

**TUKEY MORPHISMS
BETWEEN FINITE BINARY RELATIONS**

by
Rhett Barton

A thesis
submitted in partial fulfillment
of the requirements for the degree of
Master of Science in Mathematics
Boise State University

August 2021

© 2021
Rhett Barton
ALL RIGHTS RESERVED

BOISE STATE UNIVERSITY GRADUATE COLLEGE

DEFENSE COMMITTEE AND FINAL READING APPROVALS

of the thesis submitted by

Rhett Barton

Thesis Title: Tukey Morphisms between Finite Binary Relations

Date of Final Oral Examination: 06 July 2021

The following individuals read and discussed the thesis submitted by student Rhett Barton, and they evaluated his presentation and response to questions during the final oral examination. They found that the student passed the final oral examination.

Samuel Coskey, Ph.D.	Chair, Supervisory Committee
Paul Ellis, Ph.D.	Member, Supervisory Committee
John Clemens, Ph.D.	Member, Supervisory Committee
Marion Scheepers, Ph.D.	Member, Supervisory Committee

The final reading approval of the thesis was granted by Samuel Coskey, Ph.D., Chair of the Supervisory Committee. The thesis was approved by the Graduate College.

To Abby, Max, and The Third

ACKNOWLEDGMENTS

Thank you to John Clemens and Marion Scheepers for your ready willingness to serve on the supervisory committee.

Thank you to Jens Harlander, Samuel Coskey, and Paul Ellis, who have been excellent instructors and mentors. You have helped me feel at ease in an often-intimidating world.

Thank you to my employer, Blue Cross of Idaho, and to my direct managers, Marc Roberts and Dave Allgrunn, who have made it possible for me to achieve this goal while pursuing my career. I will forever be grateful for your support and flexibility.

Thank you to my high school math teacher, Phil Johnson, who helped me see beauty in the subject and was an ever-encouraging presence.

Above all, thank you to my wonderful wife Lauren, for your patience, support, sacrifice, and love.

ABSTRACT

Let $\mathbf{A} = (A, A_+, A)$ and $\mathbf{B} = (B, B_+, B)$ be relations. A *morphism* is a pair of maps $\varphi : B \rightarrow A$ and $\varphi_+ : A_+ \rightarrow B_+$ such that for all $b \in B$ and $a \in A_+$, $\varphi(b)Aa \Rightarrow bB\varphi_+(a)$. We study the existence of morphisms between finite relations. The ultimate goal is to identify the conditions under which morphisms exist. In this thesis we present some progress towards that goal. We use computation to verify the results for small finite relations.

TABLE OF CONTENTS

DEDICATION	iv
ACKNOWLEDGMENTS	v
ABSTRACT	vi
LIST OF TABLES	ix
LIST OF FIGURES	x
1 Introduction	1
2 Relations and Morphisms	4
2.1 Background	4
2.2 Finite Relations	10
2.3 Bimorphic Relations	12
3 Partial Results on the Morphism Decision Problem	14
3.1 Homomorphisms between Relations	14
3.2 Incremental Transformations	16
3.3 Skeleton Bimorphic Form	23
3.4 Special Cases	26
3.4.1 Ladder Relations	27
3.4.2 Further Special Results on Morphism Existence	32

4	Classification of Small Finite Relations	35
4.1	$\mathbf{R}_{5,5}$ and Computation	35
4.2	Discussion of Hasse Diagram	38
	REFERENCES	43
A	R Code	44
B	Morphisms between Skeleton Relations	54

LIST OF TABLES

4.1	$\mathbf{R}_{5,5}$ Skeleton Bimorphic Forms	40
4.2	Duals and Dominating Numbers for $\mathbf{R}_{5,5}$ Skeleton Bimorphic Forms . .	41

LIST OF FIGURES

2.1	Example of a finite relation and its dual, represented as bipartite graphs	5
2.2	Example of a dominating family that contains no smaller dominating family, but is not of least cardinality (circled in red). For contrast, a dominating family of least cardinality is circled in blue.	6
2.3	Morphism diagram	7
2.4	Example: A morphism between two finite relations	8
2.5	Counter Example: There is no morphism from \mathbf{A} to \mathbf{B} despite $\delta(\mathbf{A}) = \delta(\mathbf{B}) = 3$ and $\delta(\mathbf{A}^?) = \delta(\mathbf{B}^?) = 2$	9
2.6	Illustration of maximal points (red), minimal points (blue), and twins (green)	11
3.1	Example: A minus-surjective homomorphism	15
3.2	Illustration of Lemma 3.2.1. Here we add edges to \mathbf{A} to obtain \mathbf{A}° and find a morphism from \mathbf{A} to \mathbf{A}°	17
3.3	Illustration of Lemma 3.2.3. Here we delete points from \mathbf{A} to obtain \mathbf{A}° and find a morphism from \mathbf{A} to \mathbf{A}°	18
3.4	Illustration of Lemma 3.2.4. Here we delete non-minimal points from \mathbf{A} to obtain \mathbf{A}° and find a morphism from \mathbf{A}° to \mathbf{A}	19
3.5	Illustration of Lemma 3.2.5. Here we delete points from \mathbf{A}_+ to obtain \mathbf{A}° and find a morphism from \mathbf{A}° to \mathbf{A}	20

3.6	Illustration of Lemma 3.2.6. Here we delete non-maximal points from A_+ to obtain \mathbf{A}° and find a morphism from \mathbf{A} to \mathbf{A}°	21
3.7	Illustration of Lemma 3.2.7. Here we delete twin points from \mathbf{A} to obtain \mathbf{A}° and find morphisms from \mathbf{A} to \mathbf{A}° and from \mathbf{A}° to \mathbf{A}	23
3.8	Finding the Skeleton Bimorphic Form of a Relation. Maximal points in A_+ are circled in red. Minimal point in A are circled in blue.	27
3.9	When $\delta(\mathbf{B}) = 1$, there is a morphism from \mathbf{A} to \mathbf{B}	28
3.10	Example of Lemma 3.4.3. There is a morphism from the 2-ladder to any relation with dominating number 2.	29
3.11	Example of Lemma 3.4.4. There is a morphism from the 3-ladder to the 2-ladder because $3 \geq 2$	30
3.12	Counter Example from Figure 2.5 continued. Each subrelation has a dominating number < 3	33
4.1	Hasse Diagram of $\mathbf{R}_{5,5}$ Skeleton Bimorphic Forms	42

CHAPTER 1

INTRODUCTION

In this thesis, we study Tukey morphisms between finite relations. Before proceeding to the topic in earnest, we give a brief outline of related concepts that give context as to the origins and motivation for this area of research.

If (A, \leq_A) and (B, \leq_B) are two partial orders, then a *Galois connection* between these is a pair of monotone functions: $\phi : B \rightarrow A$ and $\psi : A \rightarrow B$, such that for all $a \in A$ and $b \in B$, we have

$$\phi(b) \leq_A a \iff b \leq_B \psi(a)$$

Galois connections generalize the correspondence between subgroups and subfields investigated in Galois theory [3].

A directed partial order (X, \leq_X) is a partial order such that every pair of elements has an upper bound. If (A, \leq_A) and (B, \leq_B) are directed partial orders, then a map $\phi : B \rightarrow A$ is called *Tukey* if the preimage of each bounded subset of A is bounded in B . A map $\psi : A \rightarrow B$ is called *cofinal* if it maps cofinal subsets of A to cofinal subsets of B [5].

The existence of Tukey map from B to A is equivalent to the existence of a cofinal map from A to B [4]. Assuming that ϕ is a Tukey map from B to A and ψ is a corresponding cofinal map from A to B , the following implication holds for all $a \in A$

and $b \geq_B B$:

$$\phi(b) \leq_A a \Rightarrow b \leq_B \psi(a)$$

These maps are referred to as *Galois-Tukey connections*. As the name suggests and can be seen from the implication above, they are related to Galois connections. In a sense they are more general since they relax the “if and only if” to “implies”.

If such a Tukey map from B to A exists, we write $B \leq_T A$ and say that B is *Tukey reducible* to A . If both $A \leq_T B$ and $B \leq_T A$, we write $A \equiv_T B$ and say that A and B are *Tukey equivalent* [5].

The original motivation for Galois-Tukey connections comes from the Moore-Smith theory of convergence in general topological spaces [5]. However, Galois-Tukey connections have found applications in comparing complexities of various directed sets or, more generally, partial orders [4].

These comparisons can reveal useful information. For example, Tukey reducibility downward preserves calibre-like properties, such as c.c.c., property K, precalibre \aleph_1 , σ -linked, and σ -centered [2].

Galois-Tukey connections can be generalized by letting (A, \leq_A) and (B, \leq_B) be relations, rather than partial orders. However, in that context it is not generally true that a given ϕ gives rise to a corresponding ψ such that $\phi(b) \leq_A a \Rightarrow b \leq_B \psi(a)$. It then becomes necessary to explicitly give two mappings, ϕ and ψ , that satisfy the condition

$$\phi(b) \leq_A a \Rightarrow b \leq_B \psi(a)$$

Pairs of maps between relations that satisfy this condition are referred to as *generalized Galois-Tukey connections*. Throughout this thesis, we use the convention established by Blass and refer to them as *morphisms* [1].

In Definition 2.1.4 we introduce the notion of a *dominating number* of a relation. Informally, it is the minimum size of a subset of the codomain such that every element of the domain is related to a member of that subset. The definitions of many cardinal characteristics amount to dominating numbers of specific relations. For example, the cardinal characteristic \mathfrak{d} is the least cardinality of a set of functions from ω to ω such that every function from ω to ω is eventually dominated by a member of that set. Then \mathfrak{d} is the dominating number of the \leq^* relation on ω^ω .

By employing the fact that many cardinal characteristics are dominating numbers, morphisms have found extensive use in the proofs of many cardinal characteristic inequalities. Once the underlying relations for two cardinal characteristics are determined, Theorem 2.1.7 from Blass shows that if a morphism can be found between these relations, then an inequality exists between the cardinal characteristics [1].

With the usefulness and interest of morphisms established, the focus of this thesis is not the *application* of morphisms, but rather what can be said about the *existence* of morphisms. In particular, we explore under which conditions a morphism exists between finite relations.

Chapter 2 outlines definitions and results, drawn largely from Blass, which provide a foundation for the following chapters.

Chapter 3 shows progress in our understanding of when morphisms exist, particularly in the finite case.

Chapter 4 provides a classification for small finite relations. We apply the results from Chapter 3 to these examples, as well as provide computational checks.

CHAPTER 2

RELATIONS AND MORPHISMS

2.1 Background

We begin with some fundamental definitions, drawing largely from Blass [1]. The basic objects of inquiry are *relations* and their *duals*.

Definition 2.1.1. (Blass 4.1 [1]) A triple $\mathbf{A} = (A_-, A_+, A)$ consisting of two nonempty sets A_- , A_+ , and a binary relation $A_- \times A_+ \rightarrow A$ is called a **relation**.

Definition 2.1.2. (Blass 4.3 [1]) If $\mathbf{A} = (A_-, A_+, A)$ then the **dual** of \mathbf{A} is the relation $\mathbf{A}^\circ = (A_+, A_-, : \check{A})$ where $:$ denotes the complement and \check{A} is the converse of A , thus $x : \check{A}y$ if and only if $y \check{A}x$.

Figure 2.1 provides an illustration of a finite relation and its dual, represented as bipartite graphs. Colors have been added to aide in the visualization. In particular, for a relation $\mathbf{A} = (A_-, A_+, A)$, for a given $a \in A_+$, the same color is used for all $(x, a) \in A$. This convention is used in many of the figures throughout the thesis.

We now introduce the notion of *dominating families*. Dominating families are subsets of A_+ that “cover” all of A_- . The smallest such subset is a *minimal dominating family* and its cardinality is called the *dominating number*.

Definition 2.1.3. (Blass 4.2 [1]) An **A-dominating family** is a subset Y of A_+ such that for every $a \in A_-$ there exists $\alpha \in Y$ such that $aA\alpha$.

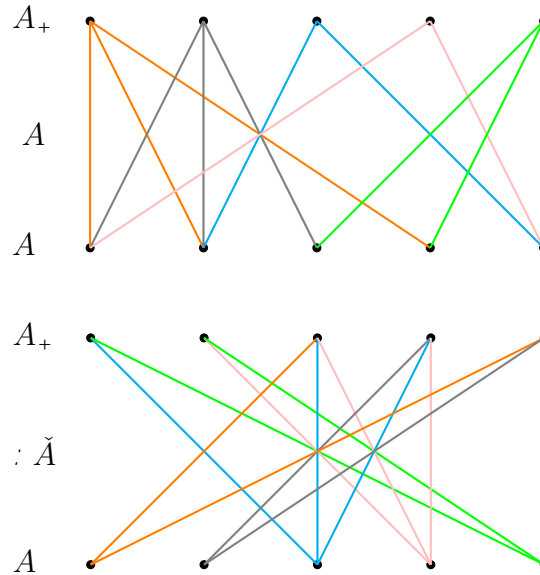


Figure 2.1: Example of a finite relation and its dual, represented as bipartite graphs

Definition 2.1.4. (Blass 4.2 [1]) The **dominating number** $\delta(\mathbf{A})$ of a relation \mathbf{A} is the smallest cardinality of any \mathbf{A} -dominating family. If no \mathbf{A} -dominating family exists, we say that $\delta(\mathbf{A}) = \top$. We refer to an \mathbf{A} -dominating family that has cardinality $\delta(\mathbf{A})$ as a **minimal \mathbf{A} -dominating family**.

The use of “minimal” in Definition 2.1.4 differs slightly from what the reader might expect. Here we use it to mean “an \mathbf{A} -dominating family *of least cardinality*”. This definition is not equivalent to “an \mathbf{A} -dominating family that contains no smaller \mathbf{A} -dominating family”. The former definition implies the latter, but not vice versa. Figure 2.2 gives an illustration of why they are not equivalent.

In Lemma 2.1.5 we see that if no \mathbf{A} -dominating family exists, then \mathbf{A}^\top has a point in its codomain that relates to all points in its domain, i.e. $\delta(\mathbf{A}^\top) = 1$. We return to this special edge case in Lemma 3.4.1.

Lemma 2.1.5. *Let \mathbf{A} be a relation. There does not exist an \mathbf{A} -dominating family if*

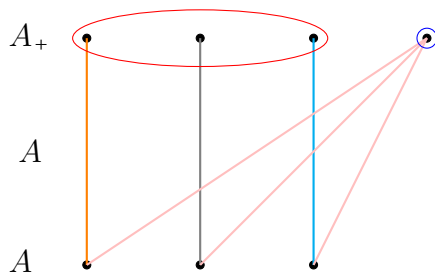


Figure 2.2: Example of a dominating family that contains no smaller dominating family, but is not of least cardinality (circled in red). For contrast, a dominating family of least cardinality is circled in blue.

and only if $\delta(\mathbf{A}^?) = 1$.

Proof. Suppose there does not exist an \mathbf{A} -dominating family. Then there exists a 2^A such that $a \notin \mathcal{A}$ for all $\alpha \in A_+$. Then $\alpha: \check{A}a$ for all $\alpha \in A_+$. Then fag is an $\mathbf{A}^?$ -dominating family and $\delta(\mathbf{A}^?) = 1$. \square

Definition 2.1.6 provides the formal definition for *morphisms*. Morphisms are the primary objects by which we compare relations throughout this thesis.

Definition 2.1.6. (Blass 4.8 [1]) A (Tukey) **morphism** from one relation $\mathbf{A} = (A_-, A_+, A)$ to another $\mathbf{B} = (B_-, B_+, B)$ is a pair of functions $\varphi = (\varphi_-, \varphi_+)$ such that:

- $\varphi_- : B_- \rightarrow A_-$
- $\varphi_+ : A_+ \rightarrow B_+$
- for all $b \in B_-$ and $a \in A_+$, $\varphi_-(b) \in A_- \Rightarrow b \in B_- \varphi_+(a)$

In this thesis, we denote “there is a morphism from \mathbf{A} to \mathbf{B} ” as $\mathbf{A} \leq \mathbf{B}$.

Figure 2.3 gives a diagram to help visualize this definition. The definition implies that the diagram “commutes.” However, in this diagram the dotted arrows represent relations, rather than functions, so it is only an illustration.

$$\begin{array}{ccc}
 A_+ & \xrightarrow{\varphi_+} & B_+ \\
 \hat{A} & & \hat{B} \\
 \vdots & & \vdots \\
 A & \xleftarrow{\varphi} & B
 \end{array}$$

Figure 2.3: Morphism diagram

See Figure 2.4 for an example of a morphism between two small finite relations. Dashed arrows represent the functions and, as before, bipartite graphs represent the relations. We use this visualization approach throughout the thesis.

If $\varphi = (\varphi_-, \varphi_+)$ is a morphism from \mathbf{A} to \mathbf{B} , then $\varphi^\circ = (\varphi_+, \varphi_-)$ is a morphism from \mathbf{B}° to \mathbf{A}° . We can see this by taking the contrapositive of the implication and applying Definition 2.1.2:

$$\begin{aligned}
 (\varphi_-(b)Aa \Rightarrow bB\varphi_+(a)) & \Rightarrow (b\mathbf{B}\varphi_+(a) \Rightarrow \varphi_-(b)\mathbf{A}a) \\
 & \Rightarrow (\varphi_+(a): \check{B}b \Rightarrow a: \check{A}\varphi_-(b))
 \end{aligned}$$

This is also illustrated in Figure 2.4.

Theorem 2.1.7 from Blass [1] provides the motivation for the definition of morphism.

Theorem 2.1.7. (Blass 4.9 [1]) If $\mathbf{A} \dashv \mathbf{B}$ then $\delta(\mathbf{A}) = \delta(\mathbf{B})$ and $\delta(\mathbf{A}^\circ) = \delta(\mathbf{B}^\circ)$.

Proof. Suppose (φ_-, φ_+) is a morphism from \mathbf{A} to \mathbf{B} . Let $D_{\mathbf{A}} \subseteq A_+$ be a minimal \mathbf{A} -dominating family. Let $Y = \varphi_+(D_{\mathbf{A}})$. Note that $Y \subseteq B_+$ and $\delta(\mathbf{A}) = |Y|$. We will show that Y is a \mathbf{B} -dominating family: For $b \in B$, consider $\varphi_-(b) \in A$. Because $D_{\mathbf{A}}$ is a \mathbf{A} -dominating family, there exists $a \in D_{\mathbf{A}}$ such that $\varphi_-(b)Aa$. Then

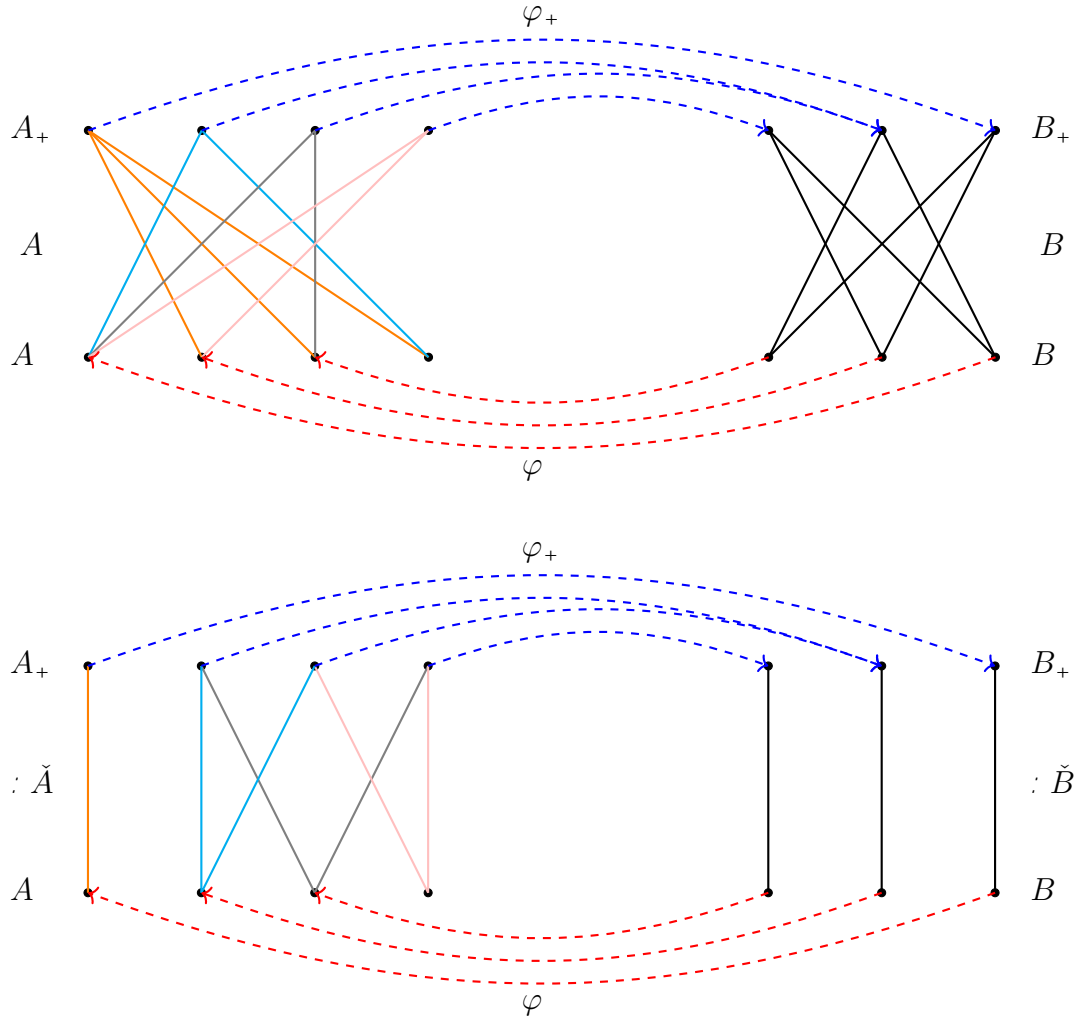


Figure 2.4: Example: A morphism between two finite relations

by the definition of morphism, $bB\varphi_+(a)$. In particular, $\varphi_+(a) \supseteq Y$. Then Y is a \mathbf{B} -dominating family. Then $\delta(\mathbf{A}) = |Y| = \delta(\mathbf{B})$.

Since (φ_+, φ) is a morphism from $\mathbf{B}^?$ to $\mathbf{A}^?$, the same argument can be used to show $\delta(\mathbf{B}^?) = \delta(\mathbf{A}^?)$.

If no such \mathbf{A} -dominating family $D_{\mathbf{A}}$ exists, then by Lemma 2.1.5 $\delta(\mathbf{A}) = 1$ and $\delta(\mathbf{A}^?) = 1$. Then $\delta(\mathbf{A}) = \delta(\mathbf{B})$ and $\delta(\mathbf{A}^?) = \delta(\mathbf{B}^?)$. \square

This leads us to wonder if the converse is true, i.e. is a morphism between two arbitrary relations \mathbf{A} and \mathbf{B} guaranteed if $\delta(\mathbf{A}) = \delta(\mathbf{B})$ and $\delta(\mathbf{A}^?) = \delta(\mathbf{B}^?)$? Rather than give the reader a false sense of hope, we immediately give a counterexample to this conjecture in Figure 2.5. This counter example follows from Lemma 3.4.8.

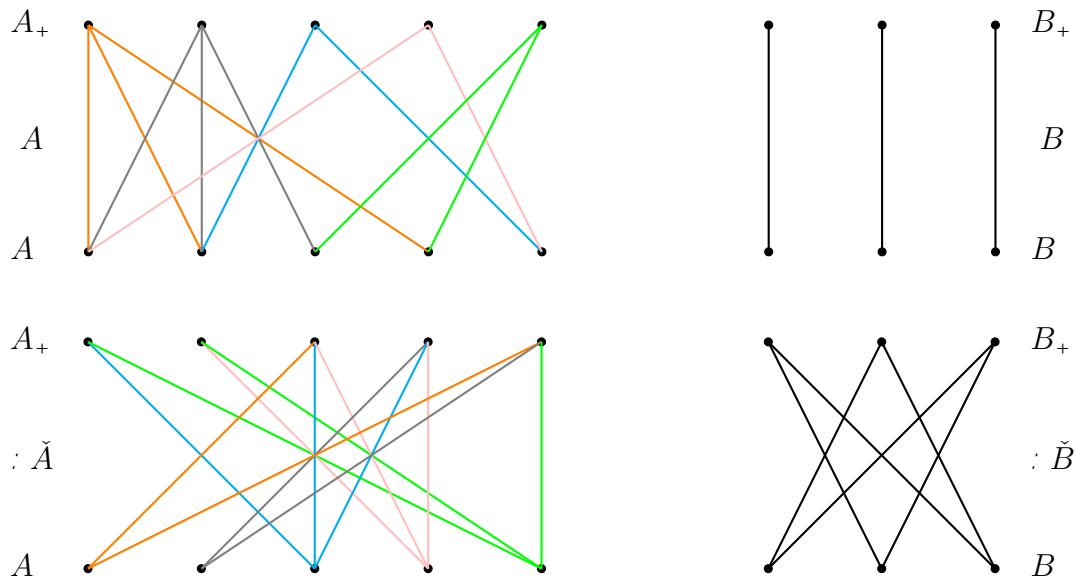


Figure 2.5: Counter Example: There is no morphism from \mathbf{A} to \mathbf{B} despite $\delta(\mathbf{A}) = \delta(\mathbf{B}) = 3$ and $\delta(\mathbf{A}^?) = \delta(\mathbf{B}^?) = 2$

Given this counter example, the question of, “When does there exist a morphism between two relations?” becomes interesting. The quest to answer this question in the finite context is the motivation for the following chapters.

The contrapositive of Theorem 2.1.7 gives us our first sufficient condition for morphism non-existence.

Corollary 2.1.8. *If $\delta(\mathbf{A}) \neq \delta(\mathbf{B})$ or $\delta(\mathbf{A}^?) \neq \delta(\mathbf{B}^?)$ then $\mathbf{A} \not\leq \mathbf{B}$.*

2.2 Finite Relations

Before proceeding on the topic of morphisms, we outline some key definitions and properties of finite relations.

As we saw previously, a finite relation can be thought of as a bipartite graph. As such, we borrow the concept of a *neighborhood* from graph theory.

Definition 2.2.1. The **neighborhood** of a point a in a relation $\mathbf{A} = (A_-, A_+, A)$ is the set of all points that relate to a , $\{b \in A_- \mid aAb\}$, denoted as $N_{\mathbf{A}}(a)$.

Definition 2.2.2. Let \mathbf{A} be a relation. For $a, b \in A_+$, define the **neighborhood relation** with respect to \mathbf{A} , denoted $\leq_{\mathbf{A}}$, as $a \leq_{\mathbf{A}} b$ if and only if $N_{\mathbf{A}}(a) \subseteq N_{\mathbf{A}}(b)$.

Lemma 2.2.3. *Given a relation \mathbf{A} , the neighborhood relation is a preorder on A_+ and A_- , i.e. it is reflexive and transitive.*

Proof. This is obvious by the reflexivity and transitivity of the subset relation:

- Reflexive: $N_{\mathbf{A}}(a) \subseteq N_{\mathbf{A}}(a)$ for all $a \in A_+$.
- Transitive: If $N_{\mathbf{A}}(a) \subseteq N_{\mathbf{A}}(b)$ and $N_{\mathbf{A}}(b) \subseteq N_{\mathbf{A}}(c)$, then $N_{\mathbf{A}}(a) \subseteq N_{\mathbf{A}}(c)$ for all $a, b, c \in A_+$.

These arguments work similarly for A_- . □

Note that the neighborhood relation is not anti-symmetric, so it is not a partial order: $N_{\mathbf{A}}(a) \subseteq N_{\mathbf{A}}(b)$ and $N_{\mathbf{A}}(b) \subseteq N_{\mathbf{A}}(a)$ imply that $N_{\mathbf{A}}(a) = N_{\mathbf{A}}(b)$ but not that $a = b$.

We now define the notions of “maximal”, “minimal”, and “twin” points with respect to the neighborhood preorder. Figure 2.6 provides an illustration.

Definition 2.2.4. Let \mathbf{A} be a relation and let $X = A$ or $X = A_+$. A point $a \in X$ is **A-maximal** in the neighborhood preorder on X if there does not exist $b \in X$ such that $N_{\mathbf{A}}(a) \subset N_{\mathbf{A}}(b)$, i.e. the neighborhood of a is not a proper subset of any other neighborhood.

Definition 2.2.5. Let \mathbf{A} be a relation and let $X = A$ or $X = A_+$. A point $a \in X$ is **A-minimal** in the neighborhood preorder on X if there does not exist $b \in X$ such that $N_{\mathbf{A}}(b) \subset N_{\mathbf{A}}(a)$, i.e. the neighborhood of a is not a proper superset of any other neighborhood.

Definition 2.2.6. Let \mathbf{A} be a relation and let $X = A$ or $X = A_+$. Two points $a, b \in X$ are said to be **A-twins** if $N_{\mathbf{A}}(a) = N_{\mathbf{A}}(b)$, i.e. they have the same neighborhoods.

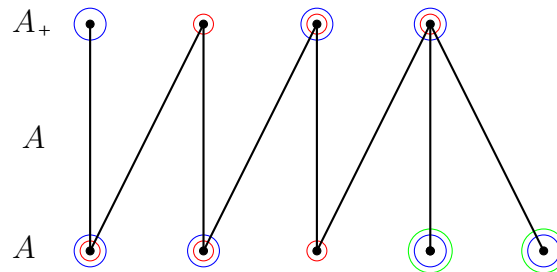


Figure 2.6: Illustration of maximal points (red), minimal points (blue), and twins (green)

A minimal point in \mathbf{A} is maximal in $\mathbf{A}^?$. Similarly, a maximal point in \mathbf{A} is minimal in $\mathbf{A}^?$.

Lemma 2.2.7. Let \mathbf{A} be a relation. A point $a \in A$ is **A-minimal** if and only if it is **A[?]-maximal**. A point $a \in A$ is **A-maximal** if and only if it is **A[?]-minimal**.

Proof. Let $a \in A$ be **A-minimal**. By Definition 2.2.5, there does not exist $b \in A$ such that $N_{\mathbf{A}}(b) \subset N_{\mathbf{A}}(a)$. If there exists $c \in A$ such that $N_{\mathbf{A}}(c) \supset N_{\mathbf{A}}(a)$, then a

and c are twins. Then by definition of the subset relation, if there exists $c \supseteq A$ such that $cAx \Rightarrow aAx$ for all $x \supseteq A_+$, then a and c are twins. By the contrapositive, if there exists $c \supseteq A$ such that $a \not\supseteq Ax \Rightarrow c \not\supseteq Ax$ for all $x \supseteq A_+$, then a and c are twins. Then by Definition 2.1.1, if there exists $c \supseteq A$ such that $x: \check{A}a \Rightarrow x: \check{A}c$ for all $x \supseteq A_+$, then a and c are twins. Then if there exists $c \supseteq A$ such that $N_{\mathbf{A}}(a) \subseteq N_{\mathbf{A}}(c)$, then a and c are twins. Then there does not exist $b \supseteq A$ such that $N_{\mathbf{A}}(a) \not\subseteq N_{\mathbf{A}}(b)$, which is the definition of \mathbf{A}^\varnothing -maximal. A similar argument shows that a point $a \supseteq A$ is \mathbf{A} -maximal if and only if it is \mathbf{A}^\varnothing -minimal. \square

Corollary 2.2.8. *Let \mathbf{A} be a relation. A point $a \supseteq A_+$ is \mathbf{A} -minimal if and only if it is \mathbf{A}^\varnothing -maximal. A point $a \supseteq A_+$ is \mathbf{A} -maximal if and only if it is \mathbf{A}^\varnothing -minimal.*

Proof. Let $\mathbf{B} = \mathbf{A}^\varnothing$. Then $B = A_+$, $B_+ = A$, $B = : \check{A}$, and $\mathbf{B}^\varnothing = \mathbf{A}$. By Lemma 2.2.7, $b \supseteq B$ is \mathbf{B} -minimal if and only if it is \mathbf{B}^\varnothing -maximal. Then $b \supseteq A_+$ is \mathbf{A}^\varnothing -minimal if and only if it is \mathbf{A} -maximal. Similarly, by Lemma 2.2.7, $b \supseteq B$ is \mathbf{B} -maximal if and only if it is \mathbf{B}^\varnothing -minimal. Then $b \supseteq A_+$ is \mathbf{A}^\varnothing -maximal if and only if it is \mathbf{A} -minimal. \square

2.3 Bimorphic Relations

Lemma 2.3.1 proves that $\mathbf{A} ! \mathbf{B} \wedge \mathbf{B} ! \mathbf{C} \Rightarrow \mathbf{A} ! \mathbf{C}$ (transitivity). This fact is employed widely throughout the remainder of the thesis.

Lemma 2.3.1. (Transitivity) Let \mathbf{A} , \mathbf{B} , and \mathbf{C} be relations. If $\varphi = (\varphi_-, \varphi_+)$ is a morphism from \mathbf{A} to \mathbf{B} and $\vartheta = (\vartheta_-, \vartheta_+)$ is a morphism from \mathbf{B} to \mathbf{C} , then $\varsigma = (\varphi_-, \vartheta_-, \vartheta_+, \varphi_+)$ is a morphism from \mathbf{A} to \mathbf{C} .

Proof. If φ is a morphism from \mathbf{A} to \mathbf{B} then for all $b \in B$ and $a \in A_+$, $\varphi(b)Aa \Rightarrow bB\varphi_+(a)$. Similarly, since ϑ is a morphism from \mathbf{B} to \mathbf{C} then for all $c \in C$ and $d \in B_+$, $\vartheta(c)Bd \Rightarrow cC\vartheta_+(d)$.

Suppose that $\varphi(\vartheta(c))Aa$ for some $c \in C$ and $a \in A_+$. Then because φ is a morphism, $\vartheta(c)B\varphi_+(a)$. Then because ϑ is a morphism, $cC\vartheta_+(\varphi_+(a))$. \square

It is shown in Lemma 3.2.1 that $\mathbf{A} \sim \mathbf{A}$ (reflexivity). However, it should be noted that symmetry does not hold generally, i.e. $\mathbf{A} \sim \mathbf{B} \not\Rightarrow \mathbf{B} \sim \mathbf{A}$. If $\mathbf{A} \sim \mathbf{B} \wedge \mathbf{B} \sim \mathbf{A}$, \mathbf{A} and \mathbf{B} are said to be *bimorphic*.

The full power of transitivity can be used with bimorphic relations because it allows us to consider any two bimorphic relations to be “equivalent” with respect to the morphism relation.

Lemma 2.3.2. *Suppose \mathbf{A} and \mathbf{B} are bimorphic relations. Then for a relation \mathbf{C} , $\mathbf{A} \sim \mathbf{C}$ if and only if $\mathbf{B} \sim \mathbf{C}$. Additionally, $\mathbf{C} \sim \mathbf{A}$ if and only if $\mathbf{C} \sim \mathbf{B}$.*

Proof. Suppose $\mathbf{A} \sim \mathbf{C}$. Then since $\mathbf{B} \sim \mathbf{A}$ and by Lemma 2.3.1, $\mathbf{B} \sim \mathbf{C}$. By a similar argument, $\mathbf{B} \sim \mathbf{C} \Rightarrow \mathbf{A} \sim \mathbf{C}$.

Now, suppose $\mathbf{C} \sim \mathbf{A}$. Then since $\mathbf{A} \sim \mathbf{B}$ and by Lemma 2.3.1, $\mathbf{C} \sim \mathbf{B}$. By a similar argument, $\mathbf{C} \sim \mathbf{B} \Rightarrow \mathbf{C} \sim \mathbf{A}$. \square

This equivalence proves useful in simplifying our quest: instead of determining the existence or non-existence of a morphism between any two finite relations, we can consider bimorphic classes of relations. In particular, we can choose to consider a representative from each bimorphic class that is of “minimal complexity.” Finding bimorphic forms of minimal complexity is a focus of Chapter 3. These minimal forms serve as the building blocks for the classification of small finite relations in Chapter 4.

CHAPTER 3

PARTIAL RESULTS ON THE MORPHISM DECISION PROBLEM

The goal of the morphism decision problem is to find sufficient conditions for $\mathbf{A} \leq \mathbf{B}$ and sufficient conditions for $\mathbf{A} \not\leq \mathbf{B}$.

Section 3.1 provides an alternative structure (minus-surjective homomorphisms) for finding morphisms between relations. The focus of Section 3.2 is finding, for a given \mathbf{A} , a variety of relations \mathbf{A}^0 such that $\mathbf{A} \leq \mathbf{A}^0$ or $\mathbf{A}^0 \leq \mathbf{A}$. We can then use transitivity to apply these moves in sequence. Section 3.3 provides one such application to result in bimorphic forms of “reduced complexity”. Section 3.4 outlines several sufficient conditions for morphism existence or non-existence.

3.1 Homomorphisms between Relations

We start by defining a homomorphism between relations.

Definition 3.1.1. A **homomorphism** between relations $\mathbf{A} = (A_-, A_+, A)$ and $\mathbf{B} = (B_-, B_+, B)$ is a function $f = g \upharpoonright h$ where

$$g : A_- \rightarrow B_-$$

$$h : A_+ \rightarrow B_+$$

such that if xAy then $f(x)Bf(y)$ for all $x \in A$ and $y \in A_+$.

We impose further restriction on the homomorphism in Definition 3.1.2. This definition is custom-designed to yield a natural morphism, as seen in Lemma 3.1.3. Figure 3.1 gives an example of such a homomorphism.

Definition 3.1.2. A **minus-surjective homomorphism** between relations $\mathbf{A} = (A, A_+, A)$ and $\mathbf{B} = (B, B_+, B)$ is a homomorphism $f = g \circ h$ where

$$g : A \rightarrow B$$

$$h : A_+ \rightarrow B_+$$

such that g is surjective.

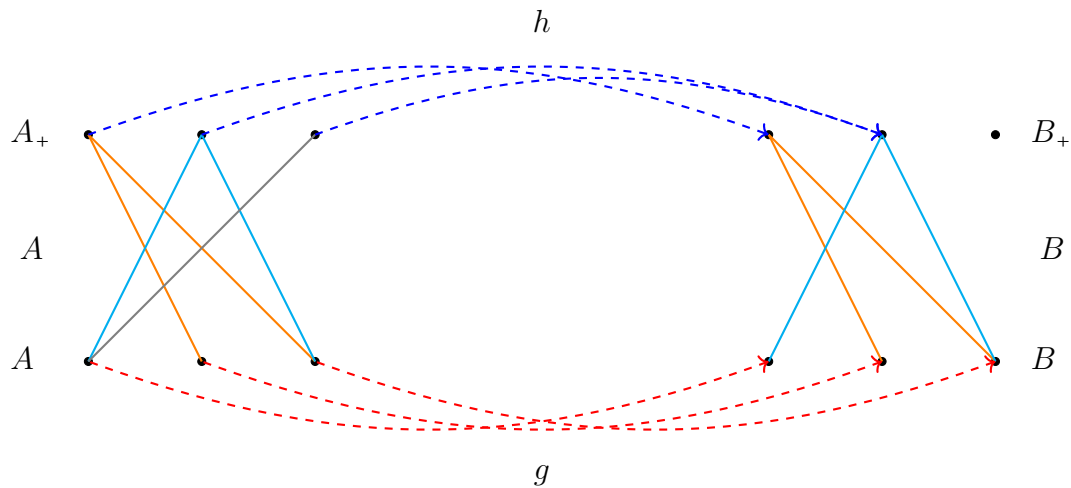


Figure 3.1: Example: A minus-surjective homomorphism

Lemma 3.1.3. Let \mathbf{A}, \mathbf{B} be relations and suppose $f = g \circ h$ is a minus-surjective homomorphism from \mathbf{A} to \mathbf{B} . Let $j : B \rightarrow A$ satisfy $g \circ j = id_B$. Then $\varphi = (j, h)$ is a morphism from \mathbf{A} to \mathbf{B} .

Proof. Suppose that $j(b)Aa$ for some $b \in B$, $a \in A_+$. Then by Definition 3.1.1, $f(j(b))Bf(a)$, which simplifies to $bBh(a)$. \square

Definition 3.1.4 and Lemma 3.1.5 give us a tool for comparing certain closely-related relations using homomorphisms.

Definition 3.1.4. Let $\mathbf{A} = (A, A_+, A)$ and $\mathbf{B} = (B, B_+, B)$ be relations such that $A \subseteq B$ and $A_+ \subseteq B_+$. The **inclusion function from \mathbf{A} to \mathbf{B}** $i = j \sqcup k$ is defined such that:

- $j : A \rightarrow B$
- $k : A_+ \rightarrow B_+$
- $j(a) = a$ for all $a \in A$
- $k(a) = a$ for all $a \in A_+$

Lemma 3.1.5. *The inclusion function from $\mathbf{A} = (A, A_+, A)$ to $\mathbf{B} = (B, B_+, B)$ is a homomorphism if and only if $A \subseteq B$.*

Proof. Suppose that $A \subseteq B$ and aAb . Then aBb , which implies that $i(a)Bi(b)$. Then i is a homomorphism.

Now suppose that i is a homomorphism and aAb . Then $i(a)Bi(b)$, which implies that aBb and $A \subseteq B$. \square

3.2 Incremental Transformations

Lemma 3.2.1 shows that adding edges to a relation \mathbf{A} results in a relation \mathbf{A}° such that $\mathbf{A} \sqsubseteq \mathbf{A}^\circ$. Letting $A^\circ = A$ proves that the morphism relation has the reflexive property, i.e. $\mathbf{A} \sqsubseteq \mathbf{A}$.

Lemma 3.2.1. *Let $\mathbf{A} = (A, A_+, A)$ and $\mathbf{A}^\circ = (A, A_+, A^\circ)$ be relations such that $A \subseteq A^\circ$. Then there exists a morphism from \mathbf{A} to \mathbf{A}° .*

Proof. Since $A \subseteq A^\circ$, by Lemma 3.1.5, the inclusion function $i = j \circ k$ from \mathbf{A} to \mathbf{A}° is a homomorphism. Since $j : A \rightarrow A$ is bijective, it has a unique inverse. Then by Lemma 3.1.3, (j^{-1}, k) is a morphism from \mathbf{A} to \mathbf{A}° . \square

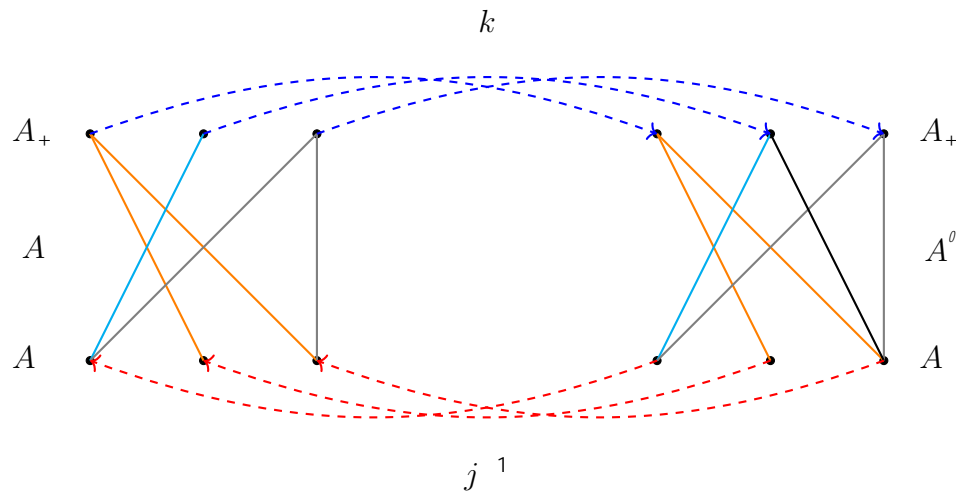


Figure 3.2: Illustration of Lemma 3.2.1. Here we add edges to \mathbf{A} to obtain \mathbf{A}° and find a morphism from \mathbf{A} to \mathbf{A}° .

In Definition 3.2.2 we define the induced subrelation.

Definition 3.2.2. Suppose \mathbf{A} is a relation and $S \subseteq A$ and $R \subseteq A_+$. The **induced subrelation** $\mathbf{A}[S, R]$ is the relation whose points consist of S and R and for any two points $s \in S$ and $r \in R$, s and r are related in $\mathbf{A}[S, R]$ if and only if s and r are related in \mathbf{A} . That is, $\mathbf{A}[S, R] = fS, R, A^\circ g$, where $A^\circ = A \cup S \cup R$.

Effectively, induced subrelations can be used to “delete” points. When points are deleted from A , there is a morphism from the original relation to the induced subrelation. This is illustrated in Figure 3.3.

Lemma 3.2.3. *Suppose \mathbf{A} is a relation and $\mathbf{A}^\circ = \mathbf{A}[S, A_+] = (S, A_+, A^\circ)$ is an induced subrelation of \mathbf{A} . Then there exists a morphism from \mathbf{A} to \mathbf{A}° .*

Proof. Let $i : S \rightarrow A$ and $i_+ : A_+ \rightarrow A_+$ be the inclusion maps. Suppose $i(x)Ay$ for some $x \in S$ and $y \in A_+$. Since i is the inclusion function, xAy , i.e. $(x, y) \in A$. Then by definition of A° , $xA^\circ y$. Because $i_+ : A_+ \rightarrow A_+$ is the inclusion function, $xA^\circ i_+(y)$. □

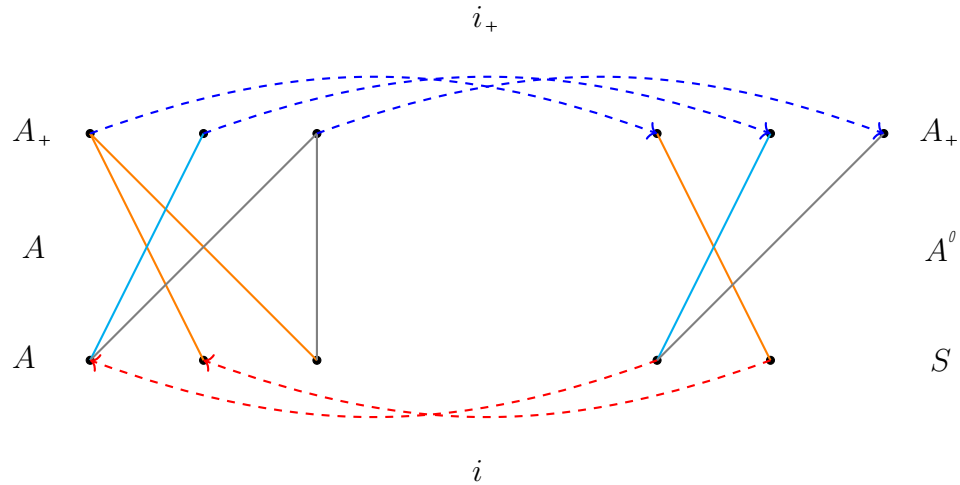


Figure 3.3: Illustration of Lemma 3.2.3. Here we delete points from A to obtain A° and find a morphism from \mathbf{A} to \mathbf{A}° .

However, if only non-minimal points are deleted from A , there is also a morphism that goes the other direction. This is illustrated in Figure 3.4.

Lemma 3.2.4. *Let $\mathbf{A} = (A, A_+, A)$ and $M \subseteq A$ be the set of all \mathbf{A} -minimal points in A . Let $\mathbf{A}^\circ = \mathbf{A}[M, A_+]$, where $M \subseteq N \subseteq A$. Then there exists a morphism from \mathbf{A}° to \mathbf{A} .*

Proof. Define $\varphi : A \rightarrow N$ as follows: For all $a \in A$,

$$\varphi(a) = \begin{cases} a & \text{when } a \in M \\ b & \text{when } a \notin M \text{ for some } b \in M \text{ chosen such that } N_{\mathbf{A}}(b) \subseteq N_{\mathbf{A}}(a) \end{cases}$$

Let i_+ be the identity function from A_+ to A_+ . We will show that (φ, i_+) is a morphism from \mathbf{A}° to \mathbf{A} : Suppose that $\varphi(x)A^\circ y$ for some $x \in A$ and $y \in A_+$. If $x \in M$, then $xA^\circ y$. Since $A^\circ \subseteq A$, this implies that xAy . Then $xAi_+(y)$ also. If $x \notin M$, then $\varphi(x) = z$ for some $z \in M$ chosen such that $N_{\mathbf{A}}(z) \subseteq N_{\mathbf{A}}(x)$. Then $zA^\circ y$ and zAy . This implies that xAy , and so $xAi_+(y)$. \square

Then by Lemma 3.2.4 and Lemma 3.2.3, the resulting relation obtained by removing non-minimal points from A is bimorphic with the original relation.

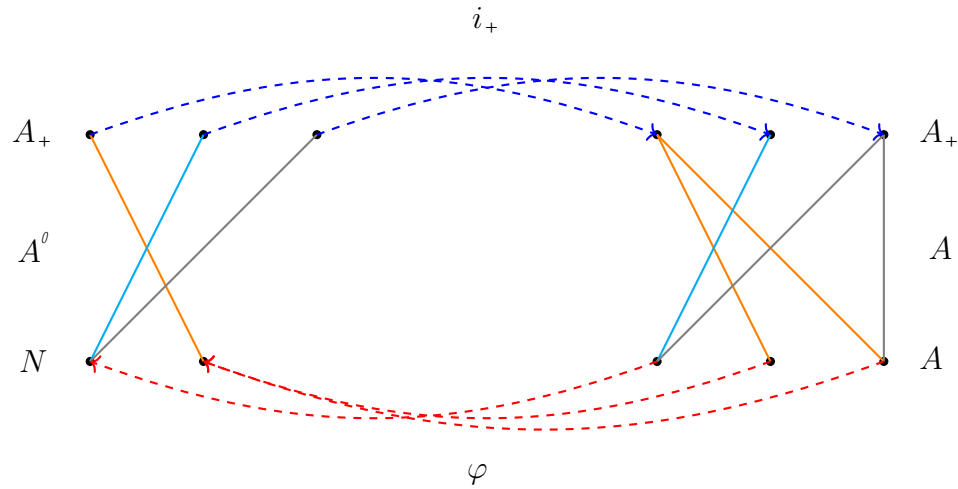


Figure 3.4: Illustration of Lemma 3.2.4. Here we delete non-minimal points from A to obtain A° and find a morphism from A° to A .

The dual notion is deleting non-maximal points from A_+ . Deleting points from A_+ results in a new relation that can morphism onto the original. This is illustrated in Figure 3.5.

Lemma 3.2.5. *Suppose \mathbf{A} is a relation and $\mathbf{A}^\circ = \mathbf{A}[A_-, R] = fA_-, R, A^\circ g$ is an induced subrelation of \mathbf{A} . Then there exists a morphism from \mathbf{A}° to \mathbf{A} .*

Proof. Note that $A_- = A_-, R = A_+,$ and $A^\circ = A_-$. Then the inclusion function from \mathbf{A}° to \mathbf{A} is a homomorphism (Lemma 3.1.5). Further, the inclusion function is minus-surjective, since $A_- = A_-$. Then by Lemma 3.1.3, there is a morphism from \mathbf{A}° to \mathbf{A} . □

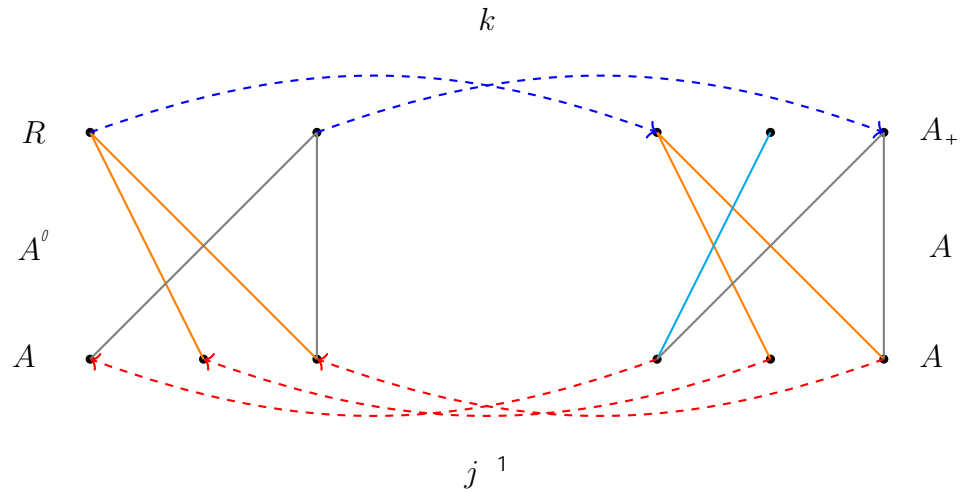


Figure 3.5: Illustration of Lemma 3.2.5. Here we delete points from A_+ to obtain \mathbf{A}° and find a morphism from \mathbf{A}° to \mathbf{A} .

However, if only non-maximal points are deleted from A_+ , there is also a morphism that goes the other direction. This is illustrated in Figure 3.6.

Lemma 3.2.6. *Let $\mathbf{A} = (A_-, A_+, A)$ and $M \subseteq A_+$ be the set of all \mathbf{A} -maximal points in A_+ . Let $\mathbf{A}^\circ = \mathbf{A}[A_-, N]$, where $M \cap N = \emptyset$. Then there exists a morphism from \mathbf{A} to \mathbf{A}° .*

Proof. Define $\varphi_+ : A_+ \rightarrow N$ as follows: For all $a \in A_+$,

$$\varphi_+(a) = \begin{cases} a & \text{when } a \in M \\ b & \text{when } a \notin M \text{ for some } b \in M \text{ chosen such that } N_{\mathbf{A}}(a) \subseteq N_{\mathbf{A}}(b) \end{cases}$$

Let i be the identity function from A to A . We will show that (i, φ_+) is a morphism from \mathbf{A} to \mathbf{A}^0 : Suppose that $i(x)Ay$ for some $x \in A$ and $y \in A_+$. Then xAy . If $y \in M$, then xA^0y by definition of \mathbf{A}^0 . Then $xA^0\varphi_+(y)$. If $y \notin M$, then $\varphi_+(y) = z$ for some $z \in M$ chosen such that $N_{\mathbf{A}}(y) \subseteq N_{\mathbf{A}}(z)$. This implies xAz . Then xA^0z and so $xA^0\varphi_+(y)$. \square

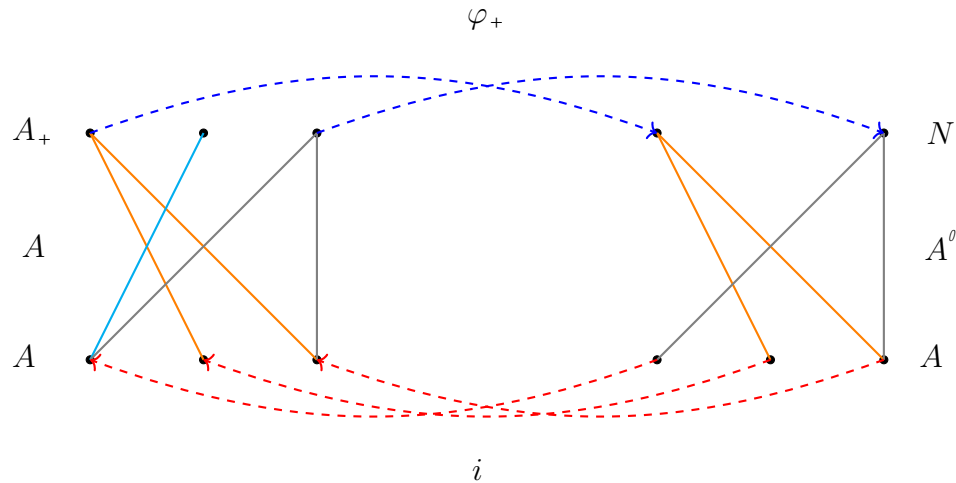


Figure 3.6: Illustration of Lemma 3.2.6. Here we delete non-maximal points from A_+ to obtain \mathbf{A}^0 and find a morphism from \mathbf{A} to \mathbf{A}^0

Twin points can be seen as duplicate points in the relation, and as such can be removed to create a relation that is bimorphic with the original. This is illustrated in Figure 3.7.

Lemma 3.2.7. *Let \mathbf{A} be a relation and let $R \subseteq A$ and $S \subseteq A_+$ be sets of points such that each point in $A \setminus R$ has an \mathbf{A} -twin in R and each point in $A_+ \setminus S$ has an \mathbf{A} -twin in S . Let $\mathbf{A}^\circ = \mathbf{A}[R, S]$. Then there is a morphism from \mathbf{A} to \mathbf{A}° and from \mathbf{A}° to \mathbf{A} .*

Proof. Let $i_- : R \hookrightarrow A$ be the inclusion function and define $\varphi_+ : A_+ \rightarrow S$ as

$$\varphi_+(x) = \begin{cases} x & \text{when } x \in S \\ y & \text{when } x \in A_+ \setminus S \text{ for } y \in S \text{ such that } y \text{ is an } \mathbf{A}\text{-twin of } x \end{cases}$$

Then we see that (i_-, φ_+) is a morphism from \mathbf{A} to \mathbf{A}° : Suppose $i_-(b)Aa$ for some $b \in R$ and $a \in A_+$. Then bAa . Either $a \in S$ or $a \in A_+ \setminus S$. If $a \in S$, then $bA^\circ a$ and $bA^\circ \varphi_+(a)$. If $a \in A_+ \setminus S$, then it has an \mathbf{A} -twin $c \in S$ and bAc . This implies that $bA^\circ c$ and $bA^\circ \varphi_+(a)$.

Now, let $i_+ : S \hookrightarrow A_+$ be the inclusion function and define $\varphi_- : A \rightarrow R$ as

$$\varphi_-(x) = \begin{cases} x & \text{when } x \in R \\ y & \text{when } x \in A \setminus R \text{ for } y \in R \text{ such that } y \text{ is an } \mathbf{A}\text{-twin of } x \end{cases}$$

Then we see that (φ_-, i_+) is a morphism from \mathbf{A}° to \mathbf{A} : Suppose $\varphi_-(b)A^\circ a$ for some $b \in A \setminus R$ and $a \in S$. Either $b \in R$ or $b \in A \setminus R$. If $b \in R$, then $bA^\circ a$. Then bAa and $bAi_+(a)$. If $b \in A \setminus R$, then it has an \mathbf{A} -twin $c \in R$ and cAa . This implies that bAa and $bAi_+(a)$. □

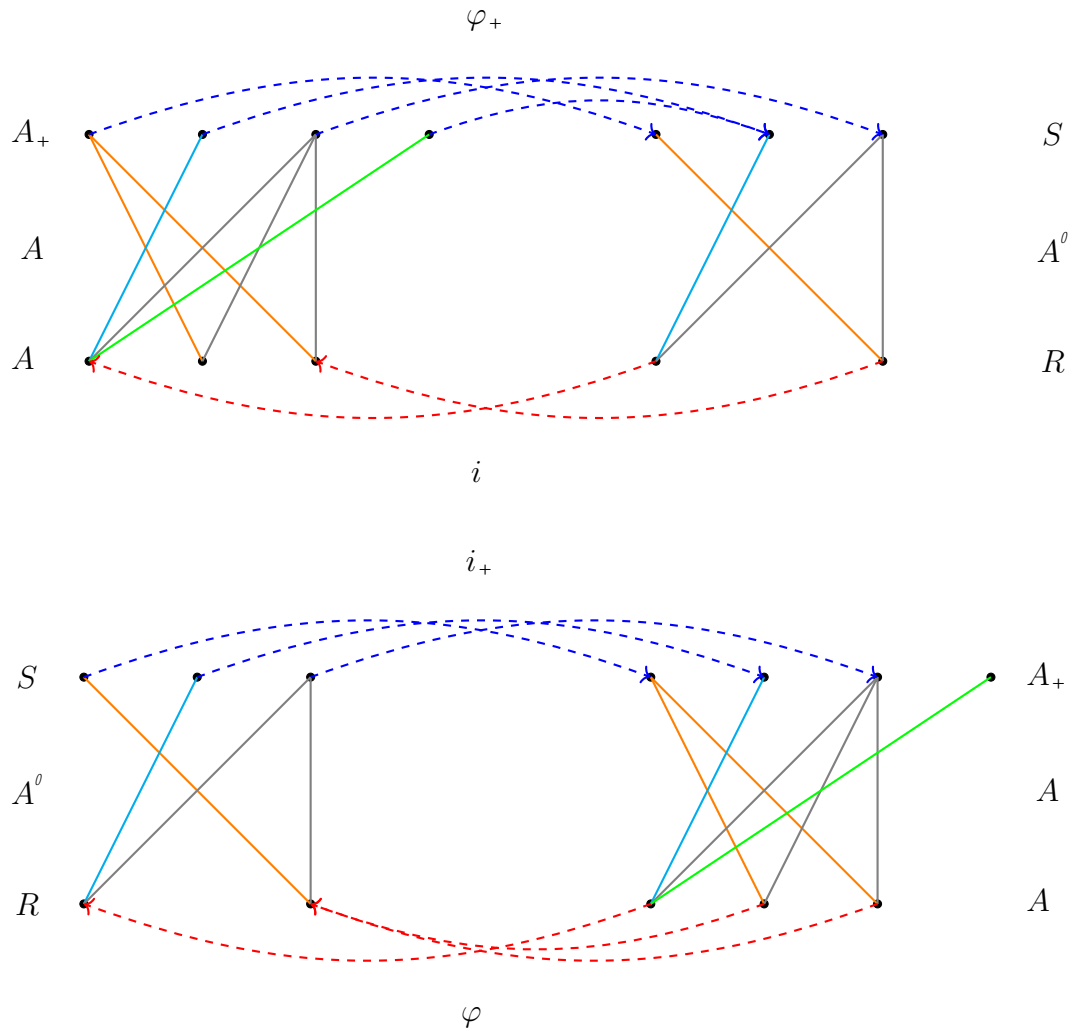


Figure 3.7: Illustration of Lemma 3.2.7. Here we delete twin points from A to obtain A^0 and find morphisms from A to A^0 and from A^0 to A .

3.3 Skeleton Bimorphic Form

The operations of deleting twins, deleting non-minimal points from A , and deleting non-maximal points from A_+ allow us to transform a relation into one which is less complex, but still bimorphic with the original. We can then repeat these operations until they are no longer applicable to obtain a bimorphic form of “minimal” size.

The question arises whether the order of these operations matters. The status of the minimality or maximality of a point can change if other points are deleted first. However, the next two lemmas state that if a non-minimal or non-maximal point becomes minimal or maximal through the deletion of other points, it will become the twin of an existing point. In particular, once a point becomes “deletable,” it stays deletable.

Lemma 3.3.1. *Let $\mathbf{A} = (A_-, A_+, A)$ be a relation and let $x \in A_-$ be a non \mathbf{A} -minimal point. Let $\mathbf{A}^\circ = \mathbf{A}[A_-, A_+ \cap \bar{f}yg]$, for some $y \in A_+$. Then x is either non \mathbf{A}° -minimal or is an \mathbf{A}° -twin of an \mathbf{A} -minimal point.*

Proof. Because x is non \mathbf{A} -minimal, there exists $z \in A_-$ such that $N_{\mathbf{A}}(z) \subset N_{\mathbf{A}}(x)$. Without loss of generality, suppose z is \mathbf{A} -minimal. Suppose $x: Ay$. Then $z: Ay$. Then $N_{\mathbf{A}^\circ}(x) = N_{\mathbf{A}}(x)$ and $N_{\mathbf{A}^\circ}(z) = N_{\mathbf{A}}(z)$. Then $N_{\mathbf{A}^\circ}(z) \subset N_{\mathbf{A}^\circ}(x)$ and x is non \mathbf{A}° -minimal. Now suppose that xAy . Then either zAy or $z: Ay$.

First, suppose zAy . Then $N_{\mathbf{A}^\circ}(x) = N_{\mathbf{A}}(x) \cap \bar{f}yg$ and $N_{\mathbf{A}^\circ}(z) = N_{\mathbf{A}}(z) \cap \bar{f}yg$. Then $N_{\mathbf{A}^\circ}(z) \subset N_{\mathbf{A}^\circ}(x)$ and x is non \mathbf{A}° -minimal.

Second, suppose $z: Ay$. Then $N_{\mathbf{A}^\circ}(x) = N_{\mathbf{A}}(x) \cap \bar{f}yg$ and $N_{\mathbf{A}^\circ}(z) = N_{\mathbf{A}}(z)$. Then $N_{\mathbf{A}^\circ}(z) \subset N_{\mathbf{A}^\circ}(x)$. Then either $N_{\mathbf{A}^\circ}(z) \subset N_{\mathbf{A}^\circ}(x)$ and x is non \mathbf{A}° -minimal or $N_{\mathbf{A}^\circ}(z) = N_{\mathbf{A}^\circ}(x)$ and x is an \mathbf{A}° -twin of an z . \square

Lemma 3.3.2. *Let $\mathbf{A} = (A_-, A_+, A)$ be a relation and let $x \in A_+$ be a non \mathbf{A} -maximal point. Let $\mathbf{A}^\circ = \mathbf{A}[A_- \cap \bar{f}yg, A_+]$, for some $y \in A_-$. Then x is either non \mathbf{A}° -maximal or is an \mathbf{A}° -twin of an \mathbf{A} -maximal point.*

Proof. Because x is non \mathbf{A} -maximal, there exists $z \in A_+$ such that $N_{\mathbf{A}}(z) \supset N_{\mathbf{A}}(x)$. Without loss of generality, suppose z is \mathbf{A} -maximal. Suppose $y: Az$. Then $y: Ax$.

Then $N_{\mathbf{A}^\circ}(x) = N_{\mathbf{A}}(x)$ and $N_{\mathbf{A}^\circ}(z) = N_{\mathbf{A}}(z)$. Then $N_{\mathbf{A}^\circ}(z) \supset N_{\mathbf{A}^\circ}(x)$ and x is non \mathbf{A}° -maximal. Now suppose that yAz . Then either yAx or $y: Ax$.

First, suppose yAx . Then $N_{\mathbf{A}^\circ}(x) = N_{\mathbf{A}}(x) \cap \bar{y}yg$ and $N_{\mathbf{A}^\circ}(z) = N_{\mathbf{A}}(z) \cap \bar{y}yg$. Then $N_{\mathbf{A}^\circ}(z) \supset N_{\mathbf{A}^\circ}(x)$ and x is non \mathbf{A}° -maximal.

Second, suppose $y: Ax$. Then $N_{\mathbf{A}^\circ}(x) = N_{\mathbf{A}}(x)$ and $N_{\mathbf{A}^\circ}(z) = N_{\mathbf{A}}(z) \cap \bar{y}yg$. Then $N_{\mathbf{A}^\circ}(z) \supset N_{\mathbf{A}^\circ}(x)$. Then either $N_{\mathbf{A}^\circ}(z) \supset N_{\mathbf{A}^\circ}(x)$ and x is non \mathbf{A}° -maximal or $N_{\mathbf{A}^\circ}(z) = N_{\mathbf{A}^\circ}(x)$ and x is an \mathbf{A}° -twin of an z . \square

Putting this all together, we obtain a simple algorithm for finding a less complex bimorphic form of a relation.

Definition 3.3.3. The **skeleton bimorphic form** of a relation \mathbf{A} is the relation resulting from the following algorithm:

1. Create an induced subrelation by retaining all the \mathbf{A} -maximal points in A_+ and all the \mathbf{A} -minimal points in A_- .
2. Repeat Step 1 until every point in A_+ is \mathbf{A} -maximal and every point in A_- is \mathbf{A} -minimal.
3. Create an induced subrelation by retaining a single representative for each set of twins.

The simplifying power of this algorithm from Definition 3.3.3 can be seen in Figure 3.8. The algorithm's usefulness lies in the fact that each operation yields a relation that is bimorphic with the original, justifying the name.

Lemma 3.3.4. *A relation \mathbf{A} is bimorphic with its skeleton bimorphic form.*

Proof. By Lemmas 3.2.3 and 3.2.6, there is a morphism from \mathbf{A} to the relation resulting from Step 1 of Definition 3.3.3. By Lemmas 3.2.5 and 3.2.4, there is a morphism from the relation resulting from Step 1 of Definition 3.3.3 to \mathbf{A} . Then they are bimorphic. Then by induction, \mathbf{A} is bimorphic with the relation resulting after Step 2. Finally, by Lemma 3.2.7, \mathbf{A} is bimorphic with the relation resulting after Step 3. \square

3.4 Special Cases

This section will explore the morphism decision problem for special cases of \mathbf{A} and \mathbf{B} . The case when $\delta(\mathbf{B}) = 1$ turns out to be very simple.

Lemma 3.4.1. *Let \mathbf{A}, \mathbf{B} be finite relations. If $\delta(\mathbf{B}) = 1$ then there exists a morphism from \mathbf{A} to \mathbf{B} .*

Proof. Let φ_+ map all points in A_+ to the element of a minimal \mathbf{B} -dominating family. Then $bB\varphi_+(a)$ is true for all $b \in B$ and $a \in A_+$. Then the implication $\varphi_-(b)Aa \Rightarrow bB\varphi_+(a)$ is true for all $\varphi_-(b) \in A$, $a \in A_+$, and $b \in B$. Then for any $\varphi_- : B \rightarrow A$, $\varphi = (\varphi_-, \varphi_+)$ is a morphism from \mathbf{A} to \mathbf{B} . \square

Figure 3.9 example illustrates Lemma 3.4.1. Note that in this example, $\delta(\mathbf{A}) = 2$ and $\delta(\mathbf{B}) = 1$. If φ_+ maps all values of A_+ to the single dominating element in B_+ , then the morphism is guaranteed to work because the consequent $bB\varphi_+(a)$ of the implication $\varphi_-(b)Aa \Rightarrow bB\varphi_+(a)$ is always true. This can also be seen by examining the dual, in which case the antecedent $\varphi_+(a): \check{B}b$ of the implication is always false, guaranteeing that the implication $\varphi_+(a): \check{B}b \Rightarrow a: \check{A}\varphi_-(b)$ will be true.

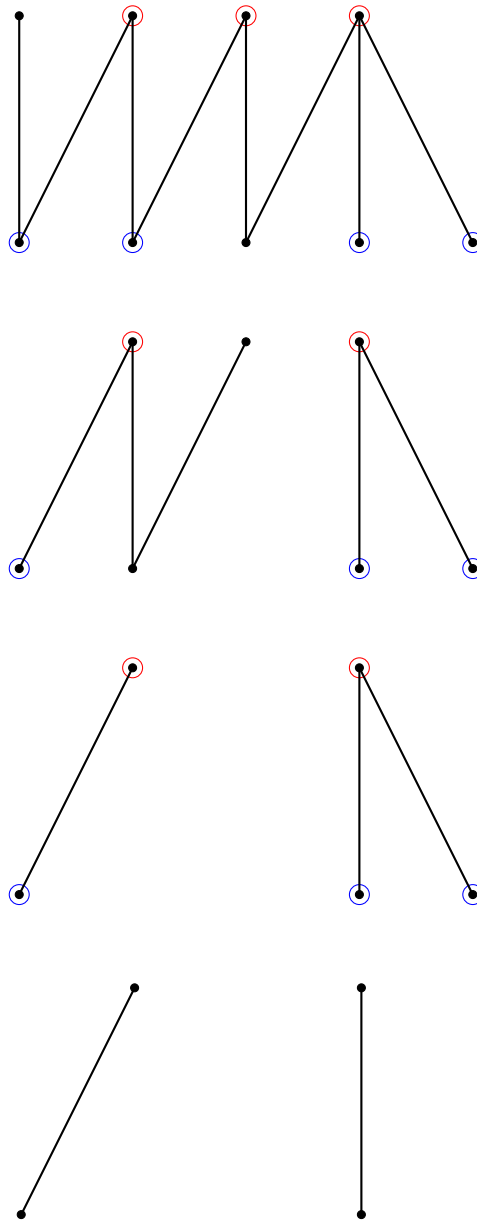


Figure 3.8: Finding the Skeleton Bimorphic Form of a Relation. Maximal points in A_+ are circled in red. Minimal point in A_- are circled in blue.

3.4.1 Ladder Relations

For a given dominating number, there is a certain class of special skeleton bimorphic forms whose bipartite graphs are shaped like a ladder, motivating the Definition

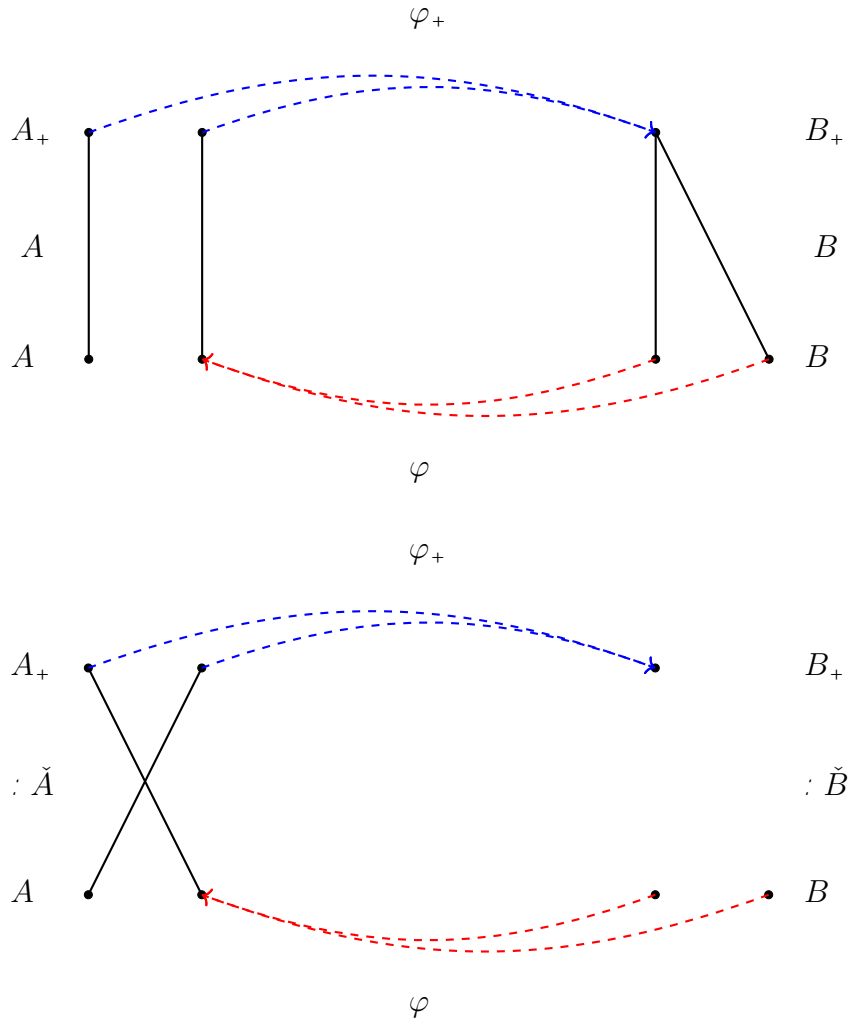


Figure 3.9: When $\delta(B) = 1$, there is a morphism from A to B .

3.4.2.

Definition 3.4.2. A ladder relation L is a finite relation such that for each $x \in L$ there exists a unique $y \in L_+$ such that xLy and for each $y \in L_+$ there exists a unique $x \in L$ such that xLy . A ladder relation with dominating number n is referred to as an **n-ladder**.

The relation A in Figure 3.9 is an example of a 2-ladder.

Ladder relations are special in the sense that they can morphism onto any relation that shares their dominating number. In this way they are also the least complex relation with regards to the morphism relation for a given dominating number. Figure 3.10 provides an illustration.

Lemma 3.4.3. *Let \mathbf{A} be a ladder relation and \mathbf{B} be a relation such that $\delta(\mathbf{A}) = \delta(\mathbf{B})$. There is a morphism from \mathbf{A} to \mathbf{B} .*

Proof. Let $D_{\mathbf{B}}$ be a minimal \mathbf{B} -dominating family and let $\varphi_+ : A_+ \rightarrow D_{\mathbf{B}}$ be a bijection. For each $b \in B$, $bB\beta$ for some $\beta \in D_{\mathbf{B}}$. Define $\varphi(b) = a$ for $a \in A$ such that $aA(\varphi_+)^{-1}(\beta)$.

Suppose $\varphi(b)A\alpha$ for some $b \in B$ and $\alpha \in A_+$. Then $\varphi(b)A(\varphi_+)^{-1}(\beta)$ for some $\beta \in D_{\mathbf{B}}$. By definition of φ , $bB\beta$. Then $bB\varphi_+(\alpha)$, which implies that (φ, φ_+) is a morphism from \mathbf{A} to \mathbf{B} . □

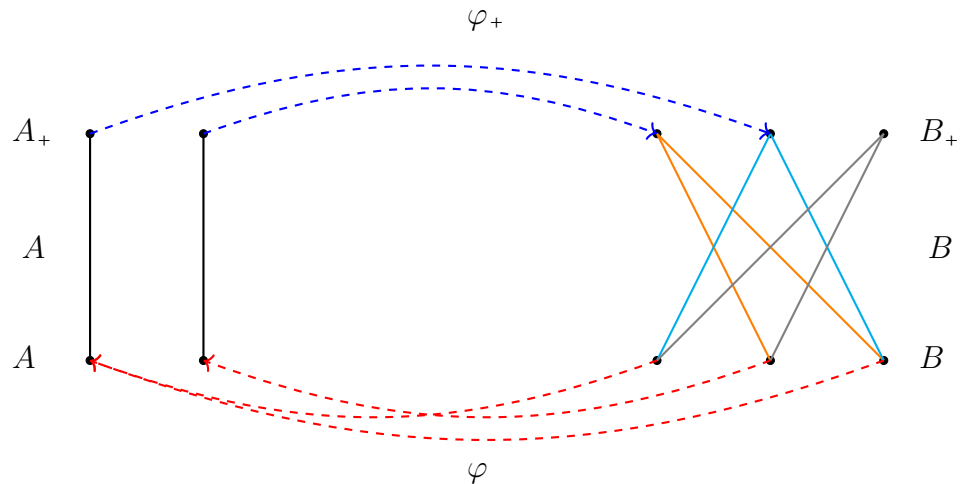


Figure 3.10: Example of Lemma 3.4.3. There is a morphism from the 2-ladder to any relation with dominating number 2.

Ladder relations also morphism onto other ladder relations of lesser size. An example can be seen in Figure 3.11. Then by transitivity, ladders morphism onto any relation with a dominating number less than or equal to their own.

Lemma 3.4.4. *If \mathbf{A} and \mathbf{B} are ladder relations such that $\delta(\mathbf{A}) \geq \delta(\mathbf{B})$, then there is a morphism from \mathbf{A} to \mathbf{B} .*

Proof. Let $\varphi : B \rightarrow A$ be an injective function. For each $\alpha \in A_+$, define $\varphi_+(\alpha)$ as follows: For the unique $a \in A$ such that $aA\alpha$, if $a = \varphi(b)$ for some $b \in B$, then define $\varphi_+(\alpha) = \beta$, where $bB\beta$. Otherwise, α can be mapped to any point in B_+ .

Suppose $\varphi(b)A\alpha$ for some $b \in B$ and $\alpha \in A_+$. Then by definition of φ_+ , $\varphi_+(\alpha) = \beta$, where $bB\beta$. Then $bB\varphi_+(\alpha)$, which implies that (φ, φ_+) is a morphism from \mathbf{A} to \mathbf{B} . \square

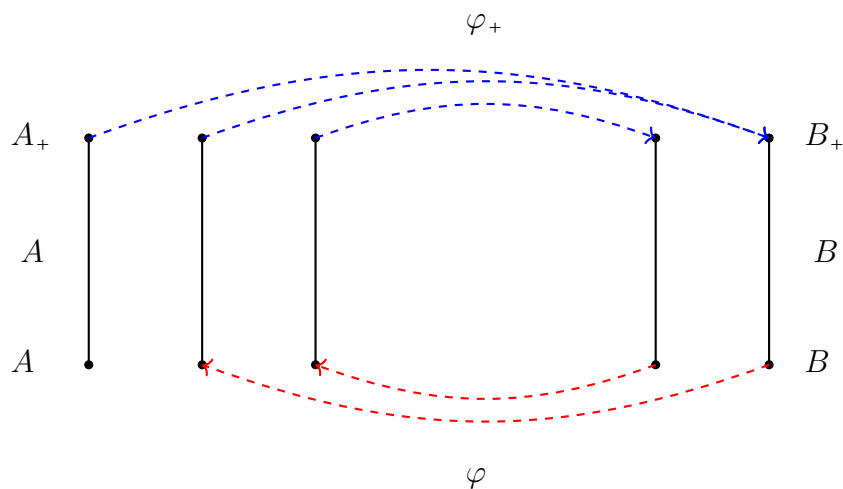


Figure 3.11: Example of Lemma 3.4.4. There is a morphism from the 3-ladder to the 2-ladder because $3 \geq 2$.

2-Ladders

The 2-ladder is particularly interesting because it is isomorphic to its dual. This leads to some interesting results that suggest the centrality of the 2-ladder in finite relations.

Lemma 3.4.5. *Let \mathbf{A} be a relation. If $\delta(\mathbf{A}^\circ) = 2$, then there is a morphism from \mathbf{A} to the 2-ladder.*

Proof. Let \mathbf{B} be a 2-ladder. Note that \mathbf{B}° is also a 2-ladder. Then by Lemma 3.4.3, there is a morphism from \mathbf{B}° to \mathbf{A}° . Then there exists a morphism from \mathbf{A} to \mathbf{B} . \square

In particular, this implies that any relation with a dominating number of 2 that has a dual relation with dominating number of 2 is bimorphic with the 2-ladder.

Corollary 3.4.6. *Let \mathbf{A} be a relation. If $\delta(\mathbf{A}) = \delta(\mathbf{A}^\circ) = 2$, then \mathbf{A} is bimorphic with the 2-ladder.*

Proof. Let \mathbf{B} be a 2-ladder. By Lemma 3.4.3, there is a morphism from \mathbf{B} to \mathbf{A} . By Lemma 3.4.5, there is a morphism from \mathbf{A} to \mathbf{B} . \square

Another implication of Lemma 3.4.5 is that \mathbf{A} is comparable via morphism to \mathbf{A}° as long as at least one of them have a dominating number of 2.

Corollary 3.4.7. *For a relation \mathbf{A} , if $\delta(\mathbf{A}^\circ) = 2$, then there is a morphism from \mathbf{A} to \mathbf{A}° .*

Proof. By Lemma 3.4.5, there is a morphism from \mathbf{A} to the 2-ladder. By Lemma 3.4.3 there is a morphism from a 2-ladder to \mathbf{A}° . Then by transitivity, there is a morphism from \mathbf{A} to \mathbf{A}° . \square

3.4.2 Further Special Results on Morphism Existence

Informally, the idea behind Lemma 3.4.8 is that if φ has to be a non-surjective function, then the points that aren't included in the image of φ can be thought of as having been deleted.

Lemma 3.4.8. *Let \mathbf{A} and \mathbf{B} be relations such that $jA = jB$. There exists a morphism from \mathbf{A} to \mathbf{B} if and only if there exists $\mathbf{A}^\circ = \mathbf{A}[C, A_+]$, where $C \subseteq A$ and $jC = jB$, such that there exists a morphism from \mathbf{A}° to \mathbf{B} .*

Proof. Suppose $\varphi = (\varphi_-, \varphi_+)$ is a morphism from \mathbf{A} to \mathbf{B} . Note that $j\varphi(B) = jB$, so $\varphi(B) \subseteq C$ for some $C \subseteq A$ and $jC = jB$. Then let $\mathbf{A}^\circ = \mathbf{A}[C, A_+]$ and let $\varphi^\circ : B \rightarrow C$ be defined as $\varphi^\circ(x) = \varphi(x)$ for all $x \in B$. Then $(\varphi^\circ, \varphi_+)$ is obviously a morphism from \mathbf{A}° to \mathbf{B} .

Now suppose that there exists $\mathbf{A}^\circ = \mathbf{A}[C, A_+]$, where $C \subseteq A$ and $jC = jB$, such that there exists a morphism from \mathbf{A}° to \mathbf{B} . By Lemma 3.2.3, there is a morphism from \mathbf{A} to \mathbf{A}° . Then by transitivity, there is a morphism from \mathbf{A} to \mathbf{B} . \square

We can now prove that the relations in Figure 2.5 provide a counter example to the conjecture that $\delta(\mathbf{A}) \wedge \delta(\mathbf{B}) \wedge \delta(\mathbf{A}^\circ) \wedge \delta(\mathbf{B}^\circ) \Rightarrow \mathbf{A} \rightarrow \mathbf{B}$. As can be seen in Figure 3.12, each subrelation of \mathbf{A} that retains 3 points in A has a dominating number < 3 . Then by Corollary 2.1.8 there does not exist a morphism from any of these subrelations to \mathbf{B} . Then by Lemma 3.4.8, there cannot exist a morphism from \mathbf{A} to \mathbf{B} .

Lemma 3.4.9 provides a sufficient condition for a morphism to exist. Informally, the idea behind this lemma is that if two larger relations are made up of smaller, disjoint relations, then the larger relations will have a morphism if the smaller ones do.

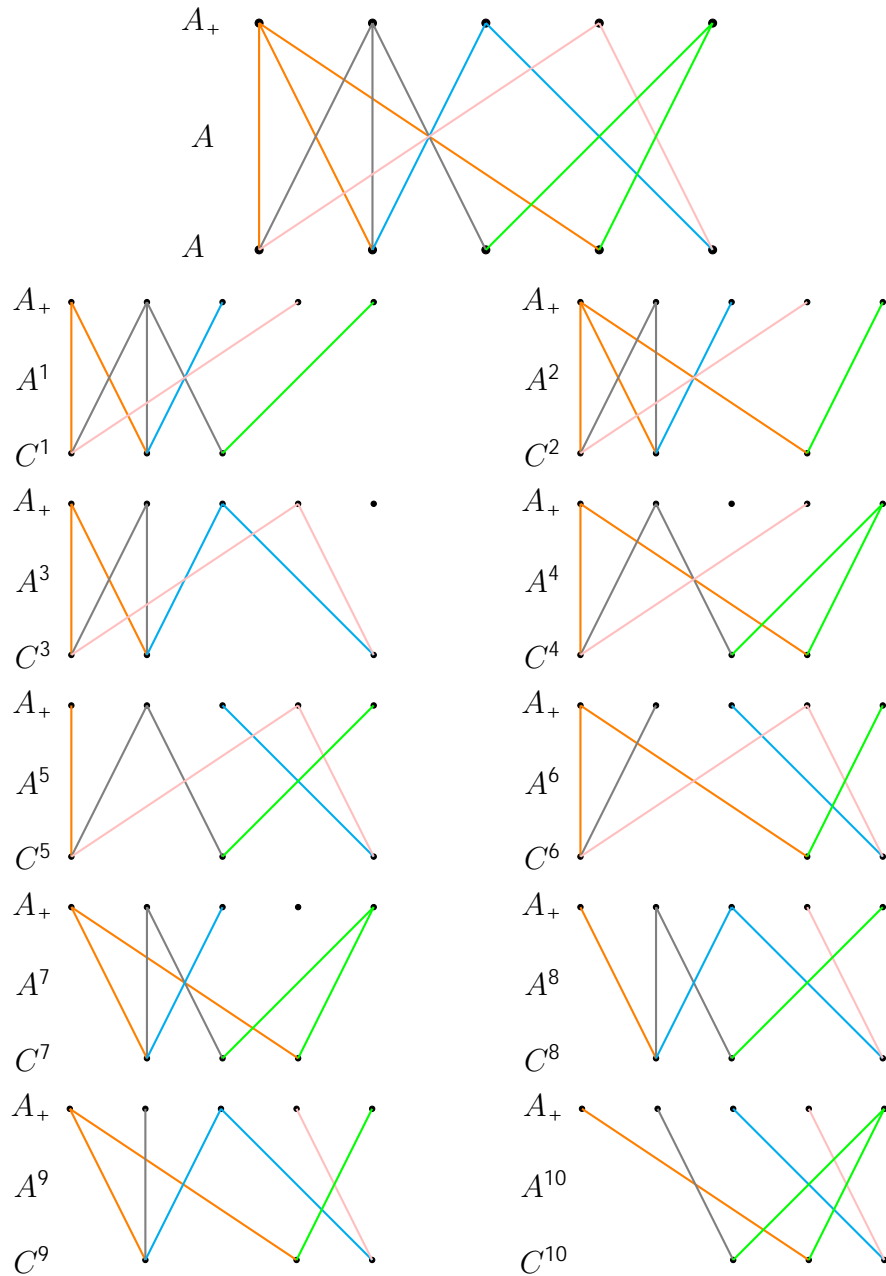


Figure 3.12: Counter Example from Figure 2.5 continued. Each subrelation has a dominating number < 3 .

Lemma 3.4.9. *Let $A, B, C,$ and D be disjoint relations, and let X and Y be relations such that $X = A \upharpoonright C$, $X_+ = A_+ \upharpoonright C_+$, $X = A \upharpoonright C$, $Y = B \upharpoonright D$, $Y_+ = B_+ \upharpoonright D_+$,*

$Y = B \sqcup D$. If there exists a morphism from \mathbf{A} to \mathbf{B} and from \mathbf{C} to \mathbf{D} , then there exists a morphism from \mathbf{X} to \mathbf{Y} .

Proof. Suppose $\varphi = (\varphi_-, \varphi_+)$ is a morphism from \mathbf{A} to \mathbf{B} and $\vartheta = (\vartheta_-, \vartheta_+)$ is a morphism from \mathbf{C} to \mathbf{D} . We will show that $\varsigma = (\varsigma_-, \varsigma_+)$, where $\varsigma_- = \varphi_- \sqcup \vartheta_-$ and $\varsigma_+ = \varphi_+ \sqcup \vartheta_+$, is a morphism from \mathbf{X} to \mathbf{Y} : Suppose for some $y \in Y$ and $x \in X_+$ that $\varsigma_-(y) \neq x$. Then either $y \in B$, $x \in A_+$ and $\varphi_-(x) = y$ or $y \in D$, $x \in C_+$ and $\vartheta_-(x) = y$. This implies either $x \in \varphi_+^{-1}(y)$ or $x \in \vartheta_+^{-1}(y)$. Either way, $x \in \varsigma_+^{-1}(y)$. \square

CHAPTER 4

CLASSIFICATION OF SMALL FINITE RELATIONS

Definition 4.0.1. The set of relations with m elements in the domain and n elements in the codomain will be denoted as $\mathbf{R}_{m,n}$. i.e.

$$\mathbf{R}_{m,n} = \{f(A_-, A_+, A) : |A_-| = m \wedge |A_+| = n\}$$

We observe that if $m' \leq m$ and $n' \leq n$, for all $\mathbf{A} \in \mathbf{R}_{m',n'}$, there exists $\mathbf{A}^0 \in \mathbf{R}_{m,n}$ such that \mathbf{A} and \mathbf{A}^0 are bimorphic. This is because a smaller relation can be “padded” with twins to create a morphism that is bimorphic with the original (see Lemma 3.2.7). Rather than study morphisms between all relations in $\mathbf{R}_{m',n'}$ such that $m' \leq m$ and $n' \leq n$, it suffices to study the relations just in $\mathbf{R}_{m,n}$.

4.1 $\mathbf{R}_{5,5}$ and Computation

This section focuses on classifying the morphism problem for all relations in $\mathbf{R}_{m,n}$ such that $m \leq 5$ and $n \leq 5$. Even though we could just consider the relations in $\mathbf{R}_{5,5}$, we include the smaller relations due to the ease of computation. The computation is done using the R programming language. The full code is included in Appendix A.

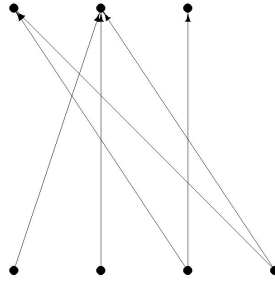
In the code, we represent the relations primarily as *incidence matrices*. For a given $\mathbf{A} = (A_-, A_+, A)$, the elements of A_- are represented as rows and the elements

of A_+ as columns of the matrix. For each element x_{ij} of the matrix, $x_{ij} = 1$ if the corresponding elements of A and A_+ relate to each other. Otherwise, $x_{ij} = 0$. For example, the code generates the following 4×3 matrix, representing a relation with 4 elements in A and 3 elements in A_+ :

```
> matrices_1to5_x_1to5$Size_4x3[[101]]
      [,1] [,2] [,3]
[1,]    0    1    0
[2,]    0    1    0
[3,]    1    0    1
[4,]    1    1    0
```

We can also convert these matrix objects to directed graph objects. This is helpful for visualization and for checking for isomorphisms between relations. Using the same matrix from the previous example:

```
> graph_from_matrix(matrices_1to5_x_1to5$Size_4x3[[101]])
IGRAPH 59f9fa7 D--B 7 6 --
+ attr: type (v/l)
+ edges from 59f9fa7:
[1] 1->6 2->6 3->5 3->7 4->5 4->6
```



Using this framework, we generated all finite relations in $\mathbf{R}_{m,n}$ such that $m = 5$ and $n = 5$. We then ran all of these relations through the algorithm defined in Definition 3.3.3 to get their corresponding skeleton bimorphic form. This resulted in 32 unique (up to isomorphism) relations. The graphical output for each of these can be seen in Table 4.1.

This set of 32 is closed with respect to the dual operation, that is, for each \mathbf{A} in the set, \mathbf{A}° is also in the set. This is explained by Lemma 2.2.7 and Corollary 2.2.8, which state that points in A_- that are \mathbf{A} -minimal are \mathbf{A}° -maximal and points in A_+ that are \mathbf{A} -maximal are \mathbf{A}° -minimal (and vice versa).

For each relation in Table 4.1, the dual and the associated dominating numbers are included in Table 4.2. This is output from the code.

To check for morphisms between each of the 32 relations, we ran the following algorithm for each pair of relations. The corresponding function in R code is called *morphism_from_AtoB* and can be seen in Appendix A.

1. Input two adjacency matrices, \mathbf{A} and \mathbf{B} , representing the relations. (Rows of \mathbf{A} represent the elements of A_- , Columns of \mathbf{A} represent the elements of A_+ .)
2. Find all the possible functions $\varphi : B \rightarrow A_-$ by getting all permutations (with repetition) of the rows of \mathbf{A} of length $|B|$.

3. Find all the possible functions $\varphi_+ : A_+ \rightarrow B_+$ by getting all permutations (with repetition) of the columns of \mathbf{B} of length $|A_+|$.
4. For each combination of φ_- and φ_+ , do the following:
 - (a) Create a $|B| \times |A_+|$ matrix representing the relation $\varphi_-(b_i)Aa_j$. Call this matrix \mathbf{M}_1 .
 - (b) Create a $|B| \times |A_+|$ matrix representing the relation $b_iB\varphi_+(a_j)$. Call this matrix \mathbf{M}_2 .
 - (c) Compare \mathbf{M}_1 and \mathbf{M}_2 coordinate-wise. If each element of \mathbf{M}_2 is greater than or equal to its corresponding element of \mathbf{M}_1 , then (φ_-, φ_+) is a morphism from \mathbf{A} to \mathbf{B} .

Each pair of \mathbf{A} and \mathbf{B} such that there is a morphism from \mathbf{A} to \mathbf{B} is output from the code and can be seen in Appendix B.

The Hasse Diagram in Figure 4.1 is created by using the output in Appendix B. A line that goes *downward* from a relation \mathbf{A} to a relation \mathbf{B} signifies that there exists a morphism from \mathbf{A} to \mathbf{B} . Relations that are bimorphic are included on the same line and are separated by comma. The dominating number information from Table 4.2 is also included.

4.2 Discussion of Hasse Diagram

The Hasse Diagram allows us to observe experimentally many of the behaviors that were described in Chapter 3.

As stated in Lemma 3.4.4, ladders of larger size morphism onto ladders of smaller size. As stated in Lemma 3.4.3, ladders morphism onto relations that have the same dominating number.

As stated in Lemma 3.4.5, if $\delta(\mathbf{A}^?) = 2$ then \mathbf{A} morphisms onto the 2-ladder and, as stated in Corollary 3.4.7, \mathbf{A} morphisms onto $\mathbf{A}^?$. As stated in Corollary 3.4.6, all relations \mathbf{A} such that $\delta(\mathbf{A}) = \delta(\mathbf{A}^?) = 2$ are bimorphic with the 2-ladder. These results, along with duality, provide explanation for the symmetry of the diagram.

The 3-ladder (4) is bimorphic with 16. This was anticipated by Lemma 3.4.9 since $4 = 3 \uparrow 2$ and $16 = 9 \uparrow 2$ (note that 3 and 9 are bimorphic).

Question 4.2.1. *Can we find general lemmas that justify the existence (and non-existence) of all the morphisms in the Hasse diagram?*

Question 4.2.2. *What does the classification look like for larger relations?*

Question 4.2.3. *What structural properties does the morphism order satisfy? (infinite chains, antichains, etc.)*

Question 4.2.4. *How does the number of skeleton bimorphic relations grow in $\mathbf{R}_{k,k}$ for $k > 5$?*

Question 4.2.5. *Can we find a "simpler" bimorphic form? i.e. one that has the property that if \mathbf{A} and \mathbf{B} are bimorphic then they are isomorphic? What is the algorithm to arrive at such a form?*

Question 4.2.6. *What is the distribution of dominating numbers for finite relations up to a certain size? What is the bivariate distribution of $\delta(\mathbf{A})$ and $\delta(\mathbf{A}^?)$?*

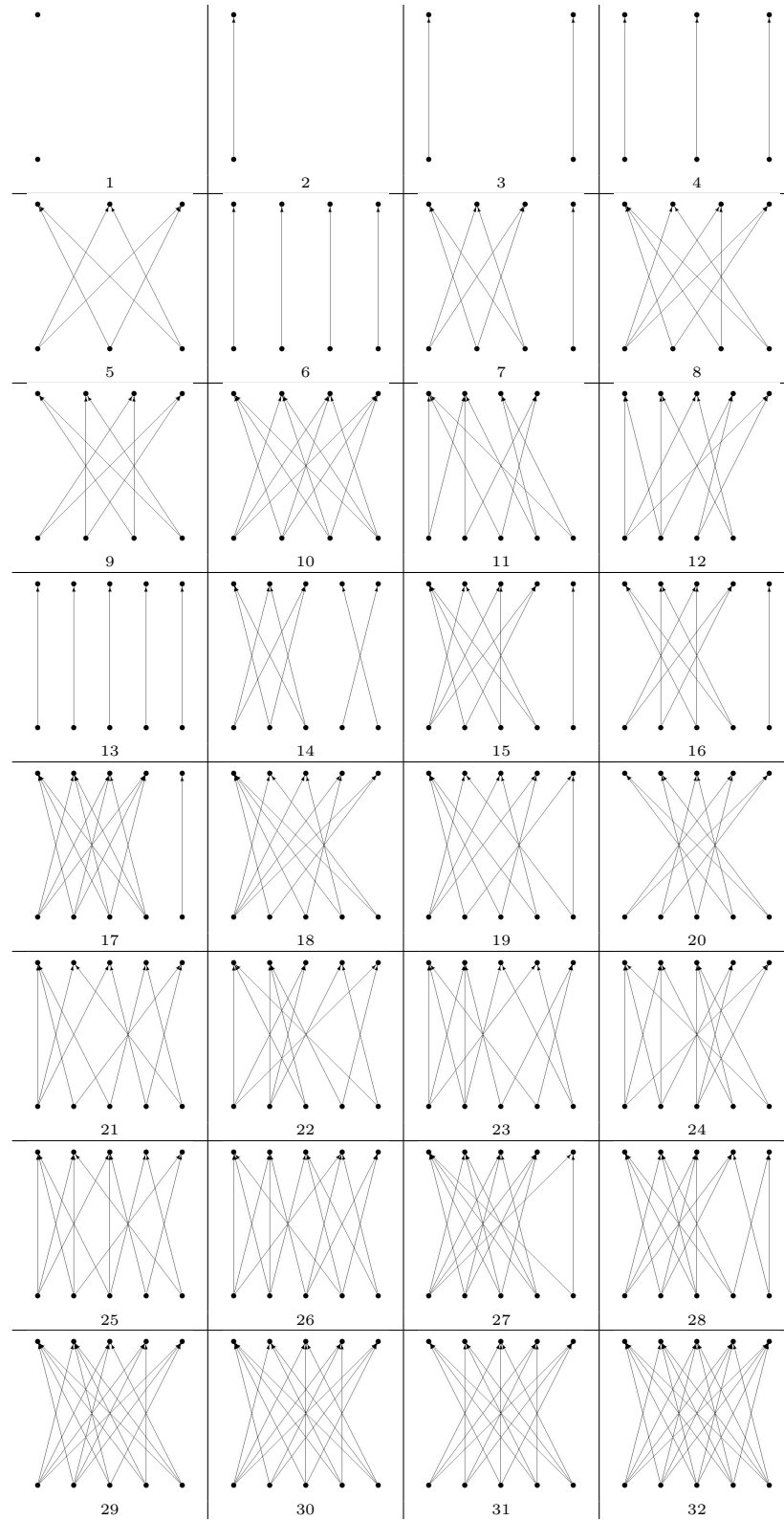
Table 4.1: $R_{5,5}$ Skeleton Bimorphic Forms

Table 4.2: Duals and Dominating Numbers for $R_{5,5}$ Skeleton Bimorphic Forms

A	$A^?$	$\delta(A)$	$\delta(A^?)$
1	2	7	1
2	1	1	7
3	3	2	2
4	5	3	2
5	4	2	3
6	10	4	2
7	8	3	2
8	7	2	3
9	9	2	2
10	6	2	4
11	12	2	2
12	11	2	2
13	32	5	2
14	29	4	2
15	27	3	2
16	30	3	2
17	18	3	2
18	17	2	3
19	28	2	2
20	31	3	2
21	26	3	2
22	25	2	2
23	24	3	2
24	23	2	3
25	22	2	2
26	21	2	3
27	15	2	3
28	19	2	2
29	14	2	4
30	16	2	3
31	20	2	3
32	13	2	5

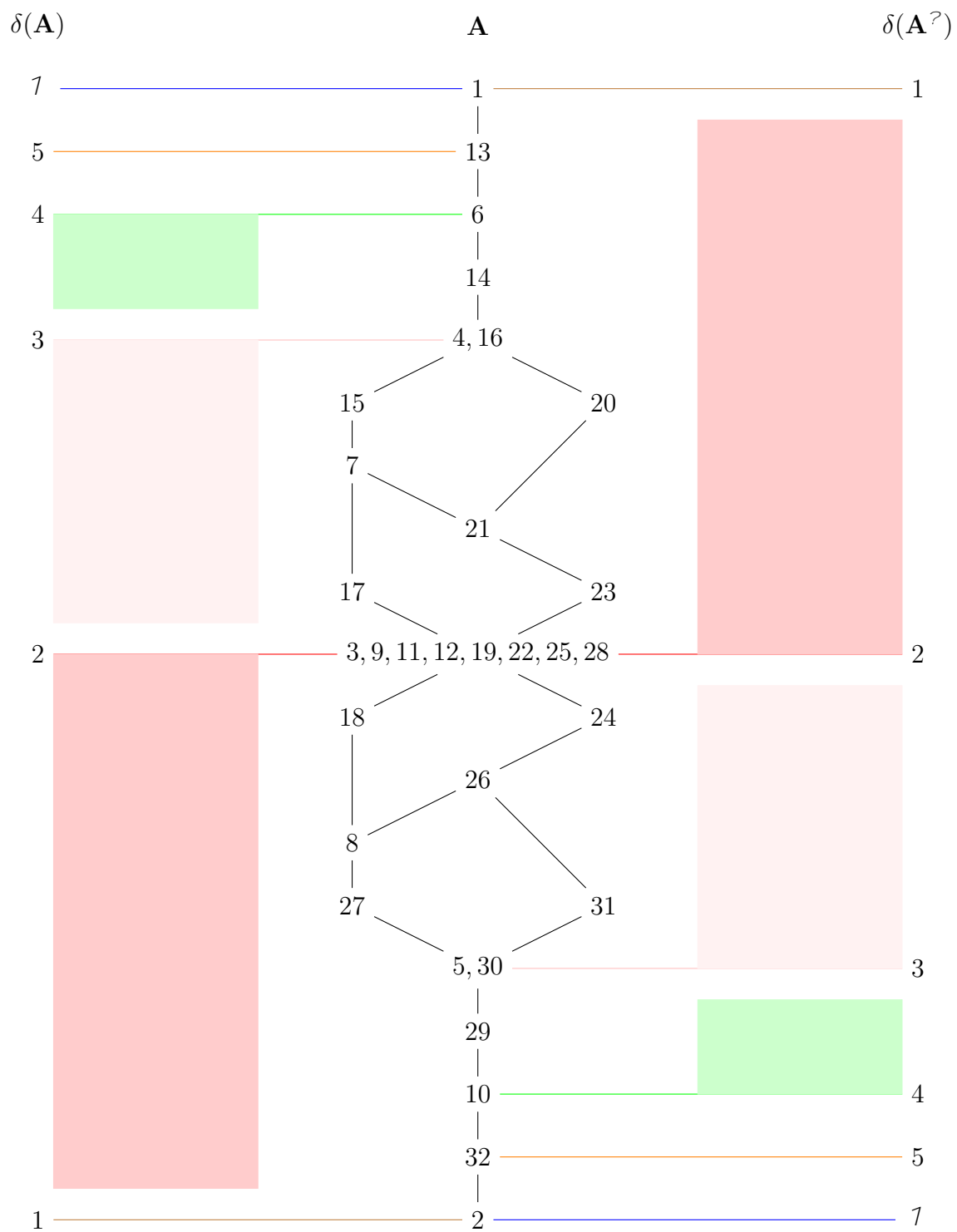


Figure 4.1: Hasse Diagram of $R_{5,5}$ Skeleton Bimorphic Forms

REFERENCES

- [1] Andreas Blass. *Combinatorial Cardinal Characteristics of the Continuum*. In: *Foreman M., Kanamori A. (eds)*. Handbook of Set Theory. Springer, Dordrecht., 2010.
- [2] Natasha Dobrinen and Stevo Todorćević. Tukey types of ultrafilters, 2011.
- [3] M. Ern , J. Koslowski, A. Melton, and G. Strecker. A primer on galois connections. *Annals of the New York Academy of Sciences*, 704, 1993.
- [4] Slawomir Solecki and Stevo Todorćević. Cofinal types of topological directed orders. *Annales de l'Institut Fourier*, 54(6):1877–1911 (2005), 2004.
- [5] John W. Tukey. *Convergence and Uniformity in Topology*. (AM-2). Princeton University Press, 1940.

APPENDIX A

R CODE

```

install.packages(c("igraph", "R.utils", "gtools", "data.table", "RcppGreedySetCover"))

#Define Functions

graph_from_matrix <- function(M, plotGraph = T, graphName = "", saveGraph = F) {
  #Convert incidence matrices to graphs
  library(igraph)
  graphs_temp <- as.directed(graph.incidence(M), mode = "arbitrary")
  if(plotGraph == T) {
    if(length(M) == 1) {
      i <- 1
      j <- 1
    } else {
      i <- nrow(M)
      j <- ncol(M)
    }
    layout <- matrix(c(seq(0, i-1), seq(0, j-1), rep(0, i), rep(1, j)), byrow = F, nrow = i+j, ncol = 2)
    par(mar = c(0.5, 0.5, 0.5, 0.5))
    plot(graphs_temp, layout = layout, vertex.color = 'black',
         vertex.label = NA, vertex.size = 7,
         edge.arrow.size=1, edge.color = 'black', main = paste(graphName, sep=''))
  }
  if(saveGraph == T) {
    assign(graphName, graphs_temp, envir = .GlobalEnv)
  }
  return(graphs_temp)
}

```

```

generate_finite_relations <- function(Usizes, Vsizes,
                                     saveMatrices = T, saveGraphs = T,
                                     matricesListName = 'matrices', graphsListName = 'graphs'
) {
  library(R.utils)
  #Generate finite relations of a given size
  library(igraph)
  library(gtools)
  grid <- expand.grid(Usizes, Vsizes)
  #Create all possible binary vectors of length j
  #2^j possibilities at each step
  perm <- lapply(X = Vsizes, function(k) permutations(n= 2, r = k, v = 0:1,
                                                    repeats.allowed = TRUE))

  #Get "all" i x j matrices by combining (with replacement)
  #the vectors from the previous step i times
  #The zero vectors can be left out, since we are only interested in relations
  #that have dominating families
  #((2^j-1)+i-1)!/(i!((2^j-1)-1)!) possibilities at each step
  combi <- lapply(X = 1:nrow(grid),
                 function(k) combinations(n = nrow(perm[[match(grid[k, 2], Vsizes)]])-1,
                                         r = grid[k, 1],
                                         v = 2:nrow(perm[[match(grid[k, 2], Vsizes)]]),
                                         repeats.allowed = TRUE))

  matrix_temp <- lapply(X = 1:length(combi),
                      function(k) lapply(X = 1:nrow(combi[[k]]),
                                         function(n) matrix(perm[[match(grid[k, 2], Vsizes)]]
                                                             [combi[[k]][n, ]],
                                                             byrow = F, nrow = grid[k, 1],
                                                             ncol = grid[k, 2])))

  if(saveMatrices == T) {
    names(matrix_temp) <- lapply(X = 1:nrow(grid),
                                function(k) paste0("Size_", grid[k, 1], "x", grid[k, 2]))
    matrix_name <- paste(matricesListName)
    assign(matrix_name, matrix_temp, envir = .Global Env)
  }

  if(saveGraphs == T) {
    graphs_temp <- lapply(X = matrix_temp,
                        function(M) as.directed(graph.incidence(M), mode = "arbitrary"))
  }
}

```



```

graphs_name <- paste(graphsListName)
assign(graphs_name, graphs_temp, envir = .GlobalEnv)
}
}

count_of_graph_generation <- function(Usizes, Vsizes) {
  n <- 0
  for(j in Vsizes){
    #2^j possibilities at each step
    for(i in Usizes) {
      #Get "all" i x j matrices by combining (with replacement) the vectors from
      #the previous step i times
      #The zero vectors can be left out, since we are only interested
      #in relations that have dominating families
      #((2^j -1)+i -1)!/(i!((2^j -1)-1)!) possibilities at each step
      n <- n + factorial((2^j -1)+i -1)/(factorial(i)*factorial((2^j -1)-1))
    }
  }
  return(n)
}

dual_relation <- function(A) {
  #Find the dual of a finite relation
  A dual <- t(1- A)
  return(A dual)
}

setsystem_from_matrix <- function(M){
  library(data.table)
  #Convert incidence matrix to two column data frame (set system):
  j = ncol(M)
  S1 <- c()
  S2 <- c()
  for (k in 1:j) {
    # - 1st column is points in A+ (columns in the incidence matrix)
    S1 <- c(S1, rep(k, length(which(M[, k] == 1))))
    # - 2nd column contain the points in A- that they relate to
    S2 <- c(S2, which(M[, k] == 1))
  }
}

```

```

}
S <- data.table("set" = S1, "element" = S2)
return(S)
}

dominating_number <- function(M) {
  #Find the (approximate) dominating number of a finite relation
  #Test for if a dominating number exists
  if(0 %in% rowSums(M)) {return(NA)}
  #Test for dominating number = 1
  if(nrow(M) %in% colSums(M)) {return(1)}
  #Test for dominating number = 2
  combs <- t(combn(x = 1:ncol(M), m = 2))
  comp_value <- apply(X = combs, MARGIN = 1,
                     function(i nd) pmax(M[, i nd[1]], M[, i nd[2]]))
  nrow(M) %in% colSums(comp_value)
  if(nrow(M) %in% colSums(comp_value)) {return(2)}
  #Test for dominating number = 3
  combs <- t(combn(x = 1:ncol(M), m = 3))
  comp_value <- apply(X = combs, MARGIN = 1,
                     function(i nd) pmax(M[, i nd[1]], M[, i nd[2]], M[, i nd[3]]))
  nrow(M) %in% colSums(comp_value)
  if(nrow(M) %in% colSums(comp_value)) {return(3)}
  #Test for dominating number = 4
  combs <- t(combn(x = 1:ncol(M), m = 4))
  comp_value <- apply(X = combs, MARGIN = 1,
                     function(i nd) pmax(M[, i nd[1]], M[, i nd[2]], M[, i nd[3]], M[, i nd[4]]))
  nrow(M) %in% colSums(comp_value)
  if(nrow(M) %in% colSums(comp_value)) {return(4)}
  #Otherwise use a greedy set cover algorithm
  library(RcppGreedySetCover)
  S <- setsystem_from_matrix(M)
  invisible(capture.output(res <- greedySetCover(S, TRUE)))
  D <- uniqueN(res[, 1])
  return(D)
}

maximal_points_from_matrix <- function(M, RowsOrCols = "Cols"){

```

```

#Identify maximal points from a matrix
if(RowsOrCols == "Cols") {
  z <- unlist(lapply(X = 1:ncol(M),
                    function(k) all(unlist(lapply(X = 1:ncol(M),
                                                  function(n) any(M[,k,drop = FALSE] > M[,n,drop = FALSE]) |
                                                  all(M[,k,drop = FALSE] == M[,n,drop = FALSE]))))))))
}
if(RowsOrCols == "Rows") {
  z <- unlist(lapply(X = 1:nrow(M),
                    function(k) all(unlist(lapply(X = 1:nrow(M),
                                                  function(n) any(M[k,,drop = FALSE] > M[n,,drop = FALSE]) |
                                                  all(M[k,,drop = FALSE] == M[n,,drop = FALSE]))))))))
}
return(z)
}

minimal_points_from_matrix <- function(M, RowsOrCols = "Rows"){
  #Identify maximal points from a matrix
  if(RowsOrCols == "Cols") {
    z <- unlist(lapply(X = 1:ncol(M),
                      function(k) all(unlist(lapply(X = 1:ncol(M),
                                                    function(n) any(M[,k,drop = FALSE] < M[,n,drop = FALSE]) |
                                                    all(M[,k,drop = FALSE] == M[,n,drop = FALSE]))))))))

    return(z)
  }
  if(RowsOrCols == "Rows") {
    z <- unlist(lapply(X = 1:nrow(M),
                      function(k) all(unlist(lapply(X = 1:nrow(M),
                                                    function(n) any(M[k,,drop = FALSE] < M[n,,drop = FALSE]) |
                                                    all(M[k,,drop = FALSE] == M[n,,drop = FALSE]))))))))

    return(z)
  }
}

remove_twins <- function(M){
  #Remove twin points from a matrix
  A <- M
  row_dups <- duplicated(A)

```

```

col_dups <- duplicated(t(A))
A <- A[!row_dups,!col_dups, drop = FALSE]
return(A)
}

skeleton_bimorphic_form <- function(M){
  A <- M
  maximal_points <- maximal_points_from_matrix(A, RowsOrCols = "Cols")
  minimal_points <- minimal_points_from_matrix(A, RowsOrCols = "Rows")
  while(sum(maximal_points) < length(maximal_points) ||
        sum(minimal_points) < length(minimal_points)) {
    A <- A[which(minimal_points), which(maximal_points), drop = FALSE]
    maximal_points <- maximal_points_from_matrix(A, RowsOrCols = "Cols")
    minimal_points <- minimal_points_from_matrix(A, RowsOrCols = "Rows")
  }
  A <- remove_twins(A)
  return(A)
}

canonical_form <- function(M) {
  i <- nrow(M)
  j <- ncol(M)
  I <- rowSums(M)
  J <- colSums(M)
  if(identical(I, rep(1, i)) & identical(J, rep(1, j))) {
    return(diag(i))
  }
  else if (identical(I, rep(j-1, i)) & identical(J, rep(i-1, j))) {
    return(1-diag(i))
  }
  else {
    return(M[order(rowSums(-M)), order(colSums(-M)), drop = FALSE])
  }
}

morphism_from_AtoB <- function(A, B, printMorphisms = F) {
  library(R.utils)
  #Explicitly Test for Morphism

```

```

Ami n <- 1:nrow(A)
Bmi n <- 1:nrow(B)
Apl <- 1:ncol(A)
Bpl <- 1:ncol(B)

phi mi n <- permutations(n = length(Ami n), r = length(Bmi n), v = Ami n, repeats.allowed = TRUE)
phi pl <- permutations(n = length(Bpl), r = length(Apl), v = Bpl, repeats.allowed = TRUE)

rowsandcols <- unlist(lapply(X = 1:nrow(phi mi n),
                           functi on(k) lapply(X = 1:nrow(phi pl),
                                                functi on(n) list(phi mi n[k, ], phi pl [n, ]))), recursive = FALSE)

M1 <- lapply(X = 1:length(rowsandcols), functi on(k) A[rowsandcols[[k]][[1]], ])
M2 <- lapply(X = 1:length(rowsandcols), functi on(k) B[, rowsandcols[[k]][[2]])
morphi sm <- unlist(lapply(X = 1:length(rowsandcols), functi on(k) all(M2[[k]] >= M1[[k]])))

if(printMorphi sms == T) {
  z <- rowsandcols[whi ch(morphi sm)]
}
else{z <- length(whi ch(morphi sm)) > 0}

return(z)
}

#Generate all finite relations graphs of size 5x5 or less
generate_fi ni te_relati ons(Usi zes = 1:5, Vsi zes = 1:5,
                           saveMatrices = T, saveGraphs = F,
                           matricesLi stName = "matrices_1to5_x_1to5",
                           graphsLi stName = "")

#Create Skel eton Relati ons
#Get the canonical form (order by degree) to make the duplicates more obvious
skel eton_matrices_1to5_x_1to5 <- lapply(X = 1:length(matrices_1to5_x_1to5),
                                         functi on(k) lapply(X = matrices_1to5_x_1to5[[k]],
                                                             functi on(M) canoni cal_form(skel eton_bi morphi c_form(M))))

```

```

#Find Unique (up to isomorphism) Skeleton Relations
#Remove exact duplicates
unique_skeleton_matrices_1to5_x_1to5_stage <- c(list(matrix(0)),
      unique(unlist(skeleton_matrices_1to5_x_1to5,
        recursive = FALSE)))
unique_skeleton_graphs_1to5_x_1to5_stage <- lapply(X = unique_skeleton_matrices_1to5_x_1to5_stage,
      function(M) graph_from_matrix(M, plotGraph = F))

#Find all graphs that are isomorphic
isomorphic_list <- lapply(X = unique_skeleton_graphs_1to5_x_1to5_stage,
      function(G1) unlist(lapply(X = unique_skeleton_graphs_1to5_x_1to5_stage,
        function(G2) isomorphic(G1, G2, method = 'vf2'))))
isomorphism_classes <- unique(lapply(X = isomorphic_list, function(L) which(L)))

#Find all matrices that are symmetric
symmetric_list <- unlist(lapply(X = unique_skeleton_matrices_1to5_x_1to5_stage, function(M) t(isSymmetric(M))))
symmetric_indices <- which(symmetric_list)

#Choose a representative for each isomorphic class. Choose a symmetric relation if one exists.
unique_skeleton_relations_final_indices <- unlist(lapply(X = isomorphism_classes,
      function(C) {if(length(intersect(C, symmetric_indices)) == 0)
        {C[1]} else{intersect(C, symmetric_indices)[1]}}))

unique_skeleton_matrices_1to5_x_1to5 <-
  unique_skeleton_matrices_1to5_x_1to5_stage[unique_skeleton_relations_final_indices]
unique_skeleton_graphs_1to5_x_1to5 <-
  unique_skeleton_graphs_1to5_x_1to5_stage[unique_skeleton_relations_final_indices]

#Check that skeleton relations are closed under dual
unique_skeleton_matrices_1to5_x_1to5_duals <- lapply(X = unique_skeleton_matrices_1to5_x_1to5,
      function(M) dual_relation(M))
unique_skeleton_graphs_1to5_x_1to5_duals <- lapply(X = unique_skeleton_matrices_1to5_x_1to5_duals,
      function(M) graph_from_matrix(M, plotGraph = F))
isomorphic_list_duals <- lapply(X = unique_skeleton_graphs_1to5_x_1to5_duals,
      function(G1) unlist(lapply(X = unique_skeleton_graphs_1to5_x_1to5,
        function(G2) isomorphic(G1, G2, method = 'vf2'))))
Ai ndex <- 1:length(unique_skeleton_matrices_1to5_x_1to5)
dual Index <- unlist(lapply(X = isomorphic_list_duals, function(L) if(length(which(L)) == 0){NA} else{which(L)}))

```

```

identical(sort(dualIndex), Aindex)#returns TRUE

#Calculate the Dominating Numbers, Dual Dominating Numbers, Size of A- and A+
Ami nus <- unlist(lapply(X = unique_skeleton_matrices_1to5_x_1to5, function(M) nrow(M)))
Apl us  <- unlist(lapply(X = unique_skeleton_matrices_1to5_x_1to5, function(M) ncol(M)))
D      <- unlist(lapply(X = unique_skeleton_matrices_1to5_x_1to5, function(M) dominating_number(M)))
Ddual  <- unlist(lapply(X = unique_skeleton_matrices_1to5_x_1to5, function(M) dominating_number(dual_relation(M))))
skeleton_characteristics <- cbind(Aindex, Ami nus, Apl us, D, Ddual , dual Index)
colnames(skeleton_characteristics) <- c("Aindex", "Ami nus", "Apl us", "d(A)", "d(A dual)", "dual Index")
skeleton_characteristics

#Print out all skeleton graphs
lapply(X = 1:length(unique_skeleton_matrices_1to5_x_1to5),
       function(k) {
         M <- unique_skeleton_matrices_1to5_x_1to5[[k]]
         mypath <- file.path("C:/Users/Jared/Desktop/Rhett Thesis",
                             paste0("[", k, "].jpg", sep = ""))
         jpeg(file=mypath)
         graph_from_matrix(M, plotGraph = T, saveGraph = F)
         dev.off()
       })

#Classification of morphisms for graphs up to 5x5
#Create a data frame with all combinations of skeleton relations up to 5x5
morphisms_1to5_x_1to5 <- merge(data.frame(skeleton_characteristics = skeleton_characteristics),
                              data.frame(skeleton_characteristics = skeleton_characteristics),
                              by = NULL)
names(morphisms_1to5_x_1to5) <- c("Aindex", "Ami nus", "Apl us", "dA", "dA_dual", "A_dual Index",
                                "Bindex", "Bmi nus", "Bpl us", "dB", "dB_dual", "B_dual Index")

#Explicit morphism checking
MorphismsFromAtoB <-
  lapply(X = 1:nrow(morphisms_1to5_x_1to5),
        function(k) morphism_from_AtoB(unique_skeleton_matrices_1to5_x_1to5[[morphisms_1to5_x_1to5[k, 1]]],
                                       unique_skeleton_matrices_1to5_x_1to5[[morphisms_1to5_x_1to5[k, 7]]],
                                       printMorphisms = T))

MorphismsFromAtoB <- unlist(lapply(1:length(MorphismsFromAtoB),

```

```
function(k) if(length(MorphismsFromAtoB[[k]]) == 0){FALSE} else{TRUE}))

morphisms_1to5_x_1to5 <- cbind(morphisms_1to5_x_1to5, MorphismsFromAtoB)
morphisms_1to5_x_1to5[which(morphisms_1to5_x_1to5$MorphismsFromAtoB == TRUE), c("Index", "BiIndex")]
```


APPENDIX B

MORPHISMS BETWEEN SKELETON RELATIONS

```
> morphisms_1to5_x_1to5[which(morphisms_1to5$MorphismsFromAtoB == TRUE), c("Aindex", "Bindex")]
```

	Aindex	Bindex						
			58	26	2	97	1	4
1	1	1	59	27	2	100	4	4
33	1	2	60	28	2	102	6	4
34	2	2	61	29	2	109	13	4
35	3	2	62	30	2	110	14	4
36	4	2	63	31	2	112	16	4
37	5	2	64	32	2	129	1	5
38	6	2	65	1	3	131	3	5
39	7	2	67	3	3	132	4	5
40	8	2	68	4	3	133	5	5
41	9	2	70	6	3	134	6	5
42	10	2	71	7	3	135	7	5
43	11	2	73	9	3	136	8	5
44	12	2	75	11	3	137	9	5
45	13	2	76	12	3	139	11	5
46	14	2	77	13	3	140	12	5
47	15	2	78	14	3	141	13	5
48	16	2	79	15	3	142	14	5
49	17	2	80	16	3	143	15	5
50	18	2	81	17	3	144	16	5
51	19	2	83	19	3	145	17	5
52	20	2	84	20	3	146	18	5
53	21	2	85	21	3	147	19	5
54	22	2	86	22	3	148	20	5
55	23	2	87	23	3	149	21	5
56	24	2	89	25	3	150	22	5
57	25	2	92	28	3	151	23	5

			247	23	8	303	15	10
152	24	5	248	24	8	304	16	10
153	25	5	249	25	8	305	17	10
154	26	5	250	26	8	306	18	10
155	27	5	252	28	8	307	19	10
156	28	5	257	1	9	308	20	10
158	30	5	259	3	9	309	21	10
159	31	5	260	4	9	310	22	10
161	1	6	262	6	9	311	23	10
166	6	6	263	7	9	312	24	10
173	13	6	265	9	9	313	25	10
193	1	7	267	11	9	314	26	10
196	4	7	268	12	9	315	27	10
198	6	7	269	13	9	316	28	10
199	7	7	270	14	9	317	29	10
205	13	7	271	15	9	318	30	10
206	14	7	272	16	9	319	31	10
207	15	7	273	17	9	321	1	11
208	16	7	275	19	9	323	3	11
225	1	8	276	20	9	324	4	11
227	3	8	277	21	9	326	6	11
228	4	8	278	22	9	327	7	11
230	6	8	279	23	9	329	9	11
231	7	8	281	25	9	331	11	11
232	8	8	284	28	9	332	12	11
233	9	8	289	1	10	333	13	11
235	11	8	291	3	10	334	14	11
236	12	8	292	4	10	335	15	11
237	13	8	293	5	10	336	16	11
238	14	8	294	6	10	337	17	11
239	15	8	295	7	10	339	19	11
240	16	8	296	8	10	340	20	11
241	17	8	297	9	10	341	21	11
242	18	8	298	10	10	342	22	11
243	19	8	299	11	10	343	23	11
244	20	8	300	12	10	345	25	11
245	21	8	301	13	10	348	28	11
246	22	8	302	14	10	353	1	12

			496	16	16	588	12	19
355	3	12	513	1	17	589	13	19
356	4	12	516	4	17	590	14	19
358	6	12	518	6	17	591	15	19
359	7	12	519	7	17	592	16	19
361	9	12	525	13	17	593	17	19
363	11	12	526	14	17	595	19	19
364	12	12	527	15	17	596	20	19
365	13	12	528	16	17	597	21	19
366	14	12	529	17	17	598	22	19
367	15	12	545	1	18	599	23	19
368	16	12	547	3	18	601	25	19
369	17	12	548	4	18	604	28	19
371	19	12	550	6	18	609	1	20
372	20	12	551	7	18	612	4	20
373	21	12	553	9	18	614	6	20
374	22	12	555	11	18	621	13	20
375	23	12	556	12	18	622	14	20
377	25	12	557	13	18	624	16	20
380	28	12	558	14	18	628	20	20
385	1	13	559	15	18	641	1	21
397	13	13	560	16	18	644	4	21
417	1	14	561	17	18	646	6	21
422	6	14	562	18	18	647	7	21
429	13	14	563	19	18	653	13	21
430	14	14	564	20	18	654	14	21
449	1	15	565	21	18	655	15	21
452	4	15	566	22	18	656	16	21
454	6	15	567	23	18	660	20	21
461	13	15	569	25	18	661	21	21
462	14	15	572	28	18	673	1	22
463	15	15	577	1	19	675	3	22
464	16	15	579	3	19	676	4	22
481	1	16	580	4	19	678	6	22
484	4	16	582	6	19	679	7	22
486	6	16	583	7	19	681	9	22
493	13	16	585	9	19	683	11	22
494	14	16	587	11	19	684	12	22

			756	20	24	816	16	26
685	13	22	757	21	24	817	17	26
686	14	22	758	22	24	819	19	26
687	15	22	759	23	24	820	20	26
688	16	22	760	24	24	821	21	26
689	17	22	761	25	24	822	22	26
691	19	22	764	28	24	823	23	26
692	20	22	769	1	25	824	24	26
693	21	22	771	3	25	825	25	26
694	22	22	772	4	25	826	26	26
695	23	22	774	6	25	828	28	26
697	25	22	775	7	25	833	1	27
700	28	22	777	9	25	835	3	27
705	1	23	779	11	25	836	4	27
708	4	23	780	12	25	838	6	27
710	6	23	781	13	25	839	7	27
711	7	23	782	14	25	840	8	27
717	13	23	783	15	25	841	9	27
718	14	23	784	16	25	843	11	27
719	15	23	785	17	25	844	12	27
720	16	23	787	19	25	845	13	27
724	20	23	788	20	25	846	14	27
725	21	23	789	21	25	847	15	27
727	23	23	790	22	25	848	16	27
737	1	24	791	23	25	849	17	27
739	3	24	793	25	25	850	18	27
740	4	24	796	28	25	851	19	27
742	6	24	801	1	26	852	20	27
743	7	24	803	3	26	853	21	27
745	9	24	804	4	26	854	22	27
747	11	24	806	6	26	855	23	27
748	12	24	807	7	26	856	24	27
749	13	24	809	9	26	857	25	27
750	14	24	811	11	26	858	26	27
751	15	24	812	12	26	859	27	27
752	16	24	813	13	26	860	28	27
753	17	24	814	14	26	865	1	28
755	19	24	815	15	26	867	3	28

			918	22	29	961	1	31
868	4	28	919	23	29	963	3	31
870	6	28	920	24	29	964	4	31
871	7	28	921	25	29	966	6	31
873	9	28	922	26	29	967	7	31
875	11	28	923	27	29	969	9	31
876	12	28	924	28	29	971	11	31
877	13	28	925	29	29	972	12	31
878	14	28	926	30	29	973	13	31
879	15	28	927	31	29	974	14	31
880	16	28	929	1	30	975	15	31
881	17	28	931	3	30	976	16	31
883	19	28	932	4	30	977	17	31
884	20	28	933	5	30	979	19	31
885	21	28	934	6	30	980	20	31
886	22	28	935	7	30	981	21	31
887	23	28	936	8	30	982	22	31
889	25	28	937	9	30	983	23	31
892	28	28	939	11	30	984	24	31
897	1	29	940	12	30	985	25	31
899	3	29	941	13	30	986	26	31
900	4	29	942	14	30	988	28	31
901	5	29	943	15	30	991	31	31
902	6	29	944	16	30	993	1	32
903	7	29	945	17	30	995	3	32
904	8	29	946	18	30	996	4	32
905	9	29	947	19	30	997	5	32
907	11	29	948	20	30	998	6	32
908	12	29	949	21	30	999	7	32
909	13	29	950	22	30	1000	8	32
910	14	29	951	23	30	1001	9	32
911	15	29	952	24	30	1002	10	32
912	16	29	953	25	30	1003	11	32
913	17	29	954	26	30	1004	12	32
914	18	29	955	27	30	1005	13	32
915	19	29	956	28	30	1006	14	32
916	20	29	958	30	30	1007	15	32
917	21	29	959	31	30	1008	16	32

1009	17	32
1010	18	32
1011	19	32
1012	20	32
1013	21	32
1014	22	32
1015	23	32
1016	24	32
1017	25	32
1018	26	32
1019	27	32
1020	28	32
1021	29	32
1022	30	32
1023	31	32
1024	32	32