_2 = \emptyset$ because the $\mathcal{J}_n$ is disjoint. So $(J_1 \times K_1) \bigcap (J_2 \times K_2) = \emptyset$. Similarly, $(J_1 \times K_1) \bigcap (J_2 \times K_2) = \emptyset$ if $K_1 \neq K_2$.
Claim 3: $\bigcup \mathcal{V}_n$ covers $G \times H$. Consider (g, h) as an element of $C \times H$. Since $\bigcup \mathcal{J}_n$ is a large cover of G the set $S_1 = \{n : (\exists J \in \mathcal{J}_n)(g \in J)\}$ is infinite and there is an N such that $S_2 = \{n : (\exists K \in \mathcal{K}_n)(h \in K)\} \supseteq \{n : n \geq N\}$. Pick an $n \in S_1 \cap S_2$. Pick $J \in \mathcal{J}_n$ with $g \in J$ and $K \in \mathcal{K}_n$ with $h \in K$. Then $(g, h) \in J \times K \in \mathcal{V}_n$.$\diamondsuit$

Corollary 8. *Let $(G, *_1)$ and $(H, *_2)$ be metrizable topological groups such that $(G, *_1)$ has $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ and H is zero-dimensional. Then $(G \times H, *)$ is a group with property $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$.*

Proof: We show that $(H, *)$ has $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O}^{cgp})$. The reason for this is that since H is zerodimensional, each open cover of it has a refinement by a disjoint open cover. Thus for a given sequence $(\mathcal{U}_n : n < \infty)$ from $\mathcal{O}_{\mathsf{nbd}}$ for H we can choose for each n a disjoint open refinement $\mathcal{V}_n$ which covers H. Clearly $\cup_{n<\infty} \mathcal{V}_n$ is c-groupable.$\diamondsuit$

To illustrate: Let $\mathbb{P}$ denote the set of irrational numbers. E. Pol has shown under CH[2] that there is a metrizable space X with property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ such that $X \times \mathbb{P}$ does not have $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. Now $\mathbb{P}$ is homeomorphic to a closed subset of the zerodimensional group $(\mathbb{Z}^{\mathbb{N}}, +)$. Thus $X \times \mathbb{Z}^{\mathbb{N}}$ also does not have $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. But for any topological group $(G, *)$ with property $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$, the group $G \times \mathbb{Z}^{\mathbb{N}}$ still has $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$.

Hattori, Yamada and independently Rohm, have proven the following product theorem for $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$:

Theorem 9 (Hattori-Yamada, Rohm)**.** *If X is σ-compact and if X and Y both have the property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$, then $X \times Y$ has the property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$.*

[2]For a new proof using a weaker hypothesis, see [19] and [20].

We shall prove an analogous theorem, Theorem 11, for topological groups. Since $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ is weaker than $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$ (see the remarks following Conjecture 3), we are able to use a weaker restriction than σ-compact. We use the following result in our proof:

Lemma 10 ([7]). *The following statements are equivalent:*

(1) *X has the Hurewicz property and property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$.*
(2) *For each sequence $(\mathcal{U}_n : n < \infty)$ of open covers of X there is a sequence $(\mathcal{V}_n : n < \infty)$ such that:*
  (a) *Each $\mathcal{V}_n$ is a finite collection of open sets;*
  (b) *Each $\mathcal{V}_n$ is pairwise disjoint;*
  (c) *Each $\mathcal{V}_n$ refines $\mathcal{U}_n$;*
  (d) *there is a sequence $n_1 < n_2 < \cdots < n_k < \cdots$ of positive integers such that each element of X is in all but finitely many of the sets $\cup(\cup_{n_k \leq j < n_{k+1}} \mathcal{V}_j)$.*

Theorem 11. *Let $(G, *)$ be a group which has property $\mathsf{S}_c(\mathcal{O}_{nbd}, \mathcal{O})$ as well as the Hurewicz property. Then for any topological group $(H, *)$ satisfying $\mathsf{S}_c(\mathcal{O}_{nbd}, \Lambda)$, $G \times H$ also satisfies $\mathsf{S}_c(\mathcal{O}_{nbd}, \mathcal{O})$.*

Proof: Let $(\mathcal{O}(U_n \times V_n) : n < \infty)$ be a sequence of $\mathcal{O}_{nbd}$-covers of $G \times H$. Then each $\mathcal{O}(U_n)$ is an $\mathcal{O}_{nbd}$-cover of G and each $\mathcal{O}(V_n)$ is an $\mathcal{O}_{nbd}$-cover of H.

Since $(G, *)$ has the Hurewicz property and $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$, it has by Theorem 5 the property $\mathsf{S}_c(\mathcal{O}, \mathcal{O})$. Letting $(\mathcal{O}_{\mathsf{nbd}}(U_n) : n < \infty)$ be the sequence of open covers in (2) of Lemma 10, let $(\mathcal{V}_n : n < \infty)$ be the corresponding sequence provided by (2) of that lemma, and fix $n_1 < n_2 < \cdots < n_{k+1} < \cdots$ as there.

For each k define $W_k = \cap_{n_k \leq j < n_{k+1}} V_j$. Then consider the sequence $(\mathcal{O}_{\mathsf{nbd}}(W_k) : k < \infty)$ for H. Since $(H, *)$ has property $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \Lambda)$ choose for each k a pairwise disjoint refinement $\mathcal{R}_k$ of $\mathcal{O}_{\mathsf{nbd}}(W_k)$, consisting of open sets, such that each $h \in H$ is contained in infinitely many of the sets $\cup\mathcal{R}_k$. Notice that for each k, $\mathcal{R}_k$ is a disjoint refinement of each $\mathcal{O}_{\mathsf{nbd}}(V_j)$ for $n_k \leq j < n_{k+1}$.

For each j define $\mathcal{K}_j$ as follows: Find k with $n_k \leq j < n_{k+1}$ and put

$$\mathcal{K}_j = \{V \times R : V \in \mathcal{V}_j \text{ and } R \in \mathcal{R}_k\}.$$

Claim 1: $\mathcal{K}_j$ is a refinement of $\mathcal{O}_{\mathsf{nbd}}(U_j \times V_j)$:
Proof: Consider $V \times R \in \mathcal{K}_j$: Since $V \in \mathcal{V}_j$, choose a member A_j of $\mathcal{O}_{\mathsf{nbd}}(U_j)$ with $V \subset A_j$. Choose $g_j \in G$ with $A_j = g_j * U_j$. Next, since $R \in \mathcal{R}_k$, choose a $B_k \in \mathcal{O}_{\mathsf{nbd}}(W_k)$ with $R \subseteq B_k$. Choose $h_j \in H$ with $B_k = h_j * W_k$. Then in particular we have $R \subseteq B_k \subseteq h_j * V_j$. But this implies that $V \times R \subset (g_j, h_j) * (U_j \times V_j)$, an element of $\mathcal{O}_{\mathsf{nbd}}(U_j \times V_j)$.

Claim 2: $\mathcal{K}_j$ is a disjoint family of open sets:
Proof: This is clear.

Claim 3: $\cup_{j<\infty}\mathcal{K}_j$ is a cover of $G \times H$:
Proof: To see this, consider $(g,h) \in G \times H$. Choose N so large that for each $k \geq N$ we have $g \in \cup(\cup_{n_k \leq j < n_{k+1}}\mathcal{V}_j)$. Then choose a $k > N$ with $h \in \cup\mathcal{R}_k$. It follows that for a j with $n_k \leq j < n_{k+1}$ we have (g,h) in $\cup\mathcal{K}_j$.
This completes the proof. $\diamondsuit$

Theorem 12. *Let $(G,*)$ be a non-discrete metrizable topological group*[3]*. Then $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ is equivalent to $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \Lambda)$.*

Proof: Let $(\mathcal{O}(U_n) : n < \infty)$ be a sequence in $\mathcal{O}_{\mathsf{nbd}}(G)$. Choose a sequence $\epsilon_n : n < \infty)$ such that $\epsilon_i > \epsilon_{i+1}$ for all $i < \infty$ and $diam_d(U_1 \cap U_2 \cap \cdots \cap U_n) > \epsilon_n$ for all n. Define $(\mathcal{O}(V_n) : n < \infty)$ such that $diam_d(V_i) = \epsilon_i$ for $i = 1, 2, \cdots, n$. Write $\mathbb{N} = \bigcup_{m<\infty} I_m$ where each I_m is infinite, and for $m \neq k$, $I_m \cap I_k = \emptyset$. Apply $S_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ to the sequence $(\mathcal{O}(V_n) : n \in I_m)$ for all m. Let T_n be a pairwise disjoint family refining $\mathcal{O}(V_n)$, $n \in I_m$ such that $\cup\{T_n : n \in I_m\}$ covers G. We will show that $\cup\{T_n : n \in \mathbb{N}\}$ is a large cover. Take an element $x \in G$ and pick $m_1 \in I_1$ with $x \in \cup\ T_{m_1}$. Next, pick $W_1 \in T_{m_1}$ with $x \in W_1$ and N_1 so large that for all $n \geq N_1$ we have $\epsilon_n < diam_d(W_1)$. Then pick i_2 so large that the smallest element of I_{i_2} is larger than N_1. Now choose $m_2 \in I_{i_2}$ with $x \in \bigcup T_{m_2}$. Pick $W_2 \in T_{m_2}$ with $x \in W_2$. Since $m_2 \geq N_1$, $\epsilon_{m_2} < diam_d(W_1)$, and by definition of $\mathcal{O}(V_{m_2})$, $diam_d(W_2) \leq diam_d(V_{m_2}) \leq \epsilon_{m_2} < diam_d(W_1)$. Next pick N_2 so large that for all $n \geq N_2$ we have $\epsilon_n < diam_d(W_2)$ and continue the same way as we did with N_1. Continuing like this we find $W_1, W_2, W_3, \cdots$ infinitely many distinct elements of $\cup\{T_n : n < \infty\}$ covering x.$\diamondsuit$

Note in particular that if for each n $\mathcal{V}_n$ is a disjoint family of open sets, and if $\cup_{n<\infty}\mathcal{V}_n$ is a large cover of G, then for each $g \in G$ the set $\{n : g \in \cup\mathcal{V}_n\}$ is infinite. This is because for each n there is at most one set in $\mathcal{V}_n$ that might contain g.

Corollary 13. *Let $(G,*)$ be a group which has property $\mathsf{S}_c(\mathcal{O}_{nbd}, \mathcal{O})$ as well as the Hurewicz property. Then for any metrizable topological group $(H,*)$ satisfying $\mathsf{S}_c(\mathcal{O}_{nbd}, \mathcal{O})$, $G \times H$ also satisfies $\mathsf{S}_c(\mathcal{O}_{nbd}, \mathcal{O})$.*

Proof: Use Theorems 11 and 12. $\diamondsuit$

Corollary 14. *Let $(G,*)$ be a metrizable group which has property $\mathsf{S}_c(\mathcal{O}_{nbd}, \mathcal{O})$ as well as the Hurewicz property. Then all finite powers of $(G,*)$ have the property $\mathsf{S}_c(\mathcal{O}_{nbd}, \mathcal{O})$.*

Proof: Use Corollary 13. $\diamondsuit$

It is not clear that the full Hurewicz property is needed in Theorem 11 or Corollaries 13 and 14: maybe Hurewicz-boundedness is enough.

Problem 4. *In Theorem 11, can we replace the requirement that G has the Hurewicz property with the weaker requirement that $(G,*)$ has the property $\mathsf{S}_1(\Omega_{\mathsf{nbd}}, \mathcal{O}^{gp})$?*

[3]In this context, non-discrete is equivalent to having no isolated points.

In light of results of E. and R. Pol - [19] - we conjecture that neither Menger boundedness, nor the Menger property is enough to obtain Theorem 11:

Conjecture 5. *There is a metrizable Menger bounded group* $(G, *)$ *with property* $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$*, such that* G^2 *is Menger bounded but does not have* $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$*.*

Conjecture 6. *There is a metrizable group* $(G, *)$ *which has the property* $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$*, and* G^2 *has the Menger property but does not have* $\mathsf{S}_c(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$*.*

4. Games

The following game, denoted $\mathsf{G}_c(\mathcal{A}, \mathcal{B})$, is naturally associated with $\mathsf{S}_c(\mathcal{A}, \mathcal{B})$: Players ONE and TWO play as follows: They play an inning for each natural number n. In the n-th inning ONE first chooses $\mathcal{O}_n$, a member of $\mathcal{A}$, and then TWO responds with $\mathcal{T}_n$ refining $\mathcal{O}_n$. A play $(\mathcal{O}_1, \mathcal{T}_1, \cdots, \mathcal{O}_n, \mathcal{T}_n, \cdots)$ is won by TWO if $\cup_{n<\infty}\mathcal{T}_n$ is a member of $\mathcal{B}$; else, ONE wins. Versions of different length of this game can also be considered: For an ordinal number α let $\mathsf{G}_c^{\alpha}(\mathcal{A}, \mathcal{B})$ be the game played as follows: in the β-th inning ($\beta < \alpha$) ONE first chooses $\mathcal{O}_\beta$, a member of $\mathcal{A}$, and then TWO responds with a pairwise disjoint $\mathcal{T}_\beta$ which refines $\mathcal{O}_\beta$. A play

$$\mathcal{O}_0, \mathcal{T}_0, \cdots, \mathcal{O}_\beta, \mathcal{T}_\beta, \cdots \ \beta < \alpha$$

is won by TWO if $\cup_{\beta<\alpha}\mathcal{T}_\beta$ is a member of $\mathcal{B}$; else, ONE wins. Thus the game $\mathsf{G}_c(\mathcal{A}, \mathcal{B})$ is $\mathsf{G}_c^{\omega}(\mathcal{A}, \mathcal{B})$.

Theorem 15. *Let* $(G, *)$ *be a metrizable group. Then the following statements hold:*

1. *If* $dim(G) \leq n$ *then TWO has a winning strategy in* $\mathsf{G}_c^{n+1}(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$*.*
2. *If TWO has a winning strategy in* $\mathsf{G}_c^{n+1}(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$*, then the* $dim(G) \leq n$*.*
3. *If* G *is countable dimensional, then TWO has a winning strategy in* $\mathsf{G}_c^{\omega}(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$*.*
4. *If TWO has a winning strategy in* $\mathsf{G}_c^{\omega}(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$*, then* G *is countable dimensional.*

Proof: We prove 3 and 4. The proofs of 1 and 2 are similar.

Proof of 3**:**

Suppose that G is countable dimensional. We define the following strategy for TWO: Write $G = \cup_{n<\infty}G_n$ where each G_n is zero-dimensional. Let $\mathcal{U}$ be an element of $\mathcal{O}_{\mathsf{nbd}}$. For $\mathcal{U} = \mathcal{O}(U)$ of G and $n < \infty$, consider $\mathcal{U}$ as a cover of G_n. Since G_n is zero-dimensional, find a pairwise disjoint family $\mathcal{V}_n$ of subsets of G_n open in G_n such that $\mathcal{V}_n$ covers G_n and refines $\mathcal{O}(U)$. Choose a pairwise disjoint family $\sigma(\mathcal{U}, n)$ refining $\mathcal{O}(U)$ such that each element V of $\mathcal{V}_n$ is of the form $U \cap G_n$ for some $U \in \sigma(\mathcal{U}, n)$. Now TWO plays as follows: In inning 1 ONE plays $\mathcal{U}_1$, and TWO responds with $\sigma(\mathcal{U}_1, 1)$, thus covering G_1. When ONE has played $\mathcal{U}_2$ in the second inning TWO responds with

$\sigma(\mathcal{U}_2, 2)$, thus covering G_2, and so on. And in the n-th inning, when ONE has chosen the cover $\mathcal{U}_n$ of G TWO responds with $\sigma(\mathcal{U}_n, n)$, covering G_n. This strategy evidently is a winning strategy for TWO.

Proof of 4: Let σ be a winning strategy for TWO. Choose a neighborhood basis $(U_n : n < \infty)$ of the identity element e of G so that $diam_d(U) < \frac{1}{n}$ for all n. Consider the plays of the game in which in each inning ONE chooses for some n a cover $\mathcal{U}_n$ of G of the form $\mathcal{O}(U_n)$.

Define a family $(C_\tau : \tau \in {}^{<\omega}\mathbb{N})$ of subsets of G as follows:

(1) $C_\emptyset = \cap\{\cup\sigma(\mathcal{U}_n) : n < \infty\}$;

(2) For $\tau = (n_1, \cdots, n_k)$, $C_\tau = \cap\{\cup\sigma(\mathcal{U}_{n_1}, \cdots, \mathcal{U}_{n_k}, \mathcal{U}_n) : n < \infty\}$

Claim 1: $G = \cup\{C_\tau : \tau \in {}^{<\omega}\mathbb{N}\}$:

For suppose on the contrary that $x \notin \cup\{C_\tau : \tau \in {}^{<\omega}\mathbb{N}\}$. Choose an n_1 such that $x \notin \sigma(\mathcal{U}_{n_1})$. With $n_1, \cdots, n_k$ chosen such that $x \notin \sigma(\mathcal{U}_{n_1}, \cdots, \mathcal{U}_{n_k})$, choose an n_{k+1} such that $x \notin \sigma(\mathcal{U}_{n_1}, \cdots, \mathcal{U}_{n_{k+1}})$, and so on. Then

$$\mathcal{U}_{n_1}, \sigma(\mathcal{U}_{n_1}), \mathcal{U}_{n_2}, \sigma(\mathcal{U}_{n_1}, \mathcal{U}_{n_2}), \cdots$$

is a σ-play lost by TWO, contradicting the fact that σ is a winning strategy for TWO.

Claim 2: Each C_τ is zero-dimensional.

For consider an $x \in C_\tau$. Say $\tau = (n_1, \cdots, n_k)$. Thus, x is a member of $\cap\{\cup\sigma(\mathcal{B}_{n_1}, \cdots, \mathcal{U}_{n_k}, \mathcal{U}_n) : n < \infty\}$. For each n choose a neighborhood $V_n(x) \in \sigma(\mathcal{U}_{n_1}, \cdots, \mathcal{U}_{n_k}, \mathcal{U}_n)$. Since for each n we have $diam_d(V_n(x)) < \frac{1}{n}$, the set $\{V_n(x) \cap C_\tau : n < \infty\}$ is a neighborhood basis for x in C_τ. Observe also that each $V_n(x)$ is also closed in C_τ because: The set $V = \cup\sigma(\mathcal{U}_{n_1}, \cdots, \mathcal{U}_{n_k}, \mathcal{U}_n) \setminus V_n(x)$ is open in G and so $C_\tau \setminus V_n(x) = C_\tau \cap V$ is open in C_τ. Thus each element of C_τ has a basis consisting of clopen sets. Also note that for each n, C_τ is a disjoint union of clopen sets each of diameter $\leq \frac{1}{n}$. $\diamondsuit$

5. Remarks and acknowledgment.

Regarding Theorem 2: For a left invariant metric d let $\mathcal{U}_d$ be a the family of sets U_ϵ, $\epsilon > 0$ where we define $U_\epsilon = \{(x, y) \in G \times G : d(x, y) < \epsilon\}$. The family $\mathcal{U}_d$ generates the left-uniformity of the topological group G. Refer to [9] Chapters III §3 and IX §3 and [10] Chapter 8.1 regarding these facts. Let $\mathcal{O}_d$ denote the collection of open covers of the form $\{U_\epsilon(x) : x \in G\}$ where $U_\epsilon(x) = \{y : (x, y) \in U_\epsilon\}$. The referee pointed out that a third equivalence can be added in Theorem 2:

(3) For each left-invariant metric d, $S_c(\mathcal{O}_d, \mathcal{O})$ holds.

If additionally it is assumed that G has the Hurewicz property, then yet another equivalence can be added (see [3], Theorem 5):

(4) For some left-invariant metric d, $S_c(\mathcal{O}_d, \mathcal{O})$ holds.

It is not clear that (3) and (4) are equivalent for all metrizable groups. In light of the example of E. Pol and R. Pol in connection with Conjecture 3, it seems likely that (3) and (4) are not equivalent.

However, note that in Corollary 13, if we assume that the one metrizable group has the Hurewicz property, and if we assume each of the two metrizable groups has the Haver property in all equivalent left invariant metrics, then the product group also has the Haver property in all equivalent left invariant metrics, even though this product need not have the Hurewicz property. For example, let $(\mathbb{R},+)$ be the one group, and let $(\mathbb{Z}^{\mathbb{N}},+)$ be the other group. Each is metrizable, $(\mathbb{R},+)$ has the Hurewicz property and $(\mathbb{Z}^{\mathbb{N}},+)$ does not have the Hurewicz property. But $\mathbb{R}\times\mathbb{Z}^{\mathbb{N}}$ is finite dimensional and so has $\mathsf{S}_c(\mathcal{O},\mathcal{O})$. Thus by Theorem 1 this product has the Haver property in all equivalent metrics but does not have the Hurewicz property.

Regarding Theorem 5: There is a more general theorem. Let $\mathcal{U}$ be uniformity on X generating the topology $\tau_{\mathcal{U}}$. For $V\subset X\times X$, define $V(x)=\{y\in X:(x,y)\in V\}$. We say that an open cover of $(X,\tau_{\mathcal{U}})$ is uniform with respect to $\mathcal{U}$ if it is of the form $\{V(x):x\in X\}$, for some $V\in\mathcal{U}$. Define $\mathcal{O}_{\mathcal{U}}=\{\{V(x):x\in X\}:V\in\mathcal{U}\}$.

Theorem 16. *Let $\mathcal{U}$ be a uniformity generating the topology $\tau_{\mathcal{U}}$ on the set G. Assume that the topological space $(G,\tau_{\mathcal{U}})$ has the Hurewicz property. Then $S_c(\mathcal{O}_{\mathcal{U}},\mathcal{O})$ is equivalent to $S_c(\mathcal{O},\mathcal{O})$.*

The proof of this theorem is very similar to the proof of Theorem 5.

I thank the referee for the useful remarks and E. Pol and R. Pol for communicating their result on Conjecture 3 to me.

References

[1] D.F. Addis and J.H. Gresham, *A class of infinite dimensional spaces. Part I: Dimension Theory and Alexandroff's problem*, **Fundamenta Mathematicae** 101:3 (1978), 195 - 205.
[2] L. Babinkostova, *Selective versions of screenability*, **Filomat** 17:2 (2003), 127 - 134.
[3] L. Babinkostova, *When does the Haver property imply selective screenability?* **Topology and its Applications** 154 (2007), 1971 - 1979
[4] L. Babinkostova, *Metrizable groups and strict o-boundedness*, **Mathematichki Vesnik** 58:3-4 (2006), 131 - 138.
[5] L. Babinkostova, Lj.D.R. Kočinac and M. Scheepers, *Combinatorics of open covers (XI): Topological groups*, **Topology and its Applications** 154:7 (2007), 1269 - 1280.
[6] L. Babinkostova and M. Scheepers, *Selection principles and countable dimension*, **Note di Matematica**, to appear.
[7] L. Babinkostova and M. Scheepers, *Products and selection principles*, **Topology Proceedings**, to appear.
[8] P. Borst, *Some remarks on C spaces*, **Topology and its Applications** 154 (2007), 665 - 674.
[9] N. Bourbaki, *General Topology*, **Addison-Wesley Publishing Company**, (1966).
[10] R. Engelking, *General Topology*, **Polish Scientific Publishers**,(1977).
[11] Y. Hattori and K. Yamada, *Closed pre-images of C-spaces*, **Mathematica Japonica** 34 (1989), 555 - 561.
[12] W.E. Haver, *A covering property for metric spaces*, **Springer Lecture Notes in Mathematics** 375 (1974), 108 - 113.
[13] W. Hurewicz, *Über eine Verallgemeinerung des Borelschen Theorems*, **Mathematische Zeitschrift** 24 (1925), 401 – 421.

[14] E. Hewitt and K.A. Ross, *Abstract Harmonic Analysis I*, **Springer-Verlag**, 1963.

[15] Lj.D.R. Kočinac and M. Scheepers, *Combinatorics of open covers (VII): groupability*, **Fundamenta Mathematicae**.

[16] M. Machura and B. Tsaban, *The combinatorics of the Baer-Specker group*, **Israel Journal of Mathematics** (to appear)

[17] E. Pol, *A weakly infinite dimensional space whose product with the irrationals is strongly infinite dimensional*, **Proceedings of the American Mathematical Society** 98:2 (1986), 349 - 352.

[18] E. Pol, *Spaces whose n-th power is weakly infinite dimensional, but whose $n+1$-th power is not*, **Proceedings of the American Mathematical Society** 117:3 (1993), 871 - 876.

[19] E. Pol and R. Pol, *On metric spaces with the Haver property which are Menger spaces*, preprint.

[20] E. Pol and R. Pol, *A metric space with the Haver property whose square fails this property*, preprint.

[21] R. Pol, *A weakly infinite dimensional compactum which is not countable-dimensional*, **Proceedings of the American Mathematical Society** 82:4 (1981), 634 - 636.

[22] D.M. Rohm, *Products of infinite-dimensional spaces*, **Proceedings of the American Mathematical Society** 108:4 (1990), 1019- 1023.

[23] B. Tsaban, *o-Bounded groups and other topological groups with strong combinatorial properties*, **Proceedings of the American Mathematical Society**,134 (2006), 881-891.

Contact Information:

Liljana Babinkostova
Department of Mathematics
Boise State University
Boise, ID 83725

e-mail: liljanab@diamond.boisestate.edu
fax: 208 - 426 - 1356
phone: 208 - 426 - 2896