COMMUNICATING MULTIPLICATIVE REASONING THROUGH SEMIOTIC RESOURCES

by

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The following individuals read and discussed the dissertation submitted by student Emilie N. Eisenberger, and they evaluated the student's presentation and response to questions during the final oral examination. They found that the student passed the final oral examination.

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DEDICATION

I would like to dedicate my dissertation to my amazing support system! To my husband, Curtis. From the minute I said I wanted to do this you have supported me every step of the way. Even when I doubted I could handle it all, you gave me unwavering support and encouragement to keep going. To my children Elise and Max. This was not easy. I would have much rather been with you guys than in “the cave.” Thank you for your understanding and the “are you done yets” which pushed me to finish. I hope you see that you can accomplish anything you put your heart and mind to. To my brother and mom, the three of us have persevered together for a long time. I know Gram is proud of where we all are now. And to all my family, particularly the ones farther away. You have been my models for unconditional love and support and have shaped me into the person I am today. We work hard and play hard, and no one does love and celebrating like we do.

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ABSTRACT

The importance of fostering in students the requisite language to understand what is being communicated and how to communicate their understanding requires educators to conceptualize themselves as teachers of language and content. It is possible to engage in activities of the mathematics classroom and through that participation engage in language practices and mathematical practices simultaneously. The purpose of this study was to explore the use of semiotic resources, and modality, with a student-generated tool on students' communication of multiplicative reasoning.

The study design was a qualitative case study that included a single third-grade class with an in-depth look at six students of varying knowledge levels. Two students, one male and one female, were randomly selected from Beyond, On, and Approaching levels. Discourse analysis served dual purposes for the data collected: first, it explored a socially constructed multi-modal tool utilized as an activity to enhance language use individually and interactively during mathematical discourse; second, it supported investigating the language used by participants during the studied activities and how they relate to Communication About and Communication In multiplication.

The findings support the utilization of semiotic resources, inclusive of visual representations, signs, symbolic notations, and receptive and expressive language elements as fundamental to the learning and communication we are asking of our students. Through the interplay of semiotic resources, a multimodal student-generated

tool can support students in summarizing their learning, individually and interactively, enhancing their means of communicating discursively in mathematics.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

LIST OF ABBREVIATIONS

ALM	Academic Literacy in Mathematics
BICS	Basic Interpersonal Communication Skills
CALP	Cognitive Academic Language Proficiency
CCSS	Common Core State Standards
DMTI	Developing Mathematical Thinking Institute
ELL	English Language Learner
NCTM	National Council of Teachers of Mathematics
NGA	National Governors Association
SOLO	Structure of Observed Learning Outcomes
UNESCO	United Nations Educational and Scientific Organization
ZDM	Zentralblatt für Didaktik der Mathematik

CHAPTER ONE: INTRODUCTION

Background of the Study and Current Problem

Gone are the days of the silent mathematics classroom where computation reigned as the means to convey understanding. Communicating is a part of learning, and we have typically not considered how students do that in a mathematics classroom. Once classified as the subject of universal language, conveying understanding has increased the language demands and become an integral part of the mathematics classroom. Beyond the universal language of number, today's mathematics classroom necessitates the construction of a solid foundation of knowledge to portray both conceptual and procedural understanding through language and communicative discourse. This calls for creating a language-rich environment that allows students to engage in content area reading about mathematics with the curriculum supports needed to communicate their experiences. Without integrating language into the teaching and practice in daily mathematics lessons we leave students without the requisite skills to thrive in the mathematics classroom.

Language and thought are effectively bound as the former drives the development of cognitive skills one needs to communicate and participate in mathematics activities and on high-stakes assessments. As evidenced by the National Assessment of Educational Progress [NAEP], only 41% of the nation's fourth graders performed at or above proficiency levels in mathematics increasing 1% from the percentage of proficiency in 2017 (U.S. Department of Education, 2019). Idaho students' fourth-grade proficiency level is at 43%. Eighth graders in the United States have a proficiency level of 34%, indicating that our nation's scores are not improving as students get older (U.S Department of Education, 2019). According to the National Center for Education Statistics [NCES], disparities are amplified in grade levels amongst students of varying ethnicities and multi-lingual and socio-economic statuses (U.S. Department of Education, 2019). Language impacts both mono- and multi-lingual learners, and their mathematical scores and language proficiency can inhibit not only student understanding of the task but how students demonstrate knowledge (Erath et al., 2018; Prediger et al., 2015). These impacts could be apparent in their communicative participation and through content reading obstacles in classroom text and on assessments (Bailey et al., 2015; Prediger et al., 2015; Nagy et al., 2012).

Communication plays an essential role in the habits we attempt to cultivate in the mathematics classroom. An essential challenge of the discipline is to assist students in shifting from "every day, informal ways of constructing knowledge to the technical and academic ways that are necessary for disciplinary learning in all subjects" (Schleppegrell, 2007, p. 140). Communicating in the everyday, or social language, and more technical schooling language, has been described by Cummins (1979) as Basic Interpersonal Communication Skills (BICS) and Cognitive Academic Language Proficiency (CALP). Students come to school established in the language of their homes and communities and not the specialized academic knowledge and vocabulary associated with the content domains, like mathematics. Social language outpaces the development of academic language; however, the use of both conceptualizes classroom discourse across curriculum content. Nagy et al. (2012) synthesize academic language and vocabulary research to define the academic language register as the "specialized language, both oral and written of academic settings that facilitates communication and thinking about disciplinary content" (p. 92). This has become of extreme importance to the mathematics classroom as the increased demands of classroom discourse and written response become part of the standards (see *Common Core State Standards [CCSS]*, National Governor's Association [NGA], 2010), and practices in mathematics (National Council of Teachers of Mathematics, [NCTM], 2000). Snow and Uccelli (2010), contend that "academic language is intrinsically more difficult than other language registers and that thinking about the educational experiences that promote its development is a crucial task for educators of all students" (p. 114). The importance of fostering in students the requisite language to understand what is being communicated and how to communicate their understanding requires educators to conceptualize themselves as teachers of language and content. Development of this nature is not intended to be unilateral from the educator to the student but co-constructed in the classroom community.

Research Questions

This study investigated third grade students' communication of multiplicative reasoning through social semiotic resources. A considerable amount of research has been compiled on the connection between language and mathematics (see Erath et al., 2021, p. 966). Earlier research focused on correlating reading ability to mathematical abilities, analyzing text and problem-solving comprehension which separates language and mathematics (Aiken, 1972; Monroe & Engelhart, 1931). In the last twenty years there has been a shift towards the interconnectedness of mathematics and language and how we engage them simultaneously when learning content (Halliday, 1978, 1993; Moschkovich, 2015a; Moschkovich & Zahner, 2018; O'Halloran, 1998, 2010, 2015; Schleppegrell, 2007). Recent recommendations have iterated foci for future research: language development in mathematics for mono- and multi-lingual learners, research regarding more specific views of language and how they relate to content strands within mathematics, and the content of the communicative discourse (Erath et al., 2021; Moschkovich, 2018; Planas & Schütte, 2018). For the purpose of this study, I will utilize a social semiotic framework for generating communication about and in mathematics. Through this lens, mathematics content and language development are actualized within a "broader sense of literacy as participation in practices and discourses. These discursive practices involve multiple aspects of mathematical proficiency, multiple symbol systems (written text, numbers, graphs, tables, etc.), and multiple modes of communication (oral, written, receptive, productive)" (Moschkovich, 2015a, p. 45). The development of mathematical literacy can be viewed through a multitude of resources and socially situated meanings within a community as students interact and communicate in and about their mathematical knowledge.

The studies purpose explored the use of semiotic resources, and modality, with a student-generated tool on students' communication of multiplicative reasoning. The research questions are as follows:

How do semiotic resources and student-generated tools enhance students' abilities to communicate their multiplicative reasoning?

1. How does the utilization of the student-generated tool assist students in communicating mathematically?

2. What semiotic resources are evident in student activity and how are students utilizing them?
3. How does the student-generated tool influence the expression of multiplicative understanding?

Significance of the Study

The study's central question, *How do semiotic resources and student-generated tools enhance students' abilities to communicate their multiplicative reasoning?* benefits prior and future research. First, the findings of this study can contribute to the ongoing discussion of language and mathematics as interwoven content strands. Second, by exploring the use of a student-generated multi-modal tool implications for integrating and acknowledging semiotic resources as a means of representing thought and generating communication could be instituted into classroom practice by educators and students. And finally, initiating discourse practices is an important aspect of classroom teaching and learning; however, analyzing the kinds and depth of communication will provide educators a means of evaluating the meaningfulness of the discourse opportunities which will contribute to research recommendations in language and mathematics.

Organization of the Study

The study is organized into five chapters. Chapter one is comprised of the introduction to the research questions and an overview of the study. Chapter two investigates the conceptual framework of language and mathematics and the connection of language to expressing knowledge within the content of multiplication. The theoretical framework, also addressed in chapter two, explores the connection of semiotic resources, or modality, and the social construction and communication of resources students utilize to participate discursively. In chapter three, a description of the qualitative case study and analysis methods for examining the data will be explained. Chapter four presents the findings from the analysis within a communication framework for an in-depth look at the content of the Discourse and semiotic resources evidenced in communication. The fifth and final chapter, discusses the findings from chapter four in connection with the conceptual and theoretical frameworks and research questions of this study. Recommendations for future research will be addressed in this chapter.

CHAPTER TWO: LITERATURE REVIEW

Communication

Communication can be distinguished by its receptive (listening and reading) and expressive (speaking and writing) language qualities. The language components: reading, writing, listening, and speaking, offer the opportunity to share, display, and consider the reasonings and justifications for the mathematics students are conceptualizing. However, not all students successfully recognize how to interpret mathematics concepts and communicate mathematical thought processes in the classroom (Barwell, 2005; Moschkovich, 2002; Schlepegrell, 2007). Expressing mathematical ideas through language has become an integral component of the transition towards a more communicatively oriented mathematics community.

Mathematical Register

Considerable research has been devoted to investigating the relationship between language and mathematics. In Aiken's (1972) review of literature, he summarized over forty years of research on language factors and their effects on mathematics, including a previous review of literature by Monroe and Engelhart (1931) identifying reading ability and its relationship to mathematics ability. Language factors addressed in recommendations for future research emphasized vocabulary development and textbook analysis, verbal and non-verbal problem-solving schema development, and discovery and exposition learning. Aiken's (1972) final recommendation for future research acknowledged the introduction of a linguistic lens to analyze mathematics learning in a deeper syntactic and grammatical manner.

The United Nations Educational and Scientific Organization (UNESCO, 1974) symposium amassed linguists and mathematicians with an objective "to contribute to the systematization within the field of those difficulties in mathematics education which pertain to linguistics, and further to analyze these difficulties and their mutual relationships" (p. 8). Comparable to other school subjects such as social studies, science and language arts, mathematics has its own subject-specific language often referred to as the "mathematics register" (Halliday, 1974). Halliday (1974) defines a mathematics register

> as a set of meanings that belong to the language of mathematics (the mathematical use of natural language) and that a language must express if it is used for mathematical purposes. We should not think of a mathematical register as constituting solely terminology, or of the development of a register as simply a process of adding new words (p. 65).

The inclusion of specialized and general academic words from Halliday's (1974) initial formulation implicitly acknowledged that the language of mathematics includes the technical vocabulary to construct meaning but more so the "meanings, styles, and modes of argument" necessary for mathematics (Halliday, 1974, p. 65) which is emphasized by Pimm (1987). The mathematics register advances the conception of mathematics as more than a number and computation discipline, but a content rich in its own vocabulary, grammatical structures and means of reasoning that necessitate elaboration. Halliday (1993) also addressed the grammatical structures inherent in the composite of language and mathematics. One type of grammatical structure is the mathematics symbols utilized to convey meaning, through a grammatical symbolic notation. The grammar of mathematical symbolism, (O'Halloran, 1998, 2010, 2015), links the symbolic and visual representations allowing for the language of what is happening in symbolic notation to be written or verbalized formally. For example, symbols for operations that are delineated through specialized vocabulary and parenthesis indicate the sequence, or processes for performing the mathematical task.

When constructing the mathematics register, an overemphasis on vocabulary and precise grammatical phrases can constrain the resources educators teach with and restrict student engagement to formalized modes of expression. An emphasis on vocabulary has inevitably led to lists of mathematical words, deemed essential, for each domain and subconstruct of mathematics. The complexity of these words can be situated as unfamiliar words, unfamiliar phrases, and words with multiple meanings (Maher et al., 2018; Moschkovich, 2015a). The following components exemplify the mathematics register: a highly technical vocabulary, semi-technical terms, dense noun phrases, complex subordinated clauses, conjunctions with precise meanings, and implicit logical relationships (Abedi & Lord, 2001; Moschkovich, 2015a; Schleppegrell, 2001, 2007; Schütte, 2019; Wilkinson, 2019). In mathematical situations, the register includes "specialized vocabulary that is exclusively mathematical (binomial); everyday vocabulary that is re-purposed as specialized (table, product); and dense noun phrases that express specialized meaning (area under a curve)" (Wilkinson, 2018, p. 169). The individual terms are highly specialized to the discipline, which carries no meaning outside of mathematics; they are compounded by repurposed words with the technical meaning inside mathematics and in other disciplines; and phrases utilized as description of processes in the discipline (Sigley & Wilkinson, 2015; Wilkinson, 2018).

Vocabulary Acquisition

Numerous studies have addressed vocabulary acquisition in mathematics as something that must be taught explicitly; positing that mathematics learning, nor understanding, cannot take place without the acquisition of specialized vocabulary first (Kovarik, 2010; Monroe & Pendergrass, 1997; Monroe & Panchyshyn, 1995; Pierce & Fontaine, 2009; Riccomini et al., 2015). To teach vocabulary explicitly, Riccomini et al. (2015), recommend using purposeful word instruction and practice, with the opportunity to use words in context. Suggestions from the literature also include formulating cross-curriculum connections to literacy strategies for use in mathematics (Riccomini et al., 2015); and utilizing general and specialized graphic organizers in mathematics (Bruun et al., 2015; Dexter & Hughes, 2011; Monroe & Panchyshyn, 1995; Monroe & Pendergrass, 1997).

Scaffolding can be described as "different types of adult guidance, with different purposes, in multiple settings, and across various time scales" (Moschkovich, 2015b, p. 1067). Scaffolding transpires with the utilization of tools (Prediger & Hein, 2017), through revoicing (Enyedy et al., 2008; Moschkovich, 2002, 2015b), or classroom activity (Aineamani, 2019). "Revoicing," noted by Moschkovich (2002, 2015), allows for validation of the students' ideas while reframing with more formalized academic vocabulary. This allows for taking students' ideas seriously while pushing students conceptually (Brendefur et al., 2015; Sherin, 2002) to weave conversational and formal academic language registers. The development of an activity which engages students in reading, writing, listening, and speaking is an important way to formalize language and can be achieved by interacting with text (Alshwaikh & Morgan, 2018), tools that deepen activity (Morgan et al., 2014; Sfard, 2008), and collaboration (Francisco, 2013). Thus, engaging students in the interaction and communication necessary to be successful mathematicians. Specialized vocabulary plays a significant role as a feature of the mathematics register. However, the hierarchal placement of specialized vocabulary acquisition preceding meaning-making implies understanding cannot be conveyed without it. The divergence initiates a dilemma for educators regarding how and when they should initiate new vocabulary, and whether it should be completed explicitly or implicitly (Turner et al., 2019). Mathematical tasks can be dense with the technical language and semantic features mentioned above. This imposes challenges to students as they create and convey meaning through more than words, phrases and sentences, to enhance their language output to the meaning, modes and arguments necessary to communicate mathematically.

Mathematical Literacy

Consequently, to meet the standards in mathematics, communicating mathematical understanding has become an integral component of the teaching and learning in the classroom. Students routinely engage in content literacy development in mathematics as they participate in classroom conversations about their ideas or the ideas of others and generate explanations or justifications in written form for their work. Literacy in mathematics is more than reading and interpreting word problems to demonstrate knowledge but the broadening of minimal reading and writing to engage in deeper expressions of literacy in the mathematics classroom. The Academic Literacy in

Mathematics framework (ALM) (Moschkovich, 2015a, Moschkovich & Zahner, 2018) proposes three integrated components for constructing mathematical literacy: mathematical proficiency, mathematical practices, and mathematical discourse. This integrates the cognitive (reasoning, metacognitive) and the socio-cultural by emphasizing the development of language and mathematics simultaneously. Through this integrated approach to instruction, Moschkovich (2015a) states the development of academic literacy,

> Allows students to use multiple modes of communication, symbol systems, registers, and languages as resources for mathematical reasoning, and supports students in negotiating situated meanings for mathematical language that is grounded in mathematical activity (p 45).

Language is important to meaning making but is so enmeshed in the learning and understanding of mathematics as to be not hierarchal but inclusive in every aspect. In an ALM framework, modalities, and semiotic resources, assist in navigating the mathematics practices for explaining, reasoning, justifying, and generating claims. Situated activities within the mathematics community support sense-making, whether informally or formally and with the requisite tools through modalities for participation (Moschkovich, 2015a; Moschkovich & Zahner, 2018). Student communication can then encapsulate the conceptual and procedural understandings of their mathematics classroom and produce that knowledge in a multitude of ways.

Semiotics and Modality

Further research in the language and mathematics domain considers the mathematics discipline's linguistic implications, through a functional linguistic and semiotic lens (Halliday, 1978, 1993; O'Halloran, 2015; Schleppegrell, 2007). The extension of functional linguistics has shifted the focus to a more semiotic system of language, as mentioned above, inclusive of models and visuals to analyze and extend our understanding of the mathematics register (O'Halloran, 1998, 2010, 2015; Schleppegrell, 2007). Halliday (1993) defines the semiotic process as "the distinctive characteristics of human learning" and is the process of making meaning (p. 93). Language becomes the means by which we interpret meaning individually and within social situations. Halliday (1993) suggests the essentiality of experience through language, not as a separate entity to the mathematical situations we research. The impetus for this shift is to demonstrate the relationship amongst mathematics, language, and visual representations to "extend the typological resources of natural language to enable it to connect to the more topological meanings made with visual representations" (Lemke, 2003, p. 1). In mathematics, for example, texts often include unfamiliar vocabulary and complex sentence structures and are also often multimodal, incorporating diagrams, tables, graphs, images, and mathematical expressions. In defining the multimodal framework, O'Halloran (2015) states

> The multimodal (or multi-semiotic) makeup of mathematics means three different meaning potentials are accessed to construct mathematical reality: namely, linguistic, symbolic, and visual forms of representation, each of which have developed specific grammatical features to fulfill the functions they are required to serve. That is, language is used to reason about the mathematical results in a discourse of argumentation in which mathematical processes are related to each other and interpreted (p. 71).

Multimodality and its influences on communication were theorized by Kress, et al. (2001) in his research regarding modality in the science classroom. In studying the evolving theories of semiotics and modality Kress et al. (2001) noted, “our sense of the interrelations of the modes became clearer, as did our general understanding of human semiosis - how humans make meanings, represent and respond to these meanings, and rework the meanings of others” (p. 9). This perspective offers a transitional focus that facilitates connections between the visual and linguistic aspects of mathematics and allows students to interpret the mathematical meaning and express aspects of mathematical thought through varying forms of language.

Studies utilizing a semiotic framework have focused on analyzing the mathematical and everyday registers to highlight how language and mathematics integrate through modality. This progresses beyond distinct integrations of reading, writing, listening, and speaking as a means for mathematical output but as a construct for linguistic communication. This enlists the expressive and receptive qualities of language along with visual and non-linguistic forms of communication, such as gestures (Chen & Herbst, 2013; Fernandes et al., 2017; Shein, 2012), tools (Brinkmann, 2003; Haneda, 2014; Kolloffel et al., 2011; Prediger and Hein, 2017), and manipulatives, which are utilized to express informal and academic language for sense-making. Shein (2012) examined the role of gestures and discourse on fifth-grade English Language Learner (ELL) students’ participation in error analysis for finding the area of geometric shapes. As students interacted, the classroom educator employed gestures to engage students in the mathematics discussion and facilitated students’ reasoning using representations and gestures to explain their models. Students participated in modalities to make meaning, and convey meaning, through spoken and gestural discourse. Chen and Herbst (2013) recognized the importance of modalities as students practice geometric conjectures with diagrams, gestures, and verbal discourse. The study reports that diagrams were an integral method for assisting students in making conjectures and are an important tool, along with gestures, in assisting students to reason and interact mathematically within the classroom. Adapting Toulmin's argumentation tool, Prediger and Hein (2017), analyzed how scaffolding can support students in developing multi-step argumentation in grades nine and ten on theorems of angles. The tool used for scaffolding, *materialized argumentation structure,* combines a graphic aid to sequence ideas for argumentation. Graphic aids and written explanation were utilized as part of the tool. However, expressing their mathematical argument aloud remained a challenge in part due to word and noun phrases necessary to connect and portray argument and evidence.

Connecting the visual and linguistic facets of mathematics allows students to express and interpret mathematical meaning, not only through expressive and receptive language but also through varying forms of modality, generating a resource-rich toolkit in which students can engage to communicate mathematically. Brenner (1994, 1998) classified mathematics classroom communication into three categories: Communication about mathematics – reflecting, describing, and reasoning about their processes and the processes of others; Communication in mathematics – register and representations; Communication with mathematics – use of mathematics as a tool to explore real world contexts. Language and modalities are embedded in every aspect of the framework and support a focus on communicating themes in mathematics. Brenner’s framework encompasses the communicative aspects of language expression and mathematical knowledge as interconnected constructs. This engages the learner in selecting the modality that best conveys their understanding for themselves and to their classmates. The nature of establishing a mathematics community allows students and educators to be equal partners in the generating and sharing of understanding rather than work in isolation. Cobb & Bowers (1999) attend to the engagement of all students in the activity by looking at ways individual students are reasoning and participating. This will be important to research as we use a multimodal tool to engage students across diverse classroom structures.

Communication Summary

Language development in mathematics has advanced from a localized mathematics register of specified vocabulary knowledge and contextual problem comprehension (Aiken, 1972; Halliday, 1974) to a more inclusive emphasis of language and mathematics development. The earlier scope of language development in mathematics emphasized the mode of reading and speaking in the formal mathematics register. The semiotic frameworks (O'Halloran, 2010, 2015; Schleppegrell, 2007), have focused on utilizing the mathematical and everyday registers to emphasize how language and mathematics integrate and express learning through modalities. With focused strategies and modalities, educators can enhance student communication. It becomes increasingly important that academic language cease being perceived as developing content-specific vocabulary but the expanding of students' registers to communicate in school settings and beyond them. According to Moschkovich (2013), instruction for multilingual students should go beyond developing a set vocabulary for a mathematics skill and instead should "provide opportunities for students to actively use mathematical language to communicate about and negotiate meaning for mathematical situations" (p. 19). Although stated for multilingual learners, it is an important contention for all students in today's mathematics classroom. These opportunities will assist students in bridging the everyday and technical vocabulary if used interchangeably during mathematics discourse. This progresses beyond the distinct utilization of reading, writing, listening, and speaking as a means for mathematical output, but as a construct for linguistic communication, which enlists those expressive and receptive modes along with representations, visuals, and non-linguistic forms of communication, such as gestures, for sense-making. Educators engaged in enriching language support students in navigating mathematics at a deeper level through the strategies of revoicing and interactions with tools, texts, and classmates to strengthen communication. Based on these studies, it is important to examine a more comprehensive tool for communicating mathematics knowledge through modality and its influence on multiplicative reasoning.

Mathematics Knowledge

Communication and reasoning have become an integral part of the mathematics classroom. Mathematics is no longer a matter of just knowing, or doing computation, with students locally and nationally, increasingly asked to demonstrate an in-depth knowledge of content. In response, policy and content standards were enacted with suggestions for increasing understanding and participation in their mathematics learning to support United States students competing in a global economy. In Principles and Standards for School Mathematics, (2000) NCTM suggest that math programs "allow students to (1) organize and consolidate their mathematical thinking through communication; (2) communicate their mathematical thinking coherently and clearly to peers, teachers, and others; (3) analyze and evaluate the mathematical thinking and strategies of others; and (4) use the language of mathematics to express mathematical ideas precisely" (Kostos & Shin, 2010, p.223). Language is used to conceptualize mathematical concepts learned in the classroom. With the enactment of the Standards for Mathematical Content and the Standards for Mathematical Practice (CCSS, NGA, 2010), students are expected to progressively communicate their mathematical content knowledge. Actively engaging students in conceptualizing mathematical understanding is intertwined with demonstrating procedural competency and communicating through language modalities in today's state and national assessments.

Conceptual vs. Procedural Knowledge

Scholars have debated the emphasis of conceptual versus procedural learning and which should take precedence during instruction to maximize outcomes (Hiebert & Lefevre, 1986). Within these dueling learning styles, the prevalent pedagogy extends major implications for classroom instruction, curricular materials, and assessments throughout our nation's classrooms. When discussing procedural and conceptual knowledge, there is an effort to emphasize how we acquire knowledge, delve into the relationship between conceptual and procedural knowledge, and how we present that knowledge. Hiebert and Lefevre (1986) define conceptual knowledge as "knowledge that is rich in relationships. It can be thought of as a connected web of knowledge, a network in which the linking relationships are as prominent as the discrete pieces of information" (pp. 3-4) in the seminal book edited by Hiebert (1986). Procedural knowledge as defined is "made up of two distinct parts. One part is composed of the formal language, or symbol representation system, of mathematics. The other part consists of the algorithms, or rules, for completing mathematical tasks" (p. 6). As thoroughly as these definitions are cited, the authors acknowledge flaws clearly defining this construct as "not all knowledge fits nicely into one class or the other. Some knowledge lies at the intersection" (Hiebert & Lefevre, 1986, p. 9). Although there is an attempt to place each in a position of prominence it is important to note how symbiotic the two are to developing understanding. Hiebert and Lefevre, (1986) acknowledge the distinction between procedural and conceptual knowledge in definition while highlighting how mutually beneficial establishing a correlation between the two is. In a review of conceptual understanding literature, Crooks and Alibali (2014) found a thorough understanding of conceptual knowledge difficult to encapsulate as there is no accord on how to define or measure it. With research designating meaning for conceptual knowledge in myriad ways, the problem becomes in identifying conceptual understanding, the relationship between conceptual and procedural understanding, and how to use research to guide practice (Baroody et al., 2007; Crooks & Alibali, 2014). Crooks and Alibali (2014) suggest enacting a common framework to actualize conceptual understanding to allow for many of the definitions noted in the review of literature to be subsumed into two categories: knowledge of general principles and knowledge of the principles underlying procedures. Crooks and Alibali (2014) state, that

> general principles knowledge involves understanding of mathematical ideas without relation to specific problems or procedures. Knowledge of principles underlying procedures, on the other hand, involves connecting concepts to specific procedures; for example, knowing why certain procedures work for certain problems or knowing the purpose of each step in a procedure (p. 71).

The authors believe a framework for discerning conceptual knowledge will guide efforts to research this construct across specific mathematical domains and direct appropriate research measurements.

Connecting Conceptual and Procedural Knowledge

Current research in mathematical understanding can be viewed as broadening the characterization of procedural knowledge and noting its interrelated and iterative relationship with conceptual knowledge (Barmby et al., 2009; Baroody et al., 2007; Crooks & Alibali, 2014; Kilpatrick et al., 2002; Rittle Johnson et al., 2001; Rittle-Johnson & Schneider, 2014; Star, 2012). The narrow scope of defined procedural knowledge has led to measures of fact recall and memorization. This has resulted in the view of procedural knowledge as secondary to, or less important than procedural knowledge (Star, 2012). Baroody et al. (2007) and Star (2012) discuss reconceptualizing procedural knowledge to discuss the type and quality of connection. Part of establishing procedural and conceptual knowledge as parallel constructs is to understand that one does not mean richer connections than the other. One can have weak conceptual knowledge and high procedural knowledge, just as there can be high conceptual knowledge with weak procedural knowledge (Baroody et al., 2007). Learning is designated as either strong or weak schema development noted as deep conceptual knowledge or meaningful procedural knowledge. Baroody et al. (2007) suggest "increasing integration with corresponding conceptual knowledge increases the accuracy, versatility, duration and generality of strategy choice and adaptability" (p. 126). Rittle-Johnson et al. (2001) conducted experiments with fifth and sixth grade students learning decimal fractions. Their research indicated that pre-test conceptual knowledge predicted gains in procedural knowledge and procedural knowledge gains improved conceptual knowledge. This signifies a rich connection between the two without compromising one for the other but constructs that influence each other.

Mathematical Proficiency

Kilpatrick et al. (2002), also view the symbiotic relationship between conceptual understanding and procedural fluency as a necessity for successfully increasing the learning of mathematics. Neither procedural nor conceptual knowledge is designated as the epitome but part of the existing paradigm of mathematical proficiency. Through analysis of research, experience in teaching and learning mathematics, Kilpatrick et al. (2002), define mathematical proficiency as the "mathematical knowledge, understanding and skill people need to have successful learning in mathematics" and conceptual understanding as "comprehension of mathematical concepts, operations and relations" (p. 5). Kilpatrick et al. (2002), recognize conceptual understanding as one of five key strands of the broader construct of mathematical proficiency which are interwoven and interdependent: conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition (see figure 1). Students can make and see connections, flexibly represent their mathematical thought, justify the reasonableness of answers, and commit less content to memory as they are able to connect content easily (Kilpatrick et al., 2002).

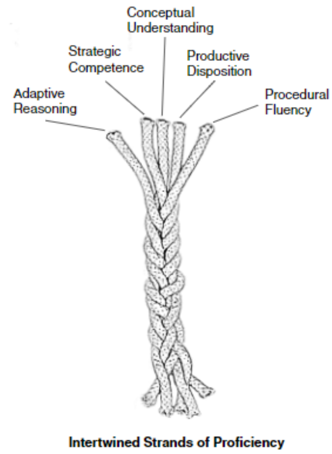
Conceptual understanding – comprehension of mathematical concepts, operations, and relations	
Procedural fluency – skill in carrying out procedures, flexibly, accurately, efficiently, and appropriately	
Strategic competence – ability to formulate, represent, and solve mathematical problems	
Adaptive reasoning – capacity for logical thought, reflection, explanation, and justification	
Productive disposition – habitual inclination to see mathematics as sensible, useful, and worthwhile, coupled with a belief in diligence and one's own efficacy.	

Figure 1 Strands of Proficiency

Representing and connecting knowledge is a major component of deep understanding. Variations of representations of their mathematical thought assist students in discussions regarding similarities and differences amongst solution methods and making connections within their own learning and the learning of others. The emphasis on procedural fluency as the existence in "flexible as well as efficient and appropriate application of procedures" embody the enactment of high schema connections (Baroody et al., 2007, p.120)**.** In illustrating the interconnectedness of procedural fluency and conceptual understanding, Kilpatrick et al. (2002) state "understanding makes learning skills easier, less susceptible to common errors, and less prone to forgetting. By the same token, a certain level of skill is required to learn many mathematical concepts with understanding and using procedures can help strengthen and develop that understanding" (p. 122).

<u>Proficiency and Practice Through Communication</u>

If the CCSS Standards for Mathematical Content (NGA, 2010) are the "what" students need to know, then the Standards for Mathematical Practice are the "how". The eight Practice Standards support the processes (NCTM): problem solving, reasoning and proof, communication, representation, and connections with proficiencies (Kilpatrick et al., 2002): conceptual understanding, procedural fluency, strategic competence, adaptive reasoning, and productive disposition, which are necessary for the development of mathematics learning. Students engage in the components of the mathematical practices when expressing their understandings through discourse opportunities in the classroom contributing to mathematics' literacy (Moschkovich, 2015a; Moschkovich & Zahner, 2018). Practices involve thinking and reasoning, while students communicate their understanding in the social environs of their classroom community through modalities for sense making and meaning. Through the development of academic literacy, students are provided tasks which allow us to observe their conceptual understanding through explanations and reasonings while engaging in multimodal means of communication to share that understanding (Moschkovich, 2015a). It becomes less about which practices students engage in and more about how they are participating in the practices.

Engaging students in explanations, discussions, and proof bring to light the justification and reasoning that assist in practicing the mathematics registers and give students opportunities to engage in the practice standards for mathematics through varying modalities. They often provide the evidence one uses to show their knowledge and are part of the mathematical practices as stated in the CCSS (NGA, 2010) and NCTM's Principles to Action (2014). The eight practices are as follows: 1. Make sense of problems and persevere in solving them. 2. Reason abstractly and quantitatively. 3. Construct viable arguments and critique the reasoning of others. 4. Model with mathematics. 5. Use appropriate tools strategically. 6. Attend to precision. 7. Look for and make use of structure. 8. Look for and express regularity in repeated reasoning (For a more thorough description see CCSS, NGA, 2010, www.corestandards.org/Math/Practices). *Student participation in the mathematical* practices provide a glimpse into how students are developing mathematical understanding and how that understanding evolves through classroom engagement. This constitutes the enactment of tasks rich in engaging students in thinking and reasoning about mathematics, without which student learning is affected (Henningsen & Stein, 1997). As Niss (2003) specified and was emphasized by Lithner (2017) the importance in assisting students towards mathematical competence which is the "ability to understand, judge, do and use mathematics" inclusive of problem solving and justifying choices and conclusion through reasoning (p. 938). Generalizing and justification of thought processes for solving or using a particular strategy are ways to demonstrate reasoning. Speaking and writing are prominent forms of expressive language. We will examine each of these modalities and their connection to communicating understanding through mathematical practices while also considering integration of multiple modes which strengthen communication and understanding.

Mathematical Practices and Modality

Speaking

In mathematics, we are shifting from a focus on communicating the correct answer to emphasizing understanding, sharing different methods and models for problem solving and communicating our thinking to others. Discourse is often used to conceptualize mathematical concepts learned in the classroom and has been researched extensively (see Ryve, 2019 for a review of discourse in mathematics literature). For the purpose of this paper, we will focus on Gee's Discourse (big D) (2004) which is defined as “ways of combining and integrating language, actions, interactions, ways of thinking, believing, valuing, and using various symbols, tools, and objects to enact a particular sort of socially recognizable identity” (p. 29). The continual building of knowledge through language and the use of other objects, tools and technologies establish “language-in-action”(Gee, 1999, p. 13). The process of reading and writing, advances both oral and written forms of mathematics Discourse by facilitating interaction and language practice among students. Interaction can contextualize discussions and assist students in practicing the language of their mathematics community through modalities (Gee, 2004, 2014; Moschkovich, 2003, 2015a; Moschkovich & Zahner, 2018; O’Halloran, 2015; Schleppegrell, 2007; Sfard, 2001, 2008). The cooperative nature is especially supportive of those who may have linguistic, cultural, ethnic, and skill-level differences (Barwell, 2005; Baxter et al., 2005; Schleppegrell, 2007; Snow & Uccelli, 2009 Zwiers, 2007).

Research on the effects of engaging students in mathematical practices of discourse have observed the impact of student collaboration and cooperative learning (Francisco, 2013; Slavin et al., 2009), teacher moves (Kazemi & Hintz, 2014; Jacobs & Empson, 2016; Lithner, 2017; Maher et al., 2018; Sfard, 2008; Stein et al., 1996, 2008; Zwiers & Hamerla, 2018), discourse competence (Erath et al., 2018; Moschkovich, 2015b), and positioning (Andersson & Wagner, 2019). Slavin et al., (2009) found that studies of cooperative learning amongst students in elementary school programs have a large impact on achievement measures with a weighted mean ES=+0.42 in 9 randomized or quasi-randomized experiments (p. 43). A study of high school students engaged in a probability task show that collaborative activity promotes understanding while engaging students in the practices to refine and examine their ideas socially (Francisco, 2013).

Ingram et al., (2020), view student interaction and how they enable justification and reasoning of mathematical thinking to be socially constructed through analyzing participation in discourse.

Interaction through collaboration, or cooperative learning, can become the gateway to students' conceptions, models, and strategies. Educators play an important role in enacting meaningful discourse routines (Kazemi & Hintz, 2014; Sfard, 2008; Stein et al., 2008; Zwiers & Hamerla, 2018) as a powerful way to unpack and scaffold language students will encounter. Students speak with purpose to improve their conversational skills and mathematics register simultaneously. Zwiers and Hamerla (2018) define academic conversations as "sustained and purposeful conversations about school topics" (p. 1). The practice of engaging in academic conversations is an intentionally scaffolded process of enacting purposeful communication amongst students within a classroom (Moschkovich, 2015b). Tasks, which engage students at a high level (Stein et al., 1996) allow for pressing students conceptually to justification, explanations, and meaning making (Brendefur et al., 2013). These tasks allow for students to explore mathematics "with worthwhile tasks and ask students to share their work and reflect on necessary elements within the mathematics" (Brendefur et al., 2013, p. 65). The ability of educators, and students, to engage in discourse and mathematical practices in meaningful ways impacts the presenting of their mathematical proficiency.

A situated perspective on the mathematics classroom recognizes participation and contributions to the development of mathematical practices socially, supporting students in progressing mathematically (Cobb & Bowers, 1999; Yackel & Cobb, 1996, Cobb et al, 2011). Cobb and Bower's (1999) work are situated in the symbiotic relationship between instructional design and classroom research in the mathematics classroom. Close attention is paid to the engagement in students reasoning and communal practice. The compulsory relationship between students' contributions and the transformation of classroom practices, establish their engagement in the socially constructed situation and participation in the Community of Practice. Ethical implications as posited by Cobb and Bowers (1999), recognize level of participation by students may vary in this framework. Adjustments must be made to accentuate participation by students to actuate their possibility of learning (Cobb & Bowers, 1999; Stein et al., 1996). Non-participation does not lie solely on the student but on how educators facilitate, and encourage, engagement in the practice of mathematics as a valued member of the community (Bailey, 2015; Barwell, 2005; Cobb & Bowers, 1999).

A student's ability to verbalize a connection among concepts and representations, is a means for educators to look upon demonstration of conceptual understanding. However, the evidence may not need to be strictly verbal as we often have students which understand before they can verbalize understanding (Crooks & Alibali, 2014; Kilpatrick et al., 2002). Communication involves more than writing and speaking and is inclusive of other symbols and artifacts which can be utilized to connect meaning (Gee, 2004, 2014; Moschkovich, 2015a; Moschkovich & Zahner, 2018). In Shein's (2012) study of fifth grade English Language Learners (ELLs), the use of gestures and discourse were reported to communicate understanding of geometric area calculations. The use of gestures by students and teachers enhanced the discourse during these lessons and was increased by the teacher's use of revoicing and representations (Moschkovich, 1999; Shein, 2012). Chen and Herbst (2013) also investigate students' multimodal communication of geometric reasoning with gestures, speech, and diagrams. They conclude that the use of all triangulate communication amongst the three modalities and assist students in making reasoned conjectures. Fernandes et al. (2017), examined how multimodality assisted bilingual sixth through eighth grade students on an area measurement task selected from the National Assessment of Educational Progress (NAEP). Through interviews, the students were observed using manipulatives to form diagrams and gesturing to make claims and explain their reasoning. The authors view modality as an approach for assessing what our students understand broadening the language expression beyond speaking for reasoning.

Writing

Writing has been implemented in the mathematics classroom in a variety of ways, such as learning logs, journals, mathematical autobiography, writing of problems and test questions, freewriting, creative writing, proof writing and formal response papers (Kostos & Shin, 2010; Bosse & Faulconer, 2008; Baxter et al., 2005; Burns, 2004; Fried & Amit, 2003; Pugalee, 2001; Countryman, 1992). According to Firmender et al. (2017), it is imperative to reflect on how we are asking students to write. Are students being asked to "*write about math* (e.g., math autobiography, writing about one's feelings about math), which foregrounds the literacy aspects" (p. 86) as opposed to *writing in mathematics* to communicate learning (Firmender et al., 2017)? Although educators recognize the need for attending to writing in the mathematics classroom, time integration and reconciling mathematical content and writing ability can be daunting. Research has reported low implementation of writing by educators outside simplified versions of writing such as note taking and single sentence response with an answer (Bakewell, 2008; Kosko, 2016;

Ntenza, 2004). Educator perceptions of mathematics writing play a key role in its implementation. In interviews of mathematics educators, Ntenza (2004) found that teachers "did not seem to appreciate the relationship between writing and mathematics, nor reached a common understanding amongst and with themselves on how to collect evidence of students' learning through writing" (p. 18). Besides not understanding the importance of integrated writing in mathematics, educators also report time constraints for writing in class (Bakewell, 2008).

Writing can contextualize discussions and assist students in practicing the language of their mathematics community, grammatical structures of the English language and writing forms (Barwell, 2005). Through reading and writing, language and content can be of benefit to each other. Students written response can show the interconnectedness of language and content learning (Kostos & Shin, 2010; Schleppegrell, 2007; Zwiers, 2007) and be utilized for formative assessment (Kostos & Shin, 2010; Barwell, 2005; Burns, 2004; Fried & Amit, 2003; Countryman, 1992). With numerous types of writing possibilities for the mathematics classroom, how can we be sure that we are writing to learn and deepen mathematical thinking and understanding (Bosse & Faulconer, 2008). The Elementary Mathematical Writing Task Force (Casa et al., 2016), sought to clarify the types of writing by adhering to two goals: "for students *to reason* mathematically and *to communicate ideas"* (p. 3). The task force then recommended four types of writing for mathematics and their purposes to meet their goals. These types of writing are explanatory, to help make personal sense of the situation; informative/explanatory, to describe and explain; argumentative, to construct and critique an argument; and mathematically creative, showcase ideas and problem-solving situations (Casa et al., 2016). Educators can generate writing prompts accompanied with the sentence stems and frames, for students to engage and respond to writing with appropriate scaffolds. This entails facilitation of their ideas for writing about mathematics rather than prompting students with rules and shortcuts (Casa et al., 2016; Bosse & Faulconer, 2008). The task force recommended the following for consideration when integrating writing:

- Writing can be developed across a continuum, based on student grade and linguistic level. Sometimes students will write just to write, and not fulfill the complete writing process.
- Audience influences student mathematical understanding. Vary task prompts so students can write to friends and family members and in different forms.
- Mathematical writing may take multiple forms which can be represented through paragraphs, sentences, notes, or letters. (Casa et al., 2016, p. 4-5)

These recommendations from the Elementary Writing Task Force allow educators to focus on very specific kinds of writing. However, considering how writing can be integrated into daily or weekly lesson plans requires a closer inspection of curriculum and materials.

An additional tool used to develop literacies in mathematics is the mathematics journal (Kostos & Shin, 2010; Baxter et al., 2005; Pugalee, 2004; Fried & Amit, 2003) which reports increased student proficiencies for students. Baxter et al. (2005), designated writing as a form of communication for seventh grade general mathematics students. Observation noted minimal participation of students during classroom interactions; however, analysis of written work in their journals show students ability to explain their mathematical thinking, use representations and interact with the content. In a meta-analysis of mathematics writing literature by Powell et al. (2017), a synthesis of articles from January 1990 to January 2016, yielded twenty-nine studies that analyzed student writing through outcomes from peer reviewed journals. Of the twenty-nine studies, seven focused on elementary students from first through sixth grade. All focused on explanatory forms of writing in mathematics. This study highlights the need for more information on how the modality of writing influences mathematical understanding.

A combination of modalities also affords students a valuable means of communicating their mathematical reasoning. Bjuland et al. (2008) researched the use of written representations in combination with gesture with sixth grade students during a geometry unit. Using gesture in conjunction with the representation and dialogue assisted students in collaborative group work and participation in their mathematics community. Oviatt & Cohen (2013) analyzed high school students use of multimodality which consisted of integrated technology (digital pens), representations, and writing. Findings revealed students' written work predicted their success with their mathematical task by ninety-six percent. Suggestions for future research include examining different content domains and observation of situated classroom interactions.

Mathematics Knowledge Summary

The intersection between the content standards engendering understanding and the practices and their focus on communicating processes and proficiencies seek to develop the expertise of mathematicians. These curriculum reform transitions are supported by mathematics research. Hiebert and Grouws (2007), in their reconciliation of a definition for conceptual understanding from a sociocultural perspective, state that understanding is an activity which would mean "participating in a community of people who are becoming adept at doing and making sense of mathematics as well as coming to value such activity" (p. 382). Sociocultural theory emphasizes learning that takes place among experts and novices collaboratively among a common goal (Tharp, 1994; Vygotsky, 1978). Understanding can be co-created through content conversations in shared activity with the teacher and communally with other students. Progressing conceptual understanding through shared experiences and discourse, unify content language and everyday communication. Tharp states (1994), it is through the interface of everyday experience and content meaning that the highest order of understanding can be achieved and verbal tools "can be manipulated for the solution of practical problems of the experienced world" (p. 155) through practices in discourse. Students engaged in this process interact in social practices and utilize their shared knowledge whether in face-to-face interactions within the mathematics classroom, or in isolation, and engage in mathematics content (Greeno, 1997). Teaching and learning are conceptualized as an activity rich in interaction amongst a developing community of learners "where the ways of thinking, modes of inquiry, communicative convention, values, and beliefs characteristic of the wider mathematical community can be progressively enacted and appropriated" (Goos, 1999, p. 4).

Hiebert and Lefevre (1986) suggest several reasons for establishing a relationship between conceptual and procedural knowledge. Linking a procedure to conceptual knowledge helps with understanding of a concept but also retrieving information later. Generating rich instruction integrating each type of knowledge benefits students in their learning in presenting them with numerous models, strategies, and language to utilize.

Although a focus for mathematics research, the definition and measurement of conceptual knowledge, can be broad and daunting for identifying and measuring. Crooks and Alibali (2014) state, “the term “conceptual knowledge” has come to denote a wide array of constructs, making it difficult to understand the major findings in the field, the ways in which conceptual knowledge relates to procedural knowledge, and the most effective ways to utilize current research to guide instructional practices” (p. 345). According to Kilpatrick et al., (2002) there are five strands for mathematical proficiency inclusive of conceptual and procedural understanding which interweave to highlight and impart understanding. Yackel and Hanna (2003) suggest when students can justify their own thinking with language and other modalities, they further their own understandings. Educators play a crucial role in assisting students in providing evidence for their reasonings to support their understandings with strategies and procedures. In utilizing the ALM framework, Moschkovich and Zahner (2018), in an excerpt of a middle school classroom discourse, show how students use multiple representations (text, table, and graph) to demonstrate conceptual understanding, interact in small group and come to a consensus on explanations to demonstrate mathematical practices and participated in Discourse through creation of various texts and representations through multimodality (pg. 1004-1005). Semiotics, and modality, provide a lens for examining the use of tasks to support communicating mathematical practices and Discourse in the content area of multiplication.

Multiplication

Understanding of multiplicative reasoning has over relied on students' level of fact fluency (Brendefur et al., 2015; Kling & Bay-Williams, 2014; Schoenfeld, 2020; Smith, S. & Smith, M., 2006), rather than reasoning and connecting of multiplication concepts to demonstrate multiplicative understanding. Fluency fact knowledge is important; however, a shift in pedagogy, brought to the forefront by the CCSS, regard conceptual understanding of the concept of multiplication constructed through student engagement in problem solving and reasoning as equally important. To truly acknowledge the gravity of the expectations reliant on third grade to set the foundation for multiplicative reasoning, it is important to study the vertical progression of the curriculum from second grade to fourth grade. This allows us to envision how we might enhance development of multiplication in the early years and situate students in improved position for later growth in multiplication. For second grade, multiplication is introduced in the Operations and Algebraic Thinking domain which emphasizes "work with equal groups to gain foundations for multiplication" (NGA, 2010). The emphasis is on a visual model for representing multiplication, not on symbolic written notation or knowing of facts. In third grade, multiplication becomes a major work of this grade. It is essential for students to "develop an understanding of the meanings of multiplication and division of whole numbers through activities and problems involving equal-sized groups, arrays, and area models" (NGA, 2010). This spotlights the significance of developing understanding through problems solving and activities utilizing multiple representations. Multiplicative focus for fourth grade, further develops "understanding and fluency with multi-digit multiplication and developing understanding of dividing to find quotients involving multi-digit dividends" (NGA, 2010). In this grade, the emphasis is on extending multiplication with content connections to division. The combination of the content and practice standards acknowledge the necessity of beginning early to set the foundational constructs for multiplicative thinking, the use of representations and visuals to develop conceptual meaning and communicate about the construct through modalities to assist reasoning.

Multiplicative Levels

Numerous studies have been completed on students' categorical presentation of additive to multiplicative thought (Jacob & Willis, 2001, 2003; Clark & Kamii, 1996; Kouba, 1989; Sherin & Fuson, 2005) and indicators of multiplicative thought (Carrier, 2014). Kouba (1989) suggested five categories for classification of student solution strategy: *direct representation, double counting, transitional counting, additive or subtractive, and recalled number fact*. Students in grades first through third were given two equivalent set multiplication problems and four division word problems. Students' intuitive models for multiplication revolved around counting of sets and counting all the objects in the set which aligns with an additive strategy (Kouba, 1989). Jacob and Willis (2001) have labeled five phases of additive to multiplicative thought: *one-to-one counting, additive composition, many-to-one counting, multiplicative relations, and operating on the operator*. A range of multiplicative tasks (arrays, multiplication and division relationship, multiplication situations that could not be thought of as additive and grouping situations) were utilized through interview to recognize student thinking as categorically additive or multiplicative. The authors discuss representations of additive and multiplicative thinkers in their study, ways to progress students to *operating on the* *operator* and use in classrooms by educators to ascertain additive and multiplicative thinkers in the classroom are noted. Sherin and Fuson (2005) suggest six taxonomies for strategy: *count all, additive calculation, count by, pattern based, learned products and hybrid* based on interviews with third grade students before, during and after multiplication instruction. The authors reported differences in how we might categorize student strategy when the interactions between the strategies are blurred. For example, students may be able to count by a number quickly to recall a product which would make it difficult to decide whether they would classify as a *count by* or *learned product*. For this reason, Sherin and Fuson, (2005) understand the limitations to a taxonomy while also challenging our views of "assuming that the end products of learning are memorized count-by sequences or straightforwardly internalized versions of the multiplication table" for generating understandings (p. 377). Carrier (2014) narrowed categories of multiplicative thought to three: pre-multiplicative, emergent, and multiplier with up to twelve sub-categories within. Implications for this study, were the use of key words that triggered student response and as a tool for assessing student multiplicative understanding. One of the many understandings yielded from the research is the realization that there is not direct progression from additive to multiplicative thinker. Students can deploy different strategies when working with varying problem structures for multiplication and division. This constitutes the necessity of knowledge shifts in language, structure, and modeling to develop and communicate conceptual understanding of multiplication.

Multiplicative Language

As we assimilate new knowledge, connecting to familiar models, strategies and language assist in conceptualizing and communicating thinking. Adhering to the structure inherent in mathematics allows students to focus on connections between content which establishes a schema for common vocabulary (Brendefur et al., 2013). The structures integral to the conceptualization of multiplication resides in quantity ideas of units and unitizing (Anghileri, 1989; Carrier, 2014; Clark and Kamii, 1996; Park & Nunes, 2001; Simon & Blume, 1994; Ulrich, 2015, 2016) and visualizing the structure of iterating and partitioning (Barmby et al., 2009; Hurst & Linsell, 2020; Iszak, 2005; Kosko, 2020). Steffe (1994) defines units as the "countable items of one" and unitizing as "a sequence of unit items used in the formulation of an experiential counting act" (p. 14). In mathematics literature, the unitized unit, is known as the composite unit. Carrier, (2014), defines units and composite units as the recognition of the unit as a quantity of one, and as a quantity that is more than one (composite unit). Simon (2017, p. 129), define iterating and partitioning as 1) Partitioning a unit into n equal parts creates parts one of which will iterate n times to make the whole. 2) Iterating a small quantity n times, produces a specific large quantity that is n times as large. So, partitioning a unit into n equal parts creates parts of a particular size" Iterating can also be referred to by *copying* and partitioning can be *splitting evenly*. The above definition taken from Simon (2017) is regarding fractions but is useful for representations of multiplication. The importance of directing students towards the use of structural vocabulary to help make connections between different content that also use the same structural words establishes a schema for common vocabulary. Iterating and partitioning models, support our representations of multiplicative ideas and vocabulary as part of this construct.

Drawing on the work of Mulligan and Mitchelmore (1997), Jacob and Willis (2001, 2003) highlight the recognitions students must come to in multiplicative situations: groups of equal size (a multiplicand), numbers of groups (the multiplier) and a total amount (the product). "When they can construct and coordinate these aspects for both multiplication and division problems prior to carrying out the count, they are thinking multiplicatively" (Jacob & Willis, 2001, p. 307). While unit, iterating and partitioning are words associated to various mathematical content, words such as multiplier, multiplicand, and product are specialized vocabulary representative of the mathematics register specifically for multiplication. Carrier (2014) also summarized particular words from existing literature that may be early indicators of multiplicative reasoning: "(e.g., twice as big), area, split, half, one half, one third, divide, times, cut, more, less, double, larger, smaller, equal, sets, sets of sets or their synonyms" (p. 93). These mathematical word lists can be dense with multiple meaning words and phrases for use in the multiplication construct. This imposes challenges to students as they generate meaning through more than words, phrases, and sentences, to enrich their language output to the meaning, modes, and arguments necessary to communicate multiplicative knowledge.

Multiplicative Reasoning

Studies of young learners and early developments of multiplicative reasoning view patterns, subitizing and skip counting as ways young mathematicians develop the structure of units and unitizing for composite units (Baroody, 2006; Cheeseman et al., 2020; Mulligan & Mitchelmore, 2009; Papic et al., 2011; Warren et al., 2012). Young students work on patterning structures in early childhood development. Recent research suggests patterning and subitizing can act as an effective bridge for introducing the ratio concept, a form of multiplicative thinking (Warren et al., 2012). Pattern is much more than student recognition of what comes next in the patterning sequence, but that the pattern itself is a unit. The conceptual view of patterning asks that students identify the unit of repeat (yellow, yellow, blue) which entails the unitizing process (Warren et al., 2012). Patterning can be continued through the skill of subitizing, or suddenly seeing. In Kindergarten classes, subitizing is used to build student number sense around a part whole relationship. When forming mathematical reasoning around number, subitizing allows students to see a whole and the parts which compose it (Warren et al., 2012). Skip counting emphasizes the use of the composite unit and the idea that the number 3 can be something other than 1, 2, 3 and can be one unit of 3; and that unit can be repeated and counted as 3, 6, 9. Children with a rich grasp of number and mathematical patterns and relationships are more likely to become reasoners (Baroody, 2006). Investigations of multiplication and division problem solving with young students who have not had formal instruction utilized counting and grouping strategies to solve problems (Carpenter et al., 1999; Downton, 2008; Smith & Smith, 2006). Empson and Turner (2006) consider the role of partitioning through paper folding with students as young as first grade to demonstrate multiplicative thought. Their findings "suggest that given a task structure that provides multiple opportunities to test and revise ideas, shifts from additive to emergent multiplicative reasoning are likely" (Empson & Turner, 2006, p. 55). Students may need opportunities to iterate and partition unit and composite units to identify a shape and its spatial structuring using its spatial components (Battista et al., 1998). Using the idea of composing spatially, Mulligan et al. (2004) acknowledge the importance of spatial structuring and in visualizing and organizing multiplicative structures for partitioning.

It is important to understand which multiplicative strategies are present, so students can be challenged conceptually (Brendefur et al., 2015) to improve their knowledge for multiplication. Students without a developed conceptual construct for multiplication will continue to explore inefficient strategies or apply them inaccurately. This is evident in research with students over reliance on additive strategies for multiplying (Jacob & Willis, 2001, 2003; Kouba, 1989). As designated in the standards (CCSS), multiplicative reasoning is the major work of third grade laying the foundation for successive mathematical topics such as ratios, proportionality, variables, and quantification one needs to later succeed in mathematics (Carrier, 2014; Fielding-Wells et al., 2014; Van Dooren et al., 2010). Therefore, designating what constitutes as multiplicative reasoning is imperative to the facilitation of knowledge construction for multiplication which moves beyond additive thinking. Scholars have examined this role and have stressed the importance of students' ideas of quantity (Carrier, 2014; Simon & Blume, 1994), schema of correspondence (Clark & Kamii, 1996; Park & Nunes, 2001), and multiplication's co-varying role with addition (Van Dooren et al., 2010). Park and Nunes (2001) suggest that children's concept of multiplication originates in their "schema of correspondences and not in the concept of addition" (p. 764). They define the concept of multiplication as an "invariant relationship between two quantities. This constant relation, known as ratio or rate, and is a core meaning of multiplication (Park &

Nunes, 2001, 764). Carrier (2014) summarized the Park and Nunes (2001) definition as "the ratio or rate is the constant unit that is called the multiplicand and acted upon by the multiplier. Children employ the schema of correspondence to represent fixed relationships between variables and solve multiplication problems (p. 87). In order to grasp this concept MacLellan (2001) asserts "if children are genuinely to engage in multiplicative reasoning, they must appreciate that the relationship between the elements may be constant (which allows repeated addition); but they must also, critically, appreciate that the relationship may be a co-varying one" (p. 151). Inquiry based tasks also afford students the ability to progress their thinking from an absolute, or additive, structure to a more multiplicative, proportional structure as shown in a study of fourth grade students (Fielding-Wells et al., 2014). Throughout the inquiry, students' investigations were structured in conjunction with representations, small group interactions and whole class discussions (Fielding-Wells et al., 2014). The communicative aspect of the inquiry, developed throughout by teacher questioning, allowed students to challenge their reasonings and those of their classmates as they progressed to an understanding of proportionality. Unless teachers consciously assist in developing multiplicative thinking, which goes well beyond repeated addition, it may not happen for many children.

Modality and Multiplication

Understanding multiple representations of mathematical concepts is part of the CCSS (NGA, 2010) standards for mathematical practices. Bruner's modes of representation (1966): enactive, iconic, and symbolic, are of particular importance to modeling the multiplication we are learning. The modes are defined as enactive, or action; iconic, or image; and symbolic, words or language (Bruner, 1966). Brendefur et al. (2015) highlight the use of enactive, iconic and symbolic representations for multiplication learning by means of enactive models of counters and tiles and then drawing "iconic diagrams of arrays and groupings that would offer a subsequent opportunity to label the iconic representations with symbolic notation (e.g. mathematical symbols and words)" (p. 144). This will allow students to build a concrete connection for the modeling of multiplication structure and strategies to make sense of multiplication. Kilpatrick et al., (2002) acknowledge the tension amongst symbolic and concrete ways of thinking multiplicatively. The linear bar model, transformed to side-by-side bars, depicts the *rectangular array interpretation*, and when the bars are linked together with no space then becomes the *area interpretation* (p. 74). Multiple interpretations can cause tension, but more so could fixation on a singular representation without the means to reason flexibly to learn and communicate concepts (Kilpatrick et al., 2002). The importance of visualizing representations of the array model is theorized by Battista et al., (1998), and states the importance of viewing the model as a whole which will then construct a "scheme for iterating rows, columns, or layers of cubes, with layer structuring being the most efficient, both computationally and mentally" (p. 505). The authors found evidence to support students' struggle with the model based on an inability to see the rows and columns structures that compose arrays.

Previous studies have yielded important insights into students use of representations for multiplicative thought such as arrays (Barmby et al., 2009; Battista, 1998, 2020; Harries & Barmby, 2008; Izsak, 2005; Kosko, 2020), manipulatives (Hurst & Linsell, 2020; Kosko, 2020; Mills, 2019), and the ways we can use them to reason about the mathematical practices (Barwell, 2005; Barwell et al., 2005; Erath et al., 2018; Prediger and Zindel, 2017). Barmby et al., (2009), researched the use of the array as a key representation of understanding multiplication through the modality of computer-generated images and interaction amongst students ages eight to eleven. They report the importance of arrays for assisting with calculation strategies. Challenges of using the array as a representation were stated as students' unfamiliarity with the structure of the model and inability to equate dimensions to rows and columns to construct the model. This was common to finding from Battista et al. (1998). Iszak (2005), also studied the array representation with fifth grade students on computation of whole number problems. Implications for how representations are used across content areas to construct meaning are presented. Prior knowledge of the representational use of dots as a means of constructing arrays played a crucial role in students transferring those ideas to area model representations. Kosko (2020), attempted to identify third and fourth grade students' multiplicative reasoning given three types of visual representations: set, length, and area. In grouping structures, utilizing the groups of images and arrays, the possibility is present to group and visualize composite units. Length engages the ideas of a linear or continuous model for discerning multiplicative structure with number lines and bar model examples. Area representations express multiplicative relationships with a row and column structure. They most resemble arrays and can relate to square units or open area models given side dimensions (Kosko, 2020). Kosko's (2020) research supports prior studies that revealed the use of representations of set and area which most often generate a single counting unit rather than a multiplicative composite (Anghileri, 1995; Barmby et al., 2009; Battista et al., 1998; Clark and Kamii, 1996; Downton & Sullivan, 2017;

Kosko, 2020). Further research is necessary for revealing representations and their assistance in developing different levels of reasoning. Ulrich and Wilson, (2017), studied sixth grade students' multiplicative reasoning conveyed through written assessment items using both continuous (bar models and number lines) and discrete (set image) representations to ascertain their development of composite unit strategies for classification of the students' work by thought. Student written work indicated their reasoning about units using representations. Implications for continued research on the use of written assessments are suggested. Representations can be used to extend conceptual knowledge of multiplication and communicate reasoning. Connecting student representations with language allows for the use of a conceptual tool that shares meaning and assists in communicating.

Students experience with multiplication should encourage mathematics communication, inclusive of spoken discourse (Kazemi & Hintz, 2014; Sfard, 2008; Stein et al., 2008; Zwiers & Hamerla, 2018), writing (Casa et al., 2016; Cohen et al., 2015), and using formal and informal language of mathematics to reason through varying forms of modality (Moschkovich, 2015a; Moschkovich & Zahner, 2018). In effect, enhancing the discipline of mathematics own register for cultivating reading, writing, speaking, thinking, and reasoning (Schleppegrell, 2007). Rogers (2011) summarizes a definition by Kress which "draws attention to the many material resources beyond speech and writing" which postulate meaning making. Speaking and writing become just one of many modes to assist in understanding where "meaning is made in all modes separately, and at the same time, that meaning is an effect of all the modes acting jointly" (Kress et al., 2001, p. 1). Communicating through modality to express mathematical proficiency has been addressed in a variety of math topics and grade levels; however, few studies are available to connect and develop these ideas in multiplication.

Fernandes et al. (2017) examined how multimodality assisted bilingual sixth through eighth grade students on an area measurement task selected from the National Assessment of Educational Progress (NAEP). Students at differing stages of language development were observed with manipulatives to form diagrams and gesturing to make claims and explain their reasoning. The representations and gestures conveyed student understanding of the concepts of area, geometric rotations, and translations. Manipulatives allowed students to enactively construct their reasonings and strengthened the formulation of their explanations. The qualitative results submitted by the authors view modality as an approach for assessing what our students understand broadening the language expression beyond speaking and writing for assessing reasoning (Fernandes et al., 2017).

Brendefur et al. (2015) studied third, fourth, and fifth grade students developing fact fluency. Over five weeks, the treatment consisted of students using “enactive models to examine facts and facts strategies and then progressed to iconic and symbolic representations” with a social interactional approach (Brendefur et al., 2015, p. 145-146), while the comparison group activities included a daily math fact test with flash cards and mnemonics for memorizing facts. Results show that students in the treatment performed significantly better than students in the comparison group at all grade levels. The study’s results demonstrate evidence for the use of representational and social interactional frameworks to support students in making connections to strategies and relationships between multiplication facts.

Mulligan and Watson (1998) sought to understand second and third grade students' development of multiplication and division concepts drawing on Collis and Biggs, multimodal approach known as Structure of Observed Learning Outcomes (SOLO) model. The SOLO model is designated into unistructural, multi-structural and relational levels which cycle amongst each other while enactive, iconic, and symbolic modes of representation are present within each level. Student responses to multiplication word problems were designated into SOLO levels and then further categorized based on their iconic to symbolic representations of thought. Of the forty-six percent of students which iconically modeled in interview one, fifty-six percent of those were able to transfer their learning to relational symbolic representations. Mulligan and Watson's (1998) findings support the use of iconic representation to increase students use of efficient multiplicative strategies in counting composite units, as well as operating in symbolic number sentences or with written explanations in the writing mode. Implications for the use of the SOLO model to analyze important multiplicative concepts within the use of two modes, iconic representations and written symbolic representations, as important aspects of addressing how students function in the learning of multiplication and division.

Multiplication Summary

Connecting concepts and practices of multiplication is an important aspect for developing multiplicative understanding. Developing multiplicative thought begins early with informal knowledge of multiplication and the introduction of patterning to strengthen student ideas of unit (Baroody, 2006; Cheeseman et al., 2020; Mulligan & Mitchelmore, 2009; Papic et al., 2011; Warren et al., 2012). We progress in grade three to formalizing multiplicative thought with connections using composite units, connections with repeated addition, representations, and images to structure mathematical thought. Numerous studies have been completed on students' categorical presentation of additive to multiplicative thought (Jacob & Willis, 2001, 2003; Clark & Kamii, 1996; Kouba, 1989; Sherin & Fuson, 2005). Many of these studies have multiple levels allocated to the representation of additive thinking, documenting students over reliance on this construct and later pitfalls in attempting to apply it throughout their work with multiplication. Research has also stated that the road from additive to multiplicative thinker rarely progresses straight. Therefore, students may not progress through these levels distinctly, but cross through and use combinations of them to assist in problem solving depending on instruction, problem type and even number specific resources (Sherin & Fuson, 2005).

Although some studies used varying modes to communicate reasoning, the reasoning was not a communication through the modality but the result of their use of the representation. The modality represented their placement in a domain, additive, or multiplicative thinking, and less how they were integrating with and within the practices of the mathematical classroom. Few have focused on the modalities and their influence on students' communication and participation in the structure and content of their mathematics classroom. The representation, whether it be enactive (manipulative), iconic (number line, bar model, array, area model) or symbolic (number sentence or written words) becomes the communicator, or the teller of information, rather than a tool students use to operate with and instigate mathematical proficiency. I argue when modalities are enacted jointly, we have the potential to communicate more than our correctness but our processes, explanations, and justifications at the heart of understanding.

After the examination of the above studies more research is needed to explore how student expression of multiplicative understanding is enhanced through utilization of a multi-modal student generated tool. Separately researchers have concentrated on the "what" of multiplicative learning and the "how" of mathematical practices; however, little research explores the simultaneous development and demonstration of both for multiplication. The present study will add to this general body of research on multiplication and communication by analyzing how students engage in and about the mathematics to convey multiplicative understanding.

Conceptual Framework Summary

The focus of this dissertation was to examine the use of a multimodal tool on students' communication of multiplication reasoning. Research on communication and mathematics has been oriented in many ways. Academic language registers differentiate itself from the informal social language, as the language of schooling laden with the technical vocabulary of individual content areas (Nagy et al., 2012; Snow & Uccelli, 2010). Mathematics, considered a number and computation discipline, is noted by Halliday (1974) as a register with its own vocabulary, grammatical structure, and ways of reasoning. A significant feature of the mathematics register is the role of specialized vocabulary. The acquisition of specialized vocabulary has led to a dilemma for educators in terms of how and when they should initiate new vocabulary. Suggestions from the literature include purposeful word instruction and practice (Riccomini et al., 2015); and utilizing graphic organizers in mathematics (Bruun et al., 2015; Dexter & Hughes, 2011; Monroe & Panchyshyn, 1995; Monroe & Pendergrass, 1997). Mathematical tasks can be complicated with technical language and semantic features. The inherent challenge is to attempt to compose the “symbolic representations and visual images that do not match up exactly with their ‘‘translation’’ into the oral and written language used to develop the meanings they present” (Schleppegrell, 2007, p.145). Students must then create and convey meaning in the modes necessary to communicate mathematically. Research from a functional linguistic and semiotic lens analyze the implications of language in the domain of mathematics (Halliday, 1974, 1978, 1993; O’Halloran, 1998, 2011, 2015; Schleppegrell, 2007). The addition of functional linguistics has shifted the focus to a more semiotic system of language, inclusive of models and visuals to analyze and extend our understanding of the mathematics register (O’Halloran, 1998, 2011, 2015; Schleppegrell, 2007). The broadening of the semiotic system inclusive of modality as a means of communicating mathematical understanding allows for students to engage the modality (reading, writing, listening, speaking, representational, visual, gestural) that conveys meaning. A significant feature to multimodal perspectives on learning is the assumption that meanings are made through many representational and communicational resources, of which language is but one (Kress, 2000). Developing content area literacy in mathematics has become more than reading text and writing answers. Modality is a resource for students to engage in mathematical reasoning, to communicate understanding and interact in the classroom. In particular, the components of mathematical proficiency, mathematical practices, and mathematical discourse, utilize language and modality to explore conceptual proficiency through the mathematical practices and engage in situational discourse in the mathematics classroom (Moschkovich, 2015a; Moschkovich & Zahner, 2018).

A facet of the sociocultural perspective of learning and teaching necessitates engaging in the mathematics community as a mathematician to "learning the socially learned cultural traditions of what kinds of discourses and representations are useful and how to use them" (Lemke, 2001, p. 298). Language is a tool for understanding and learning and an integral part of exploring the mathematics world. In mathematics, multi-modalities inclusive of manipulatives, visuals, iconic representation, and receptive and expressive language elements are intrinsic to mathematics and mathematics learning. Hence, multimodal tools utilized to support mathematics offer a variety of possibilities reliant on the nature of the mathematical activity that is engendered to support learning and quality of the collaboration with peers. A sociocultural interpretation of teaching and learning, goes beyond the vocabulary acquisition, word problem interpretation and computation to enacting skills, sharing knowledge, and communicating mathematically with peers (Moschkovich, 2004). It is possible to engage in activities of the mathematics classroom and through that participation engage in language practices and mathematical practices simultaneously. When students engage in the discourses (informal, formal) of the mathematics community they develop the socio-mathematical norms of the classroom. Yackel and Cobb (1996) define socio-mathematical norms as the "normative aspects of mathematical discussions that are specific to students' mathematical activity." (p. 458). By exemplifying a multimodal approach, it will be useful to examine how the different modes afford learning, which mode supports the best form of communication, or whether combinations of modes afford learning (Kress, 2011). Therefore, a social semiotic framework allows researchers to examine the understanding generated by students, as they work with teachers and peers, to make sense of mathematics through representations and modalities building competencies for communicating effectively in a progressively multimodal world.

The functionality of language and mathematics has been extensively studied and provokes a critical challenge of connecting knowledge from diverse fields (applied linguistics, sociolinguistics, cultural and social psychology, discourse studies, semiotics); as well as discipline specific knowledge from mathematics research (Morgan et al., 2014; Planas & Schütte, 2018). Planas and Schütte (2018), identified similarities and differences in theoretical research as compiled for a special issue in the mathematics journal *Zentralblatt für Didaktik der Mathematik* [ZDM]. The theories listed represent the present focus of language and mathematics research: "(1) the politics of language and language diversity, (2) the modes of communication and representation in language, and (3) the interactionist dimension of language in classroom discussion" (Planas & Schütte, 2018, p. 967). Moschkovich (2018), also recommends research for language and mathematics should comprehend that “mathematical understanding involve multiple modalities and artifacts— including oral and written language, gestures, the body, inscriptions, and so on— the study of language and mathematics requires interdisciplinary approaches” (p. 38). Although for the purpose of this paper I am not focused on the construct of political language and language diverse learners, these recommendations encourage studies, like my own, of modality and interaction and how they can contribute to the understanding of subject specific content for all students.

Theoretical Framework

Social Semiotic Theory

The aim of this study is to examine the role of language in communicating students' understanding through the lens of social semiotics and modality in the mathematics content area of multiplication. According to Morgan (2006), researchers in mathematics education have brought theoretical perspective to addressing language and with that "increased attention to the nature of language and other semiotic systems used in mathematical activity and to the roles that these may play in the teaching, learning and doing of mathematics, drawing on semiotic and linguistic theories and developing them" (p. 219). An integral part of exploring mathematics is utilizing language as a tool. According to O'Halloran (2006), multi-semiotics, or modalities inclusive of manipulatives, visuals, iconic representation, and receptive and expressive language elements are intrinsic to mathematics and mathematics learning. Therefore, multimodal tasks used to support mathematics offer many possibilities enhance learning and the quality of the interaction with peers.

In its earliest definition, social semiotics has been defined as "the science of the life of signs in society" (Saussure, 1974) as cited by Hodge et al. (1988). Signs become the signifiers of meaning in differing modes, actions, and artifacts created in a social domain. "In social semiotics the focus changed from the 'sign' to the way people use semiotic 'resources' because it avoids the impression that 'what a sign stands for' is something pre-given, and not affected by its use" (Van Leeuwen, 2005, p. 3). Bezemer and Kress (2008) state that producers and users are meaning makers or sign makers; however, the making is subject to the semiotic resources available and the maker's interest and how they wish to represent their learning. The potential of the sign, or resource, is only constituted in its position in the discourse that the resource is engaged in, whether in written or spoken form, or any modality considered relevant to the user (Van Leeuwen, 2005). Both Hodge et al. (1988) and Van Leeuwen (2005) acknowledge the importance of recognizing all semiotic resources. For this study, I use the term semiotic resources as defined by Van Leeuwen (2005):

> In this book […] semiotic resources are the actions and artifacts we use to communicate, whether they are produced physiologically – with our vocal apparatus; with the muscles we use to create facial expressions and gestures, etc. – or by means of technologies – with pen, ink, and paper; with computer hardware and software; with fabrics, scissors and sewing machines, etc (p. 3).

This emerges beyond the spoken and written language and texts produced in discourse, but inclusive of activities and artifacts, which themselves can include text, representations, images, gestures, and voice to communicate in their own manner. However, the accumulation of semiotic resources is the basis for the concept of mode, which Kress (2010) defines as "a socially shaped and culturally given semiotic resource for making meaning. *Image, writing, layout, music, gesture, speech, moving image, soundtrack and 3D objects* are examples of modes used in representation and communication" (p.79). The focus on semiotic resources best assists in interpreting the research questions for this study as they pertain to the 'how' students enact these resources and the "what" knowledge they are communicating in their classroom community.

The development of language is a semiotic process and therefore, constantly evolving as new modes are introduced and the contexts in which they are learned change (Halliday, 1993). As defined by Halliday (1978, p. 2) and stated in Morgan (2006, p. 221), a social semiotic perspective recognizes that “language consists of the exchange of meanings in interpersonal contexts of one kind or another” and that this exchange is functional. Drawing on Halliday (1993), social semiotic researchers recognize three communicative meta-functions: the ideational-enact personal and social relationships, the interpersonal- related to human experience and representations, and the textual- multimodal forms of communication. For Halliday’s meta-functions utilized in research see (Alshwaikh, 2018; Alshwaikh & Morgan, 2018; Björklund Boistrup & Selander, 2009; Björklund Boistrup, 2010). This allows for classroom activity to be analyzed from three perspectives. Participating in the discursive aspects of the classroom is what an individual does to interact with their peers in a social or academic setting. Studying language must therefore consider meanings that are being exchanged, informally and formally in the embedded cultures and contexts of the classroom and the output that is being communicated. According to Morgan (2006), the context of the situation comprises activity goals, participants, and the tools available as part of the semiotic structure which is

> formed out of the socio-semiotic variables: field, tenor, and mode.
> The field of discourse may be thought of not simply as the subject matter but as the institutional setting of the activity in which a speaker and other participants are engaged. Tenor encompasses the relationships between the participants, and

> mode refers to the channel of communication (e.g., writing or speech) and other aspects of the role of language in the situation (p. 221).

Therefore, the activity is comprised of the classroom dynamics established for the purpose of the discursive mathematics community, the engagement in interactions of the students present, and the modality in which they communicate their knowledge.

O'Halloran (1998, 2006, 2011, 2015) also considers the phenomenon of semiotic resources and the multi-semiotic nature of mathematical discourse which involves mathematical symbolism, visual display, and language. She contends that the three are so symbiotically entwined that one could not investigate one without considering the others during analysis. O'Halloran summarizes the function of the three semiotic resources as follows:

> the mathematical symbolism contains a complete description of the pattern of the relationship between entities, the visual display connects our physiological perceptions to this reality, and the linguistic discourse functions to provide contextual information for the situation described symbolically and visually (p. 363).

O'Halloran (1998, 2010) utilizes research in secondary mathematics to explain the lexicogrammatical analysis of algebraic equations and how the function of mathematical symbolism affects mathematical discourse and communication. The lexicogrammatical analysis is also considered by Lemke (2003). Although these research studies focus on the systemic functional linguistics of multi-semiotics, the latter research by O'Halloran (2011, 2015) considers multi-modality and its effects on discourse analysis. See also (Bezemer & Kress, 2008; Jewitt et al., 2001; Kress, 2010, 2011).

Beyond utterance, clause, or phrase usage, the mathematical understandings students make and interpret to share within their discursive community are emphasized by the phenomenon of semiotic resources and multi-modality. In particular, a communication framework provides the lens for examining the skills necessary for mathematics thinking and learning and the affordances of modality to that communication (Brenner, 1994, 1998). By exemplifying a multi-semiotic, multi-modal approach, it will be helpful to examine how the "different modes offer different potentials for making meaning" (Kress, 2010, p. 79).

An integral aspect of the social semiotic perspective acknowledges the importance of meaning collectively with social situations and language use (Kress, 2000, 2010, 2011; Morgan, 2006; Sfard, 2001, 2008). The emphasis on the "'social' in 'social semiotics' can only come into its own when social semiotics fully engages with social theory. This kind of interdisciplinarity is an essential feature of social semiotics" (Van Leeuwen, 2005, p. 1). A facet of the sociocultural perspective of learning and teaching necessitates engaging in the mathematics community as a mathematician to "learning the socially learned cultural traditions of what kinds of discourses and representations are useful and how to use them" (Lemke, 2001, p. 298). Lemke (2001) has applied this idea to science and mathematics. A socio-cultural interpretation of teaching and learning goes beyond vocabulary acquisition, word problem interpretation, and computation to enacting skills, sharing knowledge, and communicating mathematically with peers (Moschkovich, 2004). When students can engage in the discourses (informal, formal) of the mathematics community, they then develop the socio-mathematical norms of the classroom. Cobb et al. (2011) extend ideas of norms and mathematical practice to include the "normative purpose of engaging in mathematical activity, standards of mathematical argumentation, and ways of reasoning with tools and symbols" (p. 110). It is possible to engage in mathematics classroom activities and engage in language and mathematical practices simultaneously through that participation.

As Lave and Wenger (1991) suggest, through participation, the students' knowledgeable skills evolve as they collaborate in social practices. An apprenticeship, (Lave, 1991; Lave and Wenger 1991; Vygotsky, 1978), is a way for students to learn new knowledge and engage with experts as part of a learning community. Through this Vygotsky-ian lens (1978), models for an apprentice can originate from other students in the mathematics classroom. As students are engaged in this cooperative learning process, they assimilate skills and understanding from available models. They, in turn, can become the "old timers" and participate in their mathematics classroom, sharing and modeling understanding for others. Within this perspective jointly constructed activity, among teachers and students and student to student, is central to learning in a socio-constructed perspective. The mentoring of knowledgeable teachers or peers in the classroom community assists in scaffolding interaction and learning opportunity (Bruner, 1964).

Summary of Literature Review

This study investigated third grade students' communication of multiplicative reasoning through social semiotic resources. In mathematics, classroom teachers can facilitate development and utilize numerous features of oral and written language, strategies for supporting language expression, whether informal or formal, and multiple modes of representation (gestural, oral, written, textual, or drawn). In understanding the significance of students meaning generating, tools are viewed as an integral mathematical activity for engaging reasoning "with physical materials, pictures, diagrams, computer graphics as well as with conventional written symbols" (Cobb & Bowers, 1999, p 11). In this, they acknowledge a significant connection to socially building language through interaction and the influence of tools in developing mathematical understanding.

Another perspective is the view of informal or everyday register while communicating. This interest stems from Vygotsky's (1978) recognition of the importance of language and social interaction in learning. An informal register allows engagement in the collaborative practice. Students can then use that participation to expand their understanding and formalize mathematical thought and communication. A goal for the mathematics community of practice is for teachers and peers to acknowledge informal thoughts and knowledge in the shared setting and progress to constructing more mathematical ways of expressing thought (Vygotsky, 1978). The promise of building on and modifying the everyday, informal language towards a more formal school language enriches students' experiences and accessibility for interactions in the classroom environment.

The constructs central to this research comprise the notion of communication, semiotic resources or modality, and multiplicative understanding, and the interaction of these components for supporting learning in the mathematics classroom by students. Consequently, I draw on a socially constructed semiotic perspective (Cobb & Bowers, 1999; Brenner, 1994, 1998; Gee, 2014; Greeno, 1994; Halliday, 1993; Morgan, 2006, 2014; O'Halloran, 2010; Presmeg et al., 2016; Sfard, 2001, 2008; van Leeuwen, 2005; Vygotsky, 1978) to provide a lens for exploring how students interact with and utilize multimodal tools to communicate multiplicative thinking and learning. The role of language in mathematics has been extensively studied and confronts a critical challenge of associating knowledge from diverse fields (applied linguistics, sociolinguistics, cultural and social psychology, discourse studies, semiotics); as well as discipline specific knowledge from mathematics research (Morgan et al., 2014; Planas & Schütte, 2018).

Developing mathematically literate students is more than reading and interpreting word problems. It is engaging with mathematical texts, whether formally or informally, to broaden the constraints of literacy and its' relationship to mathematics. Progressing beyond distinct utilization of reading, writing, listening, and speaking as a means for mathematical output, but as a construct for linguistic communication, which enlists those expressive and receptive modes along with representations, visuals, and non-linguistic forms of communication, such as signs, for sense-making. Educators invested in their students' mathematics and literacy advancement create opportunities for students to engage in mathematical practices and share conceptual understanding, while also attending to the types of communication and discourse students are enacting, Thus, building the more communicatively inclined community we strive for in mathematics.

CHAPTER THREE: METHODOLOGY

Introduction

Methodologies need to properly evaluate the process and the content of mathematical discourse and investigate the factors contributing to mathematical discursive communities. Researchers can use many theoretical and methodological resources to see patterns in social interactions; however, "recognizing these patterns and what they achieve provides tools for analyzing classroom processes and can also inform development of teaching practice" (Morgan et al., 2014, p. 847). This study examined several activities, through a social semiotic lens, exploring mathematical conceptual themes communicated discursively by students based on components of Brenner's (1994, 1998) communication framework. All three components of the framework support the development of mathematical understanding; however, this study focused on Communication About and Communication In mathematics as the third component, Communicating With math as a tool for real world problem-solving, was not the aim of this study. The framework will assist in actualizing conceptual knowledge and guiding efforts to research this construct across a specific mathematical domain (Crooks & Alibali, 2014)

As Morgan et al. (2014, p. 847) stated, there should be no distinction between, "doing mathematics and thinking mathematically. Detailed characterization of the nature of mathematical language thus provides a means of describing the ways in which learners are engaging in mathematical activity." With a student-generated visual, semiotic resources accumulate into a multi-modal tool to represent and communicate mathematics conceptual knowledge. The methodology chapter will address the features of a qualitative case study, participants and site selected, data collection and analysis. We will examine data collected, exploring themes communicated discursively by students and how social semiotics are weaved throughout activities. The qualitative analysis of selected interview and observation transcripts, documents, and tools will be conducted with the aim of developing descriptions of Discourse, individually and interactively, and mathematical themes across activities. The research questions guiding this study are as follows:

Research Questions

How do semiotic resources and student-generated tools enhance students' abilities to communicate their multiplicative reasoning?

1. How does the utilization of the student-generated tool assist students in communicating mathematically?
2. What semiotic resources are evident in student activity and how are students utilizing them?
3. How does the student-generated tool influence the expression of multiplicative understanding?

Study Description

This chapter explains the method of discourse analysis chosen based on social semiotic theory. The nature of mathematical discourse is multi-semiotic because it involves semiotic resources of "mathematical symbolism, visual display, and language" (O'Halloran, 1998, p. 359). Those semiotic resources encompassing language modalities of reading, writing, listening, and speaking engage images drawn by students to represent their knowledge composing multiple modalities. The tenets of social semiotics are the basis for the discourse analytic approach chosen for the qualitative case study undertaken for this dissertation. According to Gee (2014) the theory of discourse analysis can be applied to multi-modal texts composed of words, images, and other modalities, with the intent to communicate. Discourse analysis can be enacted because "discourse is about communication and we humans can communicate via other symbol systems (e.g., mathematics) or via systems composed using modalities other than language or ones composed by mixing other modalities with language" (Gee, 2014, p. 187). Therefore, discourse analysis may serve dual purposes: first, it explores a socially constructed multi-modal tool utilized as an activity to enhance language use individually and interactionally during mathematical discourse; second, it supports investigating the language used during the studied activities and how they relate to communication of mathematical knowledge during mathematical discourse.

A Discourse presents distinctive ways to communicate through modalities and interactions which help students enact and interact using various objects and tools (Gee, 2014). "Discourses (being and doing kinds of people) exist in part to allow people to carry out certain distinctive activities" (Gee, 2014, p.178). With this focus, students can establish themselves as practicing mathematicians within a classroom community. By examining the semiotic resources composed of mathematical symbols, visuals (student-generated tool), written (words or images), and spoken languages that students produced and used while reasoning, we can analyze the structure of the activities that allowed for the use of semiotic resources and what was produced socially; as well as the thematic structures of the mathematics content knowledge communicated by student mathematicians (Chapman, 1995, 1997, 2003).

As Chapman (2003) noted, the theoretical framework of social semiotics reasons that the mathematical meanings are constructed within language practices and "school learning areas are social practices in which teachers and learners use language, together with other semiotic systems, to make meanings" (p. 154). Through this methodology, Chapman utilizes analytical terms from Lemke (1985) to argue that analyzing the formation of the activity structures of the social interaction and thematic structures, which show how meaning is being used, are employed for discourse analysis. Activity structures, summarized in Chapman (2003) from Lemke (1987) are "regularly repeated and socially recognizable sequences of actions" (p. 155). Thematic structures, utilized for discourse analysis in Chapman (2003) from Lemke (1987) are defined as "systems of relations among themes. Discourse analysis of thematic structure considers how the language of text is used to develop themes, and to relate themes to each other" (p. 156). Conducting an exploration of the activity and thematic structures supports the conceptual framework of communicating multiplicative knowledge and the theoretical framework of social semiotics.

A qualitative case study is defined as an exploration of a real-life, bounded case, through detailed data collection, with rich descriptions of cases and themes (Creswell & Poth, 2018; Merriam, 1998; Merriam & Tisdell, 2015). A case study design was more appropriate for my study for several reasons. First, it allows a researcher to concentrate on a single phenomenon, or case, and aim "to uncover the interaction of significant factors characteristic of the phenomenon" (Merriam, 1998, p. 29). Those factors being communicating multiplicative thought and social semiotics. Second, as the objective of this study was to answer the "how" semiotic resources and multi-modal tools enhance communication the descriptive aspects of a case study will assist in telling the story of what students produced during activities. I conducted a descriptive exploration of how social semiotics can enhance student communication of multiplicative knowledge through activities implemented within the bounds of a third-grade classroom. By limiting this study to a single classroom and narrowing foci to one female and one male student from each of the knowledge levels (beyond, on level, approaching), I was able to analyze the discursive practices and multiplicative themes shared semiotically by groups of students within the case. This within-case analysis encompasses thematic analysis which looked for common themes that transcend the case and is rich in the "context of the case" (Creswell & Poth, 2018; Merriam, 1998; Yin 2014). As cited in Merriam (1998), Miles and Huberman stated that "by looking at a range of similar and contrasting cases, we can understand a single-case finding, grounding it by specifying *how* and *where* and, if possible, *why* it carries on as it does. We can strengthen the precision, the validity, and the stability of the findings" (p. 40).

<u>Site</u>

The site for this study was chosen in a purposive manner (Creswell, 2012) due to my work as a co-teacher, which I will address in *Role of the Researcher* section, and its use of Thinking Maps: Language for Learning resource, which will be addressed in the data collection section. The study was conducted in the 2021-2022 school year in an urban school district in the Northwest. The school district's demographics are 78.3%-

White, 11.4%-Hispanic, 4%-Black or African American, 3%-Asian and 1%-Native Hawaiian or Other Pacific Islanders.

The school chosen as part of this study is Burton Elementary School (BES; a pseudonym). BES is a school with a traditional program (English instruction) and dual-language program (Spanish and English instruction) in one building and a total of five hundred eighty-four students from Pre-Kindergarten to sixth grade. The third-grade traditional classroom was part of this study. BES is a title one school which qualifies for students' free breakfast, lunch, and snack programs. The demographics of the school are 49%-White, 36%-Hispanic, 7%-Black or African American, 5%-Two or more races, 1%-Asian and 1%-Native Hawaiian or Other Pacific Islanders.

Participants

The classroom selected as part of this bounded, case study was a third-grade classroom in a Title I school. Twenty-six students (eleven female, fifteen male) and one educator were part of this class. The classroom was selected based on their availability and their interest in participation. Specifically, six students were purposively sampled from the twenty-six students for data collection and analysis in this case study. Part of purposive sampling "involves identification and selection of individuals or groups of individuals that are proficient and well-informed with a phenomenon of interest" (Etikan et al., 2016, p. 2). Before engaging in the study, Institutional Review Board (IRB) approval was sought along with participant permission from the school district, principal, student participants and guardians (see Appendix A for IRB approval). Jansen et al. 2009, specified that students' mathematical learning goals should be shared or "mutually understood and committed to by all participants" (cited by Cobb and Jackson, 2011, p.

12). Therefore, educator participants were consulted from the studies' development. Theoretically, we wanted to explore the semiotic resources and modalities students use to communicate their mathematical development as it emerges in the social situation of the classroom in the content area of multiplication.

The students were in their seventh month of third-grade. Class demographics are 62%-White, 7%-Hispanic, 7%-Black or African American, 12%-Asian, and 7%-Two or more races. The third-grade class at Burton Elementary School (BES; a pseudonym) comprise 26 students. Of the 26 students, 20 have been part of the class since beginning the school year, starting August 2021. Before their third-grade year, their first and second-grade years of schooling were interrupted due to the Covid Pandemic. The students did not have formal instruction in multiplication before unit instruction as part of the third-grade curriculum as it is not a significant work of any prior grade.

Before beginning the study, the students completed three units of instruction that included the teaching of multiplicative reasoning. Unit two of multiplication instruction began in October 2021 with a combination of teacher-created materials and the Unit Module from the Developing Mathematical Thinking Institute curriculum resources (DMTI, 2021, https://www.dmtinstitute.com/curricular-resources). The lessons were co-planned and co-taught by the researcher (myself) and classroom educator Mrs. Brooks (a pseudonym) until December 22, 2021. See table for the sequence of instruction:

Table 1 Sequence of Instruction

Duration:	Unit Activity:
October 18-November 19, 2021	Literacy Mathematics connection Amanda Bean's Amazing Dream integrated lesson. DMTI Unit 2 Multiplication, Arrays, and Area
December 20-22, 2021and January 10-14, 2022	Problem-solving with four operations – single and multi-step problems Teacher created contexts
January 18-February 17, 2022	DMTI Fact Fluency Module – strategy card development DMTI Unit 5 Multiplication, Division, Area, and Perimeter Teacher created area and perimeter project

Data was collected on six students purposively sampled from the case's available students and whose parents gave informed consent for their participation in the study. These six students were selected based on their *I-Ready Diagnostic* (Curriculum Associates, 2017), a district required curriculum-based measure given to students three times a year (August, December, and May). The teacher and I reviewed scaled scores from students I-Ready Diagnostic and categorized the students into groups of beyond, on, and approaching grade level based on their scores. The students were assigned a number by the teacher. The numbers were then entered for the students on slips of paper and three piles of numbers based on the categories listed above were made. A number was selected randomly until we had one female and one male for each category of beyond, on, and approaching level for a total of six students.

The students participated in six activities as part of established classroom activities. To address the main research question: *How do semiotic resources and tools enhance students' abilities to explain their multiplicative reasoning?* and successive sub-questions, I analyzed the activity and thematic structures afforded by using a student-generated tool (described in more detail later), which served as an artifact for data collection. Video recordings of the lessons allowed me to analyze the social interactions and the student-generated tools impact on its development and utilization in classroom discourse considering themes and modalities present.

Semi-structured post-interviews with the same selected students provided an in-depth and personal account of the activities, modality choices, and use from different students of varying knowledge levels.

Role of the Researcher

The selection of this site and classroom as a setting for this research study were due to my role as a co-teacher at the site. In research co-teaching has been defined as "two or more people sharing responsibility for teaching all students assigned to a classroom" (Villa et al., 2013, p.3). It has now been expanded to include partnerships between the general educator and math and reading specialists, the gifted and talented teacher, and the ELL teacher (Honigsfeld & Dove, 2010, 2019; Villa et al., 2013). This approach calls for crossing the boundaries of typical collaboration and combining the skills of two highly qualified educators to share responsibility for student instruction (Bahamonde & Friend, 1999; Friend et al., 2010; Honigsfeld & Dove, 2010, 2019; Theoharis & O'Toole, 2011; Villa et al., 2013). Shared responsibility includes lesson planning, teaching, classroom management, assessment, grading, and student and parent feedback. From the shared purpose and focus on student learning, both educators must be willing to meld their teaching styles, expand their pedagogies and present themselves as equal partners during instruction.

Being an already established colleague at the school and a participant as an educator in co-teaching for the last three years has allowed me to understand the third-grade team, their long-term mathematics goals, and insight into their daily mathematics instruction. Furthermore, to present an in-depth analysis of this case study, it is necessary for the researcher to develop as much expertise in relevant topic areas as possible, collect and integrate many forms of data collection (interviews, observations, documents, audio-visuals, etc.), and utilize them to describe themes, cases and perform cross-case analysis (Creswell & Poth, 2018). As the researcher, it is important to note my positionality affords me an "inside look" at these partnerships. It will be essential to generate rich, detailed descriptions (Merriam & Tisdell, 2015), triangulate and corroborate evidence (Lincoln & Guba, 1985, Yin, 2014), and collaborate with participants consistently for accuracy of accounts, peer review, and debrief of data and process (Creswell & Poth, 2018; Lincoln & Guba, 1985; Merriam & Tisdell, 2015). As a co-teacher, I planned with every grade level weekly for 45 minutes from August to December. This allowed us to discuss curriculum, weekly learning goals, assessments, and short and long-term planning goals.

Although I was a participant-observer in the grade level co-planning sessions, I was not a part of the daily lesson delivery as a co-teacher during this study. I consider my position parallel to a design researcher who seeks to establish relationships between the research of instructional design and its practical application, student learning individually and socially, and tools which support development (Bell, 2004; Cobb, 2003; Cobb et al., 2003, Cobb & Jackson, 2011, Prediger et al., 2015). As a known educator in the building, I was not present in the classroom during lesson recordings.

Data Collection

This study reflects the principles for qualitative case study (Creswell & Poth, 2018; Merriam & Tisdell, 2015), while combining analysis perspectives from semiotic theory (Halliday, 1993; Hodge et al., 1988; Kress, 2010; O'Halloran, 2015) within socially constructed Discourse (Brenner, 1994, 1998; Gee, 2004, 2014). To address the research questions guiding this study, the data was collected in a third-grade classroom over six instructional lessons. Data collection from students included one written prompt response, two student-generated tools, and one multiplication probe. Video data consisted of video recordings of lessons (for a total of three lessons) and audio recordings of semi-structured interviews (a total of six). These data sources are detailed below (see Table 2 for data collection and timeline).

Table 2 Data Collection and Activity Timeline

Project Activity										
	Week 1					**Week 2**				
Activity 1: Prompt (no tool)	**D1**	**D2**	**D3**	**D4**	**D5**	**D6**	**D7**	**D8**	**D9**	**D10**
Prompt for student response What do you know about multiplication?	x									
Data Collection: Video-taped lesson, document	x									
Activity 2: Prompt (with tool)										
Prompt for student response What do you know about multiplication?			x							
Data collection: Video-taped lesson			x							
Activity 3: Frame Thinking (with tool)										
Continue with tool from activity two					x					
Data Collection: Video-taped lesson, artifact					x					
Activity 4: Probe (no tool)										
Multiplication fact probe						x				
Data collection: document						x				
Activity 5: Probe (with tool)										
Multiplication facts added to tool								x		
Data collection: Video-taped lesson								x		
Activity 6: Frame Thinking (with tool)										
Continue with tool from activity five										x
Data collection: Video-taped lesson, artifact										x
Interviews: Day 12-14										

Documents

The prompt for activity one initially addresses students' understandings of multiplication. Students were prompted with the question, '*What do you know about multiplication?'* The structure of activity one required students to respond to a writing prompt. Writing prompts are part of classroom instructional routines and are a cross-curricular tool for students to generate ideas about a given topic with the potential to share those ideas discursively. The interactive routine for activity one provided a time to communicate their ideas about multiplication individually, then interactively with a partner, and finally with the class. A time allotment of ten minutes was given for students to write with additional or less time provided by the teacher as she monitored the classroom. Using their responses to the prompt, the teacher commenced classroom discussion and sharing ideas. Each discourse is denoted in a different color to differentiate from individual ideas and those gathered through discourse with partners and class. The documents were collected and secured for within-case analysis with the tool constructed from activities two and five. I compared the document from students' writing prompt response to what was produced in their student-generated tool to analyze the discourse produced in the activity structure. Recorded observations provided confirmation of the activity structure and discourse of the class, and semi-structured interviews provided insight into individual student thoughts shared in the document and to triangulate data.

Another document slated for collection from activity four was a multiplication fluency probe. Students were asked to complete as many of the facts that they could recall assessing student procedural fluency of multiplication. The fluency probe gave insight into students' automaticity with facts and what kinds of facts were answered. The semi-structured interviews allowed a glimpse beyond the procedural fluency of the probe to the reasoning and strategies that are engaged. Analysis will examine student expression of their understanding through the probe and tool rather than only observing the traditional 'drill and kill' approach to multiplication fact fluency.

<u>Tool</u>

The notion of mapping thought to convey meaning and reflect learning has been popularized by the use of graphic organizers (Dexter & Hughes, 2011; Fisher & Frey, 2018; Monroe, 1998; Zollman, 2009), concept mapping (Baroody & Bartels, 2001; Brinkmann, 2003; Novak, 2006; Ryve, 2004) and Thinking Maps® (Hyerle, 1996, 2004, 2008). All have been utilized as instructional tools in classrooms across content areas, providing a visual pathway to represent and record valuable information before, during or after learning. This makes mapping a way to deliver content instruction, facilitate student thinking, and be useful for assessment (Baroody & Bartels, 2001; Brinkmann, 2003; Novak, 2006; Ryve, 2004). Thinking Maps® are a routine picture of a thought process that students can depend on to share and filter information. They are different from graphic organizers as they focus on a consistent visual that co-exists with a specific thought process (Hyerle, 1996, 2004, 2008). According to Hyerle (2008), the "visual-spatial-verbal displays of understanding support learners in *transforming static information into active knowledge* . . . uniting linguistic, numerical, and scientific languages together on the same page" (p. 7). The opportunity for students to generate a visual tool of their learning, with differing semiotic resources, will be a critical portion of the data generated that address the research questions of this study. Although there are eight Thinking Maps® visual tools, this study will focus on using one of these as a student-generated tool.

The Circle Map's (Hyerle, 1996, 2004) purpose is to define a concept with context. Other purposes are its support in activating prior knowledge, telling, brainstorming, listing, and connecting information on a given topic. The idea is stated in a small middle circle of the paper. The outside of the circle includes information about the central idea produced by the user, including words, phrases, drawings or other representations indicative of what they consider relevant about the topic. After completing the idea-generating portion of the tool, all thought is framed within an outer square, also known as the Frame of Reference (Hyerle, 1996, 2004). Still referencing the idea in the middle circle, the frame of reference extends thought to who or what influenced the ideas and forms added to the circle, and statements synthesizing "what?" the information is, and "why" it is relevant.

BES has been integrating Thinking Maps® as part of its curriculum across grade levels kindergarten through sixth grade as a schoolwide initiative since 2017. An initial full-day training was implemented in 2017 with ensuing follow-up trainings (45 minutes, periodically throughout the year) and refresher training (half-day August 2021). Initial and follow-up trainings were provided by myself and District personnel. Each grade level is responsible for formulating a plan for introducing and instructing all eight maps to students within the first quarter of each school year. Mrs. Brooks has been a participant at all trainings since the onset in 2017.

Merriam and Tisdell (2015), consider artifacts as things or objects that represent some form of communication. To address the research questions, activities two and five comprised the usage of the student-generated tool as a form of visual communication. First, students generated semiotic ideas about multiplication in the outer circle individually and then collectively through classroom interaction. The interactive routine for activity two and five provided a time to communicate their ideas about multiplication individually, then interactively with a partner, and finally with the class. Activities three and six will involve the conclusion of the tool with the inclusion of the frame of reference first individually and then through classroom interaction. Each interaction is denoted in a particular color to differentiate from individual ideas and those gathered through discourse. The central ideas for the student-generated tools are *What do you know about multiplication?* for activity two and *Multiplication Facts* for activity five. The tools were collected and secured for analysis.

Semi-Structured Interviews

Six students were chosen for semi-structured interviews. Student scaled scores on their I-Ready measure designated students as approaching, on, and beyond grade level. Two students (one female and one male) were chosen from each listed category. The interviews were audio recorded to discuss the documents and student-generated tool as created during the activities. As the tool is individualized to the students, the opportunity to explain and justify semiotic resources added to their tool and the impact of the interaction on the further development of mathematical ideas adding insight to their mathematical conceptions.

A self-developed interview protocol (see Appendix B) was used to introduce the interviewer, review the purpose of the study, list interview questions to use as a guide, and the space to record information. As highlighted by Merriam and Tisdell (2015), interviews with structured questions can obtain more specific information, while the addition of more open-ended questions encourages participants to freely respond and elaborate. The interview protocol for this study, included a variety of specific questions to obtain information about the work produced by students on the documents and the student-generated tool and open-ended questions to elicit information about the semiotic resources used to convey thought and the impact of the interaction on their conceptions.

Guided by the research questions, the semi-structured interviews were critical to exploring the questions stated in this study. Through these interviews, information was gathered from students of varying levels about their multiplication conceptions, modalities used to communicate those conceptions, and classroom interactions that contributed to conceptions. Data from these observations were transcribed verbatim for translation of speech into writing. Verbatim transcripts are most often used by qualitative researchers to capture the exact dialogue used by participants, despite how time-consuming this process can be (Lodico et al., 2006). I compared this data to the documents and tools produced in classroom lessons and recorded classroom observation during analysis through triangulation.

Video-Taped Lessons

Classroom activities were video recorded for observation and analysis as part of this study. Classroom observations of activity one (writing prompt), activity two and three (student-generated tool), and activity five and six (student-generated tool) were completed. Classroom activities were approximately twenty to thirty minutes in duration each. Analysis of video observations for multi-modality “involve repeated viewing of the data . . . honing in on excerpts . . . and viewing the data alongside the logs” in order to refine criteria for sampling data and developing analytical ideas based on the research questions (Bezemer & Jewitt, 2010, p 188).

Whether participating in classroom discursive practices with or without using the students' generated tool, the interactions are a valuable component of data connected with the social semiotic theory motivating this study. Video recordings of classroom interactions allow perspective on the collaborative setting and the opportunities for students to communicate and clarify their ideas with and without tools. These videotaped interactions during classroom discourse were transcribed verbatim. The analysis sought to observe themes across activities and the semiotic resources present in communicating mathematically during the discourse. Triangulation amongst other data collection methods was sought.

Data Analysis

According to Merriam and Tisdell (2015), the theoretical framework and research questions should guide what the researcher aims to observe. Therefore, the data collected was aligned with the research questions within the lens of communication (Brenner, 1994, 1998; Chapman, 2003; Gee, 2004, 2014; Lemke, 1987) and social semiotic theory (Halliday, 1993; Hodge et al., 1988; Kress, 2010; O'Halloran, 2015). Specifically, the gathered data provided insight into how a multi-modal student-generated tool influenced communication and how knowledge was conveyed through modality, both individually and interactively. The figure 2 shows how I initially attempted to analyze data and address the findings.

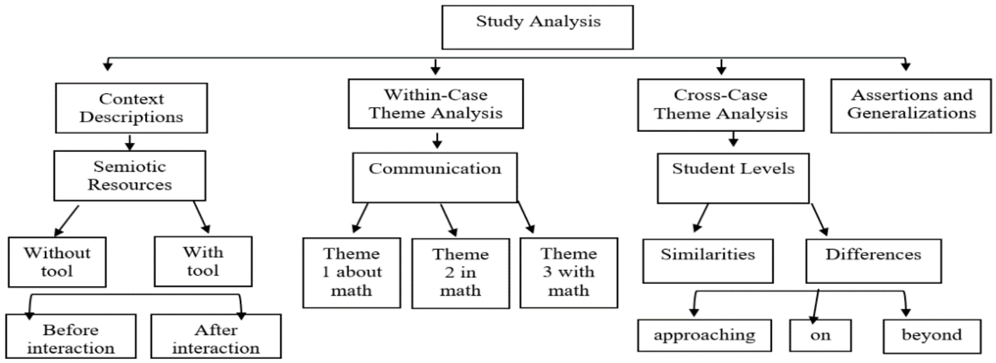


Figure 2 Initial Data Analysis

Creswell and Poth (2018, p. 185-197) refer to the analysis of data holistically and iteratively, rather than as step-by-step process through activities, as the "Data Analysis Spiral". The research process was guided by this 'spiral' which includes the procedures of memoing, coding, analysis, representing and visualizing data. The process relied on interview and observation transcripts, documents, and tools produced from the activities. I further analyzed the discourse by looking at the activity and thematic structures that were evident in the data.

The data collected investigated how the use of a student-generated tool assists in sharing multiplicative knowledge through semiotic resources. I followed the recommendations of several authors of research (Creswell & Poth, 2018; Merriam, 1998; Merriam & Tisdell, 2015; Saldana, 2013) by collecting and analyzing data to follow the inductive nature of qualitative research. As the data was collected through documents and tools, I began the process by repeated readings of the data to visualize what the data was showing and immerse in the details (Creswell & Poth, 2018). This allowed for the initial organization of data in table form and memoing process to begin. These memos allowed for primary codes and themes to emerge from the data.

Within-case analysis focused on those themes evident in the semiotic resources and student-generated tool utilized for participating discursively. The iterative process of assessing themes across data assisted in spiraling through analysis (Creswell & Poth, 2018) to formulate connections among the ways students of varying abilities are communicating multiplicatively through semiotic resources. Through many iterations of reading and reviewing data figure 3 represents a focus on discourse analysis undertaken.

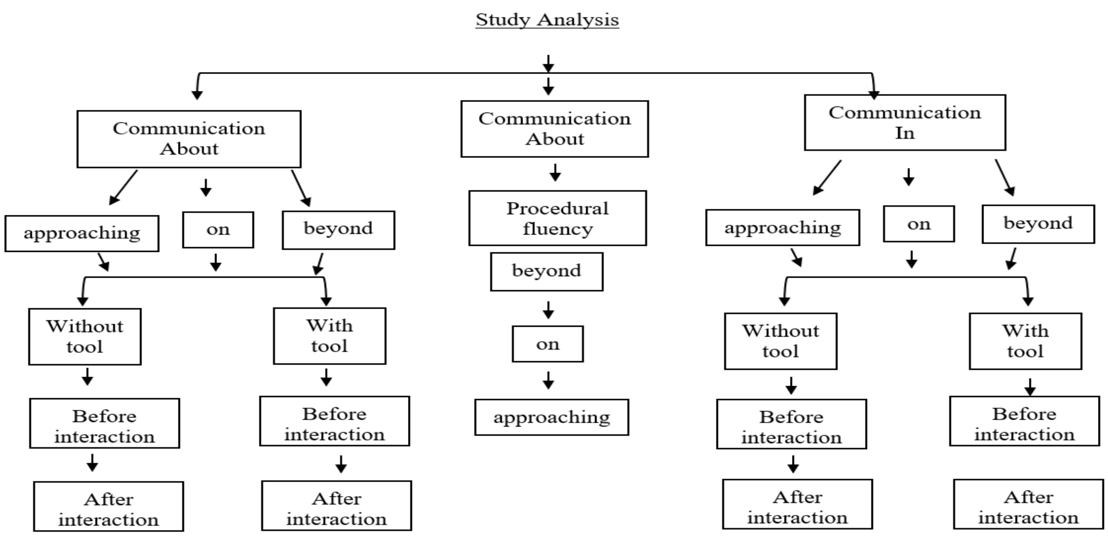


Figure 2 Actual Data Analysis

Memoing

I created document memos (Creswell & Poth, 2018) to capture developing concepts and evolving ideas which are "helpful for summarizing and identifying code categories for themes and/or comparisons across questions or data forms" (p. 189). These descriptive memos assisted in applying codes and developing themes in the initial observed document and tools. This inductive approach to analysis allowed me to see emerging themes that correlated to an existing framework for mathematical communication identified in the literature on mathematics and language. Cyclical collecting of coding and memoing allow codes to cluster and interrelate, where emerging categories are identified (Saldana, 2006), supporting the data analysis spiral (Creswell & Poth, 2018). Analytic Memos (Saldana, 2006) helped to structure the sequence of the Discourse analysis into activity and thematic structures, by organizing the analysis, and determining the continued spiral through the data towards results.

Activity Structure

The structure of the activities assisted in the organizing of data into prepared files and delineating text units based on the interactions as part of the activities. The structure of activity one required students to respond to a writing prompt. Writing prompts are part of classroom instructional routines and are a cross-curricular tool for students to generate ideas about a given topic with the potential to share those ideas discursively. The interactive routine for activity one provided a time to communicate their ideas about multiplication individually, then interactively with a partner, and finally with the class. Using their responses to the prompt, the teacher commenced classroom discussion and sharing ideas. Each discourse is denoted in a different color to differentiate from individual ideas and those gathered through discourse. This allowed for the data to be categorized into a table and sorted by those semiotic resources shared without the tool. The data was further organized by text structures following the interaction of the activity and were broken apart by individual, partner, and classroom discourse. The student-generated tool was organized in a similar way. As noted previously, the tool is an established cross-curricular instructional tool as part of the classroom involved in this study. The interactive routine for activity two also provided a time to communicate their ideas about multiplication individually, interactively with a partner, and then with the class. Each discourse is denoted in a different color to differentiate from individual ideas and those generated socially.

The data was then categorized into a table and sorted by those semiotic resources shared with the student-generated tool. These differing segments of data assist in describing the related phenomena with category names (Lodico et al., 2006). Text structures based on the individual, partner, and classroom discourse were categorized accordingly. When analyzing data, it was noted that a column for off-topic responses was needed as a category to provide the most comprehensive understanding of all student thought shared in response to the question in activity one and two.

Thematic Structure

I used an inductive approach to analysis in which thoughts from memos began the initial coding process (Merriam & Tisdell, 2015). In consistent reading and reviewing of the data before coding, student responses about multiplication were evident with the tool and without. The data can "furnish descriptive information, verify emerging hypotheses, advance new categories, offer historical understanding, track change and development," and so on (Merriam & Tisdell, 2015, p. 182). Emerging codes in the data furnished tracked changes amongst the different interactive structures, descriptive information, and advanced categories for codes.

Coding is "the process of identifying different segments of the data that describe related phenomena and labeling these parts using broad category names" (Lodico et al., 2006, p.305). Those multiplicative ideas that were identified as initial codes in the memos then became those broad category names. Data was then highlighted and coded within the text. It is recommended by Merriam and Tisdell (2015) to "review the literature that you have consulted in setting up your study" (p.177) to enhance analysis. As I viewed the multiplicative thought coded initially, themes related to communicating mathematically emerged in connection to the aims of the conceptual frameworks of communication, multiplication, and social semiotics. In particular, Brenner (1994, 1998) classified mathematics classroom communication into three categories Communication About, In, and With Mathematics. Communication With encompasses the utilization of mathematics as a tool for enacting math through real world situations and problem solving, which is beyond the scope of this paper. The components of Communication About and Communication In incorporate the understanding and language aims of the conceptual framework assisting in classifying initial codes within themes. Table 3 below outlines the initial codes and how they relate to themes and their descriptions.

Table 2 Initial Codes

	Initial Codes	Themes
Multiplicative thought	Rules Processes Strategies Reasoning	Communication About Mathematics Reflecting, describing, reasoning with others
	Vocabulary Language Symbols Representations Facts	Communication In Mathematics Register and representations

The recurring patterns in the initial codes are subsumed under these themes to convey mathematical knowledge about multiplication. Therefore, Brenner's framework embodies the communicative aspects of language expression and mathematical knowledge as interwoven constructs to help define and complete the analysis of the data to answer the research questions. Finally, detailed descriptions of semiotic resources, and modes, used by the students during activities and interviews were categorized in the framework.

I coded the data sources collectively for common themes which emerged from the initial memoing and codes to classify those themes tied to communication frameworks for mathematics to triangulate data amongst tools, documents, and transcripts. These multiple methods of data collection and analysis for triangulation function to strengthen reliability (Merriam, 1998). Recorded observations of activities one, two and three, and five and six were used as a secondary source of data to corroborate activity sequence and student interactions. Those activities were transcribed verbatim. The coding framework identified above was then applied to transcripts for coding and analysis based on the thematic structure. Student interviews triangulated data from activities and classroom observations. The semi-structured student interviews of the six students of varying levels were audio recorded and translated verbatim. The coding framework was applied to these transcripts for coding and analysis. Triangulation becomes the process of confirming evidence and comparing these different sources and perspectives from different participants (Lodico et al., 2006).

Analysis Summary

In qualitative analysis, validity can be attempted in different ways. One way is through within-case analysis and cross-case analysis. Defined by Creswell and Poth (2018) within-case analysis provides the “description of each case and themes within the case, followed by a thematic analysis across the cases, called a cross-case analysis” (p. 100). The within-case analysis is bound within the single third-grade classroom and the description of activity structures and themes coded and identified through frameworks.

The cross-case analysis observed those themes within randomly selected participants of varying knowledge levels (beyond, on, and approaching, and one female and one male from each of those categories). This provides an in-depth glimpse into the bounded case. Depth allows for rich, thick descriptions which allows readers to determine the generalizability of the research position to their own situations (Merriam, 1998).

Qualitative researchers seek to accurately represent the views and experiences of their participants through the participants own words (Lodico et al., 2006). During semi-structured interviews students reviewed their documents and tools created and collected for data and used them to share ideas and respond to questions. The modality of reading their ideas aloud during the interview verified written modalities in their documents and tools. Interviews also allowed students to express why they added ideas or resources and their importance in communicating their multiplicative knowledge. Staying true to participants' views and perspectives and sharing their ideas accurately can be validated through respondent validations, or member checks (Creswell & Poth, 2018; Merriam, 1998; Merriam & Tisdell, 2015). These member checks, defined by Merriam and Tisdell (2015), are the process involved in "taking your preliminary analysis back to some of the participants and ask[ing] whether your interpretation "rings true"" (p. 217). A member check for plausibility of findings was completed by Mrs. Brooks, the classroom teacher. I shared the data analysis figure, thematic coding structure (with initial codes and final themes), and organized tables of data. Mrs. Brooks reviewed data looking for variants that would not fit within the thematic structures of communication about and in mathematics. We jointly examined how student thought was coded within the framework and discussed particular student ideas and their representations in each theme.

Limitations

In this section, I will address the limitations of this study as they pertain to generalizability, timeline, reliability, and the COVID pandemic.

1. There is decreased generalizability due to small sample size of one classroom, with six students, from the same school district. The six students randomly chosen from the class from groups of students of varying knowledge levels, may not be generalizable to other classrooms and school settings.
2. The study took place a month after the students' final multiplication instructional unit ended. Many of the students expressed "forgetfulness" about the content since it had been so long since the instructional unit affecting reliability of data produced by participants.
3. Data was collected in the form of documents, tools, video observations and semi-structured interviews within a two-week period. Activities could have been too close together to get accurate data from different activity structures. One could also consider collecting data from the beginning of the multiplication instructional unit to the end, attempting a longitudinal study.
4. Participants reviewed their documents and tools produced during the activities in the semi-structured interviews. Mrs. Brooks, the classroom educator, also participated in a member check of the data tables and coding structure. A separate peer review could have strengthened internal validity.
5. COVID pandemic may have impacted participants in the form of learning loss if quarantining caused them to miss consecutive days during the unit.

Methodology Summary

The methodology defining this study, and the analyses and interpretations of the data, derive from general qualitative case study research (Creswell & Poth, 2018), through a semiotic and social perspective. To address the main research question, and sub-questions, attention was given “to the variety of representations developed by the students, the tools and manipulatives used, and how these interact with the class’s developing ways of talking, explaining, and justifying their thinking” (Prediger et al., 2015, p. 882). The Discourse were analyzed for dual purposes: how the multi-modal student-generated tool enhances communicative language use individually and/or through interaction and the types of mathematics communication expressed.

An inductive data analysis involved examining many small pieces of information and abstracting a connection between them (Creswell & Poth, 2018; Merriam & Tisdell, 2015). The connection was found in extant literature on communication and mathematics and themes developed to support the aims and objectives of this study. Language and modalities are embedded in every aspect of the framework and support a focus on communicating themes about and in, multiplication (Brenner, 1994, 1998). It is with this thematic lens of the Discourse that the findings are presented in answer to the research questions of this study.

CHAPTER FOUR: FINDINGS

Introduction

As stated in Chapter one, this study examined the role of semiotic resources in communicating students' understanding of the mathematics content area of multiplication using a multi-modal student-generated tool. In mathematics, educators can facilitate the development and utilization of expressive and receptive language, whether informal or formal, and semiotic resources, including tools that can represent and visualize student thinking. In understanding the significance of students' meaning-making, tools are viewed as an integral mathematical activity for engaging reasoning (Cobb & Bowers, 1999). This acknowledges a connection between building language through social interaction and the influence of tools and semiotic resources in developing mathematical understanding.

The study design was a qualitative case study that included a single third-grade class with an in-depth look at six students of varying knowledge levels. I collected data through artifacts and documents produced during classroom activities, classroom observations, and one-on-one interviews. Discourse analysis investigated how language was being used, and how meaning was constructed in different activity structures within the classroom community. In this study, the "how" pertains to the thematic structure of communicating meaning about and in mathematics with semiotic resources in the social context of a classroom. I conducted data analysis through triangulation and coding of interview transcripts, observational transcripts, and classroom artifacts and documents.

This chapter conveys the research study's findings through the summation of the communication expressed. Students synopsized their thoughts on multiplication with and without a student-generated tool, individually and socially, through interactive activities. Two dominant themes structure the findings: communication about and communication in mathematics while considering how students of varying abilities and gender utilize semiotic resources to demonstrate knowledge. Ideas shared during classroom interactions could have been added to student documents and tools. Interviews triangulated how or if students communicated those ideas. Descriptions of the themes and data extracts were selected from the Discourse analysis to portray patterns of language communication and semiotic resources utilized to address the research questions, aims, and objectives of this study. For complete data tables of each category of students' communication see Appendix C for Beyond, Appendix D for On, and Appendix E for Approaching. The research questions were as follows:

Research Questions

How do semiotic resources and student-generated tools enhance students' abilities to communicate their multiplicative reasoning?

1. How does the utilization of the student-generated tool assist students in communicating mathematically?
2. What semiotic resources are evident in student activity and how are students utilizing them?
3. How does the student-generated tool influence the expression of multiplicative understanding?

Communication About Multiplication

When communicating about mathematics (Brenner, 1994, 1998), students describe their thinking and reason about their own processes or the processes of others. This links the theme closely with the CCSS (2010) Mathematical Practice Standards, which ask students to reason and/or explain the problem-solving processes individually and socially within the classroom. Student participation in mathematical practices provide a glimpse into how students are developing mathematical understanding and how that understanding evolves through classroom interaction. It does not include specialized vocabulary as part of the mathematics register. In this theme, student ideas about the processes of doing multiplication, the steps they would take, the strategies they would use, along with justifying and reasoning when they would use certain strategies were highlighted.

The theme of communicating about multiplication provide evidence for the research questions of this dissertation in line with the aims and objectives of this study. The research questions were addressed as follows: The data in this section represent what and how students communicated mathematically with and without utilizing the tool, individually and through interaction, which is the focus of question one. In the tables below, some images from strategies, which would be Communication In, are referenced in the section for Communication About as students used the images to reason about the processes of multiplication. Semiotic resources present in images, signs, and language modalities speak to question two of the study. Question three focuses on the multiplicative communication students enacted when using the tool with the framework and corroborated with quoted student statements from interview transcripts.

The data provide a within-case analysis of students of varying knowledge levels and genders. Each level (beyond, on, approaching), shares student communication about multiplication with and without the tool. Classroom observations verified the activity structure and student interaction. The semi-structured interviews were a chance for students to extend Discourse on ideas incorporated individually or interactively.

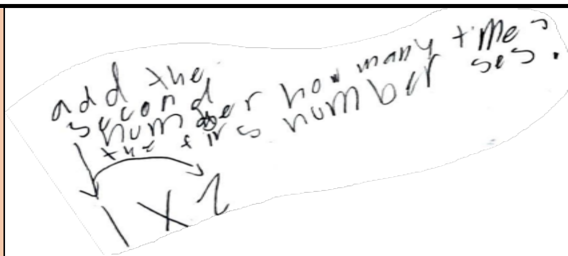
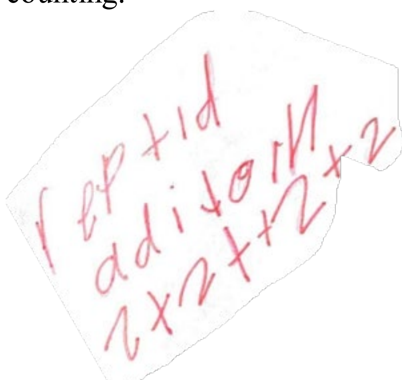
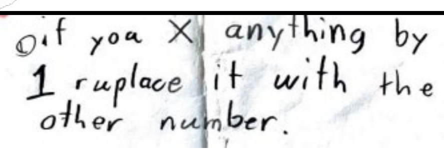
Beyond: Without and With the Tool

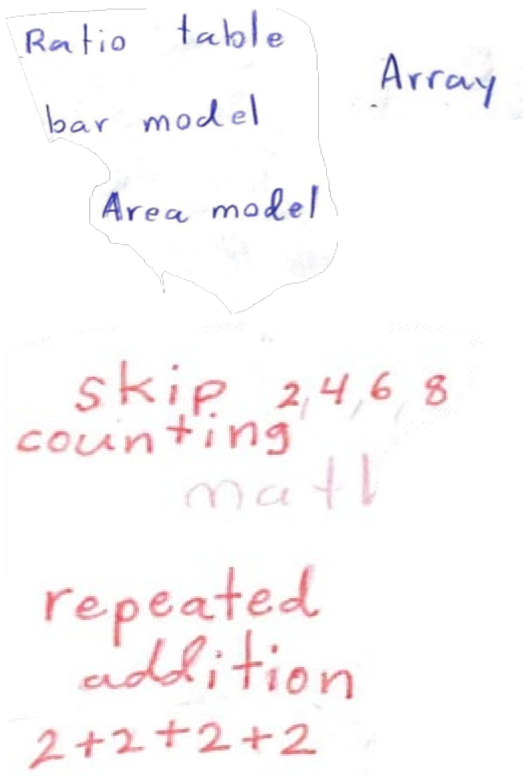
Table 3 Beyond/Communication About/Without Tool

Beyond	**Without tool**
Female:	**Male:**
Number line Bar model	5x5

Post interaction without tool:		
Female:	**Male:**	**Both:**
Repeated addition	Anything times 10 take away the 1 and add that number	Counting the first number then the second number which is 5 x 2 When you times 1 you get the number you started with Count 5 one time

Table 4 Beyond/Communication About/With Tool

Beyond	**With Tool**	**Explanation from interview**:
Female:		Add the second number, how many times the first number says “This one (points to first arrow) tells you how many times you do that one (follows arrow with finger on second arrow.”
Post interaction with tool:	Ratio table, repeated addition, skip counting. 	“I added more strategies like ratio table, repeated addition, and skip counting because that was just helping figure out more about multiplication. Because, like there are more than one way to get the answer for multiplication.”
Male:		**“**Because I think most people wouldn't think of doing that. They would keep the same number or... I don't know.”
Post interaction with tool:	Ratio table, Bar model, Area model, Array, Repeated addition, Skip counting	**Speaker 1: “**All of the things that you wrote on here, what would you consider is a resource for multiplying? Something that would help you.” **8BB:** “Probably one of these four” [points to four strategies in blue, from partner, ratio table, bar model, area model, array].” **Speaker 1: “**One of those four. Why so?” **8BB: “**Because there are other ways of solving multiplication.”

<table>
<tr><th>Beyond</th><th>With Tool</th><th>Explanation from interview:</th></tr>
<tr><td></td><td>
</td><td></td></tr>
</table>

On: Without and With Tool

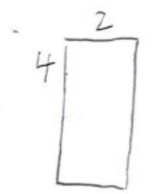
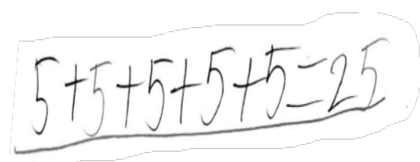
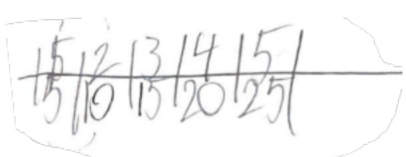
Table 5 On/Communication About/Without Tool

On	**Without tool**
Female:	**Male:**
So 12 x 2 is like 12 + 12=24 you can count by 12's or 2's 12 times	Add the number at the two times and you add the number to itself as many times as the other number

Post interaction without tool:

Female:	**Female: communication**	**Male:**
Post interaction without tool:	**Post interaction without tool:**	**Post interaction without tool:**
Repeated addition Counting the first number then the second number which is 5 x 2 Anything times 10 take away the 1 and add that number When you times 1 you get the number you started with	"Repeated addition is basically multiplication." "Anything times 10 makes the one and add the number. So six times 10, 60. So you add, and then this other one, it's the number." (attempts to explain 10 rule) "Any number times one gets the same number. So one times six gets six."	No communication about

Table 6 On/Communication About/With Tool

On	With Tool	Explanation from interview:
Female:	use for area model 	"So this would be four. Two... And then you'd have to multiply this by this [points to 4 and 2] ... to get what's inside of the box. [shades inside with finger]"
Post interaction with tool:	Repeated addition	"Because 12 times something, or 12 times two, you have to add 12 plus 12." "Repeated addition for five times, because you can just count five, 10, 15."
Male:	 	"These are all for 5 X 5. A ratio table, so for one, on the bottom is the multiplication and on the top is how many times you add it. So for this top, this is the number, how many times you have to add the number to itself. And then this is how many times. And this is what, like 5, 10, 15, 20, 25. And this is counting up by fives." "This is probably the easiest [points to repeated addition]. It's like groups. I mean, this is the easiest, but this is probably the second easiest, the ratio table and you can count over and over."
Post interaction with tool:	Counting	"Multiplication, you could count like this. [points to repeated addition equation.]"

Approaching: Without and With Tool

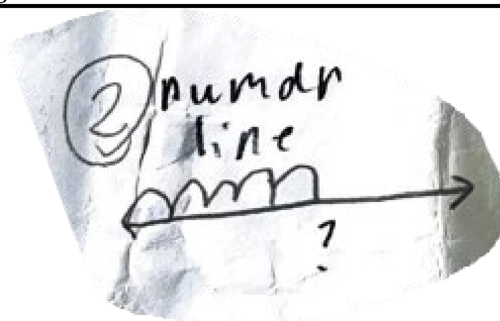
Table 7 Approaching/Communication About/Without Tool

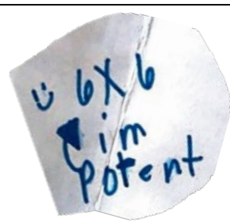
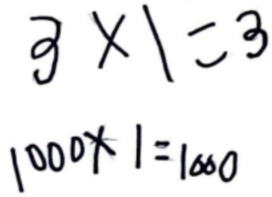
Approaching	**Without tool**
Female:	**Male:**
Ratio table	Find what you are multiplying then look at the first number like 2 then look at the next number like 5 and count 5, 2 times and will get 10
Bar model	

Post interaction without tool:

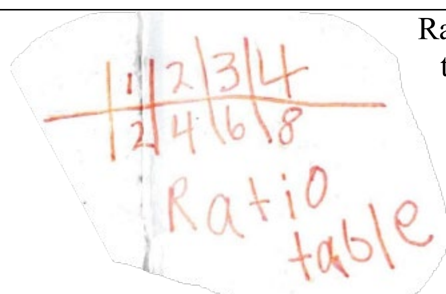
Female:	**Male:**	**Both:**
	Anything times 10 take away the 1 and add that number "So, anything times one is the one, because back to my strategy, I said two times five. I count two one time. I'm going to skip around. Two only one time."	Counting the first number then the second number which is 5 x 2 When you times 1 you get the number you started with Repeated addition Count 5 one time

Table 8 **Approaching/Communication About/With Tool**

<table>
<tr><th>Approaching</th><th>With Tool</th><th>Explanation from interview:</th></tr>
<tr><td>Female:</td><td>
</td><td>“A number line, you can use a number at the beginning. And then if you use jumps, it gets you to an even bigger number.”

“Because you can start with a number and then you can use jumps, if you use 10 and then a jump of 10, that's going to get you to 20, and then another jump of 10, that's going to get you to 30.”</td></tr>
<tr><td>Post interaction with tool:</td><td>
</td><td>“It is important because it tells you what to do, and how many you times the other number by”</td></tr>
<tr><td>Male:</td><td>
</td><td>21 BA: “BM’s idea, because I would agree with his.”

Speaker 1: “Okay.”

21 BA: “If you times anything by one, replace with the other number.”</td></tr>
<tr><td>Post interaction with tool:</td><td>

Ratio table:</td><td>Speaker 1: “What if I gave you a problem of 45 times five?”

21 BA: “45 times five?”

Speaker 1: “Which one of those would you want to do that in?”

21 BA: “You could do number line, because it has more space, but it’ll take more space. Ratio table would be better. Like I said before, you could do 1 of 45, 2 of 45 and keep going like that.”</td></tr>
</table>

Communication About Facts

Activity four and five of the study examined student procedural fluency in multiplication facts. Activity four was a fluency probe of one hundred facts ranging from times zeros to twelves. Activity five and six were a student-generated tool, that allowed students to write in any fact that they knew. The purpose of these activities was to look beyond the number of facts students could complete, but which facts they did answer and how or why the answered the facts that they did. The semi-structured interview allowed students to review facts shared in both activities and highlighted students' reasoning and solution strategies. The tables below share the reasoning for each student in each leveled category after looking at their probe and student-generated tool. The data presented share how students reasoned about the processes they used to solve multiplication facts extending student communication about multiplication beyond providing a product to the strategies students used to solve multiplication facts. For a complete listing of data see Appendix F.

Communicating About with Fact Fluency

Table 9 All Levels/Communication About Fluency

Female	Beyond Male
Reasoning in interview about facts: Umm . . . . it looks like I did the anchor facts. Yeah, I see times 2s, times 10s, times 5s, 1's	**Reasoning in interview about facts:** "Sometimes if it's a big number times a small number, I just switch them. Because eight times six is 48. So like, I would have it be six times eight."
Female	On Male:
Reasoning in interview about facts: "I would switch it. That's one of my strategies. I'll switch them around if it's easier. If it's six times five, I won't do... Or no. Five times six, I won't do six of the... Or five of the sixes because I know how to count by fives better."	**Reasoning in interview about facts:** "Oh, you take it apart, so yes. So if you use 11 + ... Oh wait, no. I was going to say 11 + 3, but that's not adding. If you use 11 X 3, so you have to add three 1s and three 10s."
Female	Approaching Male:
Reasoning in interview about facts: "If I got 3 and then 65, I'll take the three and 60 and five ones and 3 ones and then I'll put them both together."	**Reasoning in interview about facts:** "First, I know this two times nine, then I'll know four times nine. So, two times seven. It'll help two times seven because you count two more."

Communication In Mathematics

When communicating in mathematics (Brenner, 1994, 1998) students "speak" the language of the mathematics community. Communication in mathematics is a theme tied closely with the Mathematics Register (Halliday, 1974) and allows students to connect math concepts to specific formalized vocabulary, use language to define concepts and procedures of the content, and connect representations to extend the syntax, phrasing, and Discourse. It does not include problem-solving processes. In this theme, student ideas about the informal and formal registers, symbols, and representations, along with phrasing used to define concepts of multiplication, were highlighted. The theme of communicating in multiplication provides evidence for the research questions of this dissertation in line with the aims and objectives of this study. The research questions were addressed as follows: The data in this section represent the informal and formal ways students defined multiplication with and without utilizing the tool and individually and through interaction, which is the focus of question one. In the tables below, signs, symbolic notations, and representations were used by students to Communicate In multiplication. Some of the visual representations of strategies like the number line, ratio table, bar model, array model was also present in data for Communication About if students used them to explain and or justify when they would apply them. Utilizing semiotic resources of image, sign, and language modalities speak to question two of the study. Question three focuses on the multiplicative communication students enacted when using the tool with quoted student statements from interview transcripts which are evidenced throughout the communication in the section.

Beyond: Without and With Tool

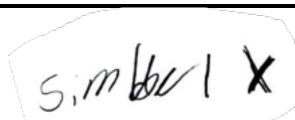
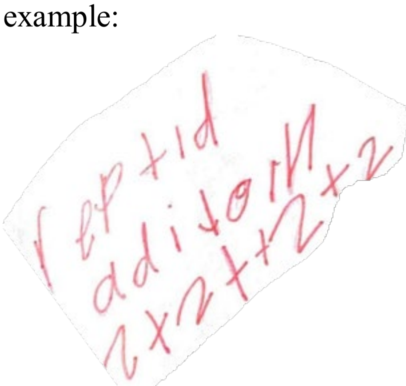
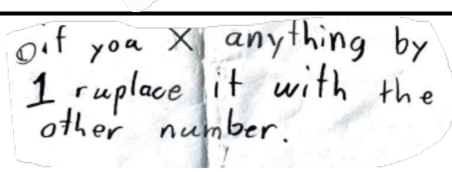
Table 10 Beyond/Communication In/Without Tool

Beyond	**Without tool**
Female:	**Male:**
The symbol is an x	5x5

Post interaction without tool:

Female:	**Male:**	**Both:**
No communication in	Like 5x 2 which is 10	

Table 11 Beyond/Communication In/With Tool

Beyond	**With Tool**	**Explanation from interview**:
Female:		**"**And this one is telling you the symbol that you can use"
	1 x 2	"Me holding something and spreading them out and using them to count, let's say one times two."
Post interaction with tool:	Repeated addition in expression example: 	"Because you can use it to do multiplication and you can add them more than once."
Male:		**identity rule of multiplication**
Post interaction with tool:	X symbol	"Because this symbol means times and the main point is to write down what multiplication is."

On: Without and With Tool

Table 12 On/Communication In/Without Tool

On — **Without tool**

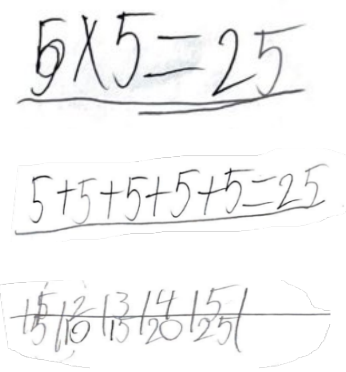
Female:	**Male:**
The x sign	No communication in
12 x 2 =24	

Post interaction without tool:

Female:	**Male:**	**Both:**
No communication in	No communication in	

Table 13 On/Communication In/With Tool

On	**With Tool**	**Explanation from interview**:
Female:	12X2=24	"It was 12 times two again. Like you can count 12, 2 times or 2's, 12 times to get 24." **Commutative property multiplication** "Because 12 times something, or 12 times two, you have to add 12 plus 12."
	numbers	24 GO: "Well, numbers is the base of it. So, you need numbers to complete it." Speaker 1: "Those numbers, is there a word we use in multiplication to talk about those numbers?" 24 FO: "Umm . . . the factors."
Post interaction with tool:	reapeted addition	"Because you have to use the same number over again. For two times 12, 12 and then 12 but don't do 12 and add two. That's not how you do it." 24 FO: "Because counting is really a part of it because you have to count over and over with the same number." Speaker 1: "What is that, when you count over and over again?" 24 FO: "Repeated addition."

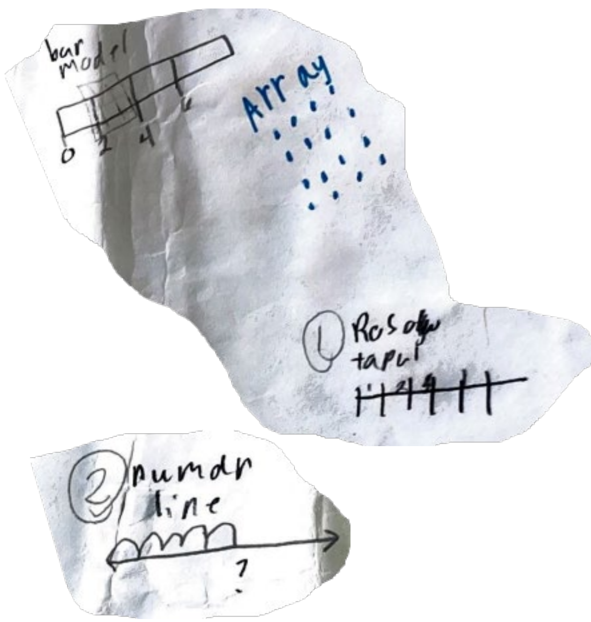
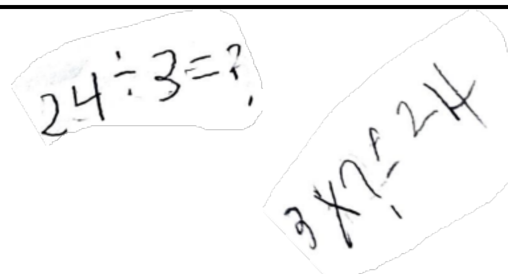
On	**With Tool**	**Explanation from interview**:
Male:		"These are all for 5 X 5.
Post interaction with tool:	Groups	"For groups, you just can have circles and put the same in each one. It is easy to count all of them from there."
	Same number over and over	20 BO: "You add the number to it over and over." Speaker 1: "Okay. Is there any other word that we use for doing something over and over again?" 20 BO: "Iterate."

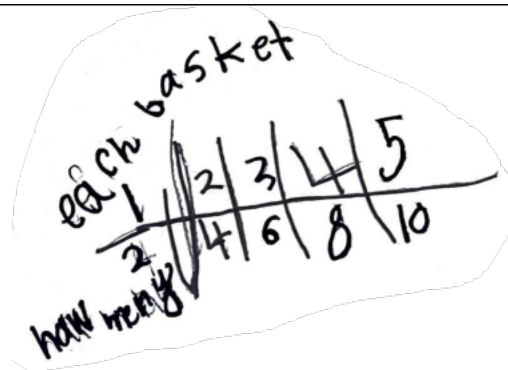
Approaching: Without and With Tool

Table 14 Approaching/Communication In/Without Tool

Approaching	**Without tool**	
Female:	**Male:**	
X	No communication in	
Post interaction without tool:		
Female:	**Male:**	**Both:**
No communication in	No communication in	

Table 15 Approaching/Communication In/With Tool

Approaching	**With Tool**	**Explanation from interview:**
Female:		"So, multiplication has many different ways you can do it"
Post interaction with tool:	X (symbol)	"The X is for the symbol of multiplication."
Male:		21 BA: "And multiplication is... Division. They go together." Speaker 1: "Okay. Can you give me an example? Like the 24 divided by 3 you have on your map."

Approaching	With Tool	Explanation from interview:
		21 BA: “But I would think like multiplication, just times. Like, 24 divided by three equals something, I think 24 would be counting three over and over until you get to 24. You count 3 . . . . . . [student counting]. That would be 8”
		“So, I usually do this first, label what I am going to do. For example, the two is going to be the apples. And the bottom is going to be how many there is. How many apples. And this part, I’m just going to write each basket. Two, four, six, eight, ten. So, it brings back to counting over and over, and to my strategy” Speaker 1: Can you tell me that context from your ratio table? “Yeah, ok. I have two apples in each basket. If I have five baskets, how many apples do I have? That is five times two.”
Post interaction with tool:	Same number over and over	Speaker 1: “I'm hearing you say the words over and over, over and over. What does that mean for multiplying, over and

Approaching	With Tool	Explanation from interview:
		over? I heard you use it a lot." 21 BA: "So, you count five two times. You copy two over and over five times. So two, four, six, eight, 10. 10, that's the answer. Iterate. Yeah. That's it."

Findings Summary

The findings indicate the student-generated tool engaged students in demonstrating knowledge through language and semiotic resources. The activity structures, both with peer and whole class interactions, offered students opportunity for socially constructing multiplicative thought. The results also showed increased Communication In and About multiplication as students exchanged ideas through partner and whole class interaction and incorporated those ideas within the tool more than they did without the tool.

The results suggest the use of a multi-modal student-generated tool broadens the language and resources applied by the students when communicating. Through interviews, students provided their own competency for communicating the ideas they procured from classmates. Data provided in the findings also support semiotic resources and modality as their own means of communicating, and also as an instrument to communicate. Visual and linguistic modes were connected in the tool and expressed in Discourse. The analysis also identified how students of varying knowledge levels, immersed themselves within the activity structure and generated a tool to engage in the classroom community.

Participants in this study shared individual ideas and also subsumed ideas from peers through interaction as indicated in the post interaction sections of the data tables. The framework assisted in illuminating student thought, not just about their own ideas, but the ideas of others. Through the thematic lens of Communication About and In mathematics, activities of this study were analyzed for the language and modalities present and the Discourse students engaged in as mathematicians. The conceptual and theoretical lens frame the discussion of this study consistent with the research questions and framework presented and analyzed in this chapter.

CHAPTER 5: DISCUSSION AND CONCLUSION

The results suggest students' language and resources for communicating are extended through the use of a multi-modal student-generated tool. The interactive activity structure allowed practicing mathematicians to accumulate semiotic resources within the tool during Discourse. These resources included image, signs, symbolic notation, and written words and phrases which could be read from the student-generated tool. Studying language must therefore consider more than single modalities of speaking and writing that are being exchanged, to the "what" and "how" thought is being exchanged in classroom contexts.

The semi-structured interviews provided students the opportunity to justify and reason about their own ideas and the ideas they acquired from others. The semiotic resources were used by students to represent their multiplicative thought, and also to communicate about processes. Students were able to connect their visual modes to expressive modes when collaborating during activity structures and within the semi-structured interviews. Students of varying knowledge levels used the tool to communicate their own knowledge of multiplication, and then were able to extend that knowledge by using the tool to communicate in the classroom community to increase their knowledge about the topic. The conceptual and social semiotic theoretical lens frame the discussion of this study consistent with the research questions and framework presented and analyzed.

Summary of Key Findings

The findings highlighted the extension of language and semiotic resources students engaged in with a student-generated tool. Classroom interactions embedded in the activity structure also increased Communication In and About multiplication. Students did not just adopt others' ideas during peer Discourse but incorporated the ideas into their own mathematical repertoire. Studying language must therefore consider meanings that are being exchanged, informally and formally, in the embedded cultures and contexts of the classroom and specific outputs that can generate productive communication.

Relating the visual and linguistic features of mathematics allows students to communicate and interpret mathematical meaning, not only through language but also through varying forms of modality, producing a tool that captures knowledge shared within the classroom community. Semiotic resources can be used to express mathematical proficiency of multiplication and communicate reasoning, as evidenced in the findings chapter, as students utilized representations and signs to speak mathematically. Connecting student representations with language allows for mathematical expression in and about multiplication from the tool. These resources are then used to participate in the discursive aspects of the classroom.

Also, the findings emphasized how students of varying knowledge levels participated in the activities and expressed multiplicative reasoning. Student participants at each level engaged with and reasoned about their mathematical thought as delineated in the findings. Although having shared less individually, they reasoned about the representations and ideas of others and interpreted them in their own way.

In summary, the qualitative case study illustrated how the visual and semiotic function within a student-generated tool enhances communication of multiplicative reasoning. In comparison to students' work without a tool in response to a prompt, students communicated more about multiplication and in multiplication using semiotic resources. Semiotic resources communicated multiplicative knowledge; however, they also became a mode for students to communicate.

Interpretation of Findings

Communication vital to the mathematics classrooms of today requires cognitive skills to express conceptual and procedural knowledge and the requisite language and thought needed to participate in discursive classroom activities and assessments (Erath et al, 2021; Ingram et al., 2020; Moschkovich & Zahner, 2018; Planas & Schütte, 2018). The central research question posed in this study was "How do semiotic resources and student-generated tools enhance students' abilities to communicate their multiplicative reasoning?" The findings suggest that student-generated tools are enhanced with semiotic resources and assist in communicating multiplicative reasoning. Each of the three sub-questions substantiate the central question with findings from the analysis. Therefore, the discussion section will examine each of the sub-questions and their relationship to the communicative themes from the Discourse Analysis, Communication About and Communication In for multiplication.

Research Question 1

The first research question proposed in this study, "*How does the utilization of the student-generated tool assist students in communicating mathematically?"* examined whether the student-generated tool aided students' mathematical communication. The question was addressed by comparing the Discourse structures of activities one and two and analyzing student Communication About and In multiplication with and without the tool. Students Communicating About multiplication described problem solving processes and reasoned about their strategies or the strategies of others; and Communicating In multiplication utilized the mathematical register and representations (Brenner 1994, 1998). The data showed all six student participants enhanced their Communication About and In multiplication with the student-generated tool. These enhanced communications included individual and interactive communication and the extension of representations beyond written modality to extend meaning-making.

The structure of activity one: prompt (without the tool), and activity two: student-generated tool (with the tool), were engaged to compare the extension of language resources and collaboration (Francisco, 2013; Morgan et al., 2014; Sfard, 2008). Rooted in this structure was the availability of processing time individually, with a partner, and then with classmates, along with the opportunity to add those collaborative ideas to their activity. Only one of the six students incorporated an idea from peer discussion from the tool; otherwise, all ideas came from class interaction and were similar across student participants. Whereas, when students utilized the tool there were individual, partner, and class interaction ideas listed for nearly every knowledge level and gender. This corroborates the importance of facilitating Discourse and interaction practices that support students in contextualizing discussion and practicing the language of their mathematics community through modality and tools (Cobb & Bowers, 1999; Gee, 2014; Moschkovich, 2015a, 2018; O'Halloran, 2015; Schleppegrell, 2007; Sfard, 2001, 2008). This is similar to research on the importance of interaction and collaborative activity to engage students in examining socially constructed ideas through Discourse allowing them to practice refining, justifying, and reasoning about those ideas (Francisco, 2013; Goos, 1999; Ingram et al., 2019).

Students showed strength in the written modality without the tool, as students wrote in more complete sentences or longer phrases. One reason for the extension of the written modality could be the consideration of the question as a writing prompt and students' use of lined paper. Yet, the output of student Communication About and In mathematics shows a disconnect between the reading of ideas from their document without the tool and the willingness to reason through the ideas. For example, all students had "Counting the first number then the second number which is 5 x 2." However, nothing in the statement sets a composite unit, or states an iteration of the second factor. With the statement alone, you would add the numbers notating an addition expression. The students included the expression 5 x 2, connecting the additive statement to the multiplicative expression. Through discourse, students did not take the opportunity to challenge this additive statement, and I missed the chance as a researcher to delve into this conception during individual interviews. This is in contrast to Fielding and Wells (2019) where students interactively challenged conceptions during investigations in small group and whole class discussions. A possible reason for this is the questioning incorporated by the educator in the Fielding and Wells study, which facilitated student challenges and reasoning during discursive activity through teacher moves. Student communication, not teacher moves, was the focus of this study, but they clearly play a crucial role in extending Discourse and will be addressed in recommendations for future research.

Student expression with the tool included less complete sentence structure, which coincides with the creating of the tool and the use of words, phrases, drawings, or other representations indicative of what they consider relevant about the topic (Hyerle 1996, 2004). As evidenced in the findings, this did not diminish the number of ideas, phrases, or representations students used to describe the topic and engage those thoughts with a partner and classmates. The written modality, in concurrence with representations and spoken modality, augmented their communicative production with the tool. Research asserts writing is a learning tool which deepens mathematical thinking and reasoning (Bosse & Faulconer, 2008; Kostos & Shin, 2010; Baxter et al., 2005; Pugalee, 2004; Fried & Amit, 2003). My findings indicate that multiple modes, beyond a single expressive mode, can be enacted to Communicate About and In mathematics. This is comparable to findings that integrated multi-modalities, inclusive of writing to extend collaborative group work (Bjuland et al., 2008) and predict math success on a task (Oviatt & Cohen 2013).

Research Question 2

The second research question proposed in this study, "*What semiotic resources are evident in student activity and how are students utilizing them?*" explored semiotic resources evident in student work and how students employed them during the activities. The data revealed that student participants evidenced semiotic resources with signs and representations and applied those representations to Communicate About and In multiplication. Students reasoned through symbolic notations with the fluency probe and the student-generated tool from activity two and five during the interview.

The potential to make meaning through semiotic resources allows for constructing mathematical conceptual and procedural knowledge by combining semiotic systems simultaneously (O'Halloran, 2005; Wilkinson, 2018, 2019). As presented in the findings, two of the six students Beyond Female and Approaching Female, used arrows and multiplicative expression to explain the process of multiplication. With the tool, the Beyond Female said "add the second number how many times the first number says," with the Approaching Female writing "important" on her tool next to the arrows and the expression. On the tool, the Beyond Female used the word "add" in her description, but in her interview used the more formal descriptor of *times* to indicate multiplication, "this one (points to first arrow) tells you how many times you do that one (follows arrow with finger) on second arrow" which added gesture to illustrate the process. The Approaching Female stated, "it is important because it tells you what to do, and how many you times the other number," but did not add a gesture to her description of the process. Both, during the interview, assert the first factor in the expression as an iterator of the second factor. Sign-making is subject to the semiotic resources available and how the maker wishes to represent their learning (Bezemer & Kress, 2008). This is reiterated by Van Leeuwen (2005), who considers the relevancy of the sign and how students position its use during discourse and in the other modalities it is engaged in. Related studies recognize the importance of diagrams, gestures, and other modalities to assist students in reasoning and interacting within the classroom and making mathematical conjectures (Chen et al., 2017; Chen & Herbst, 2013).

The data also revealed the importance of representation to express multiplicative knowledge in the student-generated tool, since all six participants used iconic representations and or symbolic notations. Without the tool, strategies were listed for repeated addition, bar model, number line, and ratio table. With the tool, strategies for each of these were listed along with groups, array, and area model with modeled representations completed individually or after the interaction. A major tenet of the CCSS Standards for Mathematical Practices (2010) is understanding multiple representations of mathematics content. Representing content enactively, iconically, and symbolically is what Bruner (1966) identified as modes of representation. These modes allow students to build connections amongst the modes, enactive to iconic or iconic to symbolic, or within a mode with multiple iconic representations. Representing and connecting knowledge is an important component of mathematical proficiency. Representational variations assist students to discuss comparisons in solution methods and connections within their own learning and the learning of others (Baroody et al., 2007; Barwell et al., 2018; Erath et al., 2018; Kilpatrick et al, 2002). Although representations are part of Communication In, students used them as an image of the strategy, what it looks like, while verbalizing the process that the image represented or reasoning about how they would use it which is Communicating About. These included semiotic representations related to previous studies which discuss the conceptualization of representations (Brendefur et al., 2015; Izsak, 2005; Kosko, 2020; Mulligan & Watson; 1998; Ulrich & Wilson, 2017).

The fluency probe from activity four consisted of one hundred facts for students to answer. Activity five asked students to write down as many multiplication facts as they could. Reconceptualizing procedural knowledge has been recommended to discuss the type and quality of connections students make conceptually to procedures (Barmby et al., 2008; Baroody et al., 2007; Crooks & Alibali, 2014; Kilpatrick et al., 2002; Rittle

Johnson et al., 2001; Rittle-Johnson & Schneider, 2015; Star, 2012). The semi-structured interviews provided insight beyond the procedural fluency to the reasoning and strategies that were engaged. Strategies explained by the students were the use of derived facts and described in their interviews as completing “anchor facts,” “doubling,” or “taking apart/decomposing.” The Approaching Female described her strategy for 6 x 4 as “six times two is 12, and then I would do that again. And then six times four would be 24.” The Beyond Male student applied a derived fact strategy to 12 x 6 “I know that three times 12 is 36, and then six times 12 is 72. Like a double.” In using a decomposing strategy, the On Male student shared, “Oh, you take it apart, so yes. So if you use 11 + ... Oh wait, no. I was going to say 11 + 3, but that's not adding. If you use 11 X 3, so you have to add three 1s and three 10s.” Making connections to “anchor facts” or facts that are easier to recall, is consistent with research on developing fact fluency (Baroody & Dowker, 2003; Brendefur et al., 2015). While it is easy to classify students by the number of facts they complete, we also tend to categorize them based on the strategy they choose (Jacob & Willis, 2001, 2003; Clark & Kamii, 1996; Kouba, 1989; Sherin & Fuson, 2005). Researchers caution these lines can be blurred based on the quickness they are able to enact a strategy or how they switch from additive types of strategies to other strategies (Carrier, 2014; Sherin & Fuson, 2005). However, it was not the aim of this study to quantify procedural knowledge through fluency but to understand how students express conceptual knowledge of symbolic notations of Communicating In to Communicate About the multiplication. The findings indicate students can reason about symbolic notations and utilize strategies to make schema connections to other facts (Baroody & Dowker, 2003; Baroody et al., 2007; Brendefur et al., 2015).

Research Question 3

The third research question proposed in this study, "*How does the student-generated tool influence the expression of multiplicative understanding?"* considered how the student-generated tool encouraged what students expressed multiplicatively. The question was addressed by conducting an in-depth look at what students Communicated About and In multiplication and their reasonings investigated in the interview. The data showed all six student participants expressed their Communication About and In multiplication with the student-generated tool. These enhanced communications included the use of visual and linguistic modalities to convey multiplicative thought and reason about processes, use of a mathematics' register, and the extension of learning opportunities for students of varying knowledge levels.

Visual Modality and Reasoning

Students worked individually and interactively to communicate understanding within their student-generated tools. The creation of a multi-modal semiotic tool required students to make sense of the practice and make meaning out of the resources used to make it (Lemke, 1990). First, students included more strategies and represented them visually, to explain multiplication processes with the tool than without. Five of the six participants added a majority of the strategies post-interaction (only the Approaching Female had more individually). As suggested by Lave and Wenger (1991), skills and knowledge evolve as students collaborate in social practices. At different points students were engaged as apprentices and experts inside the learning community and assimilated skills and understanding from available models (Lave, 1991; Lave and Wenger 1991; Vygotsky, 1978). Even though many of the strategies were acquired from others, students could reason about the processes and strategies of others. When asked what they considered resources for multiplication, many students chose strategies and reasoned that there was "more than one way to find the answer to a multiplication problem" and to use the strategies would "get you the answer."

During analysis, I looked for how students conceptualized the structure to Communicate About multiplication. To convey multiplicative reasoning, adherence to a unitized quantity, or composite unit, (Anghileri, 1989; Carrier, 2014; Clark and Kamii, 1996; Park & Nunes, 2001; Simon & Blume, 1994; Steffe, 1994; Ulrich, 2016), rather than a one-to-one counting structure is essential. Findings from the study showed that students used some inefficient strategies or relied heavily on additive thinking; nonetheless, the number of additional strategies in the tool meant there was also evidence of unitized quantity. Only two participants utilized a counting strategy for one of the many strategies that were listed. Both indicated a single count in a "grouping" strategy with one of them stating, "you can do six circles and then put little circles in each group until you get to the number, and then you can count each little circle and the big ones, and then that's how much they're in." One reason for this could be the possibility of representing each count for each group, making it easier to count in a single unit rather than unitize the units. In the literature, this is noted as a *direct representation* (Kouba, 1989), *one-to-one counting* (Jacob & Willis, 2001), and *count all* (Sherin & Fuson, 2005) classification. Many of the students emphasized a strong connection between multiplication and addition. In student explanations, this was articulated in descriptions with "add" or "like adding" in their statements which happened most often with the repeated addition and skip counting strategies and minimally with the bar model.

Research on multiplicative taxonomies considers this an over reliance on additive strategies for multiplying (Jacob & Willis, 2001, 2003; Kouba, 1989), which could cause students to struggle when they attempt to apply additive strategies to successive mathematics content in later grades.

An encouraging finding from this study was the recording of other strategies, number line, array, and area model, recognizing unitized quantity. Composite units were described in the findings as “jumps of 10,” “counting by or counting up,” and counting in rows and columns. The identification of a “scheme for iterating rows, columns” is of importance to constructing and viewing the array model (Battista et al., 1998, p. 505). Struggles with the array model as a strategy were found to be an inability to see the rows and column structure (Barmby et al., 2009; Battista, 2020) and prerequisite knowledge of dots to construct the array (Iszak, 2005). It was because of the dot structure the Beyond Male reasoned he would least likely use arrays because “it's not really good for any of them because it's like putting dots and squares everywhere and it is hard to get them straight.” Area representations express multiplicative relationships with a row and column structure that resemble arrays and can relate to square units or open area models given side dimensions (Kosko, 2020). Side dimensions of the open area model were referenced as factors for finding the area inside of the representation.

The findings show both similarities and a difference with existing research. Similarities include a flexibility by students to think both additively and multiplicatively about strategies, (Jacob & Willis, 2001, 2003; Kouba, 1989) detailed counting strategies by composite unit, (Anghileri, 1989; Carrier, 2014; Clark and Kamii, 1996; Park & Nunes, 2001; Simon & Blume, 1994; Steffe, 1994; Ulrich, 2016) and acknowledging the importance and struggles of the array model (Barmby et al., 2009; Battista et al., 1998; Iszak, 2005). A difference between these findings and extant literature is the connection to Kosko's findings of set and area models generating a single count rather than a composite unit. This was true for set models, as evidenced by the grouping data shared above, but the area model reasoned in this study correlated to a multiplicative composite. This could be because it was drawn as an open area model, so there were no square tiles to individually count. The results demonstrate how student-generated tools helped engage participants in representing strategies semiotically and reasoning about their processes as students noted no representations, less strategies, and reasoning without the tool.

Register

Communicating In mathematics entails more than a visual and symbolic notation of communicating, but includes the mathematics register or the domain-specific language of mathematics (Brenner, 1994, 1998; Halliday, 1974). Some examples of register were included with and without the tool. One idea comprised the articulation of the properties of identity and commutativity of multiplication. In Communicating About, students defined the process for multiplying by one: "If you times anything by one replace with the other number" which is a consistent definition with mathematics literature (Baroody & Dowker, 2003). Commutativity was referenced with and without the tool and with the fluency probe, but not to the extent of the identity property. Students indicated the commutative property when making statements about symbolic equations; for example, 12 x 2 =24 "Like you can count 12, 2 times or 2's, 12 times to get 24." The properties are specialized vocabulary of the domain, but not words students are expected to recall. In this instance, articulating the process is more important and students were able to do that with and without the tool. Another idea relevant across both activities was the multiplication symbol or X. Even though the addition sign was on most students' work represented in repeated addition, students stated clearly that the X is the symbol for multiplication.

Specific terms and phrases utilized as descriptors of processes in the discipline (Carrier, 2014; Sigley & Wilkinson, 2015; Wilkinson, 2018, 2019) were evident in the findings from this study. Some of the words present in student written and spoken discourse were area, times, double, bigger, and groups. These words were summarized by Carrier (2014) as early indicators of multiplicative reasoning and utilized by students to reason about representations in their tool and symbolic notation from their facts. Another grammatical structure is the link between the symbolic and visual representation that allows the language of what is happening symbolically to be written or verbalized by students; for example, 7 x 9 = 63 is seven times nine equals sixty-three (O'Halloran, 1998, 2010, 2015).

The findings show students benefitted from the use of phrases and informal register to Communicate In multiplication; however, without an emphasis on formalizing those phrases and ideas, students acting on their own was not present in this study. Students stayed in the informal register for using vocabulary individually and through interaction until formalizing some of their ideas in the interview. Informally students spoke about numbers that you use to multiply. One student formalized "numbers" to factors. The phrase "same number over and over" was prevalent in the data. Two students connected the structural word for iterating to this phrase, another used it as the definition for repeated addition. Product, the solution to a multiplication problem, was not evidenced in the Discourse. The student-generated tool grounds language and multi-modality into mathematical activity providing a situated activity to negotiate meaning socially (Moschkovich, 2015a). Students were able to engage semiotic resources to participate in the mathematics classroom community (Bailey, 2015; Barwell, 2005; Cobb & Bowers, 1999). Recommendations from literature emphasize how the everyday and mathematical registers combine to express learning and expanding students' registers to communicate in school settings and beyond them (Moschkovich, 2015a; O'Halloran, 2010, 2015; Schleppegrell, 2007). The results of this study are in line with this recommendation. Addressing how we might increase formalizing register in activity will be addressed in the recommendations.

Knowledge Levels

Students utilizing the student-generated tool made connections beyond what they had shared individually. The Approaching Male was the only student to state a relationship between multiplication and division. In his explanation for this relationship, he explains how he would solve twenty-four divided by three: "Like, 24 divided by three equals something, I think 24 would be counting three over and over until you get to 24. You count 3 . . . . . . [student counting]. That would be 8." This is consistent with results from Carrier (2014) and using different strategies for problem structures; for instance, using a multiplicative strategy to divide. Two of the students, Approaching Male and On Female, treated their student-generated tool as a "living document" inserting representations during the interview to give a visual representation of what they were sharing incorporating ideas as needed. The Approaching Male drew another representation of a ratio table so he could orally produce a contextual situation to represent what was happening in the bar model. The Approaching Female had the most comprehensive listing of strategies in her tool with images that were strictly representative of what the strategy looked like without numbers. As she read from her tool, she was able to generalize different units to speak to the processes that would happen in each one. Consistent with research on semiotic resources, the students are the meaning-makers or sign-makers, and the maker decides what resources to use and how to represent them (Bezemer & Kress, 2008). The findings show that students utilizing the tool as a sign-maker emphasized their meaning-making consistent with students at other levels and surpassing them.

In conclusion, the results from the qualitative case study support student-generated tools to enhance Communication About and In multiplication for students of varying knowledge levels. The tool allowed students to communicate semiotically by visually representing their learning and engaging other modes to convey thought. Those visual representations became an integral part of students' meaning-making and communicative processes increasing language development in the content of mathematics and providing alternate means to communicate conceptual and procedural knowledge.

Recommendations for Future Research

Communication is an integral part of the mathematics classroom and steeped in linguistic challenges. Activity structures that support language development and mathematical thought simultaneously would be of benefit to educators who should consider themselves educators of language and content. Second, tools that engage students cognitively to generate and share meaning assist students to become the creators of their own learning. The inclusion of semiotic resources involves expanding student expression. For many students, this allows them access to the mathematics community.

Future Research

One recommendation for future research is to consider a mixed-methods or quantitative study. Language and mathematics are rich in descriptive, qualitative studies. Mixed methods or quantitative studies would quantify data and establish correlations. This is consistent with recommendations for future research in language and mathematics (Erath et al., 2021).

Another recommendation would be to expand the boundaries of the case to include multiple classrooms in a single grade at one school, different grade levels in one school, or multiple classrooms in a single grade across a school district making the data more generalizable.

Another recommendation for future research is to explore Thinking Maps® (Hyerle, 1996, 2004, 2008) more thoroughly in mathematics and across content areas. The student-generated tool is part of a larger curriculum strategy that could be explored. There is minimal empirical research on their use; however, there were positive results for using them in this study.

Future research could consider the development of the student-generated tool from the beginning to end of the unit. This study was completed a month after instructional units were completed and many students mentioned “not remembering” some of the ideas they wanted to share. Expanding the timeline to incorporate instructional time might increase student conceptual knowledge or register use.

Last, future research could consider how student-generated tools enhance communication and multiplicative thought with different demographics including delving deeper into gender, special needs students, multilanguage learners, ethnicity, socio-economic status, urban versus rural, Thinking Maps® trained versus not trained. As not all students successfully recognize how to interpret mathematics concepts and communicate mathematical thought processes in the classroom (Barwell, 2005; Moschkovich, 2002; Schlepegrell, 2007).

Schools and Educators

The urban school district this study was conducted in has been attempting to implement student-generated tools, Thinking Maps® across the district for quite some time. The school as part of this study, has been implementing for five years. When surveyed about their use in different content areas, teachers who co-taught with the Multi-language Learner Co-teacher indicated higher use of them in math than those teachers who did not co-teach. Strengthening implementation at the school and district level would be an integral part of researching their effectiveness for students. One recommendation is to engage in professional development workshops for use in the mathematics class. Another recommendation for educators and coaches at the school level would be to dive deep into mathematics' standards to generate unit maps, or frameworks, with vocabulary, representations, and tool use that will be integrated across instruction.

A recommendation for educators who wish to create mathematics and literacy advancements would be to create opportunities for students to engage in mathematical practices and share conceptual understanding. This was evidenced by teacher moves, instructional routines, and classroom norms that could be integrated into practice. Teacher moves can assist students in formalizing thought and engaging students as a linguistic role model. Students need a teacher who engages with their interpretations and enables all participants to engage communicatively (Schutte, 2019).

The activity structure as part of this study was a well-developed routine. Students knew exactly how to interact with the tool. This can also be translated to the development of meaningful discourse routines (Kazemi & Hintz, 2014; Sfard, 2008; Stein et al., 2008; Zwiers & Hamerla, 2018) as a powerful way to unpack and scaffold language students will encounter. A component of enacting discourse routines and establishing a mathematics community is instituting norms for the mathematics community (Yackel & Cobb, 1996). What does your mathematics community look like, sound like, feel like?

Conclusion

Connecting a socially constructed tool with communication engages the learner in selecting the semiotic resource that best conveys understanding for themselves and to their classmates. This allows students to blend language, mathematical symbolism and visuals to construct meaning and bridge registers (Cobb & Bowers, 1999; O'Halloran, 2015; Wilkinson, 2019), engaging all semiotic systems to Communicate About and In mathematics. The communication framework allowed for a comprehensive look at student mathematical thought through Communicating About multiplication and describing problem-solving processes and reasoning; and Communicating In multiplication utilizing the mathematical register and representations (Brenner 1994, 1998). The semiotic frameworks (O'Halloran, 2010, 2015; Schleppegrell, 2007), have focused on utilizing the mathematical and everyday registers to emphasize how language and mathematics integrate and express learning through modalities. Classroom activity structures support sense-making, whether informally or formally, and assist students with the requisite tools and modalities for participation (Moschkovich, 2015a; Moschkovich & Zahner, 2018). A situated learning opportunity recognizes the development of mathematical practices socially, supporting students in their mathematical progression (Cobb & Bowers, 1999; Cobb et al., 2011; Yackel & Cobb, 1996). Students can summarize their conceptual and procedural understandings and express that through the mode that best supports their learning and communication (Kress, 2010, 2011). A significant contribution of this study is elucidating how third grade students leverage numerous resources, including tools, visual representations, and symbols to facilitate and support the development of multiplicative thought. I claim when semiotic resources are enacted, we have the potential to communicate more than our accuracy with facts or correctness in using formal vocabulary, but our processes, explanations, and justifications at the heart of understanding.

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APPENDIX A: IRB

BOISE STATE UNIVERSITY
RESEARCH AND ECONOMIC DEVELOPMENT
Research Compliance

Date: April 19, 2022

To: Keith Thiede cc: Emilie Eisenberger

From: Social & Behavioral Insitutional Review Board (SB-IRB)
c/o Office of Research Compliance (ORC)

Subject: SB-IRB Notification of Approval - Original - 101-SB22-050
Communicating Multiplicative Reasoning through Social Semiotic Resources

The Boise State University IRB has approved your protocol submission. Your protocol is in compliance with this institution's Federal Wide Assurance (#0000097) and the DHHS Regulations for the Protection of Human Subjects (45 CFR 46).

Protocol Number:	**101-SB22-050**	Received:	3/11/2022	Review:	Full Board
Expires:	**4/18/2023**	Approved:	4/19/2022		

Your approved protocol is effective until 4/18/2023. To remain open, your protocol must be renewed on an annual basis and cannot be renewed beyond 4/18/2025. For the activities to continue beyond 4/18/2025, a new protocol application must be submitted.

ORC will notify you of the protocol's upcoming expiration roughly 30 days prior to 4/18/2023. You, as the PI, have the primary responsibility to ensure any forms are submitted in a timely manner for the approved activities to continue. If the protocol is not renewed before 4/18/2023, the protocol will be closed. If you wish to continue the activities after the protocol is closed, you must submit a new protocol application for SB-IRB review and approval.

You must notify the SB-IRB of any changes to your approved protocol and the committee must review and approve these changes prior to their commencement. You should also notify the committee if your activities are complete or discontinued.

Current forms are available on the ORC website at http://goo.gl/D2FYTV

Please direct any questions or concerns to ORC at 426-5401 or humansubjects@boisestate.edu.

Thank you and good luck with your research.

1910 University Drive Boise, Idaho 83725-1139
Phone (208) 426-5401 orc@boisestate.edu
This letter is an electronic communication from Boise State University

APPENDIX B: SEMI-STRUCTURED INTERVIEW PROTOCOL

Thank you for sitting and talking with me today. I am going to ask about the work that you produced in class.

The semi-structured interview questions are as follows:

- Tell me about here (prompt).
- Can you read from your circle map.

  Why did you add this word, image, symbol, idea, resource?
- How does what you added to your tool tell what you know about multiplication?
- If you had to pick what you thought were the three most important ideas from your map, what would they be and why would you pick them?
- What are some examples of ideas you added after talking with your classmates?

  Why did you add those ideas?
- Tell me about (fluency probe) and the fact circle map.
- How did creating the tool make you feel?
- Is there anything that you want to share, or that you find important, that I have not asked you about?

APPENDIX C: BEYOND LEVEL DATA

Table C.5 Beyond/Communication About

Beyond	With Tool	Explanation from interview:
Female:	**image**	"Add the second number, how many times the first number says" "This one (points to first arrow) tells you how many times you do that one (follows arrow with finger on second arrow."
	Bar model	
	Numbers	We're using numbers and we're adding more
Post interaction with tool:	Ratio table, repeated addition skip counting.	"I added more strategies like ratio table, repeated addition, and skip counting because that was just helping figure out more about multiplication. Because, like there are more than one way to get the answer for multiplication."
	Any number times one	"And then this one is telling you if you use this number then, and then if you say one times five, then you could just have the first number up five times. So one… So we add five up one time." "Five times one that would equal five. You can use a ratio table. You can add up the same number over and over. You add the second number how many times the first number is"
Male:	If you times anything by 1 replace it with the other number	**"**Because I think most people wouldn't think of doing that. They would keep the same number or... I don't know."
Post interaction with tool: Beyond Male:	Ratio table Bar model Area model Array Repeated addition Skip counting	**Speaker 1:** "All of the things that you wrote on here, what would you consider is a resource for multiplying? Something that would help you. **8BB:** "Probably one of these four" [points to four strategies in blue, from partner work, ratio table, bar model, area model, array]." **Speaker 1: "**One of those four. Why so?" **8BB: "**Because there are other ways of solving multiplication."

		8BB: "Probably one of these four" [points to four strategies in blue, from partner work, ratio table, bar model, area model, array]." **Speaker 1: "**One of those four. Why so?" **8BB: "**Because there are other ways of solving multiplication." **Student speaking about array** "It's not really good for any of them because it's like putting dots and squares everywhere and it is hard to get them straight"

Table C.12 Beyond/Communication In

Beyond	With Tool	Explanation from interview:
Female:	Symbol X	**"**And this one is telling you the symbol that you can use"
	1 x 2	"Me holding something and spreading them out and using them to count, let's say one times two."
Post interaction with tool:	Same number over and over	
	9 x 10=90	"Nine times 10 equal 90"
	Repeated addition in expression example: 2 + 2 + 2 + 2	"Because you can use it to do multiplication and you can add them more than once."
Male:	"If you times anything by 1 replace it with the other number"	**identity rule of multiplication**
Post interaction with tool:	X symbol	"Because this symbol means times and the main point is to write down what multiplication is."
	Same number over and over	"if it's five times five, you add the same number over and over.
	Skip Counting example: 2, 4, 6, 8 Repeated addition with expression example: 2 + 2 + 2 + 2	Skip counting. Two, four, six, eight, that's math. Repeated addition, two plus two plus two, equals...

APPENDIX D: ON LEVEL DATA

Table D.7 On/Communication About

On	With Tool	Explanation from interview:
Female:	use for area model	"So this would be four. Two... And then you'd have to multiply this by this [points to 4 and 2] ... to get what's inside of the box. [shades inside with finger]"
	Like adding	"Because you have to add the numbers together. So like skip counting, but adding."
Post interaction with tool:	Use strategies	"Use those strategies to get the answer."
	To find area	"You use multiplication to find area inside, an area model, like right here. [Points to picture that was drawn]"
	Skip counting	"You're skip counting because you have to count the same one over and over"
	Repeated addition	"Because 12 times something, or 12 times two, you have to add 12 plus 12." "Repeated addition for five times, because you can just count five, 10, 15."
Male:	Communication about comes from representations. See Communication in for images	"These are all for 5 X 5. A ratio table, so for one, on the bottom is the multiplication and on the top is how many times you add it. So for this top, this is the number, how many times you have to add the number to itself. And then this is how many times. And this is what, like 5, 10, 15, 20, 25. And this is counting up by fives." "This is probably the easiest [points to repeated addition]. It's like groups. I mean, this is the easiest, but this is probably the second easiest, the ratio table and you can count over and over."
Post interaction with tool:	Counting	"Multiplication, you could count like this. [points to repeated addition equation.]"

Table D.14 On/Communication In

On	With Tool	Explanation from interview:
Female:	12 x 2 =24	"It was 12 times two again. Like you can count 12, 2 times or 2's, 12 times to get 24." Commutative property multiplication "Because 12 times something, or 12 times two, you have to add 12 plus 12."
	X	
	Multiplying	"multiplying is multiplication, but you're doing it."
	Numbers	24 OF: "Well, numbers is the base of it. So, you need numbers to complete it." Speaker 1: "Those numbers, is there a word we use in multiplication to talk about those numbers?" 24 OF: "Umm . . . the factors."
Post interaction with tool:	Times X, is other number	
	Same number over and over	"Because you have to use the same number over again. For two times 12, 12 and then 12 but don't do 12 and add two. That's not how you do it." 24 OF: "Because counting is really a part of it because you have to count over and over with the same number." Speaker 1: "What is that, when you count over and over again?" 24 OF: "Repeated addition."

On	With Tool	Explanation from interview:
	Model skip counting	"You're skip counting because you have to count the same one over and over"
Male:	5 x 5 =25 modeled as a ratio table and repeated addition	"These are all for 5 X 5. A ratio table, so for one, on the bottom is the multiplication and on the top is how many times you add it. So for this top, this is the number, how many times you have to add the number to itself. And then this is how many times. "And this is what, like 5, 10, 15, 20, 25. And this is counting up by fives." "This is probably the easiest [points to repeated addition]. It's like groups. I mean, this is the easiest, but this is probably the second easiest, the ratio table and you can count over and over."
	X	"That's the multiplication symbol." Because the symbol, if they might think it's a plus. If their paper is crooked, they might think it's a plus, so they might think it's adding, but it's supposed to be an X for multiplication.
Post interaction with tool:	Numbers	"Because in multiplication you use numbers a lot."
	Groups	"For groups, you just can have circles and put the same in each one. It is easy to count all of them from there."
	Same number over and over	20 OB: "You add the number to it over and over."

On	With Tool	Explanation from interview:
		Speaker 1: “Okay. Is there any other word that we use for doing something over and over again?” 20 OB: “Iterate.”

APPENDIX E: APPROACHING LEVEL DATA

Table E.9 Approaching/Communication About

Approaching	With Tool	Explanation from interview:
Female: Talks about all her representations So, multiplication has many different ways you can do it like a bar model, number line, or ratio table that help you a lot.	Put in groups	"You can put... If you have a really huge number, you can do six circles and then put little circles in each group until you get to the number, and then you can count each little circle and the big ones, and then that's how much they're in."
	Bar Model: Communication about comes from representations. See Communication in for images	"So, the bar model helps me a lot because there's rectangles. And then I just put numbers below it. If there was five, then I'll count up and it would get me to a number." "A bar model, it's kind of like skip counting, but with boxes."
	Ratio table: Communication about comes from representations. See Communication in for images	"I would love to use a ratio table, because if you use an easy number, it's easier to skip count with it, like one, and then six, and then two, and then you put six and six together, and then that's 12. And then you put the 12 and another 6 together for 18"
	Number line: Communication about comes from representations. See Communication in for images	"A number line, you can use a number at the beginning. And then if you use jumps, it gets you to an even bigger number." "Because you can start with a number and then you can use jumps, if you use 10 and then a jump of 10, that's going to get you to 20, and then another jump of 10, that's going to get you to 30."
	Skip count	"So you can use two plus two is four, and then you can keep

Approaching	With Tool	Explanation from interview:
		adding until you get to the number you want it to go." "Because I can skip count by any number really easily. Sometimes I get seven mixed up."
Post interaction with tool:	6 x 6 with arrow to first 6 "important modeled	"It is important because it tells you what to do, and how many you times the other number by"
	Repeated addition: Communication about comes from representations. See Communication in for images	"So, you can use two plus two is four, and then you can keep adding until you get to the number you want it to go."
	Array: Communication about comes from representations. See Communication in for images	"Array. So, how much are in a row? If there was four in a row, and then four in a column. And then you can count each row or column, and then that'll get you to a number."
	Any number times one is the other number	"So, if you use five times one, it still could be five. Because it's only one, five times."
Male:	If you times anything by one replace with the other number	21 BA: "Max's idea, because I would agree with his." Speaker 1: "Okay." 21 BA: "If you times anything by one, replace with the other number."
Post interaction with tool:	Ratio table: Communication about comes from representations. See Communication in for images	"So, a ratio table. So, one, write one, and it'll be two on the bottom. Two, four on the bottom. Three, six on the bottom. Four, eight on the bottom. So, we're counting by

Approaching	**With Tool**	**Explanation from interview**:
		two - two, four, six, eight, ratio table” Speaker 1: “What if I gave you a problem of 45 times five?” 21 BA: “45 times five?” Speaker 1: “Which one of those would you want to do that in?” 21 BA: “You could do number line, because it has more space, but it’ll take more space. Ratio table would be better. Like I said before, you could do 1 of 45, 2 of 45 and keep going like that.” “Because it shows you more about numbers in multiplication. Shows numbers counting, same number over and over. So it will give you how. It'll explain.” [points to start of ratio table where the words for the apples and baskets]
	Bar model: Communication about comes from representations. See Communication in for images	“bar model is three twos, and each one, we're going to label that one two, four, six. So it starts out six.”
	Skip counting: Communication about comes from representations. See Communication in for images	

Table E.16 Approaching/Communication In

Approaching	**With Tool**	**Explanation from interview**:
Female:	Example Equation 5 x 6=?	
	Bar model modeled	"So, the bar model helps me a lot because there's rectangles. And then I just put numbers below it. If there was five, then I'll count up and it would get me to a number." "A bar model, it's kind of like skip counting, but with boxes."
	Ratio table modeled representation communicates about	"I would love to use a ratio table, because if you use an easy number, it's easier to skip count with it, like one, and then six, and then two, and then you put six and six together, and then that's 12. And then you put the 12 and another 6 together for 18"
	Number line modeled	"A number line, you can use a number at the beginning. And then if you use jumps, it gets you to an even bigger number."
Post interaction with tool:	Array modeled	"Array. So, how much are in a row? If there was four in a row, and then four in a column. And then you can count each row or column, and then that'll get you to a number."
	X (symbol)	"The X is for the symbol of multiplication."
	Same number over and over	
	Numbers	
Male:	Example facts 1,000x1=1000; 3x1=3	"Back to [SP's] idea, and if you times anything by one, replace with the other number. The teacher said to write down your partner's idea."

Approaching	With Tool	Explanation from interview:
	From fluency tool activity 5 - 24 divided by 3	21 BA: “And multiplication is... Division. They go together.” Speaker 1: “Okay. Can you give me an example? Like the 24 divided by 3 you have on your map.” 21 BA: “But I would think like multiplication, just times. Like, 24 divided by three equals something, I think 24 would be counting three over and over until you get to 24. You count 3 . . . . . . [student counting]. That would be 8”
	Ratio table example See interview for contextual problem explanation using ratio table on map	“So, I usually do this first, label what I am going to do. For example, the two is going to be the apples. And the bottom is going to be how many there is. How many apples. And this part, I’m just going to write each basket. Two, four, six, eight, ten. So, it brings back to counting over and over, and to my strategy” Speaker 1: Can you tell me that context from your ratio table? “Yeah, ok. I have two apples in each basket. If I have five baskets, how many apples do I have? That is five times two.”
Post interaction with tool:	X (symbol)	Oh, that's a time sign.
	Ratio table example	“So, a ratio table. So, one, write one, and it'll be two on the bottom. Two, four on the bottom. Three, six on the bottom. Four, eight on the bottom. So, we're counting by two - two, four, six, eight, ratio table”

Approaching	With Tool	Explanation from interview:
	Bar model example	“bar model is three twos, and each one, we're going to label that one two, four, six. So it starts out six.”
	Skip counting example	
	Numbers	
	Same number over and over	Speaker 1: “I'm hearing you say the words over and over, over and over. What does that mean for multiplying, over and over? I heard you use it a lot.” 21 BA: “So, you count five two times. You copy two over and over five times. So two, four, six, eight, 10. 10, that's the answer. Iterate. Yeah. That’s it.”

APPENDIX F: FLUENCY REASONING

Table 16 Fluency Reasoning All Levels

<table>
<tr><th colspan="2">Beyond</th></tr>
<tr><td>Female</td><td>Male:</td></tr>
<tr><td>Reasoning in interview about fact probe:

Umm . . . . it looks like I did the anchor facts. Yeah, I see times 2s, times 10s, times 5s, 1’s</td><td>Reasoning in interview about fact probe:

“Sometimes if it's a big number times a small number, I just switch them. Because eight times six is 48. So like, I would have it be six times eight.”

“I know that three times 12 is 36, and then six times 12 is 72. Like a double”</td></tr>
<tr><th colspan="2">On</th></tr>
<tr><td>Female:</td><td>Male:</td></tr>
<tr><td>Reasoning in interview about fact probe:

“I would switch it. That's one of my strategies. I'll switch them around if it's easier. If it's six times five, I won't do... Or no. Five times six, I won't do six of the... Or five of the sixes because I know how to count by fives better.”

“So, if I did four times eight, I would do 16 plus 16.”

48 x 2
“So this one is kind of easy because eight... So I would just drop down eight and eight if it was two, and then add that 16 and then 40 and 40 would be 80. And then I take the 10 to the 80. And then I add 6 to 90 and it gets 96.”</td><td>Reasoning in interview about fact probe:

“Because this is four times if you switch it. It could be 4 X 6 and then this is 2 X 6. So 4 X 6 would be 6 + 6 is 12 and then another 6 + 6 is 12. And then if you add those 12s together, it's 24.”

“Oh, you take it apart, so yes. So if you use 11 + ... Oh wait, no. I was going to say 11 + 3, but that's not adding. If you use 11 X 3, so you have to add three 1s and three 10s.”</td></tr>
<tr><th colspan="2">Approaching: continued on next page</th></tr>
<tr><td>Female:</td><td>Male:</td></tr>
<tr><td>Reasoning in interview about fact probe:
9 GA: Okay. So I only did the ones that were easy for me.
Speaker 1: Okay. And what are you noticing about the ones that were easy for you?</td><td>Reasoning in interview about fact probe:

“First, I know this two times nine, then I'll know four times nine. So, two times seven. It'll help two times seven because you count two more.”</td></tr>
</table>

9 GA: Twos a lot. Speaker 1: Twos a lot. What else are you noticing? 9 GA: Tens. Speaker 1: Tens. So what do we call those? Twos, tens? I'm seeing- 9 GA: Fives. Speaker 1: Fives. I see ones. 9 GA: Ones. Speaker 1: What do we call ones, twos, fives, and tens? 9 GA: Anchors facts. "So six times two is 12, and then I would do that again. And then six times four would be 24" "If I got 3 and then 65, I'll take the three and 60 and five ones and 3 ones and then I'll put them both together."	