WARING RANK AND APOLARITY OF SOME SYMMETRIC POLYNOMIALS

by

Max Brian Sullivan

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Max Brian Sullivan

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Zachariah Teitler, Ph.D.	Chair, Supervisory Committee
Jens Harlander, Ph.D.	Member, Supervisory Committee
Uwe Kaiser, Ph.D.	Member, Supervisory Committee

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dedicated to Brian and Shan Sullivan

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ABSTRACT

We examine lower bounds for the Waring rank for certain types of symmetric polynomials. The first are Schur polynomials, a symmetric polynomial indexed by integer partitions. We prove some results about the Waring rank of certain types of Schur polynomials, based on their integer partition. We also make some observations about the Waring rank in general for Schur polynomials, based on the shape of their Semistandard Young Tableaux. The second type of polynomials we refer to as a Power of a Fermat-type polynomial, or a PFT polynomial. This is a Fermat type (or power sum) polynomial over n variables with degree p taken to some power k. We prove this polynomial is not compressed when $p > k$ and $k > 2$, and conjecture the result is true in general for all p. The proof takes the following form: the degree $k + 1$ annihilator ideal is examined and identified, and form of Rank-Nullity is applied, which provides a formula for the size of the degree $k + 1$ subspace of non-zero partial derivatives for that polynomial. Then we verify that this subspace is linearly independent, which gives us the dimension of the space of Derivs, and thus a lower bound for the Waring rank of that polynomial.

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CHAPTER 1

INTRODUCTION

For a homogeneous degree d polynomial, f, in n variables, a *power sum decomposition* of f is an expression $f = c_1\ell_1^d + c_2\ell_2^d + \cdots c_r\ell_r^d$ where ℓ_i are homogeneous degree 1 polynomials, and c_i are scalars. We will refer to homogeneous polynomials as *forms*, so each ℓ_i is a linear form. The length of the decomposition is the number of terms used, in this case our decomposition is length r. Here are some concrete examples of power sum decompositions.

$$xy = \frac{1}{4}\left((x+y)^2 - (x-y)^2\right)$$

$$xyz = \frac{1}{24}\left((x+y+z)^4 - (x+y-z)^4 - (x-y+z)^4 + (x-y-z)^4\right)$$

Definition 1.0.1. The **Waring rank** of a homogeneous degree d polynomial in n variables f, denoted $r(f)$, is the smallest length of a power sum decomposition.

Determining the Waring rank of an arbitrary form is often difficult. Often we search for upper and lower bounds for $r(f)$ using a variety of methods.

Explicit power sum decompositions give an upper bound for the Waring rank of forms, i.e. $r(xy) \leq 2$ and $r(xyz) \leq 4$. It is often extremely nontrivial to actually compute a power sum decomposition for an arbitrary polynomial. Computational methods exist to find the power sum decomposition, see [2]. Other methods have been developed for determining an upper bound more abstractly. For example, the upper bound can be determined by considering using facts about projective varieties, see section 5 of [7].

For a lower bound, the earliest method, and the foundation of most subsequent ones, involve *catalecticants*, first introduced by Sylvester in 1851. The catalecticant lower bound gives $r(xy) \geq 2$ and $r(xyz) \geq 3$. [14]. This method was improved in [7] giving $r(xy) \geq 2$ and $r(xyz) \geq 4$ which together with our upper bounds give the rank as $r(xy) = 2, r(xyz) = 4$.

In fact, the Waring rank of any monomial is known by [3]. The formula given by Carlini, Catalisano, and Geramita is as follows.

Theorem 1.0.2. *The Waring rank of a monomial given by* $x_0^{d_0} x_1^{d_1} \cdots x_n^{d_n}$ *where* $0 < d_0 \leq d_1 \leq \cdots \leq d_n$ *is equal to* $(d_1 + 1)(d_2 + 1) \cdots (d_n + 1)$.

The Waring rank is also known for several other types of polynomials, but only in some very specific cases. Additionally, many other types of polynomials have defined upper and lower bounds, which help us estimate their Waring rank in specific cases.

We have outlined some of the current methods on finding upper and lower bounds for the rank of homogeneous polynomials. We are interested here in using some of these current methods, and develop some strategies specific to our needs, to examine the Waring rank of several classes of symmetric polynomials.

In the next section we will provide some motivation for the topic of Waring rank, its history and applications in the fields of algebraic geometry, complexity theory, etc. Then we will define some basic terms, notations, and important results.

Sections 3 and 4 will each be dedicated each to one of the two types of polynomials we will be studying: Schur polynomials and what we call 'Power of Fermat-type' polynomials. Each will give definitions, a survey of previous results, and our main result for the bounds of the rank for these polynomials.

CHAPTER 2

BACKGROUND

This chapter will cover all the basic definitions we will utilize and depend on in our main work, as well as most previous results we will be making use of. The first section will be mainly definitions and notation, which we will make standard throughout this work. The second section will be a catalog of all the previous results we will be making use of, both specific to this field, and general ideas applied in specific ways for our purposes.

2.1 Definitions

2.1.1 Symmetric Polynomials

Let $\Bbbk$ be a field of characteristic 0 such as $\mathbb{Q}, \mathbb{R}$, or $\mathbb{C}$, and let $\Bbbk[x_1, \ldots, x_n]$ be the polynomial ring over $\Bbbk$ with variables $x_1, \ldots, x_n$.

Definition 2.1.1. A polynomial $f \in \Bbbk[x_1, \ldots, x_n]$ is *symmetric* if any of its variables can be interchanged, and the same polynomial can be obtained.

For example, $f = x_1x_2x_3 - 4x_1x_2 - 4x_1x_3 - 4x_2x_3$ and $f = x_1^3 + x_2^3 + x_3^3 + 9x_1x_2x_3$ are symmetric, since any x_i, x_j can switch places in f, and f would remain the same polynomial.

We have some common symmetric polynomials.

Definition 2.1.2. The *complete homogeneous symmetric polynomial* of degree k in n variables is a polynomial, denoted h_k, over the polynomial ring $\Bbbk[x_1, x_2 \cdots, x_n]$ of the form

$$h_k(x_1, x_2, \cdots, x_n) = \sum_{1 \le i_1 \le i_2 \cdots \le i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}$$

This type of polynomial is well studied and since it is both symmetric and homogeneous, has some nice properties. We will see it more in section 3.2.

Definition 2.1.3. The *elementary symmetric polynomial* of degree k in n variables is a polynomial denoted e_k over the polynomial ring $\Bbbk[x_1, x_2 \cdots, x_n]$ of the form

$$e_k(x_1, x_2, \cdots x_n) = \sum_{1 \le i_1 < i_2 < \cdots < i_k \le n} x_{i_1} x_{i_2} \cdots x_{i_k}$$

Definition 2.1.4. The *power sum symmetric polynomial* of degree k in n variables is a polynomial denoted p_k over the polynomial ring $\Bbbk[x_1, x_2, \cdots x_n]$ of the form

$$p_k(x_1, x_2, \cdots x_n) = \sum_{i=1}^{n} x_i^k = x_1^k + x_2^k + \cdots + x_n^k$$

This polynomial is also commonly referred to as a *Fermat-type polynomial*.

Definition 2.1.3 is important to the first type of polynomial we investigate, the Schur Polynomial. A *Schur polynomial* is a generalization of the elementary symmetric polynomial, indexed by integer partitions. We will see a more formal definition in section 3.1.

Definition 2.1.4 is important to the second polynomial, PFT polynomials. Easier to define than the Schur polynomial, a *Power of a Fermat type polynomial* or *PFT polynomial* is a Fermat type polynomial taken to a power. We investigate this type of symmetric polynomial in section 4.1.

2.1.2 Polynomial Ring Notation and Catalecticants

Now we will establish some notation for working over polynomial rings, and introduce our methods for determining bounds for the Waring rank of our symmetric polynomials.

Let $S = \Bbbk[x_1, x_2, \cdots x_n]$ denote the polynomial ring over the field $\Bbbk$ in n variables. When $n \leq 3$ we may use $\Bbbk[x], \Bbbk[x, y], \Bbbk[x, y, z]$. Otherwise, we will assume S is the polynomial ring over n variables, and we will assume our monomial ordering is lexicographic.

Let S_d be the degree d part of S, i.e. for $d = 3$ we would have

$$S_3 = \text{span}\{x_1^3, x_1^2x_2, \dots, x_{n-1}x_n^2, x_n^3\}$$

In general, the space S_d has dimension $\dim(S_d) = \binom{n+d-1}{d} = \left(\!\binom{n}{d}\!\right)$, found by counting the number of multisets of size d from the set of n variables, allowing repetition [5].

Define $\partial_{x_i} = \frac{\partial}{\partial x_i}$ as a shorthand for the partial derivative with respect to the variable x_i. For example, $\partial_{x_1}^2\partial_{x_2}^3\partial_{x_4}$ denotes a partial differential operator $(\frac{\partial}{\partial x_1})^2(\frac{\partial}{\partial x_2})^3(\frac{\partial}{\partial x_4})$.

Another convention we will sometimes utilize is given $\alpha = (a_1, a_2, \ldots, a_n)$ then $\partial^\alpha = \partial_{x_1}^{a_1} \cdot \partial_{x_2}^{a_2} \cdots \partial_{x_n}^{a_n}$. Then for $\alpha = (2, 3, 0, 4)$ we understand that $\partial^\alpha = \partial_{x_1}^2 \partial_{x_2}^3 \partial_{x_4}$.

Define $T = \mathbb{k}[\partial_{x_1}, \partial_{x_2}, \ldots, \partial_{x_n}]$ as a polynomial ring of differential operators over the variables of S. T is called the *dual ring* to S. T includes every possible partial derivative of variables in S. Let T_d denote the degree d graded part of T, similarly to S_d. So for $d = 3$

$$T_d = \operatorname{span}\{\partial_{x_1}^3, \partial_{x_1}^2 \partial_{x_2}, \ldots, \partial_{x_{n-1}} \partial_{x_n}^2, \partial_{x_n}^3\}$$

This dual ring is useful in the study of what is called the *catalecticant* map, defined next.

Definition 2.1.5. Let $f \in S_d$ and $0 \leq a \leq d$. Then the *a-th catalecticant of f* is a linear map $C_f^a : T_a \to S_{d-a}$ defined by $\partial^\alpha \mapsto C_f^a(\partial^\alpha) = \partial^\alpha(f)$.

For example, $f = x^3 + 2x^2y + 2xy^2 + y^3 \in S_3$ and a differential operator $\partial_x^2 \in T_2$ then $C_f^2(\partial_x^2) = \partial_x^2(f) = 6x + 4y$.

This notion goes back to Sylvester in 1851, see [13], and through the years, came to be written and used in this form in more contemporary work. The map is important because it is the foundation of our understanding of lower bounds for Waring rank, using the *method of partial derivatives* which we will use in this work. For more detailed info on this method see [8].

Here is a definition key to our use of this method for determining a lower bound on the Waring rank of these symmetric forms, based on the catalecticant.

Definition 2.1.6. For a polynomial f, $\operatorname{Derivs}(f)$ is the vector space spanned by all partial derivatives of f of all orders.

For example, if $f = xyz$ then $\operatorname{Derivs}(f) = \operatorname{span}\{xyz, xy, xz, yz, x, y, z, 1\}$.

Let $\operatorname{Derivs}(f)_{\leq \delta}$ be the subset of polynomials in $\operatorname{Derivs}(f)$ of degree less than or equal to δ. These are all subspaces of $\operatorname{Derivs}(f)$. Note $\operatorname{Derivs}(f)_{\leq d} = \operatorname{Derivs}(f)$ when $\deg(f) = d$, and $\operatorname{Derivs}(f)_{\leq 0} = \operatorname{span}\{1\}$. These subspaces are nested, so

$$\{0\} = \operatorname{Derivs}(f)_{\leq -1} \subsetneq \operatorname{Derivs}(f)_{\leq 0} \subsetneq \cdots \subsetneq \operatorname{Derivs}(f)_{\leq d} = \operatorname{Derivs}(f)$$

where $\operatorname{Derivs}(f)_{\leq -1} = \{0\}$ by convention. This type of nested sequence of subspaces is called a *filtration*, and we say that $\operatorname{Derivs}(f)$ is *filtered by degree*.

Another such filtration is given by *order of derivatives*. For each ϵ let $\operatorname{Derivs}(f)^{\geq \epsilon}$ be the subspace spanned by the derivatives of order ϵ or greater. In this filtration, we have

$$\{0\} = \operatorname{Derivs}(f)^{\geq d+1} \subsetneq \operatorname{Derivs}(f)^{\geq d} \subsetneq \cdots \subsetneq \operatorname{Derivs}(f)^{\geq 0} = \operatorname{Derivs}(f)$$

Observe that $\operatorname{Derivs}(f)^{\geq \epsilon} \subset \operatorname{Derivs}(f)_{\leq d-\epsilon}$ since differentiating f at least ϵ times reduces the total degree of f by at least ϵ. However, these subspaces are not necessarily equal, for example if $f = x^4 + x^2 y$ a degree 4 polynomial, then

$$\begin{aligned}
\operatorname{Derivs}(f) &= \operatorname{span}\{f, \partial_x f, \partial_y f, \partial_x^2 f, \ldots\} \\
&= \operatorname{span}\{x^4 + x^2 y, 4x^3 + 2xy, x^2, 12x^2 + 2y, 2x, 2\} \\
&= \operatorname{span}\{x^4 + x^2 y, 2x^3 + xy, x^2, y, x, 1\}
\end{aligned}$$

In particular, $\text{Derivs}(f)_{\leq 2} = \text{span}\{x^2, y, x, 1\}$. However,

$$\begin{aligned}
\text{Derivs}(f)^{\geq 2} &= \text{span}\{\partial_x^2 f, \partial_x\partial_y f, \partial_y^2 f, \partial_x^3 f, \ldots\} \\
&= \text{span}\{12x^2 + 2y, 2x, 0, 24x, 2\} \\
&= \text{span}\{6x^2 + y, x, 1\}
\end{aligned}$$

Observe $x^2, y \in \text{Derivs}(f)_{\leq 2}$ but not in $\text{Derivs}(f)^{\geq 2}$.

When f is homogeneous of degree d, then this complication does not arise so we can simplify these two filtration's into one case, and denote the space of Derivs in the following way.

Definition 2.1.7. For a homogeneous polynomial f of degree d, $\text{Derivs}(f)_\delta$ is the subspace of all homogeneous degree δ polynomials in $\text{Derivs}(f)$. Equivalently, it is the set of all order $(d-\delta)$th derivatives of f. Formally,

$$\text{Derivs}(f) = \text{Derivs}(f)_0 \oplus \text{Derivs}(f)_1 \oplus \cdots \oplus \text{Derivs}(f)_d$$

Thus we have $\text{Derivs}(f)_{\leq \delta} = \text{Derivs}(f)^{\geq d-\delta} = \text{Derivs}(f)_0 \oplus \text{Derivs}(f)_1 \oplus \cdots \oplus \text{Derivs}(f)_\delta$. That is, $\text{Derivs}(f)$ is *graded by degree* when f is homogeneous. The space of Derivs is one key to our examination of these symmetric polynomials. The other involves a less familiar, and highly related notion.

2.1.3 Apolarity

The method of partial derivatives also involves the notion of *apolarity*. The theory of apolarity was developed in the late 19th early 20th century by Clebsch, Lasker, Richmond, Sylvester, and Wakeford in their work with *canonical forms*, see [12],[15],[13].

This was later utilized by Ehrenborg and Rota to continue this work, who used it to examine the linear and algebraic dependence of forms [4], [5].

It has been shown using the notion of apolarity that for f a 'generic' ternary quartic (a polynomial in 3 variables of degree 4) cannot be written as $\ell_1^4+\ell_2^4+\ell_3^4+\ell_4^4+\ell_5^4$ iff there exists a dual ternary quartic which is apolar to $\ell_1^3, \ell_2^3, \cdots, \ell_5^3$ where ℓ_i are linear forms. In other words, under this given condition, the Waring rank of f is greater than 5, giving us a lower bound for generic homogeneous f of degree 4 in 3 variables.

The notion of apolarity gives rise to a significant object called the *Apolar algebra* of f [9].

Definition 2.1.8. For a polynomial f in the polynomial ring S, and the associated dual ring T the *Apolar Algebra* R_f is the quotient ring defined by

$$R_f = T/\operatorname{Ann}(f)$$

where $\operatorname{Ann}(f)$ is the set of all polynomial differential operators that annihilate f, called the *annihilator* of f.

$\operatorname{Ann}(f)$ is also often called the *Apolar ideal.*

Lemma 2.1.9. *Given a polynomial $f \in S$ and the ring of differential operators T, the set of polynomial differential operators that annihilate f, denoted* $\operatorname{Ann}(f)$*, is an ideal of T.*

Proof. Let f be a degree d polynomial. Then for any $\alpha = (a_1, a_2, \dots, a_n)$ such that $|\alpha| \geq d+1$ we know that $\partial^\alpha \in \text{Ann}(f)$ thus $\text{Ann}(f) \neq \emptyset$.

Given $\theta_1, \theta_2 \in \text{Ann}(f)$ then

$$(\theta_1 + \theta_2)(f) = \theta_1(f) + \theta_2(f) = 0 + 0 = 0$$

so $\theta_1 + \theta_2 \in \text{Ann}(f)$.

Given $\theta \in \text{Ann}(f)$ and $\gamma \in T$ we have

$$\theta\gamma(f) = \gamma(\theta(f)) = \gamma(0) = 0$$

So $\theta\gamma \in \text{Ann}(f)$ and therefore $\text{Ann}(f)$ is an ideal. $\square$

Similarly to the $\text{Derivs}(f)$, the annihilator is graded, so the set of all ath differential operators that annihilate f is denoted $\text{Ann}(f)_a$.

For example, $\text{Ann}(xy) = (\partial_x^2, \partial_y^2)$ since $\partial_x^2(xy) = 0$ and $\partial_y^2(xy) = 0$ so $(\partial_x^2, \partial_y^2) \subset \text{Ann}(xy)$. To see the reverse containment, we can separate $\text{Ann}(xy)$ into graded components, so $\text{Ann}(xy)_1 = \{0\}, \text{Ann}(xy)_2 = \text{span}\{\partial_x^2, \partial_y^2\}, \text{Ann}(xy)_3 = \text{span}\{\partial_x^3, \partial_x^2\partial_y, \partial_x\partial_y^2, \partial_y^3\}$ and notice that the ideal generated by $(\partial_x^2, \partial_y^2)$ contains the basis of $\text{Ann}(xy)_3$ so $\text{Ann}(xy) \subset (\partial_x^2, \partial_y^2)$ thus $\text{Ann}(xy) = (\partial_x^2, \partial_y^2)$.

We can connect the ideas of the annihilator and the space of Derivs intuitively: $\text{Derivs}(f)$ is a vector space spanned by all partial derivatives of f, and $\text{Ann}(f)$ is the ideal of all differential operators which when applied to f, result in 0.

2.2 Previous Results

We will present some previously proved theorems and facts about ranks of vector spaces, apolarity, and other ideas relating to symmetric polynomials.

Theorem 2.2.1. *(Rank-Nullity) For vector spaces V, W and a linear transformation $\phi : V \to W$ we know:*

$$\mathrm{Rank}(\phi) + \mathrm{Nullity}(\phi) = \dim V$$

where

$$\mathrm{Rank}(\phi) := \dim(\mathrm{Image}(\phi)), \qquad \mathrm{Nullity}(\phi) := \dim(\ker(\phi))$$

We can apply this theorem to the catalecticant map $C_f^a : T_{d-a} \to S_a$. The kernel is $\mathrm{Ann}(f)_{d-a}$, the homogeneous degree $d-a$ elements in $\mathrm{Ann}(f)$, and the image is $\mathrm{Derivs}(f)_a$. Therefore,

$$\dim \mathrm{Ann}(f)_{d-a} + \dim \mathrm{Derivs}(f)_a = \dim T_{d-a} \tag{2.1}$$

which we know is

$$\dim T_{d-a} = \binom{n+d-a-1}{d-a} = \left(\!\!\binom{n}{d-a}\!\!\right)$$

By examining the size of the $\mathrm{Derivs}(f)_a$ or $\mathrm{Ann}(f)_{d-a}$ we have a formula with one unknown, which can be solved. The dimensions of these subspaces are important because of the following result about the catalecticant map and Waring rank.

Theorem 2.2.2. *Given a homogeneous polynomial $f \in S$ and the catalecticant map C_f^a then the following inequality holds*

$$r(f) \geq \operatorname{rank} C_f^a$$

The proof is surprisingly quick.

Proof. Let $f = c_1\ell_1^d + \cdots c_r\ell_r^d$, then the image of C_f^a is spanned by $\partial^\alpha(f)$ for $\partial^\alpha \in T_a$ and each $\partial^\alpha(f) \in \operatorname{span}\{\partial^\alpha(\ell_i^d) : i = 1, 2, \ldots, r\}$. Each of these spanning elements is $\partial^\alpha(\ell_i^d) = c\ell_i^{d-a}$ for some $c \in \Bbbk$ therefore the image of $C_f^a \subset \operatorname{span}\{\ell_i^{d-a} : i = 1, 2, \cdots, r\}$ which implies the rank of C_f^a is at most the rank of f. $\square$

More useful results to consider about the catalecticant map rank.

Corollary 2.2.3. *Given $f \in S_d$ and $0 \leq a \leq d$ we have*

- $\operatorname{rank} C_f^a = \dim \operatorname{Derivs}(f)_a$
- $\dim \operatorname{Derivs}(f)_a = \dim \operatorname{Derivs}(f)_{d-a}$ *and the sequence of dimensions is symmetric.*

Proof. Let f be a homogeneous polynomial of degree d and let R be the apolar algebra of f (we will omit the f subscript). Observe that $\operatorname{Derivs}(f)_a \cong R_a$ and $\operatorname{Derivs}(f)_{d-a} \cong R_{d-a}$. Also observe that $R_d \cong \operatorname{Derivs}(f)_d = \operatorname{span}\{f\} \cong \Bbbk$. Taking the Cartesian product $R_a \times R_{d-a}$ gives a space of degree d polynomials, since R_k corresponds to a space of degree k polynomials. The multiplication map

$$T_a \times T_{d-a} \to T_d$$

taken modulo $\mathrm{Ann}(f)$ gives the map

$$R_a \times R_{d-a} \to R_d$$

Therefore this map is bilinear. Then each $\theta \in R_a$ gives a linear map $R_{d-a} \to \mathbb{k}$, i.e. an element of R^*_{d-a}, thus we get a map $R_a \to R^*_{d-a}$ which is linear and injective. To see this, if $\theta \in R_a$ where $\theta \neq 0$ then $\theta \notin \mathrm{Ann}(f)$, so $\theta f \neq 0$ which implies there exists some $\psi\theta \neq 0$ in R_d. Then $\psi \in R^*_{d-a}$ is an element where the linear functional given by θ is nonzero, so the map $R_a \to R^*_{d-a}$ is injective. Therefore $\dim R_a \leq \dim R^*_{d-a} = \dim R_{d-a}$. Similarly, in the other direction we get $\dim R_{d-a} \leq \dim R^*_a = \dim R_a$ so the dimensions are equal. $\square$

The consequence of this result is that the the subspaces Derivs_a are symmetric. To illustrate what this means, take the elementary symmetric polynomial $f = e_4(x_0, x_1, x_2, x_3, x_4)$. Here is the list of the dimension of $\mathrm{Derivs}(e_4(x_0, x_1, x_2, x_3, x_4))_a$ for $0 \leq a \leq 4$.

a	$\dim \mathrm{Derivs}(f)_a$
0	1
1	5
2	10
3	5
4	1

Table 2.1: Dimension of $\mathrm{Derivs}(f)_a$ **for** $f = e_4(x_1, x_2, x_3, x_4, x_5)$**.**

We will be looking at the dimension of the $\mathrm{Derivs}(f)_a$ for specific values of a, which will tell us about the annihilators and the Derivs space for all of f based on this pattern.

We will examine the catalecticant maps of our two symmetric polynomial types. Each polynomial f will be in n variables, of degree d, so as these numbers, n and d grow, so does the length of f, and the dimension of each C_f^a. Thus, our computational data will reflect the limit of our available resources.

CHAPTER 3

SCHUR POLYNOMIALS

3.1 Definition and Computation

A Schur polynomial is a type of symmetric homogeneous polynomial which is indexed by an integer partition of the degree of the polynomial.

Definition 3.1.1. Given an integer partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ where $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ and $\lambda_j \geq 0$, the function

$$a_{(\lambda_1+n-1,\lambda_2+n-2,\ldots,\lambda_n)}(x_1, x_2, \ldots, x_n) = \det \begin{bmatrix} x_1^{\lambda_1+n-1} & x_2^{\lambda_1+n-1} & \ldots & x_n^{\lambda_1+n-1} \\ x_1^{\lambda_2+n-2} & x_2^{\lambda_2+n-2} & \ldots & x_n^{\lambda_2+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^{\lambda_n} & x_2^{\lambda_n} & \ldots & x_n^{\lambda_n} \end{bmatrix}$$

is an alternating polynomial in n variables of degree d. It will be divisible by the Vandermonde determinant

$$a_{(n-1,n-2,\ldots,0)}(x_1,x_2,\ldots,x_n) = \det \begin{bmatrix} x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \ldots & x_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 \end{bmatrix} = \prod_{1\le j<k\le n} (x_j - x_k)$$

The *Schur polynomial* for the partition λ is computed by the following quotient [16].

$$s_\lambda(x_1,x_2,\ldots x_n) = \frac{a_{(\lambda_1+n-1,\lambda_2+n-2,\ldots,\lambda_n)}(x_1,x_2,\ldots,x_n)}{a_{(n-1,n-2,\ldots,0)}(x_1,x_2,\ldots,x_n)}$$

For example, let $n = 3, d = 7$ and consider the integer partition $\lambda = (4,2,1)$. Then the Schur polynomial given by λ is computed in the following way.

$$s_{(4,2,1)}(x,y,z) = \frac{a_{4+3-1,2+3-2,1}(x,y,z)}{a_{3-1,2-1,0}(x,y,z)} = \frac{\det \begin{bmatrix} x^{4+3-1} & y^{4+3-1} & z^{4+3-1} \\ x^{2+3-2} & y^{2+3-2} & z^{2+3-2} \\ x^{1+3-3} & y^{1+3-3} & z^{1+3-3} \end{bmatrix}}{\det \begin{bmatrix} x^{3-1} & y^{3-1} & z^{3-1} \\ x^{3-2} & y^{3-2} & z^{3-2} \\ x^{3-3} & y^{3-3} & z^{3-3} \end{bmatrix}}$$

Computing these two determinants we get

$$a_{6,3,1}(x,y,z) = x^6y^3z - x^3y^6z - x^6yz^3 + xy^6z^3 + x^3yz^6 - xy^3z^6$$

$$a_{2,1,0}(x,y,z) = x^2y - xy^2 - x^2z + y^2z + xz^2 - yz^2$$

Thus the Schur polynomial is given by

$$\begin{aligned} s_{(4,2,1)}(x,y,z) = {} & x^4y^2z + x^3y^3z + x^2y^4z + x^4yz^2 + 2x^3y^2z^2 + 2x^2y^3z^2 \\ & + xy^4z^2 + x^3yz^3 + 2x^2y^2z^3 + xy^3z^3 + x^2yz^4 + xy^2z^4 \end{aligned}$$

The Schur polynomial is a generalization of the elementary symmetric polynomial. Like many symmetric polynomials, it can be indexed by integer partitions. Schur polynomials have many interesting properties as a result of this, which have been well studied in algebraic combinatorics. One such property is we can examine the shape of a visual representation of these partitions in something called a *Young Diagram*.

A *Young Diagram* is a finite collection of boxes, or cells, arranged in left-justified rows, with the row lengths in non-increasing order. Listing the number of boxes in each row gives a partition λ of a non-negative integer n, the total number of boxes of the diagram. The Young diagram is said to be of shape λ, and it carries the same information as that partition.

A *Young Tableau* is a filling of a Young diagram with integers 1 through n. The integers may repeat, or not appear at all. It is semistandard if these integers are strictly increasing along columns and weakly increasing along rows. Since the integer partitions, λ, we have for these Schur polynomials could consist of a single repeated integer, $\lambda = (k, k, \ldots, k)$, or a single positive integer followed by 0's, $\lambda = (d, 0, \ldots, 0)$, we would need to use semistandard Young Tableaux to categorize these polynomials. We have an alternate definition for Schur polynomials using these semistandard YT.

Here are all the SSYT for the integer partition $(4, 2, 1)$

$$\begin{array}{|c|c|c|c|}\hline 3&3&3&3\\\hline 2&2\\\cline{1-2} 1\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 3&3&3&3\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 3&3&2&2\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 3&3&3&1\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 3&2&2&1\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|c|}\hline 3&3&3&2\\\hline 2&2\\\cline{1-2} 1\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 3&3&3&2\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 3&3&2&1\\\hline 2&2\\\cline{1-2} 1\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 3&3&2&1\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 3&3&1&1\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}$$

$$\begin{array}{|c|c|c|c|}\hline 3&3&2&2\\\hline 2&2\\\cline{1-2} 1\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 3&3&3&1\\\hline 2&2\\\cline{1-2} 1\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 3&2&2&2\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 3&3&2&1\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}\quad\begin{array}{|c|c|c|c|}\hline 3&2&1&1\\\hline 2&1\\\cline{1-2} 1\\\cline{1-1}\end{array}$$

Notice that there are 15 such semistandard Tableaux, and the sum of the coefficients on our Schur polynomial indexed by the same partition is 15. In other words, there is a one-to-one correspondence between SSYT for the partition λ and monic monomial terms in s_λ. This leads to our next definition.

Definition 3.1.2. Given the Schur polynomial $s_\lambda(x_1, x_2, \ldots, x_n)$, let $\mathcal{T}$ denote the collection of all semistandard Young Tableaux of λ, and let t_i be the weights in each SSYT, that us, t_i is the number of times i appears in the tableaux. We have

$$s_\lambda(x_1, x_2, \ldots, x_n) = \sum_{\mathcal{T}} x_i^{t_i}$$

Again, consider the Schur polynomial $s_{(4,2,1)}(x, y, z)$. Each tableaux in our list of all SSYT for $(4, 2, 1)$ corresponds to a single monomial in this Schur polynomial, where we interpret the numbers as variables. Typically $x_1, x_2, x_3, \ldots$ are represented by $1, 2, 3, \ldots$ for convenience. In this case, we can either let $x = 1, y = 2, z = 3$ or $x = 3, y = 2, z = 1$ and the result is the same. Symmetric polynomials have many convenient properties in this way! In the typical assignment of variables to weights, our first tableaux corresponds to the monomial term xy^2z^4.

The theory of Schur polynomials is vast, since they are an important example of symmetric polynomials, and their properties are very interesting to the field of algebraic combinatorics and algebraic geometry. More can be found about Schur polynomials in [10]. We are interested here in the annihilator ideal of Schur polynomials, which will largely depend, as we will see, on the integer partition λ, which can be classified by the Semi-standard Young Tableux (SSYT hereafter) for λ.

We revisit our example polynomial one more time, and examine the annihilator of it. Consider the annihilators of order 5. $\mathrm{Ann}(s_{(4,2,1)})_a \neq \{0\}$ is $a = 5$. In this case, we get $\mathrm{Ann}(s_{(4,2,1)})_5 \subset \mathrm{span}\{\partial_x^5, \partial_y^5, \partial_z^5\}$, which is fairly easy to see, since clearly taking the fifth derivative of any one variable, when the highest power on x, y, z is 4, the polynomial will be annihilated. Thus $\dim \mathrm{Ann}(f)_5 \leq 3$.

For this polynomial, there are other non-monomial annihilators of degree 5, and in fact of degree 4, $\dim \mathrm{Derivs}(f)_{\lfloor \frac{d}{2}+1 \rfloor} > 0$ where d is the degree of the polynomial. When $\dim \mathrm{Derivs}(f)_{\lfloor \frac{d}{2} \rfloor} = 0$, the Derivs is as large as possible for all degrees.

This is sometimes stated by referencing the Hilbert function of the polynomial, and when the space of $\mathrm{Derivs}(f)$ has its maximum dimension, then we say it has a *compressed* Hilbert function.

3.2 Results and Proof

We now use this background to make and prove claims about the dimension of the annihilator ideal of $s_\lambda(x_1, x_2, \ldots x_n)$ based on the partition λ.

Theorem 3.2.1. *If* $\lambda = (1, 1, \ldots, 1)$ *then* $s_\lambda = x_1 x_2 \cdots x_n$.

We will refer to this Schur polynomial with $\lambda = (1, 1, \cdots, 1)$ as the *unit-discrete* Schur polynomial.

Proof. Let $N = a_{(\lambda_1+n-1, \lambda_2+n-2, \ldots \lambda_n)}(x_1, x_2, \ldots x_n) = a_{(n, n-1, n-2, \ldots, 1)}(x_1, x_2, \ldots x_n)$ and $D = a_{(n-1, n-2, \ldots, 0)}(x_1, x_2, \ldots x_n)$. By the Jacobi bialternant formula, we know that $s_\lambda(x_1, x_2 \ldots, x_n) = \frac{N}{D}$. Let the matrix A be defined as the Vandermonde matrix:

$$\begin{bmatrix} x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \\ x_1^{n-2} & x_2^{n-2} & \ldots & x_n^{n-2} \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

Now let B be the matrix whose determinant produces N:

$$\begin{bmatrix} x_1^{n} & x_2^{n} & \ldots & x_n^{n} \\ x_1^{n-1} & x_2^{n-1} & \ldots & x_n^{n-1} \\ \vdots & \vdots & \ddots & \vdots \\ x_1 & x_2 & \ldots & x_n \end{bmatrix}$$

Finally let X be the matrix with the main diagonal containing x_1 through x_n and every other element 0:

$$\begin{bmatrix} x_1 & 0 & \dots & 0 \\ 0 & x_2 & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & x_n \end{bmatrix}$$

It is easy to see that $B = AX$, and therefore $N = \det B = \det AX = \det A \det X = D \det X$. Therefore $s_\lambda(x_1, x_2, \dots, x_n) = \frac{N}{D} = \frac{D\det X}{D} = \det X = x_1 x_2 \cdots x_n$ as desired. □

Corollary 3.2.2. *If $d = kn$ and $\lambda = (k, k, \dots, k)$ then $s_\lambda(x_1, x_2, \dots, x_n) = (x_1 x_2 \cdots x_n)^k$.*

This Schur polynomial will be referred to as the *k-discrete* Schur polynomial.

Proof. Similarly, for A as before and

$$B = \begin{bmatrix} x_1^{k+n-1} & x_2^{k+n-1} & \dots & x_n^{k+n-1} \\ x_1^{k+n-2} & x_2^{k+n-2} & \dots & x_n^{k+n-2} \\ \vdots & \vdots & \ddots & \vdots \\ x_1^k & x_2^k & \dots & x_n^k \end{bmatrix}$$

is is easy to see that $B = AX^k$ where X is the same as before. Thus $s_\lambda(x_1, x_2, \dots, x_n) = \frac{N}{D} = \frac{D\det X^k}{D} = \det X^k = x_1^k x_2^k \cdots x_n^k = (x_1 x_2 \cdots x_n)^k$ as desired. □

Therefore $\lambda = (k, k, \dots, k)$ for $f = s_\lambda(x_1, x_2, \dots, x_n)$. By Ranested-Schreyer, the rank of a monomial f of the form $f = (x_1 x_2 \cdots x_n)^k$ is known to be $r(f) = (k+1)^{n-1}$

[11]. The SSYT for the discrete Schur polynomials is a single column, and based on computed examples, has the lowest rank of any other Schur polynomials with the same parameters n, d. We now turn to the opposite extreme.

Theorem 3.2.3. *If* $\lambda = (d, 0, \ldots, 0)$ *then* $s_\lambda = s_\lambda(x_1, x_2, \ldots x_n) = \sum_{t_1+t_2+\ldots+t_n=d} x_i^{t_i}$ *and* $\operatorname{rank} C_{s_\lambda}^{\lfloor \frac{d}{2} \rfloor} = \left(\!\!\binom{n}{\lfloor \frac{d}{2} \rfloor}\!\!\right)$.

This Schur polynomial we will refer to as the *indiscrete* Schur polynomial.

For an example, start with the Schur polynomial $n = 3, d = 3$ and $\lambda = (3, 0, 0)$, over $\Bbbk[x, y, z]$. The SSYT of shape λ is a single row, with 3 columns, e.g. $\boxed{i \mid j \mid k}$ where $i, j, k \in \{1, 2, 3\}$. Since it is semistandard, the row is weakly increasing, and repetition of numbers is allowed. Then we can list all possible SSYT of this shape with these possible weights as follows. Let 111 denote the tableaux with a weight of 1 in the first, second, and third position, e.g. $\boxed{1 \mid 1 \mid 1}$ which corresponds to the term x^3. The set of all SSYT of this shape is counted by the number of ways to arrange 3 objects into 3 bins, weakly increasing and allowing repetition. This is equivalent to the number of multisets, i.e. there will be $\binom{3+3-1}{3} = \left(\!\binom{3}{3}\!\right) = 10$.

Let $\mathcal{T}_\lambda = \{ijk | i \leq j \leq k, i, j, k \in \{1, 2, 3\}\}$ denote all possible tableaux for λ. Then $\mathcal{T}_\lambda = \{111, 112, 113, 122, 123, 133, 222, 223, 233, 333\}$. Then forming the polynomial from this arrangement we get

$$s_{(3,0,0)} = x^3 + x^2y + x^2z + xy^2 + xyz + xz^2 + y^3 + y^2z + yz^2 + z^3$$

Notice that every degree 3 monomial in $\Bbbk[x, y, z]$ is present in this Schur polynomial. Recalling definition 2.1.2 we can see that this is the complete homogeneous symmetric polynomial of degree 3 in 3 variables, $h_3(x, y, z)$. This will hold in general, here is the proof.

Proof. Let $\lambda = (d, 0, 0 \ldots, 0)$. Then define $\mathcal{T}_\lambda$ as the set of all SSYT of shape λ. Namely,

$$\mathcal{T}_\lambda = \{i_1 i_2 \cdots i_n : i_1 \leq i_2 \leq \cdots \leq i_n, i_k \in [n], \sum_{k=1}^{n} i_k = d\}.$$

Elements of T_λ are length n strings whose entries come from a size n alphabet, and whose sum is equal to d. Additionally, any one weight may not appear at all. So we are counting length n strings, of non-negative weights, since our tableaux are semi-standard.

This is equivalent to the number of multisets $\left(\!\binom{n}{d}\!\right)$. Since we have already seen that each tableaux corresponds to a monomial in s_λ, and the number of degree d monomials in $\mathbb{k}[x_1, \ldots, x_n]$ is known to be $\left(\!\binom{n}{d}\!\right)$ then s_λ contains every monomial in n variables of degree d. Moreover, the tableaux for each monomial is unique, thus the coefficient on each monomial in s_λ is 1. Therefore, we can see that s_λ is the complete homogeneous symmetric polynomial of degree d in n variables.

Recall definition 2.1.2 of the complete homogeneous symmetric polynomial. Notice that for $\lambda = (d, 0, 0, \ldots, 0)$ then the Schur polynomial $s_\lambda(x_1, \ldots, x_n) = h_d(x_1, \ldots, x_n)$. Theorem 2.11 of [1] states that $h_d(x_1, \ldots, x_n)$ has a compressed Hilbert function, therefore each s_λ of this form will have a compressed Hilbert function. $\square$

We see then that in one extreme, the discrete case, has rank $(k-1)^{n-1}$. In the other extreme, the indiscrete, the Hilbert function is compressed, meaning that by 2.2.2 our lower bound for the Waring rank of the indiscrete Schur polynomial, $s_{d,0,\cdots,0} \in \mathbb{k}[x_1 \ldots, x_n]$ is rank $C_f^{\lfloor \frac{d}{2} \rfloor} = \left(\!\binom{n}{\frac{d}{2}}\!\right)$, which is as large as possible.

Every pair (n, d) will have one indiscrete Schur polynomial, however we only get a discrete Schur polynomial when $d = kn$. Therefore we classified the lower bound of one Schur polynomial for each (n, d), and a second one if $d = kn$. Generalizing this result involves examining the SSYT shape, which we will mention in the next section.

3.3 Further Questions

We examined several hundred Schur polynomials, generalizing up to 7 variables and up to degree 9. Our computer data suggests that there exists a pattern of when the Hilbert function of the Schur polynomials is compressed, that aligns with our rigorous results. By categorizing the Schur polynomials by their Semi-standard Young-Tableaux we noticed that when the Tableaux is a single row (indiscrete) the Hilbert function is compressed.

Our data suggests that if the Schur polynomial SSYT is a long row with a small tail, the Hilbert function will be compressed, and depending on the degree and number of variables, the tail can grow larger, while the HF remains compressed. Whereas, if the SSYT is a single column (discrete) we proved the Hilbert function is not compressed, and we can find the annihilator ideal. If the SSYT of a Schur polynomial is a large column with a small arm, the Hilbert function will often be uncompressed, though this case has more deviation that the former.

It is not completely clear at what point this change occurs, and if it can be generalized. It is an interesting algebraic-combinatorial problem to connect the shape of the SSYT to the rank of the annihilator ideal, or the dimension of the space of Derivs, that we would like to explore further in the future.

CHAPTER 4

POWERS OF FERMAT TYPE POLYNOMIALS

Our next type of polynomial is the PFT polynomial, which we will define below, along with some other specific background definitions and notation. Section 2 will go over some computational examples, and results, which will play a role in our larger proofs. Section 3 will be our main results for the annihilator ideal of PFT polynomials in general, and our final section will contain further questions on this subject.

4.1 Definition and Notation

Recall definition 2.1.4 of a Fermat polynomial.

Definition 4.1.1. Let $s_n^p = x_1^p + x_2^p + \cdots + x_n^p$ be the degree p Fermat-type polynomial in n variables. Then define

$$F_{(n,p,k)} := (s_n^p)^k = (x_1^p + x_2^p + \cdots + x_n^p)^k$$

be the k-th power of s_n^p. This polynomial we will call the *Power of a Fermat Type polynomial* or a *PFT polynomial*, henceforth.

The inspiration for studying this polynomial came from the papers [1] and [6] which both proved that the PFT polynomial, where $p = 2$, has compressed Hilbert function. We wanted to further generalize on the power of p, and examine the annihilator ideal of this polynomial, to determine a lower bound for its Waring rank.

Definition 4.1.2. Let f be a polynomial over S. The *support* of f, denoted $\mathrm{supp}(f)$, is the set of monomials occurring in the polynomial f.

For example, in the polynomial $f = x_1^3x_2 + 3x_3^2 - 2x_5$ the support is $\mathrm{supp}(f) = \{x_1^3x_2, x_3^3, x_5\}$. If a degree d polynomial f includes all degree $\leq d$ monomials then f is said to have *full support*. This definition of support could also be called *monomial support* to differentiate it from *variable support*, the set of variables occurring in the polynomial. When f is homogeneous, which is all f we are considering, we will use 'full support polynomial' to denote a homogeneous polynomial containing all degree d monomials.

Full support polynomials have some nice properties, the ones that will benefit us the most involve determining, based on parameters (n, p, k), what monomials exist in the partial derivatives of $f = F_{(n,p,k)}$.

We can generalize the notion of support in a way that helps us examine these PFT polynomials.

Definition 4.1.3. Let $f(x_1, x_2, \ldots, x_n)$ be a polynomial in $R = \Bbbk[x_1^p, x_2^p, \ldots, x_n^p]$. If $f(y_1, y_2, \ldots, y_n)$ has full support in the polynomial ring $R' = \Bbbk[y_1, y_2 \ldots, y_n]$ where $y_i = x_i^p$ then we say f has *p-full support*.

The main usefulness of these results is being able to determine the monomials from $F_{(n,p,k)}$ that will be present in $\mathrm{span}(\partial^\alpha)(f)$, which is the key to testing for linear independence of each of our Derivs subspaces. To make this work easier, we will introduce some notation for dealing with monomials of $\partial^\alpha(F_{(n,p,k)})$.

For $f \in S$ be an arbitrary polynomial let $Z_\theta = x_1^{p\theta_1} \cdots x_n^{p\theta_n}$ denote some monomial $Z_\theta \in \text{supp}(f)$ where $\theta = (\theta_1, \theta_2, \ldots \theta_n)$. Also let c_θ be the coefficient on Z_θ. We will see both how to determine exactly what monomials are in $\text{supp}(f)$, when f is an element of some $\text{Derivs}_\delta(F_{(n,p,k)})$, in the next section. Additionally, we will see how to find c_θ given (n,p,k), another necessary part of determining the linear independence of each subspace of the Derivs.

4.2 Computational Results and Examples

4.2.1 Useful Results

We can write a general formula for the coefficients on each PFT polynomial, for any parameters (n,p,k), using the fact that every PFT polynomial has $p-$full support, and the help of our old friend the binomial theorem.

Lemma 4.2.1. *Let $f = F_{(n,p,k)} \in S^n_{pk}$ be a PFT polynomial, and let $\theta = (\theta_1, \theta_2, \ldots, \theta_n)$ be a tuple with $0 \leq \theta_i \leq pk$. Then for any monomial $Z_\theta \in \text{supp}(f)$ the coefficient on Z_θ is given by*

$$c_\theta = \frac{k!}{\theta_1! \cdots \theta_n!}$$

Proof. The coefficient c_θ on a monomial $Z_\theta = c_\theta x_1^{p\theta_1} \cdots x_n^{p\theta_n}$ for any $Z_\theta \in \text{supp}(f)$ is given by the multinomial theorem

$$c_\theta = \binom{k}{\theta_1, \theta_2 \ldots, \theta_n} = \frac{k!}{\theta_1! \cdots \theta_n!}$$

$\square$

This allows us to easily predict coefficients on any monomial in a partial derivative of f, for some $\partial^\alpha(f) \in \text{Derivs}_\delta(f)$.

Corollary 4.2.2. *Given a differential operator $\partial^\alpha = \partial_{x_1}^{a_1} \cdots \partial_{x_n}^{a_n} \in T_\delta$ and a monomial Z_θ, the coefficient, d_θ, on $\partial^\alpha(Z_\theta)$ is given by*

$$d_\theta = \frac{k! p\theta_1! \cdots p\theta_n!}{\theta_1! \cdots \theta_n! (p\theta_1 - a_1)! \cdots (p\theta_n - a_n)!}$$

Proof. Note $\partial^\alpha(c_\theta Z_\theta) = c_\theta \partial^\alpha(Z_\theta)$ so we need only find the coefficient for $\partial^\alpha(Z_\theta)$ and multiply it by c_θ. Differentiating Z_θ by ∂^α gives us the monomial

$$\begin{aligned}
\partial^\alpha(Z_\theta) = \partial^\alpha(\prod_{i=1}^n x_i^{p\theta_i}) &= \prod_{i=1}^n \partial^{a_i}(x_i^{p\theta_i - a_i}) \\
&= \prod_{i=1}^n (p\theta_i)(p\theta_i - 1) \cdots (p\theta_i - a_i + 1) x_i^{p\theta_i - a_i} \\
&= \prod_{i=1}^n \frac{(p\theta_i)!}{(p\theta_i - a_i)!} x_i^{p\theta_i - a_i}
\end{aligned}$$

Thus the coefficient d_θ on $\partial^\alpha(Z_\theta) \in \text{supp}(\partial^\alpha(f))$ is given by

$$\begin{aligned}
d_\theta &= c_\theta \prod_{i=1}^n \frac{(p\theta_i)!}{(p\theta_i - a_i)!} \\
&= \frac{k!}{\theta_1! \cdots \theta_n!} \left(\frac{p\theta_1!}{(p\theta_1 - a_1)!} \right) \cdots \left(\frac{p\theta_n!}{(p\theta_n - a_n)!} \right) \\
&= \frac{k! p\theta_1! \cdots p\theta_n!}{\theta_1! \cdots \theta_n! (p\theta_1 - a_1)! \cdots (p\theta_n - a_n)!}
\end{aligned}$$

□

Here we have a lemma about a specific type of matrix, which we will see in some computational examples. It will be easiest to prove some facts about this matrix in general before dealing with specific cases.

Lemma 4.2.3. *For an $n \times n$ real valued matrix of the form:*

$$M = \begin{bmatrix} a & b & \dots & b \\ b & a & \dots & b \\ \vdots & \vdots & \dots & \vdots \\ b & b & \dots & a \end{bmatrix}$$

If $a + (n-1)b \neq 0$ and $a - b \neq 0$ then M is invertible.

Proof. Assume that $a + (n-1)b \neq 0$ and $a - b \neq 0$. Note that the column vector $v^T = [1, 1, \dots, 1]$ consisting of n copies of 1 is an eigenvector of M and its eigenvalue is $a + (n-1)b$. Now notice we can rewrite the matrix M as

$$M = (a-b)I + bvv^T$$

Now let w be another column vector that is orthogonal to v. Multiply by w on the right to both sides of the above equality and we can see:

$$Mw = (a-b)Iw + bvv^T w = (a-b)w + bv \cdot 0 = (a-b)w$$

because $v^T w = 0$.

So w is an eigenvector with the eigenvalue of $a - b$. So for any w orthogonal to v, w is another eigenvector with eigenvalue of $(a - b)$.

It is easy to see that there are $n-1$ linearly independent w's that will be orthogonal to v: Let

$$w_i^T = [1, x_1, \dots, x_{n-1}] \text{ where } x_i = -1, x_j = 0 \text{ for } j \neq i$$

Any vector of this form will be orthogonal to v and there are $\binom{n-1}{1} = n-1$ linearly independent w's.

Then span$\{v, w_1, w_2, \dots w_{n-1}\}$ forms a complete eigenbasis for M with eigenvalues $a+(n-1)b$ and $a-b$ (with multiplicity $n-1$). This implies the matrix is diagonalizable.

Since M is diagonalizable, then M is invertible if and only if it's eigenvalues are all nonzero. By assumption, $a + (n-1)b \neq 0$ and $a - b \neq 0$ and these are our only eigenvalues, therefore M is invertible. □

Given the polynomial f, define LM(f) as the leading monomial of f, using lex monomial ordering. For example if $f = 4x^3 + 2y^2 - 3z^4$ then LM$(f) = x^3$.

Lemma 4.2.4. *Let $f = F_{(n,p,k)}$, the (n, p, k) PFT polynomial, and let $p > k$. Consider tuples $\beta = (b_1, b_2, \dots b_n)$ and $\gamma = (c_1, c_2, \dots, c_n)$ such that $\partial^\beta, \partial^\gamma \in T_k$. Provided $\partial^\beta(f), \partial^\gamma(f) \neq 0$ we have $\beta = \gamma$ if and only if* LM$(\partial^\beta(f)) =$ LM$(\partial^\gamma(f))$.

Proof. Assume that $\partial^\beta = \partial^\gamma$. Then every term in the polynomials will be the same, hence the leading monomials will be the same, so LM$(\partial^\beta(f)) =$ LM$(\partial^\gamma(f))$.

Now assume that $\mathrm{LM}(\partial^\beta(f)) = \mathrm{LM}(\partial^\gamma(f))$. These monomials will be of the form:

$$\mathrm{LM}(\partial^\beta(f)) = x_1^{pa_1-b_1} \cdot x_2^{pa_2-b_2} \cdots x_n^{pa_n-b_n}$$

$$\mathrm{LM}(\partial^\gamma(f)) = x_1^{pa'_1-c_1} \cdot x_2^{pa'_2-c_2} \cdots x_n^{pa'_n-c_n}$$

We cannot assume that the $a_i = a'_i$.

Now using our assumption we know that

$$x_1^{pa_1-b_1} \cdot x_2^{pa_2-b_2} \cdots x_n^{pa_n-b_n} = x_1^{pa'_1-c_1} \cdot x_2^{pa'_2-c_2} \cdots x_n^{pa'_n-c_n}$$

We end up with the system

$$pa_i - b_i = pa'_i - c_i, 1 \leq i \leq n$$

Rearranging this we can see that

$$p(a_i - a'_i) = b_i - c_i$$

This implies that $p | b_i - c_i$. Since we know that $\sum b_i = \sum c_i = k$ then $b_i, c_i \leq k$ so $|b_i - c_i| \leq k < p$. Therefore $b_i - c_i = 0$ which implies $b_i = c_i$ for all $i \in \{1, 2, \ldots, n\}$. Therefore $\beta = \gamma$. □

For two tuples $\alpha = (a_1, \ldots, a_n)$ and $\beta = (b_1, \ldots, b_n)$ we say that $\alpha \equiv \beta \pmod p$ if $a_i \equiv b_i \pmod p$ for $i \in [n]$. For congruent tuples α, β we say that the differential operators $\partial^\alpha, \partial^\beta$ are 'congruent mod p', as shorthand for $\alpha \equiv \beta \pmod p$. Then we have the following result about congruent differential operators.

Lemma 4.2.5. *Suppose $\partial^{\alpha_1}, \partial^{\alpha_2} \in T_a$ are two distinct differential operators. Then for PFT polynomials $f = F_{(n,p,k)}$ if $\partial^{\alpha_1}(f), \partial^{\alpha_2}(f)$ have any monomials in common then $\alpha_1 \equiv \alpha_2 \pmod{p}$.*

Proof. Let $\partial^{\alpha_1}, \partial^{\alpha_2} \in T_{k+1}$ be two distinct differential operators. Write $\alpha_1 = p\beta_1 + \gamma_1$ and $\alpha_2 = p\beta_2 + \gamma_2$. We want to show that if $\partial^{\alpha_1}(f), \partial^{\alpha_2}(f)$ have any monomials in common, then $\gamma_1 = \gamma_2$.

Assume that $\partial^{\alpha_1}(f), \partial^{\alpha_2}(f)$ have a monomial in common, so $cx_1^{q_1}x_2^{q_2}\cdots x_n^{q_n} \in \partial^{\alpha_1}(f)$ and $dx_1^{q_1}x_2^{q_2}\cdots x_n^{q_n} \in \partial^{\alpha_2}(f)$. Let $\alpha_1 = (u_1, u_2, \cdots, u_n)$ and $\alpha_2 = (v_1, v_2, \cdots, v_n)$. Note that since $f = (x_1^p + x_2^p + \cdots + x_n^p)^k$ then every monomial of the expansion of f is of the form $x_1^{p\lambda_1}x_2^{p\lambda_2}\cdots x_n^{p\lambda_n}$.

In other words, the power on every variable is a multiple of p. Let $\beta_1 = (j_1, j_2, \cdots, j_n)$ and $\beta_2 = (l_1, l_2, \cdots, l_n)$, and let $\gamma_1 = (g_1, g_2, \cdots, g_n)$ and $\gamma_2 = (h_1, h_2, \cdots, h_n)$. Then observe the following.

$$\begin{aligned} q_i &= p\lambda_i - u_i = p\lambda_i - (pj_i + g_i) \\ &= p(\lambda_i - j_i) - g_i \equiv p - g_i \pmod{p} \end{aligned}$$

Similarly, we have

$$\begin{aligned} q_i &= p\lambda_i - v_i = p\lambda_i - (pl_i + h_i) \\ &= p(\lambda_i - l_i) - h_i \equiv p - h_i \pmod{p} \end{aligned}$$

Since $q_i \equiv p - g_i \pmod{p}$ and $q_i \equiv p - h_i \pmod{p}$ then $p - g_i \equiv p - h_i \pmod{p}$ therefore since we know $g_i, h_i < p$ then this implies $g_i = h_i$ which means $\gamma_1 = \gamma_2$ as desired. □

Note that the original λ_i's do not have to be the same for each differential operator. The common monomial term in $\partial^{\alpha_1}(f), \partial^{\alpha_2}(f)$ could have come from 2 different monomials of f, and in fact they must come from different terms of f for distinct partial derivatives.

Lemma 4.2.6. *Let $\partial^{\alpha_1}, \partial^{\alpha_2} \in T_a$ and $\alpha_1 \neq \alpha_2$. If the supports of $\partial^{\alpha_1}(f), \partial^{\alpha_2}(f)$ have a monomial, M, in common, with $\partial^{\alpha_1}(Z_\theta) = M$ and $\partial^{\alpha_2}(Z_\eta) = M$ then $Z_1 \neq Z_2$.*

Proof. Recall $Z_\theta = x_1^{p\theta_1} \cdots x_n^{p\theta_n}$ and $Z_\eta = x_1^{p\eta_1} \cdots x_n^{p\eta_n}$. Assume that $\partial^{\alpha_1}(f), \partial^{\alpha_2}(f)$ have a monomial, M, in common. By 4.2.5 this implies that $\alpha_1 \equiv \alpha_2 \pmod p$. Let the common monomial M be $M = x_1^{\zeta_1} \cdots x_n^{\zeta_n}$.

$$\partial^{\alpha_1}(Z_\theta) = x_1^{p\theta_1 - a_1} \cdots x_n^{p\theta_n - a_n} = x_1^{\zeta_1} \cdots x_n^{\zeta_n}$$
$$\partial^{\alpha_2}(Z_\eta) = x_1^{p\eta_1 - a'_1} \cdots x_n^{p\eta_n - a'_n} = x_1^{\zeta_1} \cdots x_n^{\zeta_n}$$

Then,

$$p\theta_i = \zeta_i + a_i$$
$$p\eta_i = \zeta_i + a'_i$$

Since $a_i \neq a'_i$ for some i then $p\theta_i \neq p\eta_i$, therefore $Z_\theta \neq Z_\eta$.

□

For example, when $f = F_{(3,3,2)} = x^6 + 2x^3y^3 + 2x^3z^3 + y^6 + 2y^3z^3 + z^6$ then

$$\partial_x^3(f) = 120x^3 + 12y^3 + 12z^3$$

$$\partial_y^3(f) = 12x^3 + 120y^3 + 12z^3$$

$$\partial_z^3(f) = 12x^3 + 12y^3 + 120z^3$$

These all have $\text{supp}(\partial_x^3(f)) = \text{supp}(\partial_y^3(f)) = \text{supp}(\partial_z^3(f)) = \text{span}\{x^3, y^3, z^3\}$. This is because $(3, 0, 0) \equiv (0, 3, 0) \equiv (0, 0, 3)(\mod 3)$.

Corollary 4.2.7. *For a PFT polynomial $\partial^{\alpha_1}, \partial^{\alpha_2} \in T_a$, if $\alpha_1 \equiv \alpha_2 \pmod{p}$ then $\partial^{\alpha_1}(f), \partial^{\alpha_2}(f)$ have identical monomials.*

Proof. Suppose $f = F_{(n,p,k)}$. Let $\partial^{\alpha_1}, \partial^{\alpha_2} \in T_a$ where $\alpha_1 = p\beta_1 + \gamma_1$ and $\alpha_2 = p\beta_2 + \gamma_2$, with $\gamma_1 = \gamma_2$ (this is equivalent to $\alpha_1 \equiv \alpha_2 \pmod{p}$). For ease, let $\gamma_1, \gamma_2 = \gamma$.

Let M be a monomial where $M \in \text{supp}(\partial^{\alpha_1}(f))$. This resulting monomial came from some term, $Z_1 \in \text{supp}(f)$ where $Z_1 = x_1^{p\theta_1} \cdots x_n^{p\theta_n}$. So $\partial^{\alpha_1}(Z_1) = M$. It is easy to find Z_1 based on α_1. Say $\alpha_1 = (a_1, a_2, \ldots, a_n)$ and $\alpha_2 = (a'_1, a'_2, \ldots, a'_n)$.

Then we can write α_1 as

$$\begin{aligned}\alpha_1 &= p\beta_1 + \gamma \\ &= p(b_1, b_2, \ldots, b_n) + (g_1, g_2, \ldots, g_n)\end{aligned}$$

Similarly for α_2 we have

$$
\begin{aligned}
\alpha_2 &= p\beta_2 + \gamma \\
&= p(b'_1, b'_2, \ldots, b'_n) + (g_1, g_2, \ldots, g_n)
\end{aligned}
$$

Then,

$$
\begin{aligned}
\partial^{\alpha_1}(Z_1) = M &= \partial^{\alpha_1}(x_1^{p\theta_1} \cdots x_n^{p\theta_n}) \\
M &= x_1^{p\theta_1-(pb_1+g_1)} \cdots x_n^{p\theta_n-(pb_n+g_n)} \\
M &= x_1^{p(\theta_1-b_1)-g_1} \cdots x_n^{p(\theta_n-b_n)-g_n}
\end{aligned}
$$

Let $Z_2 = x_1^{p\theta'_1} \cdots x_n^{p\theta'_n}$ be another term in f where $\theta'_i = \theta_i - b_i + b'_i$. This is always possible because we know $b_i \leq \theta_i$, so then θ'_i will always be non-negative.

Because f has $p-$full support (see 4.1.3), we are guaranteed to have $Z_2 \in \mathrm{supp}(f)$ with these given θ'_i exponents.

Then observe that

$$
\begin{aligned}
\partial^{\alpha_2}(Z_2) &= \partial^{\alpha_2}(x_1^{p\theta'_1} \cdots x_n^{p\theta'_n}) \\
&= x_1^{p(\theta_1-b_1+b'_1)-(pb'_1+g_1)} \cdots x_n^{p(\theta_n-b_n+b'_n)-(pb'_n+g_n)} \\
&= x_1^{p(\theta_1-b_1)-g_1} \cdots x_n^{p(\theta_n-b_1)-g_1} \\
&= M
\end{aligned}
$$

Therefore $M \in \operatorname{supp}(\partial^{\alpha_2} f))$ which implies $\operatorname{supp}(\partial^{\alpha_2}(f)) \subset \operatorname{supp}(\partial^{\alpha_1}(f))$. The process is similar in the other direction, therefore we can conclude that
$\operatorname{supp}(\partial^{\alpha_1}(f)) = \operatorname{supp}(\partial^{\alpha_2}(f))$.

$\square$

These results show that if $\alpha_1 \equiv \alpha_2 \pmod{p}$ then $\partial^{\alpha_1}(f), \partial^{\alpha_2}(f)$ have the same support, otherwise they will have disjoint supports (no monomials in common). This helps us with proving linear independence for the space of Derivs in our general results.

4.2.2 Computed Examples

We will examine some specific PFT polynomials and their annihilator ideals. To start, take $n = 3, p = 3, k = 2$. We have $f = F_{3,3,2} = x^6 + 2x^3y^3 + 2x^3z^3 + y^6 + 2y^3z^3 + z^6$. The dimension of $\text{Derivs}(F)_3 \subset S_3^n$ is at most $\left(\!\binom{n}{3}\!\right)$ since the number of degree 3 monomials is counted by choosing 3 variables with repetition from the set of 3 variables $\{x, y, z\}$, so $\dim S_3^3 = \left(\!\binom{n}{3}\!\right)$.

Recall the image of $C_f^a : T_3 \to S_3$ is $\text{Derivs}(f)_3$, and the nullity is $\text{Ann}(f)_3$. By 2.2.1 $\dim(T_3) = \left(\!\binom{n}{3}\!\right) = \dim(\text{Derivs}(f)_3) + \dim(\text{Ann}(f)_3)$. We claim that $\text{Ann}(f)_3 = \{\partial_x \partial_y \partial_z\}$.

First note $\partial_x \partial_y \partial_z(f) = \partial_x \partial_y \partial_z(x^6 + 2x^3y^3 + 2x^3z^3 + y^6 + 2y^3z^3 + z^6) = 0$ since there is no term in this polynomial involving all 3 variables x, y, z then $\partial_x \partial_y \partial_z(F) = 0$ therefore $\{\partial_x \partial_y \partial_z\} \subset \text{Ann}(f)_3$ so $\dim(\text{Ann}(f_3)) \geq 1$.

To show that $\dim(\text{Ann}(f)_3) = 1$, form a matrix, M, of the image of C_f^a whose columns are spanned by $\text{Derivs}(f)_3$, indexing the rows by elements of S_3 and the columns by elements of T_3. The rank of this matrix should be equal to the dimension of Derivs$(f)_3$.

So the (i, j) entry is the coefficient of the i^{th} element of S_3 on the j^{th} derivative of f. We know by the previous part that we get a column of $0's$ in the $\partial_x\partial_y\partial_z$ position. Then we need only calculate the rank of the rest of the matrix, which we can split into 2 pieces as follows.

Let M_1 be the three columns indexed by $\{\partial_x^3, \partial_y^3, \partial_z^3\}$ and let M_2 be the 6 columns indexed by $\{\partial_x^2\partial_y, \ldots, \partial_y\partial_z^2\}$, and omit the column of 0's indexed by $\partial_x\partial_y\partial_z$.

We can compute $\partial_x^3(f) = 120x^3 + 12y^3 + 12z^3$ and similarly for $\partial_y^3(f), \partial_z^3(f)$. We get the following matrix.

$$M_1 = \begin{bmatrix} 120 & 12 & 12 \\ 12 & 120 & 12 \\ 12 & 12 & 120 \end{bmatrix}$$

By Lemma 4.2.3 we can see the matrix will have full rank, so $\text{rank}(M_1) = 3$.

For $\text{rank}(M_2)$ observe that for each pair of distinct variables in $\{x, y, z\}$ there is only one term in f that contains both, therefore $\partial x^2y(f)$ is a monomial, and the monomial must be $36xy^2$ since the only term of f containing both x, y is $2x^3y^3$ and $\partial x^2y(2x^3y^3) = 36xy^2$. In fact for each $\partial x_m^2 x_n$ the image will be $\partial x_m^2 x_n(F) = 36x_m x_n^2$. Each of these monomials are unique, then this space is linearly independent and has size $2\binom{n}{2} = 6$. Therefore $\text{rank}(M_2) = 6$.

Notice that the intersection of the columns of M_1 and M_2 is 0 since the former is spanned by polynomials such that every term is in one variable alone, and M_2 is spanned by monomials such that every term is a monomial in two variables. Then $\text{rank}(M) = \text{rank}(M_1) + \text{rank}(M_2) = 3+6 = 9$. Since M is a 10×10 matrix, its nullity is $\text{nullity}(M) = 10 - 9 = 1$. So $\dim(\text{Derivs}(f)_3) = 9$, and $\dim(\text{Ann}(f) = 1$. Therefore $\text{Ann}(f)_3 = \text{span}\{\partial_x \partial_y \partial_z\}$.

The differential operator $\partial_x \partial_y \partial_z$ is what we call a *square-free* differential operator. An $a-$th order square-free differential operator ∂^α for $\alpha = (a_1, \ldots, a_k)$ is an element of T_k such that k of the a_i's are 1 and the rest are 0. These types of differential operators will be important to our results.

Now we will generalize on n, so take $f = F_{n,3,2} \in S_6$. By the multinomial theorem this function is given by

$$f = (x_1^3 + x_2^3 + \ldots + x_n^3)^2 = x_1^6 + 2x_1^3 x_2^3 + \ldots + 2x_{n-1}^3 x_n^3 + x_n^6 = \sum_{i=1}^{n} x_i^6 + \sum_{i,j \in \{1,\ldots n\}, i \neq j}^{n} 2x_i^3 x_j^3$$

We want to show that $\dim(\text{Derivs}(f)_3) = \left(\!\binom{n}{3}\!\right) - \binom{n}{3}$.

There are $\left(\!\binom{n}{3}\!\right)$ 3rd partial derivatives of f so $\dim \text{Derivs}(f)_3 \leq \left(\!\binom{n}{3}\!\right)$. Notice that $\text{span}\{\partial_{x_i} \partial_{x_j} \partial_{x_l} : i \neq j \neq l\} \subset \text{Ann}(f)_3$ since $\text{supp}(f) = \text{span}\{x_i^6, x_i^3 x_j^3 : i \neq j\}$. So the annihilator contains every square-free differential operator in T_3, thus $\dim(\text{Ann}(f))_3 \geq \binom{n}{3}$, since there are $\binom{n}{3}$ such differential operators. Now we need to establish that the remaining elements of $\text{Derivs}(f)_3$ are linearly independent.

Let M be a matrix of the image of C_f^a whose columns are spanned by $\text{Derivs}(f)_3$, indexing the rows by elements of S_3 and the columns by elements of T_3. The rank of this matrix should be equal to the dimension of $\text{Derivs}(f)_3$. We will leave off the columns of 0's created by the square-free derivatives. This matrix can be decomposed into two pieces, as before, M_1 whose columns are spanned by operators of the form $\partial_{x_i}^3$, and M_2 whose columns are spanned by $\partial_{x_i}^2 \partial_{x_j}$ where $i \neq j$.

$$M_1 = \begin{bmatrix} 120 & 12 & \cdots & 12 \\ 12 & 120 & \cdots & 12 \\ \vdots & \vdots & \vdots & \vdots \\ 12 & 12 & \cdots & 120 \end{bmatrix} \qquad M_2 = \begin{bmatrix} 36 & 0 & \cdots & 0 \\ 0 & 36 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 36 \end{bmatrix}$$

By Lemma 4.2.3 M_1 is linearly independent and invertible. For M_2 observe for any pair of variables $x_i, x_j \in \{x_1, \ldots x_n\}$ there is a unique term of F that involves both. Thus for any $\partial x_i^2 x_j$ we have $\partial x_i^2 x_j(F) = 2 \cdot 3 \cdot 2 \cdot 3 \cdot x_i x_j^2 = 36 x_i x_j^2$. This matrix is clearly invertible.

Every element of M_1 is of the form $120x_i^3 + \sum_{j=1, j\neq i}^{n} 12x_j^3$ and every element of M_2 is of the form $36x_i x_j^2$ so the intersection of their column spans is empty, therefore $\text{rank}(M) = \text{rank}(M_1) + \text{rank}(M_2)$. Since both matrices have full rank, then the remaining set of partial derivatives is linearly independent, therefore there are no additional degree 3 annihilators of f and $\text{Ann}(f)_3 = \text{span}\{\partial_{x_i}\partial_{x_j}\partial_{x_l} : i \neq j \neq l\}$.

Therefore $\dim(\text{Derivs}(f)_3) = \left(\!\binom{n}{3}\!\right) - \binom{n}{3}$. This result will continue in general, with some mild constraints on the values for p and k.

4.3 Results and Proofs

Here we present some general results about the Derivs of PFT polynomials, using the background we developed in the previous sections. Our main theorem is on the degree k and $k+1$ annihilators of general PFT polynomials, for $p > k$.

Lemma 4.3.1. *Let $f = F_{(n,p,k)}$ be the (n, p, k) PFT polynomial where $n > k$. Then $\mathrm{Ann}(f)_{k+1} \neq \{0\}$ and contains the span of all square-free monomials in T_{k+1}.*

Proof. First, note that since $n \geq k+1$ there exists at least one square-free differential operator in T_{k+1}. There are exactly $\binom{n}{k+1}$ square-free differential operators in T_{k+1}, since the number of operators is equivalent to the number of size $k+1$ subsets from a set of n elements.

Each of these square-free differential operators will annihilate f, since every monomial $Z_\theta \in \mathrm{supp}(f)$ will contain at most k distinct variables, thus differentiating with respect to $k+1$ arbitrary variables will annihilate f. Therefore $\mathrm{Ann}(f)_{k+1}$ contains the span of all these order $k+1$ differential operators. □

Theorem 4.3.2. *Let $f = F_{(n,p,k)}$ be the (n, p, k) PFT polynomial and let $p > k$. Then $\mathrm{Ann}(f)_k = \{0\}$, if $n \leq k$ then $\mathrm{Ann}(f)_{k+1} = \{0\}$ and if $n > k$ then $\mathrm{Ann}(f)_{k+1}$ is spanned by the square-free differential operators of T_{k+1}.*

Proof. First, we work over the map $C_f^k : T_k \to S_{pk-k}$. Let $\partial^\alpha \in T_k$. We know that $\partial^\alpha(f) \neq 0$ since f contains the monomial $x^{p\alpha} = x_1^{pa_1} x_2^{pa_2} \cdots x_n^{pa_n}$ with some nonzero coefficient because f has $p-$full support. Therefore $\partial^\alpha(f)$ contains $\partial^\alpha(x^{p\alpha}) = x_1^{pa_1 - a_1} x_2^{pa_2 - a_n} \cdots x_n^{pa_n - a_n}$ with some nonzero coefficient. And this term is nonzero since $p \geq 2$. By 4.2.4 we know for distinct α, β the leading monomials will be distinct. Therefore the set $\{C_f^k(\partial^\alpha) : \partial^\alpha \in T_k\}$ is linearly independent, so the matrix of C_f^k

has linearly independent columns, spanned by the partial derivatives $\partial^\alpha(f)$. Thus $\operatorname{rank} C_f^k = \dim T_k$ and $\operatorname{Ann}(f)_k = \{0\}$.

Now let $n > k$ and we work over the map $C_f^{k+1} : T_{k+1} \to S_{pk-k-1}$. We can see every square-free monomial in T_{k+1} is in $\operatorname{Ann}(f)_{k+1}$, by 4.3.1. Let ∂^α be any non-square-free differential monomial in T_{k+1}. We claim $\partial^\alpha(f) \neq 0$. Let $\alpha = (a_1, a_2, \ldots, a_n)$. Since α is non-square-free, then some $a_i \geq 2$. Without loss of generality assume that $a_1 \geq 2$. Because $\sum_{i=1}^n a_i = k+1$ then $(a_1 - 1) + \sum_{i=2}^n a_i = k$. Therefore $x_1^{p(a_1-1)} x_2^{pa_2} \cdots x_n^{pa_n}$ is a monomial with degree pk, which means it is in $\operatorname{supp}(f)$.

Then observe,

$$\partial^\alpha(x_1^{p(a_1-1)} x_2^{pa_2} \cdots x_n^{pa_n}) = c x_1^{p(a_1-1)-a_1} x_2^{pa_2-a_2} \cdots x_n^{pa_n-a_n}$$

for some constant c, and $p(a_1 - 1) - a_1 > 0$ because $p \geq 2$. Therefore $\partial^\alpha(f) \neq 0$.

Now just like in the above case, we know the sets of the form $\{\partial^\alpha(f) : \partial^\alpha \in T_{k+1}, \partial^\alpha \text{ is nonsquare-free}\}$ are linearly independent, since they will all have distinct leading monomials by 4.2.4.

The only exception is when $p = k+1$ the leading monomials of every ∂^α are distinct, except for the subset $\{\partial_{x_i}^p(f) : i \in [n]\}$, the partial derivatives of order $k+1$ which are the $k+1$ power of a single operator.

Therefore take $\operatorname{Derivs}(f)_{k+1}$ and decompose it in the following way.

$$\operatorname{Derivs}(f)_{k+1} = V_1 \oplus V_2$$

where $V_1 = \operatorname{span}\{\partial_{x_i}^p(f) : i \in [n]\}$ and V_2 is spanned by $\partial^\alpha(f)$ for all other α of total degree $k+1$ involving at least 2 variables.

The first subspace we have shown is linearly independent, by 4.2.4. The second subspace has equivalent monomials by 4.2.5 and 4.2.7. In particular, notice that the matrix for the image of $\{\partial^p_{x_i}(f) : i \in [n]\}$ will be equal to $v^T \times w$ where v is a vector of coefficients, and w is the coefficient vector for $F_{(n,p,k-1)}$. This results in the matrix of $\{\partial^p_{x_i}(f) : i \in [n]\}$. These coefficients for each $Z_\theta \in \operatorname{supp}(\partial^p_{x_i}(f))$ will be

$$\begin{cases} \left(\frac{(p\theta_i)!}{(p\theta_i - p)!}\right) & \text{if } \theta_i \neq 0 \\ p! & \text{if } \theta_i = 0 \end{cases}$$

giving two cases for each coefficient based on what variable is being differentiated, and each column will index exactly one of those differential operators. So the matrix $v^t \times w$ will look like the matrix in 4.2.3, which we have shown is invertible. Then $\{\partial^p_{x_i}(f) : i \in [n]\}$ is linearly independent.

Therefore in the matrix for the image of the map C_f^{k+1}, the columns induced by nonsquare-free monomials are linearly independent for all $p > k$, and the kernel of this map is given by $\operatorname{Ann}(f)_{k+1} = \operatorname{span} \partial^\alpha :$ squarefree monomials .

Finally, when $n \leq k$, we can observe that there are no squarefree monomials of degree $k+1$ in T_{k+1} so $\operatorname{Ann}(f)_{k+1} = \{0\}$ in this case, and the above argument shows that all the columns of the matrix of C_f^{k+1} are linearly independent.

□

This theorem implies that for any $f = F_{(n,p,k)}$ where $p > k$, we have

$$\text{rank}(f) \geq \left(\!\!\binom{n}{k+1}\!\!\right) - \binom{n}{k+1}$$

which gives a lower bound for the Waring rank of all PFT polynomials when $p > k$.

4.4 Conjecture of Generalization

We believe 4.3.2 extends to all PFT polynomials but this proof strategy fails when $p \leq k$, since 4.2.4 does not hold for PFT polynomials where $p \leq k$. Observe the following counterexample.

$$f = F_{3,3,3} = x^9 + 3x^6y^3 + 3x^6z^3 + 3x^3y^6 + 6x^3y^3z^3 + 3x^3z^6 + y^9 + 3y^6z^3 + 3y^3z^6 + z^9$$

we can take partial derivatives and observe their lead monomials: $\text{LM}(\partial_x^3(f)) = x^6, \text{LM}(\partial_y^3(f)) = x^6$. Thus the leading monomials of $\text{Derivs}_{d-k}(f)$ are not distinct for each ∂^α. This counterexample does not disprove the general result for this case, since our result on the uniqueness of the leading terms is a stronger condition than linear independence.

Conjecture 4.4.1. Let $f = F_{(n,p,k)}$ be the (n, p, k) PFT polynomial. Then theorem 4.3.2 holds for all p.

Our approach to this involved using 4.2.5 and 4.2.7 to classify each partial differential operator of T_{k+1} into congruence classes mop p, similar to our method in 4.3.2 for $p = k + 1$. We were not able to apply this method more generally, since in that case, we only had one set of congruent differential operators, single operators to the power p. The case is not so simple for $p \leq k$. For example, let $f = F_{3,3,3}$, so

$$f = x^9 + 3x^6y^3 + 3x^6z^3 + 3x^3y^6 + 6x^3y^3z^3 + 3x^3z^6 + y^9 + 3y^6z^3 + 3y^3z^6 + z^9$$

We want to examine the degree 4 partial derivatives of f, these are given by

$$\{\partial_x^4, \partial_x^3\partial_y, \partial_x^3\partial_z, \partial_x^2\partial_y^2, \partial_x^2 yz, \partial_x^2\partial_z^2, \partial_x\partial_y^3, \partial_x\partial_y^2\partial_z, \partial_x\partial_y\partial_z^2, \partial_x\partial_z^3, \partial_y^4, \partial_y^3\partial_z, \partial_y^2\partial_z^2, \partial_y\partial_z^3, \partial_z^4\}.$$

Then the congruence classes for each ∂^α are given below.

If we then look at all the monomials in $\mathrm{supp}(\partial^\alpha(f))$ for each $\partial^\alpha \in T_a$ we can verify that if two $\partial^{\alpha_1}, \partial^{\alpha_2}$ have equal γ's, then $\mathrm{supp}\{\partial^{\alpha_1}(f)\} = \mathrm{supp}\{\partial^{\alpha_2}(f)\}$. Thus 4.2.5 and 4.2.7 should aid in showing these partial derivatives are all linearly independent. Once this is established, then 4.3.2 holds without the condition $p > k$. We are confident it is possible to prove this, and hope to revisit this natural extension.

4.5 Further Questions

We were able to prove 4.3.2 for $p > k$, and we believe strongly that this theorem is true when $p \leq k$, however this case will take considerably more work, as we have seen in our original strategy. A further question we may ask is if there is a method of proving both cases together, in general. The method of partial derivatives, and apolar algebra have been useful and essential to this work, but can they be utilized in other ways, that are less computationally rigorous? Our proof is an extension of methods detailed in [1], which showed $F_{n,2,k}$ has a compressed Hilbert function. A

∂^α	$\partial^\alpha(f)$	α	β	γ
∂_x^4	$3024x^5 + 1080x^2y^3 + 1080x^2y^3$	(4,0,0)	(1,0,0)	(1,0,0)
$\partial_x^3\partial_y$	$1080x^3y^2 + 108y^5 + 108y^2z^3$	(3,1,0)	(1,0,0)	(0,1,0)
$\partial_x^3\partial_z$	$1080x^3z^2 + 108y^3z^2 + 108z^5$	(3,0,1)	(1,0,0)	(0,0,1)
$\partial_x^2\partial_y^2$	$540x^4y + 540xy^4 + 216xyz^3$	(2,2,0)	(0,0,0)	(2,2,0)
$\partial_x^2\partial_y\partial_z$	$324xy^2z^2$	(2,1,1)	(0,0,0)	(2,1,1)
$\partial_x^2\partial_z^2$	$540x^4z + 216xy^3z + 540xz^4$	(2,0,2)	(0,0,0)	(2,0,2)
$\partial_x\partial_y^3$	$108x^5 + 1080x^2y^3 + 108x^2z^3$	(1,3,0)	(0,1,0)	(1,0,0)
$\partial_x\partial_y^2\partial_z$	$324x^2yz^2$	(1,2,1)	(0,0,0)	(1,2,1)
$\partial_x\partial_y\partial_z^2$	$324x^2y^2z$	(1,1,2)	(0,0,0)	(1,1,2)
$\partial_x\partial_z^3$	$108x^5 + 108x^2y^3 + 1080x^2y^3$	(1,0,3)	(0,0,1)	(1,0,0)
∂_y^4	$1080x^3y^2 + 3024y^5 + 1080y^2z^3$	(0,4,0)	(0,1,0)	(0,1,0)
$\partial_y^3\partial_z$	$108x^3z^2 + 1080y^3z^2 + 108z^5$	(0,3,1)	(0,1,0)	(0,0,1)
$\partial_y^2\partial_z^2$	$216x^3yz + 540y^4z + 540yz^4$	(0,2,2)	(0,0,0)	(0,2,2)
$\partial_y\partial_z^3$	$108x^3y^2 + 108y^5 + 1080y^2z^3$	(0,1,3)	(0,0,1)	(0,1,0)
∂_z^4	$1080x^3z^2 + 1080y^3z^2 + 3024z^5$	(0,0,4)	(0,0,1)	(0,0,1)

Table 4.1: $\partial^\alpha \in T_4$ Applied to $f = F_{(3,3,3)}$ and Their Congruence

proof exists in [6] that shows the same result, which utilizes representation theory. This methodology could be applied here, if we continued this work starting over from a new perspective.

There exist a number of places in this work where we utilize common combinatorial methods, and there may be more results from algebraic-combinatorics which might help us better understand the origin of the results we achieved here.

Our examination of this type of polynomial started with extending the result about $F_{(n,2,k)}$ by generalizing the inside power. However, we could also extend this result by examining other symmetric polynomials taken to a power, for example given the elementary symmetric polynomial $e_{n,d}$ define a polynomial $F_{(n,d,k)} := (e_{n,d})^k$. What is the annihilator ideal of this polynomial?

We are proud of our results here, and hope to continue to expand and simplify them, as well as find connections to other similar problems and related work in Waring rank and Apolarity.

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