THE MATRIX SORTABILITY PROBLEM

by

Seth Cleaver

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Seth Cleaver

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The following individuals read and discussed the thesis submitted by student Seth Cleaver, and they evaluated his presentation and response to questions during the final oral examination. They found that the student passed the final oral examination.

Marion Scheepers, Ph.D.	Chair, Supervisory Committee
Donna Calhoun, Ph.D.	Member, Supervisory Committee
Liljana Babinkostova, Ph.D.	Member, Supervisory Committee

The final reading approval of the thesis was granted by Marion Scheepers, Ph.D., Chair of the Supervisory Committee. The thesis was approved by the Graduate College.

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ABSTRACT

Sorting is such a fundamental component of achieving efficiency that a significant body of mathematics is dedicated to the investigation of sorting. Any modern textbook on algorithms contains chapters on sorting.

One approach to arranging a disorganized list of items into an organized list is to successively identify two blocks of contiguous items, and swap the two blocks. In a fundamental paper D.A. Christie showed that a special version of block swapping, in recent times called *context directed swapping* and abbreviated **cds**, is the most efficient among block swapping strategies to achieve an organized list of items. The **cds** sorting strategy is also the most robust among block swap based sorting methods.

It has been discovered that the context directed block swap operation on a list of objects generalizes to an operation on simple graphs. In turn it has been discovered that this operation on simple graphs corresponds with an operation on the adjacency matrix of a simple graph. The adjacency matrix is a symmetric square matrix with entries 0 and 1, and all diagonal entries 0. The corresponding operation is denoted **Mcds**, abbreviating *matrix context directed swap.* The operation on the adjacency matrix naturally employs the arithmetic of $GF(2)$, the finite field of two elements. It has been speculated that the **Mcds** operation on these specific matrices over $GF(2)$ corresponds with the more than a century old *Schur complement* operation on these matrices.

In this thesis, we confirm this prior speculation about the correspondence between

Mcds and the Schur complement, in the context of $GF(2)$. We generalize the **Mcds** operation to not necessarily square matrices over arbitrary fields and we prove that the generalized **Mcds** corresponds with the Schur complement also in the more general context of all fields.

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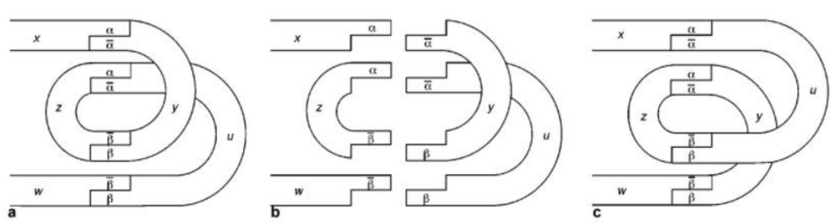
CHAPTER 1

INTRODUCTION AND BACKGROUND

1.1 Origins of Mcds

In a fundamental paper [5] by David Christie, before it was named **cds**, Christie gave the first mathematical source demonstrating the importance of the **cds** sorting operation. Further examination of [5] shows that minimal block interchanges are special cases of **cds**. The **cds** operation directly relates to a phenomenon in biology, specifically *ciliate genome maintenance*. The **cds** operation on permutations corresponds exactly to the molecular operation named *double loop aligned by alternating directs*, denoted *dlad* and described in [6]. The following is a diagram representing *dlad*, and an accompanying quote from [6]



(a) The molecule folded in a double loop aligned by alternating direct repeats. (b) Staggered cuts introduced into the aligned pointers, $\alpha\,\alpha$ and $\beta\,\beta$. (c) Recombinations between pointers.

Figure 1.1: *dlad* in Ciliates

In this thesis we consider the translation of the cds sorting operation into the world of matrices, generalizing it from the very special context of certain symmetric matrices over $GF(2)$, to all matrices (square or rectangular) over any field.
The **Mcds** operation emerged from the simpler world of permutations on n elements, and the **cds** sorting operation on those permutations. The following section explores the world of sorting permutations using specialized block swaps.

1.1.1 Permutations

Definition 1.1.1. *A* ***permutation****,* $\pi = [a_1, \ldots, a_n]$ *is an array of the integers between 1 and n inclusive, in any order, such that there are no repeats among the integers. Each integer,* $i \in \pi$*, has a left pointer and a right pointer: For an entry* i *of* π *we write* ${}^{(i-1,i)}i^{(i,i+1)}$ *where* $(i-1, i)$ *is the left pointer, and* $(i, i+1)$ *is the right pointer of* i*. [2]*

Definition 1.1.2. *Given a permutation* π *and pointers* p *and* q *that appear in*

$$p, q, p, q$$

context, there are the following possibilities:
Say $p = (x, x+1)$ *and* $q = (y, y+1)$*. Then* $x, y, x+1$ *and* $y+1$ *could appear in any one of the following relative positions in* π*:*

$$Case(1) : [\cdots x \cdots y \cdots x+1 \cdots y+1 \cdots]$$

$$Case(2) : [\cdots x \cdots y+1 \cdots x+1 \cdots y \cdots]$$

$$Case(3) : [\cdots x+1 \cdots y \cdots x \cdots y+1 \cdots]$$

$$Case(4) : [\cdots x+1 \cdots y+1 \cdots x \cdots y \cdots]$$

the corresponding α segmentations would be as follows:

$$Case(1) : [\cdots x^{(x,x+1)} \cdots y^{(y,y+1)} \cdots {}^{(x,x+1)}x+1 \cdots {}^{(y,y+1)}y+1 \cdots]$$

$$Case(2) : [\cdots x^{(x,x+1)} \cdots {}^{(y,y+1)}y+1 \cdots {}^{(x,x+1)}x+1 \cdots y^{(y,y+1)} \cdots]$$

$$Case(3) : [\cdots {}^{(x,x+1)}x+1 \cdots y^{(y,y+1)} \cdots x^{(x,x+1)} \cdots {}^{(y,y+1)}y+1 \cdots]$$

$$Case(4) : [\cdots {}^{(x,x+1)}x+1 \cdots {}^{(y,y+1)}y+1 \cdots x^{(x,x+1)} \cdots y^{(y,y+1)} \cdots]$$

and with α-notation:

$$Case(1) : [{\alpha_1}^{(x,x+1)} \frown {\alpha_2}^{(y,y+1)} \frown \alpha_3 \frown {}^{(x,x+1)}\alpha_4 \frown {}^{(y,y+1)}\alpha_5]$$

$$Case(2) : [{\alpha_1}^{(x,x+1)} \frown \alpha_2 \frown {}^{(y,y+1)}\alpha_3 \frown {}^{(x,x+1)}{\alpha_4}^{(y,y+1)} \frown \alpha_5]$$

$$Case(3) : [\alpha_1 \frown {}^{(x,x+1)}{\alpha_2}^{(y,y+1)} \frown {\alpha_3}^{(x,x+1)} \frown \alpha_4 \frown {}^{(y,y+1)}\alpha_5]$$

$$Case(4) : [\alpha_1 \frown {}^{(x,x+1)}\alpha_2 \frown {}^{(y,y+1)}{\alpha_3}^{(x,x+1)} \frown {\alpha_4}^{(y,y+1)} \frown \alpha_5]$$

In each case the ***context directed swaps on the permutation*** *π* ***with context p and q*** *is given as follows:[2]*[1]

[1]The **cds** operation always swaps α_2 and α_4 blocks.

$$Case(1) : [\alpha_1{}^{(x,x+1)} \frown {}^{(x,x+1)}\alpha_4 \frown \alpha_3 \frown \alpha_2{}^{(y,y+1)} \frown {}^{(y,y+1)}\alpha_5]$$

$$Case(2) : [\alpha_1{}^{(x,x+1)} \frown {}^{(x,x+1)}\alpha_4{}^{(y,y+1)} \frown {}^{(y,y+1)}\alpha_3 \frown \alpha_2 \frown \alpha_5]$$

$$Case(3) : [\alpha_1 \frown \alpha_4 \frown \alpha_3{}^{(x,x+1)} \frown {}^{(x,x+1)}\alpha_2{}^{(y,y+1)} \frown {}^{(y,y+1)}\alpha_5]$$

$$Case(4) : [\alpha_1 \frown \alpha_4{}^{(y,y+1)} \frown {}^{(y,y+1)}\alpha_3{}^{(x,x+1)} \frown {}^{(x,x+1)}\alpha_2 \frown \alpha_5]$$

Recall Figure 1.1. The **cds** operation and the operation given in Figure 1.1, an operation of swapping y and u sections of corresponding ciliate DNA segments exactly if we make the following correspondences:

Figure 1.1	↔	Definition 1.1.2
x	↔	α_1
y	↔	α_2
z	↔	α_3
u	↔	α_4
w	↔	α_5
α	↔	p
β	↔	q

Figure 1.2: Correspondence Between dlad and cds

The *dlad* operation in *Ciliates* interchanges y and u blocks [6], whereas the **cds** operation on permutations interchanges blocks α_2 and α_4 [2]. Example 1.1.3 and Example 1.1.5 are given to familiarize the reader with the **cds** operation:

Example 1.1.3. *Consider the following permutation:*

$$\pi_\gamma = [1, 2, 3, 4, 6, 5]$$

Besides pointers (4,5) and (5,6) no other pair p and q of pointers appears in p q p q context. Pointers $(4,5)$ *and* $(5,6)$ *are in appropriate order to perform the* ***cds*** *operation. In order to perform the* ***cds*** *operation on pointers* $p = (4,5)$ *and* $q = (5,6)$*, first identify which case of Definition 1.1.2 is applicable. In Example 1.1.3, the first occurrence of pointer* $(4,5)$ *is a right pointer, and the first occurrence of pointer* $(5,6)$ *is a left pointer. Thus* $\boldsymbol{cds}_{(4,5),(5,6)}(\pi_\gamma)$ *is an example of Case (2) by Definition 1.1.2. Next, identify the* α *segments:*

$$\begin{aligned} {\alpha_1}^{(4,5)} &= [1,2,3,4] \\ \alpha_2 &= [\{\}] \\ {}^{(5,6)}\alpha_3 &= [6] \\ {}^{(4,5)}{\alpha_4}^{(5,6)} &= [5] \\ \alpha_5 &= [\{\}] \end{aligned}$$

With the α*'s identified we can now perform the cds operation:*

$$\begin{aligned} \mathbf{cds}_{(4,5),(5,6)}(\pi_\gamma) &= \mathbf{cds}_{(4,5),(5,6)}\left([1,2,3,4,6,5]\right) \\ &= \mathbf{cds}_{(4,5),(5,6)}\left(\left[1,2,3,4^{(4,5)}, {}^{(5,6)}6, {}^{(4,5)}5^{(5,6)}\right]\right) \\ &= \mathbf{cds}_{(4,5),(5,6)}\left([{\alpha_1}^{(4,5)} \frown \alpha_2 \frown {}^{(5,6)}\alpha_3 \frown {}^{(4,5)}{\alpha_4}^{(5,6)} \frown \alpha_5]\right) \\ &= [{\alpha_1}^{(4,5)} \frown {}^{(4,5)}{\alpha_4}^{(5,6)} \frown {}^{(5,6)}\alpha_3 \frown \alpha_2 \frown \alpha_5] \\ &= [1,2,3,4,5,6] \end{aligned}$$

$\mathbf{cds}_{(4,5),(5,6)}(\pi_\gamma)$ *is an example of a* ***sorted*** permutation

$\boxdot$

Definition 1.1.4. *Let $k \in \mathbb{N}$. A permutation is called* ***sorted****, if the final result of k* ***cds*** *operations on a permutation π results in the permutation of the form:*

$$[1,\ 2,\ \ldots,\ n-1,\ n]$$

Examples 1.1.5 calculates the **cds** operation on a permutation $\pi_\sigma = [1, 5, 2, 4, 3, 6]$. For each pointer pair (p,q) from the set of pairs $\{((1, 2), (4, 5))$, $((4, 5), (2, 3))$, $((1, 2), (5, 6))$, $((2, 3), (3, 4))$, and $((4, 5), (3, 4))\}$, p and q appear in the context $p\ q\ p\ q$ or in the context $q\ p\ q\ p$ in permutation π_σ, and thus provides a context for applying the **cds** operation. Examples 1.1.5 calculates the **cds** operation to a **cds fixed point** starting from pointer pairs $((1, 2), (4, 5))$ and $((4, 5), (2, 3))$. The remainder of the pointer pairs are left as an exercise to the reader.

Example 1.1.5. *Consider the following permutation:*

$$\pi_\sigma = [1, 5, 2, 4, 3, 6]$$

Starting with $((1, 2), (4, 5))$, in order to perform the ***cds*** *operation on pointers $p = (1, 2)$ and $q = (4, 5)$, first identify which case of Definition 1.1.2 is applicable. In $[1, 5, 2, 4, 3, 6]$, the first occurrence of pointer $(1, 2)$ is a right pointer, and the first occurrence of pointer $(4, 5)$ is a left pointer. Thus $\textbf{cds}_{(1,2),(4,5)}(\pi_\sigma)$ is an example of Case (2) by Definition 1.1.2.*

Next, identify the α segments:

$$\alpha_1{}^{(1,2)} = [1]$$
$$\alpha_2 = [\{\}]$$
$${}^{(4,5)}\alpha_3 = [5]$$
$${}^{(1,2)}\alpha_4{}^{(4,5)} = [2,4]$$
$$\alpha_5 = [3,6]$$

With the α's identified we can now perform the cds operation:

$$\begin{aligned}\mathbf{cds}_{(1,2),(4,5)}(\pi_\sigma) &= \mathbf{cds}_{(1,2),(4,5)}\left([1,5,2,4,3,6]\right)\\ &= \mathbf{cds}_{(1,2),(4,5)}\left(\left[1^{(1,2)}, {}^{(4,5)}5, {}^{(1,2)}2, 4^{(4,5)}, 3, 6\right]\right)\\ &= \mathbf{cds}_{(1,2),(4,5)}\left(\left[\alpha_1{}^{(1,2)} \frown \alpha_2 \frown {}^{(4,5)}\alpha_3 \frown {}^{(1,2)}\alpha_4{}^{(4,5)} \frown \alpha_5\right]\right)\\ &= \left[\alpha_1{}^{(1,2)} \frown {}^{(1,2)}\alpha_4{}^{(4,5)} \frown {}^{(5,6)}\alpha_3 \frown \alpha_2 \frown \alpha_5\right]\\ &= [1,2,4,5,3,6] \qquad (1.1)\end{aligned}$$

The permutation in Equation (1.1) is not a **cds fixed point**.

Definition 1.1.6. *A* ***cds fixed point*** *is a permutation* π*, where there is no* p,q,p,q *or* q,p,q,p *ordering of the pointers* p *and* q *to support application of a* ***cds*** *operation. For each permutation the outcome of a maximum number of applications of the* ***cds*** *operation results in a* ***cds fixed point*** *or a* ***sorted permutation*** *[2]*

The **cds** operation on $[1,2,4,5,3,6]$ is valid for pointer pairs $((2,3),(3,4))$, $((2,3),(5,6))$ and $((3,4),(5,6))$. Example 1.1.5 continues with **cds** on pointer pair $((2,3),(5,6))$

Example 1.1.5 continued. Pointer pairs $((2,3),(5,6))$ are in correct $p\ q\ p\ q$ ordering.In order to perform the **cds** operation on pointers $p = (2,3)$ and $q = (5,6)$, first identify which case of Definition 1.1.2 is applicable . In $[1, 2, 4, 5, 3, 6]$, the first occurrence of pointer $(2, 3)$ is a right pointer, and the first occurrence of pointer $(5, 6)$ is a right pointer. Thus $\mathbf{cds}_{(2,3),(5,6)}([1, 2, 4, 5, 3, 6])$ is an example of Case (1) by Definition 1.1.2.

Next, identify the α segments:

$$\begin{aligned} {\alpha_1}^{(2,3)} &= [1, 2] \\ {\alpha_2}^{(5,6)} &= [4, 5] \\ \alpha_3 &= [\{\}] \\ {}^{((2,3)}\alpha_4 &= [3] \\ {}^{(5,6)}\alpha_5 &= [6] \end{aligned} \tag{1.2}$$

With the α's identified we can now perform the cds operation:

$$\begin{aligned} \mathbf{cds}_{(2,3),(5,6)}([1, 2, 4, 5, 3, 6]) &= \mathbf{cds}_{(2,3),(5,6)}\left(\left[1, 2^{(2,3)}, 4, 5^{(5,6)}, {}^{(2,3)}3, {}^{(5,6)}6\right]\right) \\ &= \mathbf{cds}_{(2,3),(5,6)}\left(\left[{\alpha_1}^{(2,3)} \frown {\alpha_2}^{(5,6)} \frown \alpha_3 \frown {}^{((2,3)}\alpha_4 \frown {}^{(5,6)}\alpha_5\right]\right) \\ &= \left[{\alpha_1}^{(2,3)} \frown {}^{((2,3)}\alpha_4 \frown \alpha_3 \frown {\alpha_2}^{(5,6)} \frown {}^{(5,6)}\alpha_5\right] \\ &= [1, 2, 3, 4, 5, 6] \end{aligned} \tag{1.3}$$

The permutation in Equation (1.3) is a **Sorted Permutation**.

Next, we revisit $\pi_\sigma = [1, 5, 2, 4, 3, 6]$ and calculate the **cds** from pointer pair $((4, 5), (2, 3))$ to a fixed point.

Example 1.1.7. *Consider the following permutation:*

$$\pi_\sigma = [1, 5, 2, 4, 3, 6]$$

Starting with $((4,5),(2,3))$*, in order to perform the* ***cds*** *operation on pointers* $q = (4,5)$ *and* $p = (2,3)$*, first identify which case of Definition 1.1.2 is applicable . In* $[1,5,2,4,3,6]$*, the first occurrence of pointer* $(4,5)$ *is a left pointer, and the first occurrence of pointer* $(2,3)$ *is a right pointer. Thus* $\boldsymbol{cds}_{(4,5),(2,3)}(\pi_\sigma)$ *is an example of Case (3) by Definition 1.1.2.*

Next, identify the α *segments:*

$$\begin{aligned} \alpha_1 &= [1] \\ {}^{(4,5)}\alpha_2{}^{(2,3)} &= [5, 2] \\ \alpha_3{}^{(4,5)} &= [4] \\ \alpha_4 &= [\{\}] \\ {}^{(2,3)}\alpha_5 &= [3, 6] \end{aligned} \tag{1.4}$$

With the α*'s identified we can now perform the cds operation:*

$$\begin{aligned} \mathbf{cds}_{(4,5),(2,3)}(\pi_\sigma) &= \mathbf{cds}_{(2,3),(5,6)}\left([1,5,2,4,3,6]\right) \\ &= \mathbf{cds}_{(2,3),(5,6)}\left(\left[1, {}^{(4,5)}5, 2^{(2,3)}, 4^{(4,5)}, {}^{(2,3)}3, 6\right]\right) \\ &= \mathbf{cds}_{(2,3),(5,6)}\left(\left[\alpha_1 \frown {}^{(4,5)}\alpha_2{}^{(2,3)} \frown \alpha_3{}^{(4,5)} \frown \alpha_4 \frown {}^{(2,3)}\alpha_5\right]\right) \\ &= \left[\alpha_1 \frown \alpha_4 \frown {}^{(2,3)} \frown \alpha_3{}^{(4,5)} \frown {}^{(4,5)}\alpha_2{}^{(2,3)}\alpha_5\right] \\ &= [1, 4, 5, 2, 3, 6] \end{aligned} \tag{1.5}$$

The permutation in Equation (1.5) *is not a* ***cds fixed point****. The* ***cds*** *operation on*

$[1,4,5,2,3,6]$ *is valid for pointer pairs* $((1,2),(3,4)),((1,2),(5,6))$ *and* $((3,4),(5,6))$. *Continue the* ***cds*** *operation on pointer pair* $((1,2),(3,4))$. *In order to perform the* ***cds*** *operation on pointers* $p = (1,2)$ *and* $q = (3,4)$, *first identify which case of Definition 1.1.2 is applicable . In* $[1,4,5,2,3,6]$, *the first occurrence of pointer* $(1,2)$ *is a right pointer, and the first occurrence of pointer* $(3,4)$ *is a left pointer. Thus* $\boldsymbol{cds}_{(1,2),(3,4))}(\pi_\sigma)$ *is an example of Case (2) by Definition 1.1.2.*
Next, identify the α *segments:*

$$\begin{aligned} {\alpha_1}^{(1,2)} &= [1] \\ \alpha_2 &= [\{\}] \\ {}^{(3,4)}\alpha_3 &= [4,5] \\ {}^{(1,2)}{\alpha_4}^{(3,4)} &= [2,3] \\ \alpha_5 &= [6] \end{aligned}$$

With the α*'s identified we can now perform the cds operation:*

$$\begin{aligned} \mathbf{cds}_{(1,2),(3,4)}([1,4,5,2,3,6]) &= \mathbf{cds}_{(1,2),(3,4)}\left([1^{(1,2)}, {}^{(3,4)}4, 5, {}^{(1,2)}2, 3^{(3,4)}, 6]\right) \\ &= \mathbf{cds}_{(1,2),(3,4)}\left([{\alpha_1}^{(1,2)} \frown \alpha_2 \frown {}^{(3,4)}\alpha_3 \frown {}^{(1,2)}{\alpha_4}^{(3,4)} \frown \alpha_5]\right) \\ &= [{\alpha_1}^{(1,2)} \frown {}^{(1,2)}{\alpha_4}^{(3,4)} \frown {}^{(3,4)}\alpha_3 \frown \alpha_2 \frown \alpha_5] \\ &= [1,2,3,4,5,6] \end{aligned} \tag{1.6}$$

The permutation in Equation (1.6) *is a* ***Sorted Permutation***.

Next, we compute the cds of a permutation π yet again. This π will be carried through to Section 1.2.

Example 1.1.8. *Consider the following permutation:*

$$\pi = [6, 4, 1, 3, 2, 5]$$

In order to perform the ***cds*** *operation on pointers* $q = (3, 4)$ *and* $p = (1, 2)$*, first identify which case of Definition 1.1.2 is applicable . In* π*, the first occurrence of pointer* $(3, 4)$ *is a left pointer, and the first occurrence of pointer* $(1, 2)$ *is a right pointer. Thus* $\boldsymbol{cds}_{(2,3),(5,6)}([1, 2, 4, 5, 3, 6])$ *is an example of Case (3) by Definition 1.1.2.*

Next, identify the α *segments:*

$$\alpha_1 = [6]$$
$${}^{(3,4)}\alpha_2{}^{(1,2)} = [4, 1]$$
$$\alpha_3{}^{(3,4)} = [\{\}]$$
$$\alpha_4 = [3]$$
$${}^{(1,2)}\alpha_5 = [2,\ 5]$$

With the α*'s identified we can now perform the cds operation:*

$$\begin{aligned}\mathbf{cds}_{(1,2),(3,4)}(\pi) &= \mathbf{cds}_{(1,2),(3,4)}\left(\left[6, {}^{(3,4)}4, 1^{(1,2)}, 3^{(3,4)}, {}^{(1,2)}2, 5\right]\right)\\ &= \left[6, 3^{(3,4)}, {}^{(3,4)}4, 1^{(1,2)}, {}^{(1,2)}2, 5\right]\\ &= [6, 3, 4, 1, 2, 5] = \pi_1\end{aligned}$$

Note that entries 3 and 4 of the permutation now appear in correct relative positions. Similarly, entries 1 and 2 also now appear in correct relative positions. Continuing *with the cds operation on pointers $(2,3)$ and $(4,5)$ on π_2, we obtain the following permutation:*

$$\begin{aligned}\mathbf{cds}_{(2,3),(4,5)}(\pi_1) &= \mathbf{cds}_{(2,3),(4,5)}\left(\left[6, {}^{(2,3)}3, 4^{(4,5)}, 1, 2^{2,3}, {}^{(4,5)}5\right]\right)\\ &\vdots\\ &= [6,1,2,3,4,5] = \pi_2\end{aligned}$$

*At this point, we can no longer perform cds operations, due to the fact that there is no p, q, p, q or q, p, q, p orderings of pointers left. Permutation π_2 is an example of a **cds fixed point.**. As a result, we know that π is **cds-unsortable permutation**.*

$\boxdot$

With the transition to Graph Theory and Linear Algebra in mind, we now define a **framed** permutation

Definition 1.1.9. *Given a permutation $\pi = [a_1, \ldots, a_n]$ in $\boldsymbol{S}_n$, the permutation $\pi_{framed} = [0] \frown \pi \frown [n+1]$ denotes π **framed** by $a_0 = 0$ and $a_{n+1} = n+1$.*

Thus

$$\pi_{unframed} = [a_1, \ldots, a_n]$$
$$\pi_{framed} = [0, a_1, \ldots, a_n, n+1]$$

[2]

1.2 Graph theory and Linear Algebra

Associated with each permutation is a natural graph, and associated with cds is a corresponding graph operation on the corresponding graph. From these graphs, and an operation known as graph context directed swap (**gcds**), corresponds an adjacency matrix and a corresponding matrix context directed swap [2].

We next make the transition to graph theory and to linear algebra.

1.2.1 Graph Theory

Definition 1.2.1. *A **graph** is a pair $G = (V, E)$, where V is the set of vertices and $E \subset [V]^2$ is the set of edges*[2]*.*

Definition 1.2.2. *Given a permutation $\pi = [a_1, \ldots, a_n]$, the **Breakpoint Graph** of the permutation is the undirected graph $BG(\pi) = (V, E)$, where the set of vertices, V, consists of the pointers p of π, and the set of edges, E, consists lines connecting different copies of the same pointers. [2]*

Example 1.1.8 continued. Recall $\pi = [6, 4, 1, 3, 2, 5]$ and observe π_{framed} is $[0, 6, 4, 1, 3, 2, 5, 7]$, resulting in the following breakpoint graph $BG(\pi)$:

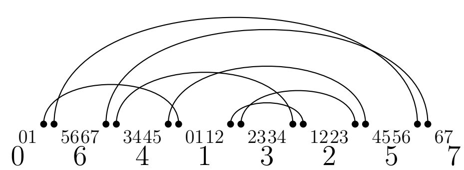


Figure 1.3: The Breakpoint Graph of π

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[2] $[V]^2$ denotes the set of all two-element subsets of the set V.

⊡

Definition 1.2.3. *Given the breakpoint graph $BG(\pi)$ of a permutation π, the* ***Overlap Graph*** *of the permutation π is the graph $OG(\pi) = (V, E)$ where vertices, V, are pointers p of π, and edges E denote overlaps between edges of $BG(\pi)$. [2]*

Example 1.1.8 continued. Referencing $BG(\pi)$, and working from left to right, we note the following overlaps:

(0,1) overlaps (6,7),(3,4),(4,5),(5,6)

(5,6) overlaps (6,7),(0,1)

(6,7) overlaps (5,6),(0,1)

(3,4) overlaps (0,1),(4,5),(1,2),(2,3)

(4,5) overlaps (0,1),(3,4)

(1,2) overlaps (3,4),(2,3)

(2,3) overlaps (3,4),(1,2)

which results in the graph in Figure 1.4

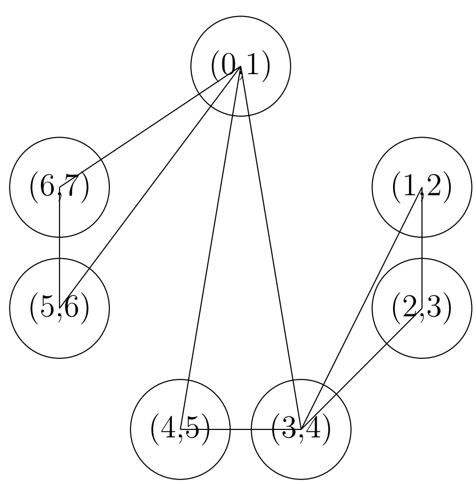

Figure 1.4: The Overlap graph of π.

$\boxdot$

1.2.2 Linear Algebra

A matrix M is defined over a specific field. All examples with matrices in this section will explicitly define the field they are in. We use the symbol $\mathbb{F}$ to denote an arbitrary field. The following are definitions of some specific finite fields.

Definition 1.2.4. $GF(2)$ *is the unique field with two elements 0 and 1. Its addition is defined as the usual addition of integers but modulo 2 and corresponds to the table below:*

+	0	1
0	0	1
1	1	0

The multiplication of $GF(2)$ *is again the usual multiplication modulo 2 and corresponds to the table below:*

×	0	1
0	0	0
1	0	1

In general finite fields are of the form $GF(p^k)$ where p is any prime number. For more details about finite fields, to consult Chapter 14 of [3]. The field we are working in governs the arithmetic for matrix computations.

From the **overlap graph**, we create a matrix called an **Adjacency Matrix**, which records where edges exist between vertices of the overlap graph.

Definition 1.2.5. *The **Adjacency Matrix** of a graph $G = (V, E)$ with vertex set V of size n and edge set E is the $n \times n$ matrix $A(G)$ defined such that, for $1 \leq i, j \leq n$,*[3]

$$A(G)_{ij} = \begin{cases} 1, & \text{if } \{v_i, v_j\} \in E \\ 0, & \text{otherwise.} \end{cases} \quad [2]$$

Example 1.1.8 continued. To reduce complexity, order pointers from "smallest" to "biggest", and have the "smallest" pointer index column/row 1, the second smallest index column/row 2 and so on until we get to the "biggest" pointer, which indexes the last column/row. Starting with the positions $A(\pi)_{1,2}$ and $A(\pi)_{2,1}$, verify if there is an $e \in E$ of $OG(\pi)$ from $(0, 1)$ and $(1, 2)$. As there is no edge, each of the entries $A(\pi)_{1,2}$ and $A(\pi)_{2,1}$ is a zero

The remaining entries can be verified, and have been filled in below:

$$A(\pi) = \begin{array}{c} \\ (0,1) \\ (1,2) \\ (2,3) \\ (3,4) \\ (4,5) \\ (5,6) \\ (6,7) \end{array} \begin{array}{c} \begin{array}{ccccccc} (0,1) & (1,2) & (2,3) & (3,4) & (4,5) & (5,6) & (6,7) \end{array} \\ \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

The adjacency matrix is a matrix of elements of the finite field $GF(2)$

$\boxdot$

[3] Adjacency matrices have entries in $GF(2)$

. The operations of the field $GF(2)$ will underly the upcoming definition of the matrix version of cds, denoted **Mcds**, for matrices over the field $GF(2)$. This limited version of **Mcds** can be found in [2].

Definition 1.2.6. ***Mcds,*** $GF(2)$

Consider $n \times n$ matrix M with i,j entry denoted $m_{i,j}$ for $i,j \leq n$, satisfying the following three requirements:

1. *Each $m_{i,j}$ is an element of $GF(2)$*

2. *For each i, $m_{i,i} = 0$ and*

3. *for each i and j, $m_{i,j} = m_{j,i}$.*

The matrix cds operation, ***Mcds****, on entries p, q is:*

$$\boldsymbol{Mcds}(M)_{p,q} = M - MI_{p,q}M$$

where I_{pq} is a matrix over $GF(2)$ with $I_{ij} = 1$ if $i = p$ and $j = q$ or $i = q$ and $j = p$ and $I_{ij} = 0$ otherwise. [2]

Example 1.1.8 continued. Recall that the adjacency matrix of the permutation π is

$$A(\pi) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

Consider $\mathbf{Mcds}(A(\pi))_{2,4}$. Recall from Definition 1.2.6 that $I_{2,4}$ is the matrix

$$\begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{bmatrix}$$

Then by Definition 1.2.6, $\mathbf{Mcds}(A(\pi))_{2,4} = M - MI_{2,4}M$=

$$\begin{bmatrix} 0&0&0&1&1&1&1\\ 0&0&1&1&0&0&0\\ 0&1&0&1&0&0&0\\ 1&1&1&0&1&0&0\\ 1&0&0&1&0&0&0\\ 1&0&0&0&0&0&1\\ 1&0&0&0&0&1&0 \end{bmatrix} - \begin{bmatrix} 0&0&0&1&1&1&1\\ 0&0&1&1&0&0&0\\ 0&1&0&1&0&0&0\\ 1&1&1&0&1&0&0\\ 1&0&0&1&0&0&0\\ 1&0&0&0&0&0&1\\ 1&0&0&0&0&1&0 \end{bmatrix} \cdot \begin{bmatrix} 0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0\\ 0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0 \end{bmatrix} \cdot \begin{bmatrix} 0&0&0&1&1&1&1\\ 0&0&1&1&0&0&0\\ 0&1&0&1&0&0&0\\ 1&1&1&0&1&0&0\\ 1&0&0&1&0&0&0\\ 1&0&0&0&0&0&1\\ 1&0&0&0&0&1&0 \end{bmatrix}$$

resulting in a matrix with each entry of row 2 and row 4 a 0, and with each entry of column 2 and column 4 a 0:[4]

[4]Governing arithmetic is that of $GF(2)$

$$\begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \tag{1.7}$$

⊡

Observation 1.2.7. *For later reference note that deleting rows/columns 2 and 4 of* $\boldsymbol{Mcds}(A(\pi))_{2,4}$ *gives the following:*

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}\begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \\ 1 \\ 3 \\ 5 \\ 6 \\ 7 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 3 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \tag{1.8}$$

1.3 Origins of Schur complement

The Schur complement is a matrix operation named after Issai Schur, and was developed independently of **Mcds**. The Schur complement is primarily used to

factorize a matrix into a product of simpler matrices. Later in this thesis, we will prove that **Mcds** and the Schur complement are related.
The following determinant formula, originally developed by Issai Schur is the origin of the Schur complement:

Theorem 1.3.1. *If M is a square matrix subdivided as P, Q, R, and S as in Definition 1.3.2, and if we let M/P denote the Schur complement of P in M, then $det(M) = det(M/P) \cdot det(P)$*

The the Schur complement appears in many branches of mathematics. A paper by Cottle [4], gives numerous manifestations of the Schur complement, ranging from statistics, to finding the inertia sets of square matrices. Inertias of matrices and graph theory are also connected, and inertias are used in efforts to solve the inverse eigenvalue problem, an open problem in mathematics.

In this thesis we provide yet another context in which the Schur complement shows up, namely in sorting permutations by block interchanges, confirming and generalizing a speculation given in an email by Dr. Robert Brijder.

In an email dated August 29, 2016, R. Brijder speculated that **Mcds** as formulated would be related to the Schur complement of an appropriately chosen 2×2 sub-matrix of a given matrix. We thank Dr. Brijder for bringing the Schur complement to our attention.

Definition 1.3.2. ***Schur Complement of a matrix*** M

Consider an $m \times n$ matrix M. Pick columns/rows p and q and relocate them to columns/rows 1 and 2 respectively, resulting in M_1. Subdivide M_1 into sub-matrices P, Q, R, and S as follows:

- *P is the 2×2 sub-matrix consisting of entries m_{pp}, m_{qq}, m_{qp} and m_{pq}*

- P^{-1} *is the inverse of matrix* P

- Q *is the* $2\times(n-2)$ *sub-matrix consisting of entries* $\{m_{px}, m_{qx} | x = \mathbb{N}\backslash\{p,q\}, x \leq (n-2)\}$

- R *is the* $(m-2)\times 2$ *sub-matrix consisting of entries* $\{m_{xp}, m_{xq} | x = \mathbb{N}\backslash\{p,q\}, x \leq (m-2)\}$

- S *is the* $(m-2)\times(n-2)$ *sub-matrix consisting of entries* $\{m_{xy} | x, y = \mathbb{N} \setminus \{p,q\}, x \leq (n-2), y \leq (m-2)\}$

The ***Schur Complement of matrix*** M ***on rows/columns*** p ***and*** q*, denoted* $\boldsymbol{SC}(M)_{p,q}$ *is defined as follows:*

$$\boldsymbol{SC}(M)_{p,q} = S - R \cdot P^{-1} \cdot Q$$

$\boldsymbol{SC}(M)_{p,q}$ *is a* $(m-2)\times(n-2)$ *matrix.* [5]

The following is an example of the Schur complement computation, using the adjacency matrix calculated earlier.[6]

Example 1.1.8 continued. Recall that the adjacency matrix of the permutation π is

[5]The Schur complement is applicable for rows and columns within the largest upper square sub-matrix of M

[6]The order we move rows and columns does not change the final result of the Schur complement, so without loss of generality, we will move columns first, and move rows second in all examples of Schur complement

$$A(\pi) = \begin{bmatrix} 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix}$$

and consider $\mathbf{SC}(A(\pi))_{2,4}$ The first step is to move the appropriate columns and rows of $A(\pi)$. Say $M = A(\pi)$ for convenience

Step 1:Move columns 2 and 4 of M to column position 1 and 2 respectively. Name the resulting matrix M_1:

$$M = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{matrix} 1 & \color{red}{2} & 3 & \color{red}{4} & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 0 & \color{red}{0} & 0 & \color{red}{1} & 1 & 1 & 1 \\ 0 & \color{red}{0} & 1 & \color{red}{1} & 0 & 0 & 0 \\ 0 & \color{red}{1} & 0 & \color{red}{1} & 0 & 0 & 0 \\ 1 & \color{red}{1} & 1 & \color{red}{0} & 1 & 0 & 0 \\ 1 & \color{red}{0} & 0 & \color{red}{1} & 0 & 0 & 0 \\ 1 & \color{red}{0} & 0 & \color{red}{0} & 0 & 0 & 1 \\ 1 & \color{red}{0} & 0 & \color{red}{0} & 0 & 1 & 0 \end{bmatrix} \end{array} \to M_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{matrix} \color{red}{2} & \color{red}{4} & 1 & 3 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} \color{red}{0} & \color{red}{1} & 0 & 0 & 1 & 1 & 1 \\ \color{red}{0} & \color{red}{1} & 0 & 1 & 0 & 0 & 0 \\ \color{red}{1} & \color{red}{1} & 0 & 0 & 0 & 0 & 0 \\ \color{red}{1} & \color{red}{0} & 1 & 1 & 1 & 0 & 0 \\ \color{red}{0} & \color{red}{1} & 1 & 0 & 0 & 0 & 0 \\ \color{red}{0} & \color{red}{0} & 1 & 0 & 0 & 0 & 1 \\ \color{red}{0} & \color{red}{0} & 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array}$$

Step 2: Move columns 2 and 4 of M_1 to column position 1 and 2 respectively. Name the resulting matrix M_2.

$$
M_1 = \begin{array}{c|ccccccc|}
 & 2 & 4 & 1 & 3 & 5 & 6 & 7 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
4 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
5 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
7 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{array}
\rightarrow M_2 = \begin{array}{c|ccccccc|}
 & 2 & 4 & 1 & 3 & 5 & 6 & 7 \\
2 & 0 & 1 & 0 & 1 & 0 & 0 & 0 \\
4 & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\
1 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\
3 & 1 & 1 & 0 & 0 & 0 & 0 & 0 \\
5 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
6 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
7 & 0 & 0 & 1 & 0 & 0 & 1 & 0
\end{array}
$$

The Schur Complement definition identifies five matrices, named

- P the 2×2 sub-matrix made of entries $m_{2,2}, m_{2,4}, m_{4,2}$ and $m_{4,4}$
- P^{-1} The inverse of matrix P
- Q the 2×5 sub-matrix made of entries $\{m_{2,x}, m_{4,x} | x = \{1,3,5,6,7\}\}$
- R the 5×2 sub-matrix made of entries $\{m_{x,2}, m_{x,4} | x = \{1,3,5,6,7\}\}$
- S the 5×5 sub-matrix made of entries $\{m_{xy} | x, y = \{1,3,5,6,7\}$

Now that the sub matrices P, Q, R and S have been identified, we continue the calculation of Schur complement [7]

[7] The governing arithmetic in Example 1.1.8 is that of $GF(2)$

$$\begin{array}{c} \\ 2 \\ 4 \\ 1 \\ 3 \\ 5 \\ 6 \\ 7 \end{array}\begin{array}{c}\begin{array}{ccccccc} 2 & 4 & 1 & 3 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} \color{blue}{0} & \color{blue}{1} & \color{orange}{0} & \color{orange}{1} & \color{orange}{0} & \color{orange}{0} & \color{orange}{0} \\ \color{blue}{1} & \color{blue}{0} & \color{orange}{1} & \color{orange}{1} & \color{orange}{1} & \color{orange}{0} & \color{orange}{0} \\ \color{green}{0} & \color{green}{1} & 0 & 0 & 1 & 1 & 1 \\ \color{green}{1} & \color{green}{1} & 0 & 0 & 0 & 0 & 0 \\ \color{green}{0} & \color{green}{1} & 1 & 0 & 0 & 0 & 0 \\ \color{green}{0} & \color{green}{0} & 1 & 0 & 0 & 0 & 1 \\ \color{green}{0} & \color{green}{0} & 1 & 0 & 0 & 1 & 0 \end{bmatrix}\end{array}$$

Figure 1.5: Matrices $\color{blue}{P}, \color{blue}{P^{-1}}, \color{orange}{Q}, \color{green}{R}$, and S Explicitly

$$S - RP^{-1}Q = \begin{array}{c} \\ 1 \\ 3 \\ 5 \\ 6 \\ 7 \end{array}\begin{array}{c}\begin{array}{ccccc} 1 & 3 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}\end{array} - \begin{array}{c} \\ 1 \\ 3 \\ 5 \\ 6 \\ 7 \end{array}\begin{array}{c}\begin{array}{cc} 2 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}\end{array} \begin{array}{c} \\ 2 \\ 4 \end{array}\begin{array}{c}\begin{array}{cc} 2 & 4 \end{array} \\ \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\end{array} \begin{array}{c} \\ 2 \\ 4 \end{array}\begin{array}{c}\begin{array}{ccccc} 1 & 3 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{bmatrix}\end{array}$$

$$= \begin{array}{c} \\ 1 \\ 3 \\ 5 \\ 6 \\ 7 \end{array}\begin{array}{c}\begin{array}{ccccc} 1 & 3 & 5 & 6 & 7 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix}\end{array} \tag{1.9}$$

$\boxdot$

Observation 1.3.3. *Injecting rows/columns of 0's in row/column positions 2 and 4 of $\boldsymbol{SC}(A(\pi_1))_{2,4}$ gives the following:*

$$\begin{array}{c} \\ \\ 1 \\ 3 \\ 5 \\ 6 \\ 7 \end{array}\begin{array}{c} \begin{matrix} 1 & 3 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array}\begin{array}{c} \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{matrix} \\ \begin{bmatrix} 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 & 1 & 0 \end{bmatrix} \end{array} = \boldsymbol{Mcds}(A(\pi_1))_{2,4} \quad (1.10)$$

At this point, we conclude the introduction of the basic concepts needed. The following sections aim to generalize **Mcds** to fields other than $GF(2)$ and gives a relationship between the recent cds sorting operation, and the over 100 year old Schur complement of a matrix.

CHAPTER 2

MOTIVATION, APPLICATION AND PURSUING GENERALIZATION

Observation 2.0.1. *By Observation 1.2.7 and 1.3.3 above, the matrix in Equation* (1.9) *is exactly the matrix obtained by deleting the* 2^{nd} *and* 4^{th} *rows and columns from the matrix in Equation* (1.7)*, resulting in the matrix in Equation* (1.8)*.*
Conversely, the matrix in Equation (1.7) *is exactly the matrix in Equation* (1.10) *expanded by injecting 0's in the* 2^{nd} *and* 4^{th} *rows and columns of Equation* (1.9)*.*

As we will soon see, the above observation is not unique to the world of graphs, nor to the world of $GF(2)$. It turns out that there is an interesting relation between **Mcds** and the Schur complement. This is explored in Chapter 4.

2.1 The connection between Mcds and Schur Complement over arbitrary fields

In the previous section, we explored the **Mcds** operation and Schur complement. However, the matrices used there were from the world of graphs. What if we ventured outside of the world of permutations and graphs, but stayed in the $GF(2)$ world- Would **Mcds** and Schur Complement retain the features of Observation 2.0.1?
Consider the following 5×5 matrix:

$$M = \begin{bmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 0 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 1 & 0 \end{bmatrix} \tag{2.1}$$

2.1.1 Computing Mcds$(M)_{3,4}$

Observation 2.1.1. *Observe that M is NOT an adjacency matrix of a graph.*

Example 2.1.2. *Consider the matrix M in Equation* (2.1)*. Computing* ***Mcds***$(M)_{3,4} = M - MI_{3,4}M$ *gives us the following:*

$$M - MI_{3,4}M = \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{2.2}$$

This concludes the computation of ***Mcds***$(M)_{3,4}$.

Observation 2.1.3. *Deleting rows and columns 3 and 4 from the matrix in Equation* (2.2) *produces the matrix:*

$$\begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccccc} 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{bmatrix} \end{array} \rightarrow \begin{array}{c} \\ 1 \\ 2 \\ 5 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 5 \end{array} \\ \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \end{array} \tag{2.3}$$

$\boxdot$

Next we compute a Schur complement of the same 5×5 matrix M.

2.1.2 Computing SC$(M)_{3,4}$

Example 2.1.2 continued. Consider the matrix M from 2.1. The first step is to move the appropriate columns and rows of M.

Step 1:Move columns and rows 3 and 4 of M to column and row position 1 and 2 respectively. Name the resulting matrix M_1. The Schur Complement definition identifies sub-matrices, named P,Q, R, and S

- P the 2×2 sub-matrix made of entries $m_{3,4}, m_{4,3}, m_{3,3}$ and $m_{4,4}$
- P^{-1} The inverse of matrix P
- Q the 2×3 sub-matrix made of entries $\{m_{3,x}, m_{4,x} | x = \{1, 2, 5\}\}$
- R the 3×2 sub-matrix made of entries $\{m_{x,3}, m_{x,4} | x = \mathbb{N} \setminus \{1, 2, 5\}\}$
- S the 3×3 sub-matrix made of entries $\{m_{xy} | x, y = \{1, 2, 5\}\}$

Now that the sub matrices P, Q, R and S have been identified, calculate the Schur complement:

$$S - RP^{-1}Q = \begin{array}{c|ccc|} & 1 & 2 & 5 \\ \hline 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 5 & 1 & 0 & 0 \\ \hline \end{array} \tag{2.4}$$

[1]

$\boxdot$

Observation 2.1.4. *Injecting rows/columns of zero's in positions 3 and 4 of* (2.4) *produces the matrix:*

$$\begin{array}{c|ccc|} & 1 & 2 & 5 \\ \hline 1 & 0 & 1 & 1 \\ 2 & 1 & 0 & 1 \\ 5 & 1 & 0 & 0 \\ \hline \end{array} \rightarrow \begin{array}{c|ccccc|} & 1 & 2 & 3 & 4 & 5 \\ \hline 1 & 0 & 1 & 0 & 0 & 1 \\ 2 & 1 & 0 & 0 & 0 & 1 \\ 3 & 0 & 0 & 0 & 0 & 0 \\ 4 & 0 & 0 & 0 & 0 & 0 \\ 5 & 1 & 0 & 0 & 0 & 0 \\ \hline \end{array} \tag{2.5}$$

Observation 2.1.5. *By Observation 2.1.3 and 2.1.4, the matrix in Equation* (2.4) *is exactly the matrix obtained by deleting the* 3^{rd} *and* 4^{th} *rows and columns from the matrix in Equation* (2.2)*, resulting in the matrix in Equation* (2.3)*.*

Conversely, the matrix in Equation (2.2) *is exactly the matrix in Equation* (2.5)

[1]The governing arithmetic in Example 2.1.2 is that of $GF(2)$

expanded by injecting 0's in the 3^{rd} *and* 4^{th} *rows and columns of the matrix in Equation* (2.4)*.*

And generally, in $GF(2)$, we observe the following

Observation 2.1.6. *The* $\boldsymbol{SC}(M)_{pq}$ *is exactly the matrix obtained by deleting the p-th and q-th rows and columns from the matrix* $\boldsymbol{Mcds}(M)_{pq}$*.*
Conversely, $\boldsymbol{Mcds}(M)_{pq}$ *is exactly the matrix* $\boldsymbol{SC}(M)_{pq}$ *expanded by adding a p-th and q-th rows of zeroes, and a p-th and q-th columns of zeroes*

The Schur complement is by definition an operation over any field the constraint is that P must be invertible. The **Mcds** operation however, is more constrained. At this point **Mcds** on a matrix M is limited by the constraints listed next in (1-4)

1. The matrix M is a square matrix
2. The matrix M is symmetric
3. Each entry of M is an element of $GF(2)$
4. Each diagonal entry of M is 0

These constraints stem from the origins of the **cds** operation on permutations, which in-turn corresponds to the **Gcds** operation on graphs, and consequently with the **Mcds** operation on matrices. The matrices that satisfy (1-4) is a small subset of the set of matrices over $GF(2)$, let alone the set of matrices over arbitrary fields. One may ask if the operation generalizes to all matrices in $GF(2)$, regardless of dimension, and for that matter all matrices over any field, or simply put, can matrices to which **Mcds** apply:

1. be $m \times n$ matrices where m and n can be distinct,

2. be not necessarily symmetric,

3. have entries in any field, and

4. have limited restrictions on its diagonal

In Chapter 3, we focus on finding a general expression for **Mcds**, and in Chapter 4 we shift our focus to finding a general expression for the Schur complement, and we establish a correspondence between **Mcds** and the Schur complement.

CHAPTER 3

A GENERALIZATION OF MCDS

3.1 A motivated generalization of Mcds

Generally speaking the **Mcds** operation performed on a matrix M has two inputs, M itself, and a second matrix, say for now C, and outputs a matrix of the form $M - MCM$, where all computations are taking place over a predetermined field $\mathbb{F}$. In Examples 1.1.8 M was assumed to be a square symmetric matrix over $GF(2)$ with a zero diagonal, and C was assumed to be a square matrix of same dimensions as M, and with two nonzero entries in specific symmetrically located off-diagonal positions. Similarly, in Example 2.1.2 ,M was assumed to be a square matrix over $GF(2)$ with a zero diagonal, and C was assumed to be a square matrix of same dimensions as M, and with two nonzero entries in specific symmetrically located off-diagonal positions. In order for M to be **Mcds** eligible for choices p and q, based on Examples 1.1.8 and 2.1.2, if the m_{pp} and m_{qq} entries are zeroes, and the m_{pq} and m_{qp} entries are non-zero (in $GF(2)$, $m_{pq} = m_{qp} = 1$). If there are no p, q for which M satisfies these properties, then M is said to be an **Mcds fixed point**.

Example 3.1.1. *Consider the following matrices:*

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 0 & 1 \end{bmatrix}$$

These are all examples of ***Mcds fixed points****. A formal definition of* ***Mcds fixed point*** *is given in Section 3.1.2, Definition 3.1.9.* ⊡

If we were to relax the condition that the matrix be square, or symmetric, or has a zero diagonal, or that the field is $GF(2)$, several considerations arise.

First, consider relaxing the requirement that the matrix be square, and that the field be $GF(2)$. In the spirit of ”reverse engineering” the **Mcds** process, let A be the outcome of the aimed at generalization of **Mcds**, where the generalized **Mcds** is applied to a matrix M. Say A is an $m \times n$ matrix. Mcds is obtained as a subtraction, $A = M - B$ for some appropriate matrix B. Since A is a $m \times n$ matrix, M and B are also $m \times n$ matrices. Next examine the matrix B in this subtraction. It is of the form MCM in Examples 1.1.8 and 2.1.2, we will retain this structure for B. As B is an $m \times n$ matrix, and M is an $m \times n$ matrix, C is forced to being an $n \times m$ matrix. Finally, we examine the factor C. Using Examples 1.1.8, and 2.1.2 as a guide, let C be a $n \times m$ matrix, with one nonzero entry, namely c_{pq}, with $p < q \leq m, n$. Multiplying an $m \times n$ matrix M with an $n \times m$ matrix C that has only one nonzero entry, MC results in a $m \times m$ dimension matrix with a specific ”nonzero” row or ”nonzero” column, p or q. Conversely, multiplying an $n \times m$ matrix C that has only one nonzero entry, with an $m \times n$ matrix M , CM results in a $n \times n$ matrix with a specific ”nonzero” row or ”nonzero” column,p or q. Continuing with the multiplication, Multiplying an $m \times m$ matrix MC,a matrix with exactly one row containing non-zeros, with an $m \times n$ matrix M, MCM results in a $m \times n$ matrix, with a formula for each entry. Conversely, Multiplying an $m \times n$ matrix M with an $n \times n$ matrix CM,a matrix with exactly one column containing non-zeros MCM results in a $m \times n$ matrix, with a explicit formula in terms of the entries of M and C.

The I_{pq} matrix defined in **Mcds** is the result of summing two matrices containing only one nonzero entry each. Specifically, $I_{pq} = C_1 + C_2$, where C_1 has an nonzero entry in position pq, and C_2 has a nonzero entry in position qp. The above work suggests the restrictions to place on M and I_{pq} for a working definition.

Definition 3.1.2. *(Working Definition)* $\boldsymbol{Mcds}_1$

Given an $m \times n$ matrix M, with restriction on entries as follows:

- $m_{pp}, m_{qq} = 0$
- $m_{pq}, m_{qp} \neq 0$
- *all other m entries are elements of the field $\mathbb{F}$*

the matrix cds operation, $\boldsymbol{Mcds}_1$, on entries p, q is given by:

$$\boldsymbol{Mcds}_1(M)_{pq} = M - MI_{pq}M$$

where I_{pq} is a $n \times m$ matrix over $\mathbb{F}$ with ij position entry nonzero if $i = p$ and $j = q$ or $i = q$ and $j = p$ and ij entry zero otherwise.[1]

[1] The $\mathbf{Mcds}_1$ operation is applicable for rows and columns within the largest upper square submatrix of M

The beauty of this definition is that entries of matrix M are no longer restricted to {0,1} and M is no longer required to be a square matrix. By Definition 3.1.2 the matrix M has the following form:

$$M = \begin{bmatrix} m_{1,1} & m_{1,2} & \cdots & m_{1,p} & \cdots & m_{1,q} & \cdots & m_{1,n} \\ m_{2,1} & m_{2,2} & \cdot & m_{2,p} & \cdot & m_{2,q} & \cdot & m_{2,n} \\ \vdots & \cdot & \ddots & \cdot & \cdot & \vdots & \cdot & \vdots \\ m_{p,1} & m_{p,2} & \ldots & 0_{pp} & \cdot & m_{pq} & \cdot & m_{pn} \\ \vdots & \cdot & \cdot & \cdot & \ddots & \cdot & \cdot & \vdots \\ m_{q,1} & m_{q,2} & \ldots & m_{qp} & \cdot & 0_{qq} & \cdot & m_{qn} \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdot & \ddots & \vdots \\ m_{m,1} & m_{m,2} & \ldots & m_{mp} & \cdot & m_{mq} & \cdot & m_{mn} \end{bmatrix}$$

Also, by Definition 3.1.2, I_{pq} has the following form:

$$I_{pq} = \begin{bmatrix} 0_{1,1} & 0_{1,2} & \cdots & 0_{1,p} & \cdots & 0_{1,q} & \cdots & 0_{1,m} \\ 0_{2,1} & 0_{2,2} & \cdot & 0_{2,p} & \cdot & 0_{2,q} & \cdot & 0_{2,m} \\ \vdots & \cdot & \ddots & \cdot & \cdot & \vdots & \cdot & \vdots \\ 0_{p,1} & 0_{p,2} & \ldots & 0_{pp} & \cdot & i_{pq} & \cdot & 0_{pm} \\ \vdots & \cdot & \cdot & \cdot & \ddots & \cdot & \cdot & \vdots \\ 0_{q,1} & 0_{q,2} & \ldots & i_{qp} & \cdot & 0_{qq} & \cdot & 0_{qm} \\ \vdots & \cdot & \cdot & \cdot & \cdot & \cdot & \ddots & \vdots \\ 0_{n,1} & 0_{n,2} & \ldots & 0_{np} & \cdot & 0_{nq} & \cdot & 0_{nm} \end{bmatrix}$$

First, we prove basic facts regarding the operation $\mathbf{Mcds}_1$ in general.

3.1.1 Mcds Calculations

Notation: We are going to use some familiar notation from linear algebra in an unfamiliar way:

- Rx_M references row x of matrix M

Example 3.1.3. *Consider the* 3×3 *matrix* $M = \begin{bmatrix} 7 & -4 & 2 \\ 0 & 22 & 1 \\ -2 & 5 & 41 \end{bmatrix}$. *Matrix* M *has three rows, each denoted by* Rx_M *, for* $x \in \{1, 2, 3\}$. *Thus:*

$$R1_M = \begin{bmatrix} 7 & -4 & 2 \end{bmatrix}$$

$$R2_M = \begin{bmatrix} 0 & 22 & 1 \end{bmatrix}$$

$$R3_M = \begin{bmatrix} -2 & 5 & 41 \end{bmatrix}$$

⊡

- Cx_M references column x of matrix M

Example 3.1.3 continued. Matrix M has three columns, each denoted by Cx_M , for $x \in \{1, 2, 3\}$. Thus:

$$C1_M = \begin{bmatrix} 7 \\ 0 \\ -2 \end{bmatrix}$$

$$C2_M = \begin{bmatrix} -4 \\ 22 \\ 5 \end{bmatrix}$$

$$C3_M = \begin{bmatrix} 2 \\ 1 \\ 41 \end{bmatrix}$$

$\boxdot$

Recall that matrix multiplication is associative, $((M \cdot I_{pq}) \cdot M = M \cdot (I_{pq} \cdot M))$, so the order we multiply these matrices does not matter in computing the final outcome of $M \cdot I_{pq} \cdot M$. Starting with $M \cdot I_{pq}$.

Lemma 3.1.4. *Consider a $m \times n$ matrix M where $n \leq m$, and let s be a row position, and t be a column position, such that $s \leq m$ and $t \leq n$*
$(M \cdot I_{pq})_{st}$ generally has form:

$$(MI_{pq})_{st} = (m_{sp} \cdot i_{pt}) + (m_{sq} \cdot i_{qt})$$

Proof. Let M and I_{pq} be as defined in Definition 3.1.2. First, consider $M \cdot I_{pq}$ as row and column operations. Let $\theta = M \cdot I_{pq}$, and consider position st of matrix θ, or θ_{st}:

$$\theta_{st} = Rs_M \cdot Ct_I$$

$$= \begin{bmatrix} m_{s1} & m_{s2} & \cdots & m_{sp} & \cdots & m_{sq} & \cdots & m_{sn} \end{bmatrix} \cdot \begin{bmatrix} i_{1t} \\ i_{2t} \\ \vdots \\ i_{pt} \\ \vdots \\ i_{qt} \\ \vdots \\ i_{nt} \end{bmatrix}$$

$$= (m_{s1} \cdot i_{1t}) + (m_{s2} \cdot i_{2t}) + \cdots + (m_{sp} \cdot i_{pt}) + \cdots + (m_{sq} \cdot i_{qt}) + \cdots + (m_{sm} \cdot i_{mt}).$$

This leaves us with sixteen cases, depending on how s and t are related to p and q. As all cases use similar arguments, we explain one case, leaving the remaining cases to the reader.:

- **Case 1:** $s = p$ and $t = p$

Let $s = p$ and $t = p$, then

$$(M \cdot I_{pq})_{pp} = Rp_M \cdot Cp_I$$
$$= (m_{p1} \cdot i_{1p}) + (m_{p2} \cdot i_{2p}) + \cdots + (m_{pp} \cdot i_{pp}) + \cdots + (m_{pq} \cdot i_{qp}) + \cdots + (m_{pn} \cdot i_{np})$$

By definition, $\{i_{1p}, i_{2p}, \cdots, , i_{pp}, \cdots, i_{np}\} = 0$ and $i_{qp} = i_{qp}$. Thus:

$$\begin{aligned}(M \cdot I_{pq})_{pp} &= (m_{p1} \cdot i_{1p}) + (m_{p2} \cdot i_{2p}) + \cdots + (m_{pp} \cdot i_{pp}) + \cdots + (m_{pq} \cdot i_{qp}) + \cdots + (m_{pn} \cdot i_{np}) \\ &= (m_{p1} \cdot (0)) + (m_{p2} \cdot (0)) + \cdots + (m_{pp} \cdot (0)) + \cdots + (m_{pq} \cdot i_{qp}) + \cdots + (m_{pn} \cdot (0)) \\ &= m_{pq} \cdot i_{qp}\end{aligned}$$

Generally we see the following:

$$\begin{aligned}\theta_{st} &= (m_{s1} \cdot i_{1t}) + (m_{s2} \cdot i_{2t}) + \cdots + (m_{sp} \cdot i_{pt}) + \cdots + (m_{sq} \cdot i_{qt}) + \cdots + (m_{sn} \cdot i_{nt}) \\ &= (m_{s1} \cdot (0)) + (m_{s2} \cdot (0)) + \cdots + (m_{sp} \cdot i_{pt}) + \cdots + (m_{sq} \cdot i_{qt}) + \cdots + (m_{sn} \cdot (0)) \\ &= (m_{sp} \cdot i_{pt}) + (m_{sq} \cdot i_{qt})\end{aligned}$$

As desired.

□

Now we can continue on with the **Mcds** calculation, which at this point has us multiply $M \cdot I_{pq}$ by M on the right side.

Lemma 3.1.5. *Consider a $m \times n$ matrix M where $n \leq m$, and let s be a row position, and t be a column position, such that $s \leq m$ and $t \leq n$.*
$(M \cdot I_{pq} \cdot M)_{st}$ generally has form:

$$(M \cdot I_{pq} \cdot M)_{st} = m_{sq} \cdot i_{qp} \cdot m_{pt} + m_{sp} \cdot i_{pq} \cdot m_{qt}$$

Proof. We are going to take a similar process as before. Continuing with the operation from Lemma 3.1.4, consider $M \cdot I_{pq} \cdot M$ as row and column operations.Let $\theta = M \cdot I_{pq}$, and $\tau = \theta \cdot M$ and consider position st of matrix τ, or τ_{st}:

$$
\begin{aligned}
(\tau)_{st} &= Rs_\theta \cdot Ct_M \\
&= \begin{bmatrix} 0_{s1} & \dots & m_{sq} \cdot i_{qp} & \cdots & m_{sp} \cdot i_{pq} & \cdots & 0_{sm} \end{bmatrix} \cdot \begin{bmatrix} m_{1t} \\ \vdots \\ m_{pt} \\ \vdots \\ m_{qt} \\ \vdots \\ m_{nt} \end{bmatrix} \\
&= 0 + \cdots + 0 + (m_{sq} \cdot i_{qp}) \cdot (m_{pt}) + \cdots + (m_{sp} \cdot i_{pq}) \cdot (m_{qt}) + \cdots + 0 \\
&= m_{sq} \cdot i_{qp} \cdot m_{pt} + m_{sp} \cdot i_{pq} \cdot m_{qt}
\end{aligned}
$$

As desired. □

Continuing with **Mcds**, we now compute $M - M \cdot I_{pq} \cdot M$

Lemma 3.1.6. *Consider a $m \times n$ matrix M where $n \leq m$, and let s be a row position, and t be a column position, such that $s \leq m$ and $t \leq n$, $(M - M \cdot I_{pq} \cdot M)_{st}$ generally has form:*

$$(M - M \cdot I_{pq} \cdot M)_{st} = m_{st} - (m_{sq} \cdot i_{qp} \cdot m_{pt} + m_{sp} \cdot i_{pq} \cdot m_{qt}) \tag{3.1}$$

Proof. Continuing with the operation from Lemma 3.1.5, M by definition is a $m \times n$, I by definition is $n \times m$. $MI_{pq}M = (m \times n)(n \times m)(m \times n) = m \times n$ Therefore subtraction is pairwise between M and $MI_{pq}M$. □

With a formula for the entries of $M - M \cdot I_{pq} \cdot M$ available, the **Mcds** operation can now be analyzed in more detail. Recall we are performing $\mathbf{Mcds}_1$ on rows and columns p and q

Theorem 3.1.7. *Let M and I_{pq} be as defined as by Definition 3.1.2. If the i_{pq} entry of I_{pq} is $\frac{1}{m_{qp}}$, and the i_{qp} entry of I_{pq} is $\frac{1}{m_{pq}}$, and all other $i_{st} = 0$, then :*

$$\begin{aligned} Rp_\beta &= \begin{bmatrix} 0_{p,1} & 0_{p,2} & \cdots & 0_{p,n-1} & 0_{p,n} \end{bmatrix} \\ Rq_\beta &= \begin{bmatrix} 0_{q,1} & 0_{q,2} & \cdots & 0_{q,n-1} & 0_{q,n} \end{bmatrix} \\ Cp_\beta &= \begin{bmatrix} 0_{1,p} \\ 0_{2,p} \\ \vdots \\ 0_{m-1,p} \\ 0_{mp} \end{bmatrix} \\ Cq_\beta &= \begin{bmatrix} 0_{1,q} \\ 0_{2,q} \\ \vdots \\ 0_{m-1,q} \\ 0_{mq} \end{bmatrix} \end{aligned} \tag{3.2}$$

I.e.,If the i_{pq} entry of I_{pq} is $\frac{1}{m_{qp}}$, and the i_{qp} entry of I_{pq} is $\frac{1}{m_{pq}}$, and all other $i_{st} = 0$, where $s \neq p, q$ or $t \neq q, p$, then each entry of rows p and q and columns p and q of β is a zero row/column.

Proof. Recall, the $\mathbf{Mcds}_1$ operation on rows/columns from Definition 3.1.2, results in a matrix where each entry of rows p and q are zero, and each entry of column p and q are zero . Therefore, by Lemma 3.1.6:

$$\text{Equation (1)}: \beta_{px} = m_{px} - (m_{pq} \cdot i_{qp} \cdot m_{px} + m_{pp} \cdot i_{pq} \cdot m_{qx}) = 0$$

$$\text{Equation (2)}: \beta_{qx} = m_{qx} - (m_{qq} \cdot i_{qp} \cdot m_{px} + m_{qp} \cdot i_{pq} \cdot m_{qx}) = 0$$

$$\text{Equation (3)}: \beta_{xp} = m_{xp} - (m_{xp} \cdot i_{pq} \cdot m_{qp} + m_{xq} \cdot i_{qp} \cdot m_{pp}) = 0$$

$$\text{Equation (4)}: \beta_{xq} = m_{xq} - (m_{xp} \cdot i_{pq} \cdot m_{qq} + m_{xq} \cdot i_{qp} \cdot m_{pq}) = 0$$

Therefore, if we solve each equation individually we see the following:

Equation (1)

$$\begin{aligned}
0 &= \beta_{px} \\
&= m_{px} - (m_{pq} \cdot i_{qp} \cdot m_{px} + m_{pp} \cdot i_{pq} \cdot m_{qx}) \\
&= m_{px} - (m_{pq} \cdot i_{qp} \cdot m_{px} + (0) \cdot i_{pq} \cdot m_{qx}) \\
&= m_{px} - (m_{pq} \cdot i_{qp} \cdot m_{px}) \\
-m_{px} &= -(m_{pq} \cdot i_{qp} \cdot m_{px}) \\
m_{px} &= m_{pq} \cdot i_{qp} \cdot m_{px} \\
1 = \frac{m_{px}}{m_{px}} &= m_{pq} \cdot i_{qp} \\
\frac{1}{m_{pq}} &= i_{qp}
\end{aligned}$$

Equation (2)

$$
\begin{aligned}
0 &= \beta_{qx} \\
&= m_{qx} - (m_{qq} \cdot i_{qp} \cdot m_{px} + m_{qp} \cdot i_{pq} \cdot m_{qx}) \\
&= m_{qx} - ((0) \cdot i_{qp} \cdot m_{px} + m_{qp} \cdot i_{pq} \cdot m_{qx}) \\
&= m_{qx} - (m_{qp} \cdot i_{pq} \cdot m_{qx}) \\
-m_{qx} &= -(m_{qp} \cdot i_{pq} \cdot m_{qx}) \\
m_{qx} &= m_{qp} \cdot i_{pq} \cdot m_{qx} \\
1 = \frac{m_{qx}}{m_{qx}} &= m_{qp} \cdot i_{pq} \\
\frac{1}{m_{qp}} &= i_{pq}
\end{aligned}
$$

Equation (3)

$$
\begin{aligned}
0 &= \beta_{xp} \\
&= m_{xp} - (m_{xp} \cdot i_{pq} \cdot m_{qp} + m_{xq} \cdot i_{qp} \cdot m_{pp}) \\
&= m_{xp} - (m_{xp} \cdot i_{pq} \cdot m_{qp} + m_{xq} \cdot i_{qp} \cdot (0)) \\
&= m_{xp} - (m_{xp} \cdot i_{pq} \cdot m_{qp}) \\
-m_{xp} &= -(m_{xp} \cdot i_{pq} \cdot m_{qp}) \\
m_{xp} &= m_{xp} \cdot i_{pq} \cdot m_{qp} \\
1 = \frac{m_{xp}}{m_{xp}} &= i_{pq} \cdot m_{qp} \\
\frac{1}{m_{qp}} &= i_{pq}
\end{aligned}
$$

Equation (4)

$$\begin{aligned}
0 &= \beta_{xq} \\
&= m_{xq} - (m_{xp} \cdot i_{pq} \cdot m_{qq} + m_{xq} \cdot i_{qp} \cdot m_{pq}) \\
&= m_{xq} - (m_{xp} \cdot i_{pq} \cdot (0) + m_{xq} \cdot i_{qp} \cdot m_{pq}) \\
&= m_{xq} - (m_{xq} \cdot i_{qp} \cdot m_{pq}) \\
-m_{xq} &= -(m_{xq} \cdot i_{qp} \cdot m_{pq}) \\
m_{xq} &= m_{xq} \cdot i_{qp} \cdot m_{pq} \\
1 = \frac{m_{xq}}{m_{xq}} &= i_{qp} \cdot m_{pq} \\
\frac{1}{m_{pq}} &= i_{qp}
\end{aligned}$$

Equation (1) corresponds to every entry in row p,Equation (2) corresponds to every entry in row q , Equation (3) corresponds to every entry in column p, and Equation (4) corresponds to every entry in column q . These 4 equations for entries p and q of $\mathbf{Mcds}_1$ give us 2 equations for entries i_{qp} and i_{pq} of matrix I_{pq}, specifically:

$$I_{pq} = \begin{bmatrix}
0_{11} & 0_{12} & \cdots & 0_{1p} & \cdots & 0_{1q} & \cdots & 0_{1m} \\
0_{21} & 0_{22} & \cdot & 0_{2p} & \cdot & 0_{2q} & \cdot & 0_{2m} \\
\vdots & \cdot & \ddots & \cdot & \cdot & \vdots & \cdot & \vdots \\
0_{p1} & 0_{p2} & \ldots & 0_{pp} & \cdot & \frac{1}{m_{qp}} & \cdot & 0_{pm} \\
\vdots & \cdot & \cdot & \cdot & \ddots & \cdot & \cdot & \vdots \\
0_{q1} & 0_{q2} & \ldots & \frac{1}{m_{pq}} & \cdot & 0_{qq} & \cdot & 0_{qm} \\
\vdots & \cdot & \cdot & \cdot & \cdot & \cdot & \ddots & \vdots \\
0_{n1} & 0_{n2} & \ldots & 0_{np} & \cdot & 0_{nq} & \cdot & 0_{nm}
\end{bmatrix}$$

□

3.1.2 Generalized Mcds Operation

With Theorem 3.1.7 proved, we can now give a generalized **Mcds** definition.

Definition 3.1.8. ***Generalized Mcds****-*

An $m \times n$ matrix M over a field $\mathbb{F}$ for which their exists a $1 \leq p < q \leq n$ such that

- $m_{pp}, m_{qq} = 0$
- $m_{pq}, m_{qp} \neq 0$

is said to be ***Mcds*** *eligible.*

It is useful to define conditions on matrices related to the **Mcds** operation.

Definition 3.1.9. *Given a matrix M, if M is not* ***Mcds*** *eligible, it is said to be a* ***Mcds fixed point***

Definition 3.1.10. *If the indices p and q witness that matrix M is* ***Mcds eligible****, the pair (p, q) is said to be an* ***Mcds context*** *for M.*

Definition 3.1.11. *If M is* ***Mcds eligible*** *and pair (p, q) is an* ***Mcds context*** *witnessing this, then β=**Mcds**$(M)_{pq}$ is the matrix cds operation,* ***Mcds****, on positions p, q where:*

$$\beta = M - MI_{pq}M$$

and I_{pq} is a $n \times m$ matrix over $\mathbb{F}$ with $I_{pq}(i,j) = \frac{1}{m_{ji}}$ if $i = p$ and $j = q$ or $i = q$ and $j = p$ and $I_{pq}(i,j) = 0$ otherwise. [2]

[2] The *Mcds* operation is applicable for rows and columns within the largest upper square sub-matrix of M

The result of one application of the **Mcds** operation to an **Mcds** eligible matrix M with a witnessing context (p, q) is a matrix of the same dimensions as M. Following previous notation, say $\beta =$**Mcds**$(M)_{pq}$. β may or may not be a **Mcds eligible**. Characterizing matrices that are **Mcds** eligible according to how many consecutive **Mcds** operations on it produces **Mcds** eligible matrices warrants investigation, and is an item for future exploration. The following definitions give direction to continue exploration in this direction.

Definition 3.1.12. *We define M to be a **Sortable Matrix** if there exists a sequence of consecutive **Mcds** operations that results in the zero matrix*

Definition 3.1.13. *Say $\beta =$**Mcds**$(M)_{pq}$, and pair (r, s) is an **Mcds context** witnessing a **Mcds eligible** matrix β, then **Mcds**$_{rs}(\beta)$ is said to be 2-sortable if the resulting matrix is either the zero matrix, or a **Mcds fixed point***

Definition 3.1.14. *A matrix is k-sortable if it permits k consecutive applications of **Mcds**, resulting in either the zero matrix, or a **Mcds fixed point***

CHAPTER 4

SCHUR COMPLEMENTS AND MCDS CORRESPONDENCE

We now turn attention to the question whether the relation between Mcds and the Schur complement observed for square matrices over $GF(2)$ hold for (not necessarily square) matrices over arbitrary fields.

4.1 Schur complement

The Schur complement does not have the following constraints, however, with the intention of finding a correlation between **Mcds** and Schur, we impose the following restrictions on the input matrix of the Schur complement operation:

- $m_{pp}, m_{qq} = 0$
- $m_{pq}, m_{qp} \neq 0$
- all other entries are elements of the field $\mathbb{F}$

We will first explore the form of the sub-matrices P, P^{-1}, Q, R and S of matrix M.

4.1.1 P, P^{-1}, Q, R and S

Lets consider a matrix M of size $m \times n$, and require M to have the value for entries:

- $m_{pp}, m_{qq} = 0$
- $m_{pq}, m_{qp} \neq 0$
- all other m entries $\in \mathbb{F}$

From M we can build the following matrices:

- M_1, the M matrix with rows/columns p and q in row/columns 1 and 2
- P the 2×2 sub-matrix made of entries m_{pp}, m_{qq}, m_{qp} and m_{pq}
- P^{-1} The inverse of matrix P
- Q the $2 \times (n-2)$ sub-matrix made of entries $\{m_{px}, m_{qx} | x = \mathbb{N} \setminus \{p, q\}, x \leq (n-2)\}$
- R the $(m-2) \times 2$ sub-matrix made of entries $\{m_{xp}, m_{xq} | x = \mathbb{N} \setminus \{p, q\}, x \leq (m-2)\}$
- S the $(m-2) \times (n-2)$ sub-matrix made of entries $\{m_{xy} | x, y = \mathbb{N} \setminus \{p, q\}, x \leq (n-2), y \leq (m-2)\}$

4.1.2 Schur Complement Generally

With the conditions above, compute position st of $S - RP^{-1}Q$ in a similar fashion as in Lemmas 3.1.4, 3.1.5 and 3.1.6.

Theorem 4.1.1. *$(S - R \cdot P^{-1} \cdot Q)_{st}$ generally has form:*

$$(S - R \cdot P^{-1} \cdot Q)_{st} = m_{st} - (m_{sq} \cdot \frac{1}{m_{pq}} \cdot m_{pt} + m_{sp} \cdot \frac{1}{m_{qp}} \cdot m_{qt})$$

Proof. Let matrices M, M_1, P, P^{-1}, Q, R and S be defined by Definition 1.3.2. Consider row operations at each step the operation $S - RP^{-1}Q$. Let θ denote $R \cdot P^{-1}$. Then generally:

$$\begin{aligned}
\theta_{st} &= Rs_R \cdot Ct_{P^{-1}} \\
&= \begin{bmatrix} m_{sp} & m_{sq} \end{bmatrix} \cdot \begin{bmatrix} (p^{-1})_{1t} \\ (p^{-1})_{2t} \end{bmatrix} \\
&= (m_{sp})((p^{-1})_{1t}) + (m_{sq})((p^{-1})_{2t})
\end{aligned}$$

However, by definition, p^{-1} only has 4 positions, and only 2 are unknown, therefore we can fill in information ($t = \{1, 2\}$), resulting in two possibilities for θ_{st}:

$$\begin{aligned}
\theta_{s1} &= Rs_R \cdot C1_{p^{-1}} \\
&= (m_{sp})((p^{-1})_{11}) + (m_{sq})((p^{-1})_{21}) \\
&= (m_{sp})(0) + (m_{sq})(\frac{1}{m_{pq}}) \\
&= \frac{m_{sq}}{m_{pq}} \\
&\text{or} \\
\theta_{s2} &= Rs_R \cdot C2_{p^{-1}} \\
&= (m_{sp})((p^{-1})_{12}) + (m_{sq})((p^{-1})_{22}) \\
&= (m_{sp})(\frac{1}{m_{qp}}) + (m_{sq})(0) \\
&= \frac{m_{sp}}{m_{qp}}
\end{aligned}$$

Continuing with the operation, let τ denote $\theta \cdot Q$, and consider τ_{st}:

$$\begin{aligned}
\tau_{st} &= Rs_{R.P^{-1}} \cdot Ct_Q \\
&= \begin{bmatrix} \frac{m_{sq}}{m_{pq}} & \frac{m_{sp}}{m_{qp}} \end{bmatrix} \cdot \begin{bmatrix} m_{pt} \\ \\ m_{qt} \end{bmatrix} \\
&= (\frac{m_{sq}}{m_{pq}})(m_{pt}) + (\frac{m_{sp}}{m_{qp}})(m_{qt}) \\
&= m_{sq} \cdot \frac{1}{m_{pq}} \cdot m_{pt} + m_{sp} \cdot \frac{1}{m_{qp}} \cdot m_{qt}
\end{aligned}$$

Continuing with the operation, $S - \tau$ is just pairwise subtraction between entries of S and τ .

Therefore:

$$(S - \tau)_{st} = m_{st} - (m_{sq} \cdot \frac{1}{m_{pq}} \cdot m_{pt} + m_{sp} \cdot \frac{1}{m_{qp}} \cdot m_{qt}) \tag{4.1}$$

□

4.1.3 When does Mcds and Schur complement correspond?

With the formulae for entries of the result of **Mcds** and of the Schur complement established, the correspondence between these two operations can be determined.

Theorem 4.1.2. *Let M be a $m \times n$ matrix that is **Mcds eligible**, let the pair (p, q) be a **Mcds context for M**, and let $\beta =$**Mcds**$(M)_{pq}$, δ=$SC(M)_{pq}$ then:*
β with rows Rp_β, Rq_β and columns Cp_β, Cq_β excluded produces δ.
Conversely, δ with rows Rp_δ, Rq_δ and columns Cp_δ, Cq_δ injected produces β

Proof. For simplicity, define **Mcds**$(M)_{pq} = \beta$ where $\beta_{st} = (\mathbf{Mcds}(M)_{pq})_{st} = m_{st} - (m_{sp} \cdot i_{pq} \cdot m_{qt} + m_{sq} \cdot i_{qp} \cdot m_{pt})$, by Theorem 3.1.7, and define β_1 to be the matrix β with rows and columns p and q excluded.

Also define $SC(M)_{pq} = \delta$ where $\delta_{st} = (SC(M)_{pq})_{st} = (S - R \cdot P^{-1} \cdot Q)_{st} = m_{st} - (m_{sp} \cdot \frac{1}{m_{qp}} \cdot m_{pt} + m_{sq} \cdot \frac{1}{m_{pq}} \cdot m_q)$ by Theorem 4.1.1, and define δ_1 to be the matrix δ with rows and columns p and q inserted in positions p and q respectively.

We show the relation that exists between β and δ, by proving that there is equality between δ and β_1 and conversely, proving that there is equality between δ_1 and β

First, consider the relation between δ and β_1. Both δ and β_1 are, by definition, $(m-2) \times (n-2)$ matrices. Also, recall by Theorem 3.1.7, $(I_{pq})_{pq} = \frac{1}{m_{qp}}$ and $(I_{pq})_{qp} = \frac{1}{m_{pq}}$. Thus:

$$\beta_{st} = m_{st} - (m_{sp} \cdot i_{pq} \cdot m_{qt} + m_{sq} \cdot i_{qp} \cdot m_{pt}) = (m_{st} - (m_{sp} \cdot \frac{1}{m_{qp}} \cdot m_{qt} + m_{sq} \cdot \frac{1}{m_{pq}} \cdot m_{pt}))$$

Consider $\delta - \beta_1$

$$\begin{aligned} \delta - \beta_1 = m_{st} - (m_{sp} \cdot \frac{1}{m_{qp}} \cdot m_{pt} + m_{sq} \cdot \frac{1}{m_{pq}} \cdot m_q) - \\ (m_{st} - (m_{sp} \cdot \frac{1}{m_{qp}} \cdot m_{qt} + m_{sq} \cdot \frac{1}{m_{pq}} \cdot m_{pt})) \\ = 0 - 0 + 0 = 0 \end{aligned}$$

Therefore, equality exists between δ and β_1.

Now, consider the relation between δ_1 and β. Both δ and β_1 are, by definition, $m \times n$ matrices. Also, recall by Theorem 3.1.7, $(I_{pq})_{pq} = \frac{1}{m_{qp}}$ and $(I_{pq})_{qp} = \frac{1}{m_{pq}}$. Thus:

$$\beta_{st} = m_{st} - (m_{sp} \cdot i_{pq} \cdot m_{qt} + m_{sq} \cdot i_{qp} \cdot m_{pt}) = (m_{st} - (m_{sp} \cdot \frac{1}{m_{qp}} \cdot m_{qt} + m_{sq} \cdot \frac{1}{m_{pq}} \cdot m_{pt}))$$

Consider $\delta_1 - \beta$

$$\delta_1 - \beta = m_{st} - (m_{sp} \cdot \frac{1}{m_{qp}} \cdot m_{pt} + m_{sq} \cdot \frac{1}{m_{pq}} \cdot m_q) - (m_{st} - (m_{sp} \cdot \frac{1}{m_{qp}} \cdot m_{qt} + m_{sq} \cdot \frac{1}{m_{pq}} \cdot m_{pt})) = 0 - 0 + 0 = 0$$

Therefore, equality exists between δ and β_1, proving the claim. □

4.1.4 Examples of Correspondence

*NOTE: Verification that matrices in this section satisfy necessary conditions to complete **Mcds** and Schur complement is left to reader*

First, consider a 3×3 matrix that is not symmetric but has $m_{pq} \neq m_{qp}$ for $p = 1$ and $q = 2$, a somewhat familiar state from the $GF(2)$ world:

Example 4.1.3. *Let*

$$M = \begin{bmatrix} 0 & 7 & 1 \\ 7 & 0 & 4 \\ 9 & 13 & 2 \end{bmatrix}$$

Consider $\beta = \mathbf{Mcds}(M)_{1,2}$ *and* $\delta = SC(M)_{1,2}$. *For* β, *the **Mcds** results in a matrix where each entry of rows* 1 *and* 2 *are zero, and each entry of columns* 1 *and* 2 *are zero , leaving only one entry to compute, namely* $\beta_{3,3}$, *which is given explicitly by Lemma 3.1.6:*

$$\begin{aligned}\beta_{3,3} &= m_{33} - (m_{32} \cdot i_{21} \cdot m_{13} + m_{31} \cdot i_{12} \cdot m_{23}) \\ &= 2 - (13 \times \frac{1}{7} \times 1 + 9 \times \frac{1}{7} \times 4) \\ &= 2 - (\frac{13}{7} + \frac{36}{7}) \\ &= 2 - \frac{49}{7} \\ &= 2 - 7 \\ &= -5\end{aligned}$$

Giving the following matrix:

$$\beta = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix} \end{array}$$

Excluding rows and columns 1 and 2 gives the 1×1 *matrix* $\begin{array}{c} \\ 3 \end{array}\!\!\begin{array}{c} 3 \\ \begin{bmatrix} -5 \end{bmatrix} \end{array}$, *call this* β_1*. Next, Consider* $\delta = SC(M)_{1,2}$*. The Schur complement of a* 3×3 *matrix on rows* 1 *and* 2 *results in the* 1×1 *matrix, where* δ_{st} *is given explicitly by Theorem 4.1.1. In this case, we are looking at* $\delta_{3,3}$*:*

$$\begin{aligned}
\delta_{3,3} &= m_{33} - (m_{32} \cdot \frac{1}{m_{21}} \cdot m_{13} + m_{31} \cdot \frac{1}{m_{12}} \cdot m_{23}) \\
&= 2 - (13 \times \frac{1}{7} \times 1 + 9 \times \frac{1}{7} \times 4) \\
&= 2 - (\frac{13}{7} + \frac{36}{7}) \\
&= 2 - \frac{49}{7} \\
&= 2 - 7 \\
&= -5
\end{aligned}$$

Giving the 1×1 *matrix* $\begin{array}{c} \\ 3 \end{array}\begin{array}{c} 3 \\ \begin{bmatrix} -5 \end{bmatrix} \end{array}$.

Injecting rows and columns of 0's in row and column positions 1 and 2 gives a 3×3 *call this* δ_1

$$\delta_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -5 \end{bmatrix} \end{array}$$

Thus we have equality between δ_1 *and* β*, and similarly* β_1 *and* δ*, as expected.*[1]

$\boxdot$

Increasing complexity, we now consider another 3×3 matrix, but this time there is no symmetry, and $m_{pq} \neq m_{qp}$ for any q, p

[1]The governing arithmetic in Example 4.1.3 is that of the $\mathbb{R}$, the natural arithmetic on real numbers

Example 4.1.4. *Let* $M = \begin{bmatrix} 0 & 3 & 7 \\ 4 & 0 & 1 \\ 4 & 2 & 18 \end{bmatrix}$ *Consider* $\beta = \mathbf{Mcds}(M)_{1,2}$ *and* $\delta = SC(M)_{1,2}$.

For β*,the* ***Mcds*** *results in a matrix where each entry of rows* 1 *and* 2 *are zero, and each entry of columns* 1 *and* 2 *are zero , leaving only one entry to compute, namely* $\beta_{3,3}$*, which is given explicitly by Lemma 3.1.6:*

$$\begin{aligned}
\beta_{3,3} &= m_{33} - (m_{32} \cdot i_{21} \cdot m_{13} + m_{31} \cdot i_{12} \cdot m_{23}) \\
&= 18 - (2 \times \frac{1}{4} \times 4 + 7 \times \frac{1}{3} \times 1) \\
&= 18 - (\frac{17}{3}) \\
&= \frac{37}{3}
\end{aligned}$$

The result is the following matrix:

$$\beta = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{37}{3} \end{bmatrix} \end{array}$$

Excluding rows and columns 1 and 2 gives the 1×1 *matrix* $\left[\frac{37}{3}\right]$ *call this* β_1.

Next, Consider $\delta = SC(M)_{1,2}$. *The Schur complement of a* 3×3 *matrix on rows* 1 *and* 2 *results in the* 1×1 *matrix, where* δ_{st} *is given explicitly by Theorem 4.1.1. In this case, we are looking at* $\delta_{3,3}$*:*

$$\begin{aligned}\delta_{3,3} &= m_{33} - (m_{32} \cdot \frac{1}{m_{21}} \cdot m_{13} + m_{31} \cdot \frac{1}{m_{12}} \cdot m_{23}) \\ &= 18 - (2 \times \frac{1}{3} \times 4 + 7 \times \frac{1}{4} \times 1) \\ &= 18 - (\frac{17}{3}) \\ &= \frac{37}{3}\end{aligned}$$

The result is the 1×1 *matrix* $\left[\frac{37}{3}\right]$.

Injecting rows and columns of 0's in row and column positions 1 and 2 gives a 3×3 *matrix, call this* δ_1

$$\delta_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \end{array}\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{37}{3} \end{bmatrix}\end{array}$$

Thus we have equality between δ_1 *and* β, *and similarly* β_1 *and* δ, *as expected.*[2]

$\boxdot$

Increasing another level of complexity, we now compute the **Mcds** and Schur Complement of a rectangular matrix, with the restrictions set in Theorem 3.1.7:

Example 4.1.5. *Let* $M = \begin{bmatrix} 0 & 7 & -6 \\ 4 & 3 & 9 \\ 2 & 3 & 0 \\ 4 & 6 & 1 \end{bmatrix}$ *Consider* $\beta = \mathbf{Mcds}(M)_{1,3}$ *and* $\delta = SC(M)_{1,3}$.

For β*,the* ***Mcds*** *results in a matrix where each entry of rows* 1 *and* 3 *are zero, and*

[2]The governing arithmetic in Example 4.1.4 is that of the $\mathbb{R}$, the natural arithmetic on real numbers

each entry of columns 1 *and* 3 *are zero , leaving only two entries to compute, namely* $\beta_{2,2}$ *and* $\beta_{4,2}$*, which is given explicitly by Lemma 3.1.6:*

$$\begin{aligned}
\beta_{2,2} &= m_{2,2} - (m_{2,3} \cdot i_{3,1} \cdot m_{1,2} + m_{2,1} \cdot i_{1,3} \cdot m_{3,2}) \\
&= 3 - (9 \cdot \frac{1}{-6} \cdot 7 + 4 \cdot \frac{1}{2} \cdot 3) \\
&= 3 - (-\frac{9}{2}) \\
&= \frac{15}{2} \\
\beta_{4,2} &= m_{4,2} - (m_{4,3} \cdot i_{3,1} \cdot m_{1,2} + m_{4,1} \cdot i_{1,3} \cdot m_{3,2}) \\
&= 6 - (1 \cdot \frac{1}{-6} \cdot 7 + 4 \cdot \frac{1}{2} \cdot 3) \\
&= 6 - (\frac{29}{6}) \\
&= \frac{7}{6}
\end{aligned}$$

The result is the following matrix:

$$\beta = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array}\!\!\begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{15}{2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{7}{6} & 0 \end{bmatrix} \end{array}$$

Excluding rows and columns 1 and 3 gives the 2×1 *matrix* $\begin{array}{c} \\ 2 \\ 4 \end{array}\!\!\begin{array}{c} 2 \\ \begin{bmatrix} \frac{15}{2} \\ \frac{7}{6} \end{bmatrix} \end{array}$ *call this* β_1*.*

Next, Consider $\delta = SC(M)_{1,3}$*. The Schur complement of a* 4×3 *on rows* 1 *and*

3 results in the 2×1 *matrix, where* δ_{st} *is given explicitly by Theorem 4.1.1, and* $s, t \neq p, q$*. In this case, we are looking at* $\delta_{2,2}$ *and* $\delta_{4,2}$*:*

$$\begin{aligned}
\delta_{2,2} &= m_{2,2} - (m_{2,3} \cdot \frac{1}{m_{1,3}} \cdot m_{1,2} + m_{2,1} \cdot \frac{1}{m_{3,1}} \cdot m_{3,2}) \\
&= 3 - (9 \cdot \frac{1}{-6} \cdot 7 + 4 \cdot \frac{1}{2} \cdot 3) \\
&= 3 - (-\frac{9}{2}) \\
&= \frac{15}{2} \\
\delta_{4,2} &= m_{4,2} - (m_{4,3} \cdot \frac{1}{m_{1,3}} \cdot m_{1,2} + m_{4,1} \cdot \frac{1}{m_{3,1}} \cdot m_{3,2}) \\
&= 6 - (1 \cdot \frac{1}{-6} \cdot 7 + 4 \cdot \frac{1}{2} \cdot 3) \\
&= 6 - (\frac{29}{6}) \\
&= \frac{7}{6}
\end{aligned}$$

The result is the 2×1 *matrix* $\begin{array}{c|c} & 2 \\ \hline 2 & \frac{15}{2} \\ 4 & \frac{7}{6} \end{array}$.

Injecting rows and columns of 0's in row and column positions 1 and 3 gives a 4×3 *matrix, call this* δ_1

$$\delta_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \end{array} \begin{array}{c} \begin{array}{ccc} 1 & 2 & 3 \end{array} \\ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{15}{2} & 0 \\ 0 & 0 & 0 \\ 0 & \frac{7}{6} & 0 \end{bmatrix} \end{array}$$

Thus we have equality between δ_1 and β, and similarly β_1 and δ, as expected.[3]

$\boxdot$

Next, before we get to more complicated examples, we give an example in $GF(5)$, of a 4×3 non symmetric matrix. Addition and multiplication are mod 5.

Example 4.1.6. *Let* $M = \begin{bmatrix} 0 & 2 & 2 \\ 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 4 & 1 \end{bmatrix}$ *Consider* $\beta = \mathbf{Mcds}(M)_{1,3}$ *and* $\delta = SC(M)_{1,3}$.
*For β,the **Mcds** results in a matrix where each entry of rows 1 and 3 are zero, and each entry of columns 1 and 3 are zero , leaving only two entries to compute, namely $\beta_{2,2}$ and $\beta_{4,2}$, which is given explicitly by Lemma 3.1.6:*

$$\begin{aligned}
\beta_{2,2} &= m_{2,2} - (m_{2,3} \cdot i_{3,1} \cdot m_{1,2} + m_{2,1} \cdot i_{1,3} \cdot m_{3,2}) \\
&= 1 - (1 \cdot \frac{1}{2} \cdot 2 + 0 \cdot \frac{1}{1} \cdot 1) \\
&= 1 - (1) \\
&= 0 \\
\beta_{4,2} &= m_{4,2} - (m_{4,3} \cdot i_{3,1} \cdot m_{1,2} + m_{4,1} \cdot i_{1,3} \cdot m_{3,2}) \\
&= 4 - (1 \cdot \frac{1}{2} \cdot 2 + 1 \cdot \frac{1}{1} \cdot 4) \\
&= 4 - (5) \\
&= -1 \\
&\equiv_5 4
\end{aligned}$$

[3]The governing arithmetic in Example 4.1.5 is that of the $\mathbb{R}$, the natural arithmetic on real numbers

The result is the following matrix:

$$\beta = \begin{array}{c|ccc|} & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 \end{array}$$

Excluding rows and columns 1 and 3 gives the 2×1 *matrix* $\begin{array}{c|c|} & 2 \\ 2 & 0 \\ 4 & 4 \end{array}$ *call this* β_1.

Next, Consider $\delta = SC(M)_{1,3}$. *The Schur complement of a* 4×3 *on rows* 1 *and* 3 *results in the* 2×1 *matrix, where* δ_{st} *is given explicitly by Theorem 4.1.1, and* $s, t \neq p, q$. *In this case, we are looking at* $\delta_{2,2}$ *and* $\delta_{4,2}$*:*

$$\begin{aligned}
\delta_{2,2} &= m_{2,2} - (m_{2,3} \cdot \frac{1}{m_{1,3}} \cdot m_{1,2} + m_{2,1} \cdot \frac{1}{m_{3,1}} \cdot m_{3,2}) \\
&= 1 - (1 \cdot \frac{1}{2} \cdot 2 + 0 \cdot \frac{1}{1} \cdot 1) \\
&= 1 - (1) \\
&= 0 \\
\delta_{4,2} &= m_{4,2} - (m_{4,3} \cdot \frac{1}{m_{1,3}} \cdot m_{1,2} + m_{4,1} \cdot \frac{1}{m_{3,1}} \cdot m_{3,2}) \\
&= 4 - (1 \cdot \frac{1}{2} \cdot 2 + 1 \cdot \frac{1}{1} \cdot 4) \\
&= 4 - (5) \\
&= -1 \\
&\equiv_5 4
\end{aligned}$$

The result is 2×1 *matrix* $\begin{array}{c|c} & 2 \\ 2 & 0 \\ 4 & 4 \end{array}$.

Injecting rows and columns of 0's in row and column positions 1 and 3 gives a 4×3 *matrix, call this* δ_1

$$\delta_1 = \begin{array}{c|ccc} & 1 & 2 & 3 \\ 1 & 0 & 0 & 0 \\ 2 & 0 & 0 & 0 \\ 3 & 0 & 0 & 0 \\ 4 & 0 & 4 & 0 \end{array}$$

Thus we have equality between δ_1 *and* β*, and similarly* β_1 *and* δ*, as expected.*[4]

⊡

For the remainder of the examples of correspondence, we will only show the final result of ***Mcds*** *and Schur complement. Verification of individual entries are left to the reader.* The following examples are of varying size, complexity, and fields. First, we give more complicated examples of the correspondences, with matrices over $\mathbb{R}$, with the natural arithmetic on real numbers.

Example 4.1.7. *Consider*

[4]The governing arithmetic in Example 4.1.6 is that of the $GF(5)$, Arithmetic mod 5

$$M = \begin{bmatrix} 2 & 2 & 4 & 4 & 1 & 0 & 4 & 0 & 5 & 4 & 4 \\ 5 & 5 & 0 & 0 & 1 & 0 & 2 & 2 & 2 & 3 & 4 \\ 4 & 5 & 0 & 2 & 2 & 4 & 3 & 0 & 2 & 3 & 4 \\ 1 & 1 & 2 & 0 & 1 & 1 & 1 & 3 & 4 & 3 & 1 \\ 3 & 2 & 4 & 4 & 3 & 4 & 1 & 5 & 5 & 0 & 0 \\ 1 & 1 & 3 & 3 & 0 & 2 & 2 & 4 & 1 & 1 & 5 \\ 3 & 5 & 4 & 4 & 3 & 5 & 4 & 2 & 2 & 4 & 3 \end{bmatrix}$$

Consider $\beta = \mathbf{Mcds}(M)_{3,4}$ *and* $\delta = SC(M)_{3,4}$. *For* β, *the* ***Mcds*** *results in a matrix where each entry of rows* 3 *and* 4 *are zero, and each entry of columns* 3 *and* 4 *are zero. Entries of* β *are given explicitly by Lemma 3.1.6, and the computation of* β *gives the following:*

$$\beta = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \begin{bmatrix} -8 & -10 & 0 & 0 & -5 & -10 & -4 & -6 & -7 & -8 & -6 \\ 5 & 5 & 0 & 0 & 1 & 0 & 2 & 2 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7 & -10 & 0 & 0 & -3 & -6 & -7 & -1 & -7 & -12 & -10 \\ \frac{-13}{2} & -8 & 0 & 0 & \frac{-9}{2} & \frac{-11}{2} & -4 & \frac{-1}{2} & -8 & -8 & \frac{-5}{2} \\ -7 & -7 & 0 & 0 & -3 & -5 & -4 & -4 & -10 & -8 & -7 \end{bmatrix} \end{array}$$

Excluding rows and columns 3 and 4 gives the 5×9 *matrix* β_1*:*

$$\beta_1 = \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \begin{bmatrix} -8 & -10 & -5 & -10 & -4 & -6 & -7 & -8 & -6 \\ 5 & 5 & 1 & 0 & 2 & 2 & 2 & 3 & 4 \\ -7 & -10 & -3 & -6 & -7 & -1 & -7 & -12 & -10 \\ \frac{-13}{2} & -8 & \frac{-9}{2} & \frac{-11}{2} & -4 & \frac{-1}{2} & -8 & -8 & \frac{-5}{2} \\ -7 & -7 & -3 & -5 & -4 & -4 & -10 & -8 & -7 \end{bmatrix} \end{array}$$

Next, Consider $\delta = SC(M)_{3,6}$. *The Schur complement of a* 7×11 *on rows* 3 *and* 4 *results in the* 5×9 *matrix, where* δ_{st} *is given explicitly by Theorem 4.1.1, and* $s, t \neq p, q$. *Computation of* δ *gives the following*

$$\delta = \begin{array}{c} \\ 1 \\ 2 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccccc} 1 & 2 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \begin{bmatrix} -8 & -10 & -5 & -10 & -4 & -6 & -7 & -8 & -6 \\ 5 & 5 & 1 & 0 & 2 & 2 & 2 & 3 & 4 \\ -7 & -10 & -3 & -6 & -7 & -1 & -7 & -12 & -10 \\ \frac{-13}{2} & -8 & \frac{-9}{2} & \frac{-11}{2} & -4 & \frac{-1}{2} & -8 & -8 & \frac{-5}{2} \\ -7 & -7 & -3 & -5 & -4 & -4 & -10 & -8 & -7 \end{bmatrix} \end{array}$$

Injecting rows and columns of 0's in row and column positions 3 and 6 gives a 9×7 *matrix, call this* δ_1

$$\delta_1 = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{array} \begin{array}{c} \begin{array}{ccccccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 & 11 \end{array} \\ \begin{bmatrix} -8 & -10 & 0 & 0 & -5 & -10 & -4 & -6 & -7 & -8 & -6 \\ 5 & 5 & 0 & 0 & 1 & 0 & 2 & 2 & 2 & 3 & 4 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -7 & -10 & 0 & 0 & -3 & -6 & -7 & -1 & -7 & -12 & -10 \\ \frac{-13}{2} & -8 & 0 & 0 & \frac{-9}{2} & \frac{-11}{2} & -4 & \frac{-1}{2} & -8 & -8 & \frac{-5}{2} \\ -7 & -7 & 0 & 0 & -3 & -5 & -4 & -4 & -10 & -8 & -7 \end{bmatrix} \end{array}$$

Thus we equality between δ_1 and β, and similarly β_1 and δ, as expected.[5]

$\boxdot$

Next, we see examples in $GF(11)$. Multiplication and addition are mod 11.

Example 4.1.8. *Let*

$$M = \begin{bmatrix} 3 & 9 & 6 & 4 & 6 & 7 & 8 \\ 6 & 4 & 7 & 7 & 4 & 0 & 10 \\ 5 & 6 & 0 & 0 & 7 & 1 & 0 \\ 9 & 8 & 2 & 9 & 10 & 7 & 8 \\ 5 & 1 & 6 & 3 & 8 & 3 & 1 \\ 3 & 2 & 8 & 8 & 9 & 0 & 8 \\ 4 & 8 & 2 & 3 & 10 & 10 & 4 \\ 5 & 3 & 0 & 0 & 5 & 3 & 4 \\ 6 & 3 & 3 & 4 & 2 & 1 & 3 \end{bmatrix}$$

[5] The governing arithmetic in Example 4.1.7 is that of the $\mathbb{R}$, the natural arithmetic on real numbers

Consider $\beta = \mathbf{Mcds}(M)_{3,6}$ *and* $\delta = SC(M)_{3,6}$*. For* β*, the* ***Mcds*** *results in a matrix where each entry of rows* 3 *and* 6 *are zero, and each entry of columns* 3 *and* 6 *are zero. Entries of* β *are given explicitly by Lemma 3.1.6, and the computation of* β *gives the following:*

$$\beta \equiv_{11} \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 \end{array} \\ \left[\begin{array}{ccccccc} 7 & 4 & 0 & 9 & 8 & 0 & 2 \\ 2 & 5 & 0 & 0 & 3 & 0 & 3 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 9 & 4 & 0 & 7 & 0 & 0 & 6 \\ 7 & 9 & 0 & 8 & 5 & 0 & 6 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 8 & 0 & 1 & 1 & 0 & 2 \\ 1 & 7 & 0 & 0 & 6 & 0 & 4 \\ 4 & 10 & 0 & 1 & 4 & 0 & 0 \end{array}\right] \end{array}$$

Excluding rows and columns 3 and 6 gives the 7×5 *matrix* β_1*:*

$$\beta_1 = \begin{array}{c} \\ 1 \\ 2 \\ 4 \\ 5 \\ 7 \\ 8 \\ 9 \end{array} \begin{array}{c} \begin{array}{ccccc} 1 & 2 & 4 & 5 & 7 \end{array} \\ \left[\begin{array}{ccccc} 7 & 4 & 9 & 8 & 2 \\ 2 & 5 & 0 & 3 & 3 \\ 9 & 4 & 7 & 0 & 6 \\ 7 & 9 & 8 & 5 & 6 \\ 0 & 8 & 1 & 1 & 2 \\ 1 & 7 & 0 & 6 & 4 \\ 4 & 10 & 1 & 4 & 0 \end{array}\right] \end{array}$$

Next, Consider $\delta = SC(M)_{3,6}$. *The Schur complement of a* 9×7 *on rows* 3 *and* 6 *results in the* 7×9 *matrix, where* δ_{st} *is given explicitly by Theorem 4.1.1, and* $s, t \neq p, q$. *computation of* δ *gives the following*

$$\delta \equiv_{11} \begin{array}{c|ccccc|} & 1 & 2 & 4 & 5 & 7 \\ \hline 1 & 7 & 4 & 9 & 8 & 2 \\ 2 & 2 & 5 & 0 & 3 & 3 \\ 4 & 9 & 4 & 7 & 0 & 6 \\ 5 & 7 & 9 & 8 & 5 & 6 \\ 7 & 0 & 8 & 1 & 1 & 2 \\ 8 & 1 & 7 & 0 & 6 & 4 \\ 9 & 4 & 10 & 1 & 4 & 0 \end{array}$$

Injecting rows and columns of 0's in row and column positions 3 and 6 gives a 9×7 *matrix, call this* δ_1

$$\delta_1 = \begin{array}{c|ccccccc|} & 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ \hline 1 & 7 & 4 & 0 & 9 & 8 & 0 & 2 \\ 2 & 2 & 5 & 0 & 0 & 3 & 0 & 3 \\ 3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 4 & 9 & 4 & 0 & 7 & 0 & 0 & 6 \\ 5 & 7 & 9 & 0 & 8 & 5 & 0 & 6 \\ 6 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 0 & 8 & 0 & 1 & 1 & 0 & 2 \\ 8 & 1 & 7 & 0 & 0 & 6 & 0 & 4 \\ 9 & 4 & 10 & 0 & 1 & 4 & 0 & 0 \end{array}$$

Thus we equality between δ_1 and β, and similarly β_1 and δ, as expected.[6]

$\boxdot$

[6]The governing arithmetic in Example 4.1.8 is that of the $GF(11)$, Arithmetic mod 11

CHAPTER 5

WHAT IS NEXT?

There is still much that is not known about **Mcds**. Below is a small sample of questions yet to be answered

1. Cardinality Questions:
   (a) How many **Mcds** eligible matrices are there over a given finite field?
   (b) For a positive integer k, How many k-sortable matrices are there over a given finite field?
2. Generalize the **Mcds** definition
   (a) Is there a sorting algorithm for matrices with nonzero entries in position pp or position qq, or
   (b) Is there a sorting algorithm for matrices with pq and qp entries non necessarily non zero
3. Tensors
   (a) Could **Mcds** be extended to a tensor cds operation?

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