TUKEY MORPHISMS
BETWEEN FINITE BINARY RELATIONS

by

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To Abby, Max, and The Third

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ABSTRACT

Let $\mathbf{A} = (A_-, A_+, A)$ and $\mathbf{B} = (B_-, B_+, B)$ be relations. A *morphism* is a pair of maps $\varphi_- : B_- \to A_-$ and $\varphi_+ : A_+ \to B_+$ such that for all $b \in B_-$ and $a \in A_+$, $\varphi_-(b)Aa \implies bB\varphi_+(a)$. We study the existence of morphisms between finite relations. The ultimate goal is to identify the conditions under which morphisms exist. In this thesis we present some progress towards that goal. We use computation to verify the results for small finite relations.

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CHAPTER 1

INTRODUCTION

In this thesis, we study Tukey morphisms between finite relations. Before proceeding to the topic in earnest, we give a brief outline of related concepts that give context as to the origins and motivation for this area of research.

If $(A, \leq_A)$ and $(B, \leq_B)$ are two partial orders, then a *Galois connection* between these is a pair of monotone functions: $\phi : B \to A$ and $\psi : A \to B$, such that for all $a \in A$ and $b \in B$, we have

$$\phi(b) \leq_A a \iff b \leq_B \psi(a)$$

Galois connections generalize the correspondence between subgroups and subfields investigated in Galois theory [3].

A directed partial order $(X, \leq_X)$ is a partial order such that every pair of elements has an upper bound. If $(A, \leq_A)$ and $(B, \leq_B)$ are directed partial orders, then a map $\phi : B \to A$ is called *Tukey* if the preimage of each bounded subset of A is bounded in B. A map $\psi : A \to B$ is called *cofinal* if it maps cofinal subsets of A to cofinal subsets of B [5].

The existence of Tukey map from B to A is equivalent to the existence of a cofinal map from A to B [4]. Assuming that ϕ is a Tukey map from B to A and ψ is a corresponding cofinal map from A to B, the following implication holds for all $a \in A$

and $b \in B$:

$$\phi(b) \leq_A a \implies b \leq_B \psi(a)$$

These maps are referred to as *Galois-Tukey connections*. As the name suggests and can be seen from the implication above, they are related to Galois connections. In a sense they are more general since they relax the "if and only if" to "implies".

If such a Tukey map from B to A exists, we write $B \leq_T A$ and say that B is *Tukey reducible* to A. If both $A \leq_T B$ and $B \leq_T A$, we write $A \equiv_T B$ and say that A and B are *Tukey equivalent* [5].

The original motivation for Galois-Tukey connections comes from the Moore-Smith theory of convergence in general topological spaces [5]. However, Galois-Tukey connections have found applications in comparing complexities of various directed sets or, more generally, partial orders [4].

These comparisons can reveal useful information. For example, Tukey reducibility downward preserves calibre-like properties, such as c.c.c., property K, precalibre $\aleph_1$, σ-linked, and σ-centered [2].

Galois-Tukey connections can be generalized by letting $(A, \leq_A)$ and $(B, \leq_B)$ be relations, rather than partial orders. However, in that context it is not generally true that a given ϕ gives rise to a corresponding ψ such that $\phi(b) \leq_A a \implies b \leq_B \psi(a)$. It then becomes necessary to explicitly give two mappings, ϕ and ψ, that satisfy the condition

$$\phi(b) \leq_A a \implies b \leq_B \psi(a)$$

Pairs of maps between relations that satisfy this condition are referred to as *generalized Galois-Tukey connections*. Throughout this thesis, we use the convention established by Blass and refer to them as *morphisms* [1].

In Definition 2.1.4 we introduce the notion of a *dominating number* of a relation. Informally, it is the minimum size of a subset of the codomain such that every element of the domain is related to a member of that subset. The definitions of many cardinal characteristics amount to dominating numbers of specific relations. For example, the cardinal characteristic $\mathfrak{d}$ is the least cardinality of a set of functions from ω to ω such that every function from ω to ω is eventually dominated by a member of that set. Then $\mathfrak{d}$ is the dominating number of the $\leq^*$ relation on ω^ω.

By employing the fact that many cardinal characteristics are dominating numbers, morphisms have found extensive use in the proofs of many cardinal characteristic inequalities. Once the underlying relations for two cardinal characteristics are determined, Theorem 2.1.7 from Blass shows that if a morphism can be found between these relations, then an inequality exists between the cardinal characteristics [1].

With the usefulness and interest of morphisms established, the focus of this thesis is not the *application* of morphisms, but rather what can be said about the *existence* of morphisms. In particular, we explore under which conditions a morphism exists between finite relations.

Chapter 2 outlines definitions and results, drawn largely from Blass, which provide a foundation for the following chapters.

Chapter 3 shows progress in our understanding of when morphisms exist, particularly in the finite case.

Chapter 4 provides a classification for small finite relations. We apply the results from Chapter 3 to these examples, as well as provide computational checks.

CHAPTER 2

RELATIONS AND MORPHISMS

2.1 Background

We begin with some fundamental definitions, drawing largely from Blass [1]. The basic objects of inquiry are *relations* and their *duals*.

Definition 2.1.1. (Blass 4.1 [1]) A triple $\mathbf{A} = (A_-, A_+, A)$ consisting of two nonempty sets A_-, A_+, and a binary relation $A \subseteq A_- \times A_+$ is called a **relation**.

Definition 2.1.2. (Blass 4.3 [1]) If $\mathbf{A} = (A_-, A_+, A)$ then the **dual** of $\mathbf{A}$ is the relation $\mathbf{A}^\perp = (A_+, A_-, \neg\breve{A})$ where $\neg$ denotes the complement and $\breve{A}$ is the converse of A, thus $x \neg\breve{A} y$ if and only if $y \not{A}\, x$.

Figure 2.1 provides an illustration of a finite relation and its dual, represented as bipartite graphs. Colors have been added to aide in the visualization. In particular, for a relation $\mathbf{A} = (A_-, A_+, A)$, for a given $a \in A_+$, the same color is used for all $(x, a) \in A$. This convention is used in many of the figures throughout the thesis.

We now introduce the notion of *dominating families*. Dominating families are subsets of A_+ that "cover" all of A_-. The smallest such subset is a *minimal dominating family* and its cardinality is called the *dominating number*.

Definition 2.1.3. (Blass 4.2 [1]) An **A-dominating family** is a subset Y of A_+ such that for every $a \in A_-$ there exists $\alpha \in Y$ such that $aA\alpha$.

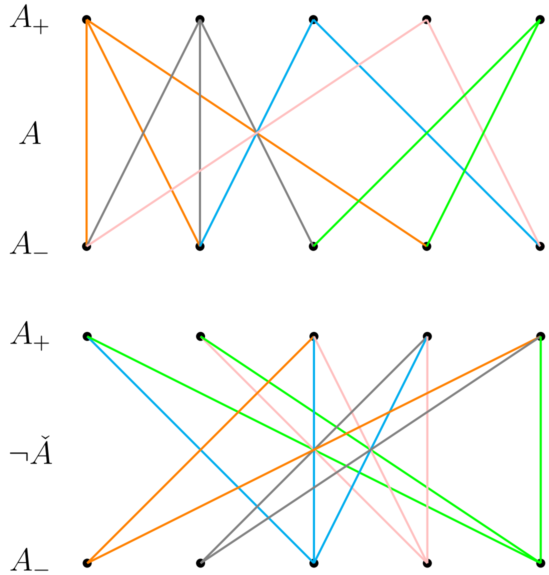


Figure 2.1: Example of a finite relation and its dual, represented as bipartite graphs

Definition 2.1.4. (Blass 4.2 [1]) The **dominating number** $\delta(\mathbf{A})$ of a relation $\mathbf{A}$ is the smallest cardinality of any $\mathbf{A}$-dominating family. If no $\mathbf{A}$-dominating family exists, we say that $\delta(\mathbf{A}) = \infty$. We refer to an $\mathbf{A}$-dominating family that has cardinality $\delta(\mathbf{A})$ as a **minimal A-dominating family**.

The use of "minimal" in Definition 2.1.4 differs slightly from what the reader might expect. Here we use it to mean "an $\mathbf{A}$-dominating family *of least cardinality*". This definition is not equivalent to "an $\mathbf{A}$-dominating family that contains no smaller $\mathbf{A}$-dominating family". The former definition implies the latter, but not vice versa. Figure 2.2 gives an illustration of why they are not equivalent.

In Lemma 2.1.5 we see that if no $\mathbf{A}$-dominating family exists, then $\mathbf{A}^\perp$ has a point in its codomain that relates to all points in its domain, i.e. $\delta(\mathbf{A}^\perp) = 1$. We return to this special edge case in Lemma 3.4.1.

Lemma 2.1.5. *Let* $\mathbf{A}$ *be a relation. There does not exist an* $\mathbf{A}$*-dominating family if*

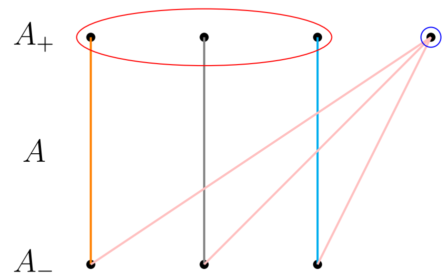


Figure 2.2: Example of a dominating family that contains no smaller dominating family, but is not of least cardinality (circled in red). For contrast, a dominating family of least cardinality is circled in blue.

and only if $\delta(\mathbf{A}^\perp) = 1$.

Proof. Suppose there does not exist an $\mathbf{A}$-dominating family. Then there exists $a \in A_-$ such that $a \not A \alpha$ for all $\alpha \in A_+$. Then $\alpha \neg \check{A} a$ for all $\alpha \in A_+$. Then $\{a\}$ is an $\mathbf{A}^\perp$-dominating family and $\delta(\mathbf{A}^\perp) = 1$. □

Definition 2.1.6 provides the formal definition for *morphisms*. Morphisms are the primary objects by which we compare relations throughout this thesis.

Definition 2.1.6. (Blass 4.8 [1]) A (Tukey) **morphism** from one relation $\mathbf{A} = (A_-, A_+, A)$ to another $\mathbf{B} = (B_-, B_+, B)$ is a pair of functions $\varphi = (\varphi_-, \varphi_+)$ such that:

- $\varphi_- : B_- \to A_-$
- $\varphi_+ : A_+ \to B_+$
- for all $b \in B_-$ and $a \in A_+$, $\varphi_-(b) A a \implies b B \varphi_+(a)$

In this thesis, we denote "there is a morphism from $\mathbf{A}$ to $\mathbf{B}$" as $\mathbf{A} \to \mathbf{B}$.

Figure 2.3 gives a diagram to help visualize this definition. The definition implies that the diagram "commutes." However, in this diagram the dotted arrows represent relations, rather than functions, so it is only an illustration.

$$\begin{tikzcd} A_+ \arrow[r, "\varphi_+"] & B_+ \\ A_- \arrow[u, dotted, "A"] & B_- \arrow[l, "\varphi_-"] \arrow[u, dotted, "B"'] \end{tikzcd}$$

Figure 2.3: Morphism diagram

See Figure 2.4 for an example of a morphism between two small finite relations. Dashed arrows represent the functions and, as before, bipartite graphs represent the relations. We use this visualization approach throughout the thesis.

If $\varphi = (\varphi_-, \varphi_+)$ is a morphism from $\mathbf{A}$ to $\mathbf{B}$, then $\varphi^\perp = (\varphi_+, \varphi_-)$ is a morphism from $\mathbf{B}^\perp$ to $\mathbf{A}^\perp$. We can see this by taking the contrapositive of the implication and applying Definition 2.1.2:

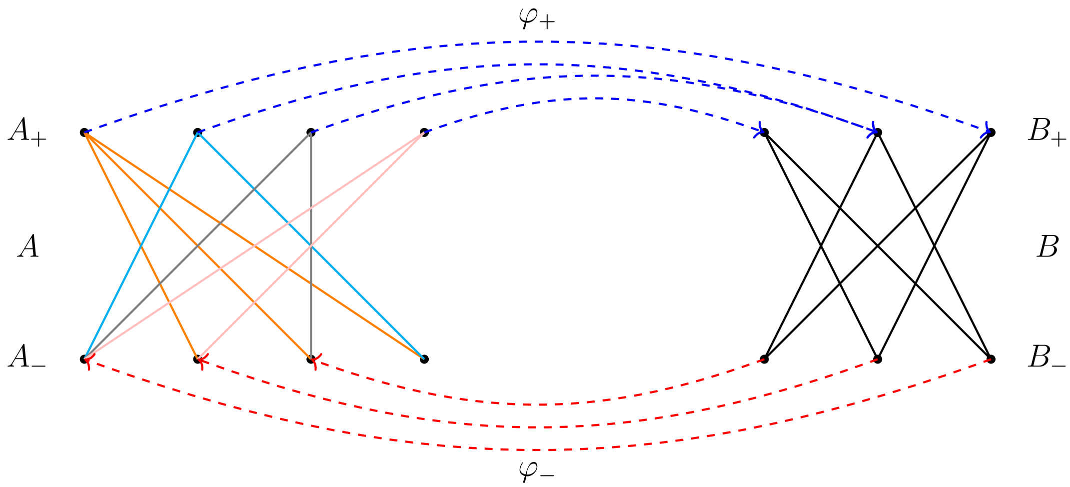
$$\begin{aligned} (\varphi_-(b)Aa \implies bB\varphi_+(a)) &\implies (b \not{B}\, \varphi_+(a) \implies \varphi_-(b) \not{A}\, a) \\ &\implies (\varphi_+(a)\neg\breve{B}b \implies a\neg\breve{A}\varphi_-(b)) \end{aligned}$$

This is also illustrated in Figure 2.4.

Theorem 2.1.7 from Blass [1] provides the motivation for the definition of morphism.

Theorem 2.1.7. (Blass 4.9 [1]) If $\mathbf{A} \to \mathbf{B}$ then $\delta(\mathbf{A}) \geq \delta(\mathbf{B})$ and $\delta(\mathbf{A}^\perp) \leq \delta(\mathbf{B}^\perp)$.

Proof. Suppose (φ_-, φ_+) is a morphism from $\mathbf{A}$ to $\mathbf{B}$. Let $D_{\mathbf{A}} \subseteq A_+$ be a minimal $\mathbf{A}$-dominating family. Let $Y = \varphi_+(D_{\mathbf{A}})$. Note that $Y \subseteq B_+$ and $\delta(\mathbf{A}) \geq |Y|$. We will show that Y is a $\mathbf{B}$-dominating family: For $b \in B_-$, consider $\varphi_-(b) \in A_-$. Because $D_{\mathbf{A}}$ is a $\mathbf{A}$-dominating family, there exists $a \in D_{\mathbf{A}}$ such that $\varphi_-(b)Aa$. Then



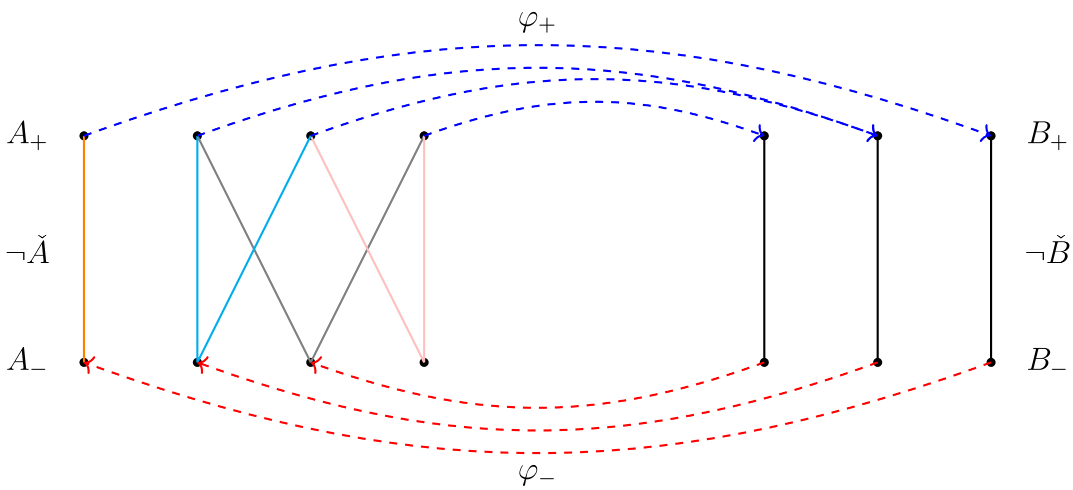


Figure 2.4: Example: A morphism between two finite relations

by the definition of morphism, $bB\varphi_+(a)$. In particular, $\varphi_+(a) \in Y$. Then Y is a $\mathbf{B}$-dominating family. Then $\delta(\mathbf{A}) \geq |Y| \geq \delta(\mathbf{B})$.

Since (φ_+, φ_-) is a morphism from $\mathbf{B}^\perp$ to $\mathbf{A}^\perp$, the same argument can be used to show $\delta(\mathbf{B}^\perp) \geq \delta(\mathbf{A}^\perp)$.

If no such $\mathbf{A}$-dominating family $D_\mathbf{A}$ exists, then by Lemma 2.1.5 $\delta(\mathbf{A}) = \infty$ and $\delta(\mathbf{A}^\perp) = 1$. Then $\delta(\mathbf{A}) \geq \delta(\mathbf{B})$ and $\delta(\mathbf{A}^\perp) \leq \delta(\mathbf{B}^\perp)$. $\square$

This leads us to wonder if the converse is true, i.e. is a morphism between two arbitrary relations $\mathbf{A}$ and $\mathbf{B}$ guaranteed if $\delta(\mathbf{A}) \geq \delta(\mathbf{B})$ and $\delta(\mathbf{A}^{\perp}) \leq \delta(\mathbf{B}^{\perp})$? Rather than give the reader a false sense of hope, we immediately give a counterexample to this conjecture in Figure 2.5. This counter example follows from Lemma 3.4.8.

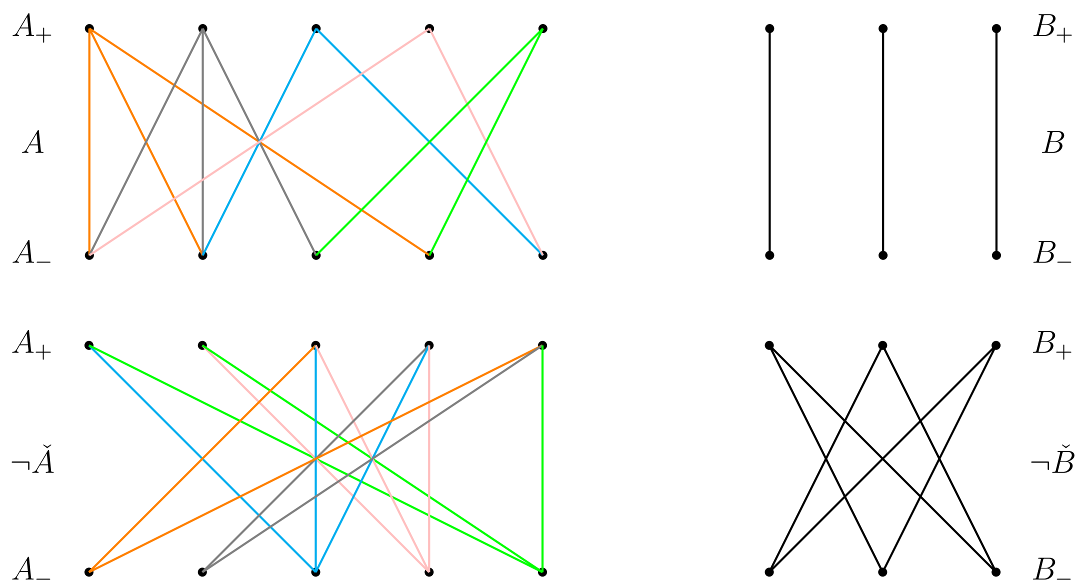


Figure 2.5: Counter Example: There is no morphism from A to B despite $\delta(\mathbf{A}) = \delta(\mathbf{B}) = 3$ **and** $\delta(\mathbf{A}^{\perp}) = \delta(\mathbf{B}^{\perp}) = 2$

Given this counter example, the question of, "When does there exist a morphism between two relations?" becomes interesting. The quest to answer this question in the finite context is the motivation for the following chapters.

The contrapositive of Theorem 2.1.7 gives us our first sufficient condition for morphism non-existence.

Corollary 2.1.8. *If* $\delta(\mathbf{A}) \ngeq \delta(\mathbf{B})$ *or* $\delta(\mathbf{A}^{\perp}) \nleq \delta(\mathbf{B}^{\perp})$ *then* $\mathbf{A} \nrightarrow \mathbf{B}$.

2.2 Finite Relations

Before proceeding on the topic of morphisms, we outline some key definitions and properties of finite relations.

As we saw previously, a finite relation can be thought of as a bipartite graph. As such, we borrow the concept of a *neighborhood* from graph theory.

Definition 2.2.1. The **neighborhood** of a point a in a relation $\mathbf{A} = (A_-, A_+, A)$ is the set of all points that relate to a, $\{b : bAa \vee aAb\}$, denoted as $N_{\mathbf{A}}(a)$.

Definition 2.2.2. Let $\mathbf{A}$ be a relation. For $a, b \in \mathbf{A}$, define the **neighborhood relation** with respect to $\mathbf{A}$, denoted $\preccurlyeq_{\mathbf{A}}$, as $a \preccurlyeq_{\mathbf{A}} b$ if and only if $N_{\mathbf{A}}(a) \subseteq N_{\mathbf{A}}(b)$.

Lemma 2.2.3. *Given a relation $\mathbf{A}$, the neighborhood relation is a preorder on A_+ and A_-, i.e. it is reflexive and transitive.*

Proof. This is obvious by the reflexivity and transitivity of the subset relation:

- Reflexive: $N_{\mathbf{A}}(a) \subseteq N_{\mathbf{A}}(a)$ for all $a \in A_+$.
- Transitive: If $N_{\mathbf{A}}(a) \subseteq N_{\mathbf{A}}(b)$ and $N_{\mathbf{A}}(b) \subseteq N_{\mathbf{A}}(c)$, then $N_{\mathbf{A}}(a) \subseteq N_{\mathbf{A}}(c)$ for all $a, b, c \in A_+$.

These arguments work similarly for A_-. □

Note that the neighborhood relation is not anti-symmetric, so it is not a partial order: $N_{\mathbf{A}}(a) \subseteq N_{\mathbf{A}}(b)$ and $N_{\mathbf{A}}(b) \subseteq N_{\mathbf{A}}(a)$ imply that $N_{\mathbf{A}}(a) = N_{\mathbf{A}}(b)$ but not that $a = b$.

We now define the notions of "maximal", "minimal", and "twin" points with respect to the neighborhood preorder. Figure 2.6 provides an illustration.

Definition 2.2.4. Let $\mathbf{A}$ be a relation and let $X = A_-$ or $X = A_+$. A point $a \in X$ is **A-maximal** in the neighborhood preorder on X if there does not exist $b \in X$ such that $N_{\mathbf{A}}(a) \subsetneq N_{\mathbf{A}}(b)$, i.e. the neighborhood of a is not a proper subset of any other neighborhood.

Definition 2.2.5. Let $\mathbf{A}$ be a relation and let $X = A_-$ or $X = A_+$. A point $a \in X$ is **A-minimal** in the neighborhood preorder on X if there does not exist $b \in X$ such that $N_{\mathbf{A}}(b) \subsetneq N_{\mathbf{A}}(a)$, i.e. the neighborhood of a is not a proper superset of any other neighborhood.

Definition 2.2.6. Let $\mathbf{A}$ be a relation and let $X = A_-$ or $X = A_+$. Two points $a, b \in X$ are said to be **A-twins** if $N_{\mathbf{A}}(a) = N_{\mathbf{A}}(b)$, i.e. they have the same neighborhoods.

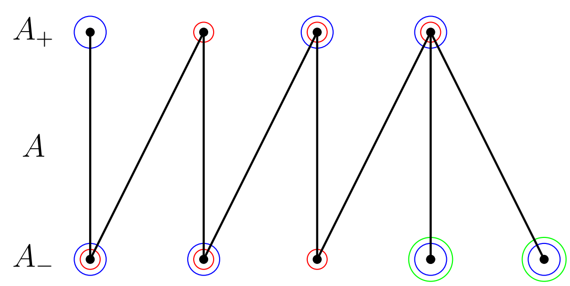


Figure 2.6: Illustration of maximal points (red), minimal points (blue), and twins (green)

A minimal point in $\mathbf{A}$ is maximal in $\mathbf{A}^\perp$. Similarly, a maximal point in $\mathbf{A}$ is minimal in $\mathbf{A}^\perp$.

Lemma 2.2.7. *Let* $\mathbf{A}$ *be a relation. A point* $a \in A_-$ *is* $\mathbf{A}$*-minimal if and only if it is* $\mathbf{A}^\perp$*-maximal. A point* $a \in A_-$ *is* $\mathbf{A}$*-maximal if and only if it is* $\mathbf{A}^\perp$*-minimal.*

Proof. Let $a \in A_-$ be $\mathbf{A}$-minimal. By Definition 2.2.5, there does not exist $b \in A_-$ such that $N_{\mathbf{A}}(b) \subsetneq N_{\mathbf{A}}(a)$. If there exists $c \in A_-$ such that $N_{\mathbf{A}}(c) \subseteq N_{\mathbf{A}}(a)$, then a

and c are twins. Then by definition of the subset relation, if there exists $c \in A_-$ such that $cAx \implies aAx$ for all $x \in A_+$, then a and c are twins. By the contrapositive, if there exists $c \in A_-$ such that $a \not A\, x \implies c \not A\, x$ for all $x \in A_+$, then a and c are twins. Then by Definition 2.1.1, if there exists $c \in A_-$ such that $x\neg\breve{A}a \implies x\neg\breve{A}c$ for all $x \in A_+$, then a and c are twins. Then if there exists $c \in A_-$ such that $N_{\neg\breve{\mathbf{A}}}(a) \subseteq N_{\neg\breve{\mathbf{A}}}(c)$, then a and c are twins. Then there does not exist $b \in A_-$ such that $N_{\neg\breve{\mathbf{A}}}(a) \subsetneq N_{\neg\breve{\mathbf{A}}}(b)$, which is the definition of $\mathbf{A}^\perp$-maximal. A similar argument shows that a point $a \in A_-$ is $\mathbf{A}$-maximal if and only if it is $\mathbf{A}^\perp$-minimal.

$\square$

Corollary 2.2.8. *Let* $\mathbf{A}$ *be a relation. A point* $a \in A_+$ *is* $\mathbf{A}$*-minimal if and only if it is* $\mathbf{A}^\perp$*-maximal. A point* $a \in A_+$ *is* $\mathbf{A}$*-maximal if and only if it is* $\mathbf{A}^\perp$*-minimal.*

Proof. Let $\mathbf{B} = \mathbf{A}^\perp$. Then $B_- = A_+$, $B_+ = A_-$, $B = \neg\breve{A}$, and $\mathbf{B}^\perp = \mathbf{A}$. By Lemma 2.2.7, $b \in B_-$ is $\mathbf{B}$-minimal if and only if it is $\mathbf{B}^\perp$-maximal. Then $b \in A_+$ is $\mathbf{A}^\perp$-minimal if and only if it is $\mathbf{A}$-maximal. Similarly, by Lemma 2.2.7, $b \in B_-$ is $\mathbf{B}$-maximal if and only if it is $\mathbf{B}^\perp$-minimal. Then $b \in A_+$ is $\mathbf{A}^\perp$-maximal if and only if it is $\mathbf{A}$-minimal. $\square$

2.3 Bimorphic Relations

Lemma 2.3.1 proves that $\mathbf{A} \to \mathbf{B} \wedge \mathbf{B} \to \mathbf{C} \implies \mathbf{A} \to \mathbf{C}$ (transitivity). This fact is employed widely throughout the remainder of the thesis.

Lemma 2.3.1. (Transitivity) Let $\mathbf{A}$, $\mathbf{B}$, and $\mathbf{C}$ be relations. If $\varphi = (\varphi_-, \varphi_+)$ is a morphism from $\mathbf{A}$ to $\mathbf{B}$ and $\vartheta = (\vartheta_-, \vartheta_+)$ is a morphism from $\mathbf{B}$ to $\mathbf{C}$, then $\varsigma = (\varphi_- \circ \vartheta_-, \vartheta_+ \circ \varphi_+)$ is a morphism from $\mathbf{A}$ to $\mathbf{C}$.

Proof. If φ is a morphism from $\mathbf{A}$ to $\mathbf{B}$ then for all $b \in B_-$ and $a \in A_+$, $\varphi_-(b)Aa \implies bB\varphi_+(a)$. Similarly, since ϑ is a morphism from $\mathbf{B}$ to $\mathbf{C}$ then for all $c \in C_-$ and $d \in B_+$, $\vartheta_-(c)Bd \implies cC\vartheta_+(d)$.

Suppose that $\varphi_-(\vartheta_-(c))Aa$ for some $c \in C_-$ and $a \in A_+$. Then because φ is a morphism, $\vartheta_-(c)B\varphi_+(a)$. Then because ϑ is a morphism, $cC\vartheta(\varphi_+(a))$. □

It is shown in Lemma 3.2.1 that $\mathbf{A} \to \mathbf{A}$ (reflexivity). However, it should be noted that symmetry does not hold generally, i.e. $\mathbf{A} \to \mathbf{B} \not\Leftrightarrow \mathbf{B} \to \mathbf{A}$. If $\mathbf{A} \to \mathbf{B} \wedge \mathbf{B} \to \mathbf{A}$, $\mathbf{A}$ and $\mathbf{B}$ are said to be *bimorphic*.

The full power of transitivity can be used with bimorphic relations because it allows us to consider any two bimorphic relations to be "equivalent" with respect to the morphism relation.

Lemma 2.3.2. *Suppose* $\mathbf{A}$ *and* $\mathbf{B}$ *are bimorphic relations. Then for a relation* $\mathbf{C}$*,* $\mathbf{A} \to \mathbf{C}$ *if and only if* $\mathbf{B} \to \mathbf{C}$*. Additionally,* $\mathbf{C} \to \mathbf{A}$ *if and only if* $\mathbf{C} \to \mathbf{B}$*.*

Proof. Suppose $\mathbf{A} \to \mathbf{C}$. Then since $\mathbf{B} \to \mathbf{A}$ and by Lemma 2.3.1, $\mathbf{B} \to \mathbf{C}$. By a similar argument, $\mathbf{B} \to \mathbf{C} \implies \mathbf{A} \to \mathbf{C}$.

Now, suppose $\mathbf{C} \to \mathbf{A}$. Then since $\mathbf{A} \to \mathbf{B}$ and by Lemma 2.3.1, $\mathbf{C} \to \mathbf{B}$. By a similar argument, $\mathbf{C} \to \mathbf{B} \implies \mathbf{C} \to \mathbf{A}$. □

This equivalence proves useful in simplifying our quest: instead of determining the existence or non-existence of a morphism between any two finite relations, we can consider bimorphic classes of relations. In particular, we can choose to consider a representative from each bimorphic class that is of "minimal complexity." Finding bimorphic forms of minimal complexity is a focus of Chapter 3. These minimal forms serve as the building blocks for the classification of small finite relations in Chapter 4.

CHAPTER 3

PARTIAL RESULTS ON THE MORPHISM DECISION PROBLEM

The goal of the morphism decision problem is to find sufficient conditions for $\mathbf{A} \to \mathbf{B}$ and sufficient conditions for $\mathbf{A} \not\to \mathbf{B}$.

Section 3.1 provides an alternative structure (minus-surjective homomorphisms) for finding morphisms between relations. The focus of Section 3.2 is finding, for a given $\mathbf{A}$, a variety of relations $\mathbf{A}'$ such that $\mathbf{A} \to \mathbf{A}'$ or $\mathbf{A}' \to \mathbf{A}$. We can then use transitivity to apply these moves in sequence. Section 3.3 provides one such application to result in bimorphic forms of "reduced complexity". Section 3.4 outlines several sufficient conditions for morphism existence or non-existence.

3.1 Homomorphisms between Relations

We start by defining a homomorphism between relations.

Definition 3.1.1. A **homomorphism** between relations $\mathbf{A} = (A_-, A_+, A)$ and $\mathbf{B} = (B_-, B_+, B)$ is a function $f = g \cup h$ where

$$g : A_- \to B_-$$

$$h : A_+ \to B_+$$

such that if xAy then $f(x)Bf(y)$ for all $x \in A_-$ and $y \in A_+$.

We impose further restriction on the homomorphism in Definition 3.1.2. This definition is custom-designed to yield a natural morphism, as seen in Lemma 3.1.3. Figure 3.1 gives an example of such a homomorphism.

Definition 3.1.2. A **minus-surjective homomorphism** between relations $\mathbf{A} = (A_-, A_+, A)$ and $\mathbf{B} = (B_-, B_+, B)$ is a homomorphism $f = g \cup h$ where

$$g : A_- \to B_-$$

$$h : A_+ \to B_+$$

such that g is surjective.

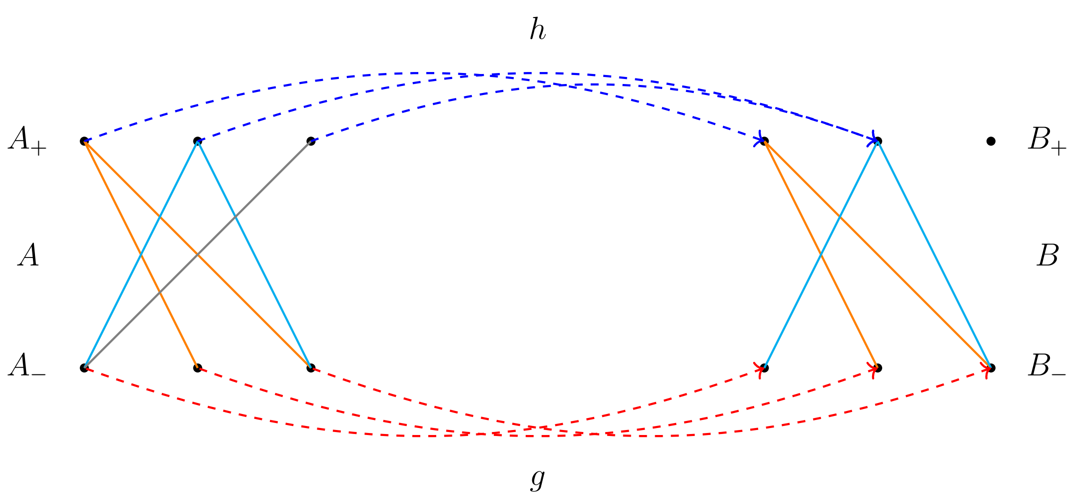


Figure 3.1: Example: A minus-surjective homomorphism

Lemma 3.1.3. *Let* $\mathbf{A}$, $\mathbf{B}$ *be relations and suppose* $f = g \cup h$ *is a a minus-surjective homomorphism from* $\mathbf{A}$ *to* $\mathbf{B}$. *Let* $j : B_- \to A_-$ *satisfy* $g \circ j = id_{B_-}$. *Then* $\varphi = (j, h)$ *is a morphism from* $\mathbf{A}$ *to* $\mathbf{B}$.

Proof. Suppose that $j(b)Aa$ for some $b \in B_-$, $a \in A_+$. Then by Definition 3.1.1, $f(j(b))Bf(a)$, which simplifies to $bBh(a)$. $\square$

Definition 3.1.4 and Lemma 3.1.5 give us a tool for comparing certain closely-related relations using homomorphisms.

Definition 3.1.4. Let $\mathbf{A} = (A_-, A_+, A)$ and $\mathbf{B} = (B_-, B_+, B)$ be relations such that $A_- \subseteq B_-$ and $A_+ \subseteq B_+$. The **inclusion function from A to B** $i = j \cup k$ is defined such that:

- $j : A_- \to B_-$
- $k : A_+ \to B_+$
- $j(a) = a$ for all $a \in A_-$
- $k(a) = a$ for all $a \in A_+$

Lemma 3.1.5. *The inclusion function from* $\mathbf{A} = (A_-, A_+, A)$ *to* $\mathbf{B} = (B_-, B_+, B)$ *is a homomorphism if and only* $A \subseteq B$*.*

Proof. Suppose that $A \subseteq B$ and aAb. Then aBb, which implies that $i(a)Bi(b)$. Then i is a homomorphism.

Now suppose that i is a homomorphism and aAb. Then $i(a)Bi(b)$, which implies that aBb and $A \subseteq B$. $\square$

3.2 Incremental Transformations

Lemma 3.2.1 shows that adding edges to a relation $\mathbf{A}$ results in a relation $\mathbf{A}'$ such that $\mathbf{A} \to \mathbf{A}'$. Letting $A' = A$ proves that the morphism relation has the reflexive property, i.e. $\mathbf{A} \to \mathbf{A}$.

Lemma 3.2.1. *Let* $\mathbf{A} = (A_-, A_+, A)$ *and* $\mathbf{A}^{'} = (A_-, A_+, A^{'})$ *be relations such that* $A \subseteq A^{'}$. *Then there exists a morphism from* $\mathbf{A}$ *to* $\mathbf{A}^{'}$.

Proof. Since $A \subseteq A^{'}$, by Lemma 3.1.5, the inclusion function $i = j \cup k$ from $\mathbf{A}$ to $\mathbf{A}^{'}$ is a homomorphism. Since $j : A_- \to A_-$ is bijective, it has a unique inverse. Then by Lemma 3.1.3, (j^{-1}, k) is a morphism from $\mathbf{A}$ to $\mathbf{A}^{'}$. □

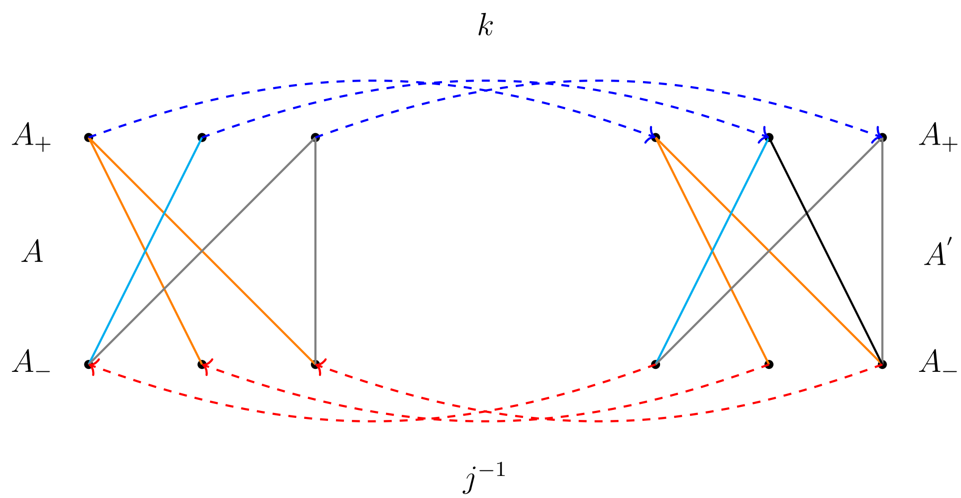


Figure 3.2: Illustration of Lemma 3.2.1. Here we add edges to A to obtain $\mathbf{A}^{'}$ and find a morphism from A to $\mathbf{A}^{'}$.

In Definition 3.2.2 we define the induced subrelation.

Definition 3.2.2. Suppose $\mathbf{A}$ is a relation and $S \subseteq A_-$ and $R \subseteq A_+$. The **induced subrelation** $\mathbf{A}[S, R]$ is the relation whose points consist of S and R and for any two points $s \in S$ and $r \in R$, s and r are related in $\mathbf{A}[S, R]$ if and only if s and r are related in $\mathbf{A}$. That is, $\mathbf{A}[S, R] = \{S, R, A^{'}\}$, where $A^{'} = A \restriction S \times R$.

Effectively, induced subrelations can be used to "delete" points. When points are deleted from A_-, there is a morphism from the original relation to the induced subrelation. This is illustrated in Figure 3.3.

Lemma 3.2.3. *Suppose* $\mathbf{A}$ *is a relation and* $\mathbf{A}^{'} = \mathbf{A}[S, A_{+}] = (S, A_{+}, A^{'})$ *is an induced subrelation of* $\mathbf{A}$*. Then there exists a morphism from* $\mathbf{A}$ *to* $\mathbf{A}^{'}$*.*

Proof. Let $i_{-} : S \to A_{-}$ and $i_{+} : A_{+} \to A_{+}$ be the inclusion maps. Suppose $i_{-}(x)Ay$ for some $x \in S$ and $y \in A_{+}$. Since i_{-} is the inclusion function, xAy, i.e. $(x, y) \in A$. Then by definition of $A^{'}$, $xA^{'}y$. Because $i_{+} : A_{+} \to A_{+}$ is the inclusion function, $xA^{'}i_{+}(y)$. $\square$

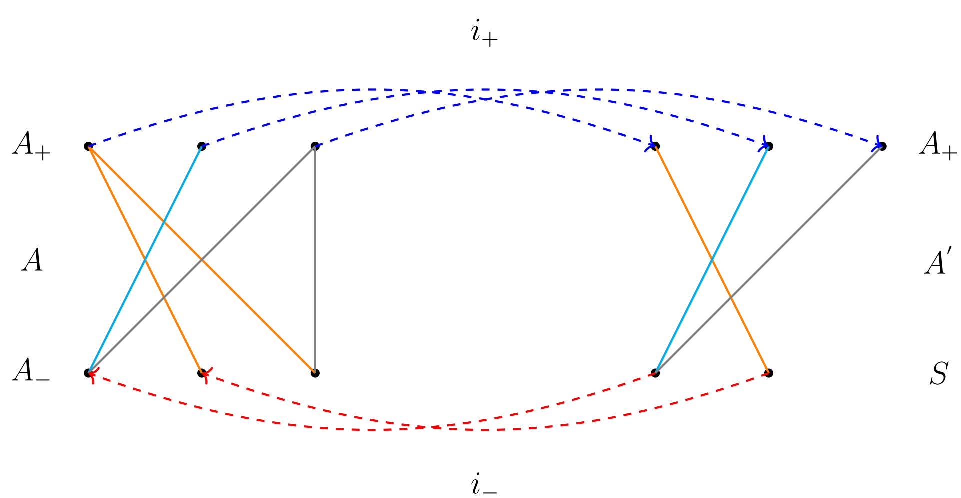


Figure 3.3: Illustration of Lemma 3.2.3. Here we delete points from A_{-} to obtain $\mathbf{A}^{'}$ and find a morphism from $\mathbf{A}$ to $\mathbf{A}^{'}$.

However, if only non-minimal points are deleted from A_{-}, there is also a morphism that goes the other direction. This is illustrated in Figure 3.4.

Lemma 3.2.4. *Let* $\mathbf{A} = (A_{-}, A_{+}, A)$ *and* $M \subseteq A_{-}$ *be the set of all* $\mathbf{A}$*-minimal points in* A_{-}*. Let* $\mathbf{A}^{'} = \mathbf{A}[N, A_{+}]$*, where* $M \subseteq N \subseteq A_{-}$*. Then there exists a morphism from* $\mathbf{A}^{'}$ *to* $\mathbf{A}$*.*

Proof. Define $\varphi_- : A_- \to N$ as follows: For all $a \in A_-$,

$$\varphi_-(a) = \begin{cases} a & \text{when } a \in M \\ b & \text{when } a \notin M \text{ for some } b \in M \text{ chosen such that } N_{\mathbf{A}}(b) \subsetneq N_{\mathbf{A}}(a) \end{cases}$$

Let i_+ be the identity function from A_+ to A_+. We will show that (φ_-, i_+) is a morphism from $\mathbf{A}'$ to $\mathbf{A}$: Suppose that $\varphi_-(x)A'y$ for some $x \in A_-$ and $y \in A_+$. If $x \in M$, then $xA'y$. Since $A' \subseteq A$, this implies that xAy. Then $xAi_+(y)$ also. If $x \notin M$, then $\varphi_-(x) = z$ for some $z \in M$ chosen such that $N_{\mathbf{A}}(z) \subsetneq N_{\mathbf{A}}(x)$. Then $zA'y$ and zAy. This implies that xAy, and so $xAi_+(y)$. $\square$

Then by Lemma 3.2.4 and Lemma 3.2.3, the resulting relation obtained by removing non-minimal points from A_- is bimorphic with the original relation.

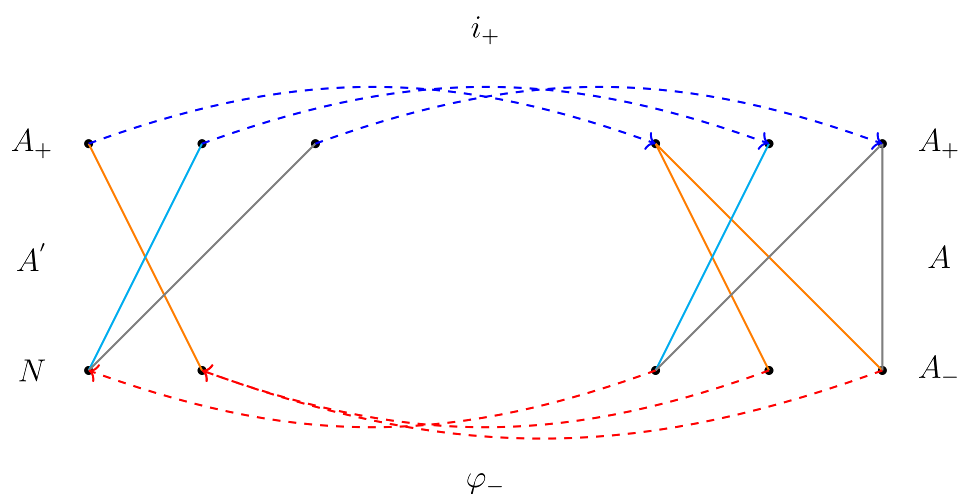


Figure 3.4: Illustration of Lemma 3.2.4. Here we delete non-minimal points from A_- to obtain $\mathbf{A}'$ and find a morphism from $\mathbf{A}'$ to $\mathbf{A}$.

The dual notion is deleting non-maximal points from A_+. Deleting points from A_+ results in a new relation that can morphism onto the original. This is illustrated in Figure 3.5.

Lemma 3.2.5. *Suppose* $\mathbf{A}$ *is a relation and* $\mathbf{A}' = \mathbf{A}[A_-, R] = \{A_-, R, A'\}$ *is an induced subrelation of* $\mathbf{A}$*. Then there exists a morphism from* $\mathbf{A}'$ *to* $\mathbf{A}$*.*

Proof. Note that $A_- \subseteq A_-$, $R \subseteq A_+$, and $A' \subseteq A$. Then the inclusion function from $\mathbf{A}'$ to $\mathbf{A}$ is a homomorphism (Lemma 3.1.5). Further, the inclusion function is minus-surjective, since $A_- = A_-$. Then by Lemma 3.1.3, there is a morphism from $\mathbf{A}'$ to $\mathbf{A}$. □

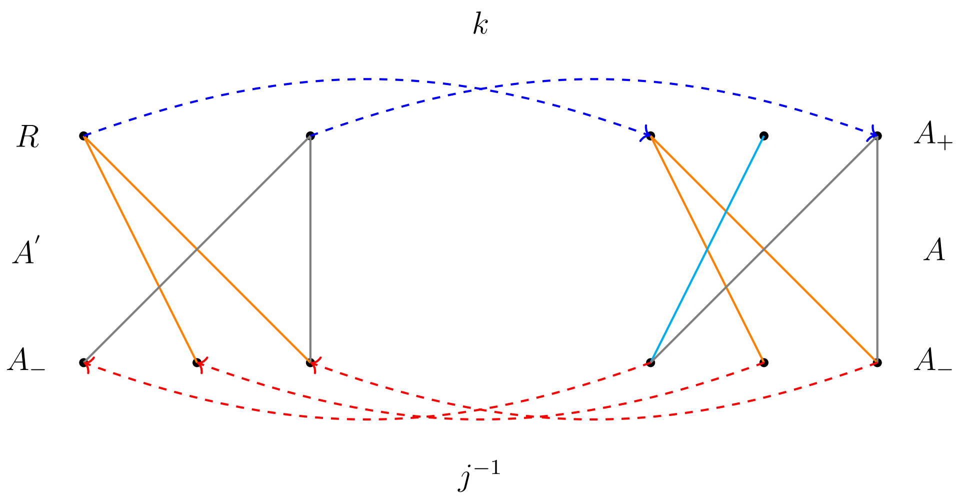


Figure 3.5: Illustration of Lemma 3.2.5. Here we delete points from A_+ to obtain $\mathbf{A}'$ and find a morphism from $\mathbf{A}'$ to $\mathbf{A}$.

However, if only non-maximal points are deleted from A_+, there is also a morphism that goes the other direction. This is illustrated in Figure 3.6.

Lemma 3.2.6. *Let* $\mathbf{A} = (A_-, A_+, A)$ *and* $M \subseteq A_+$ *be the set of all* $\mathbf{A}$*-maximal points in* A_+*. Let* $\mathbf{A}' = \mathbf{A}[A_-, N]$*, where* $M \subseteq N \subseteq A_+$*. Then there exists a morphism from* $\mathbf{A}$ *to* $\mathbf{A}'$*.*

Proof. Define $\varphi_+ : A_+ \to N$ as follows: For all $a \in A_+$,

$$\varphi_+(a) = \begin{cases} a & \text{when } a \in M \\ b & \text{when } a \notin M \text{ for some } b \in M \text{ chosen such that } N_{\mathbf{A}}(a) \subsetneq N_{\mathbf{A}}(b) \end{cases}$$

Let i_- be the identity function from A_- to A_-. We will show that (i_-, φ_+) is a morphism from $\mathbf{A}$ to $\mathbf{A}'$: Suppose that $i_-(x)Ay$ for some $x \in A_-$ and $y \in A_+$. Then xAy. If $y \in M$, then $xA'y$ by definition of $\mathbf{A}'$. Then $xA'\varphi_+(y)$. If $y \notin M$, then $\varphi_+(y) = z$ for some $z \in M$ chosen such that $N_{\mathbf{A}}(y) \subsetneq N_{\mathbf{A}}(z)$. This implies xAz. Then $xA'z$ and so $xA'\varphi_+(y)$. $\square$

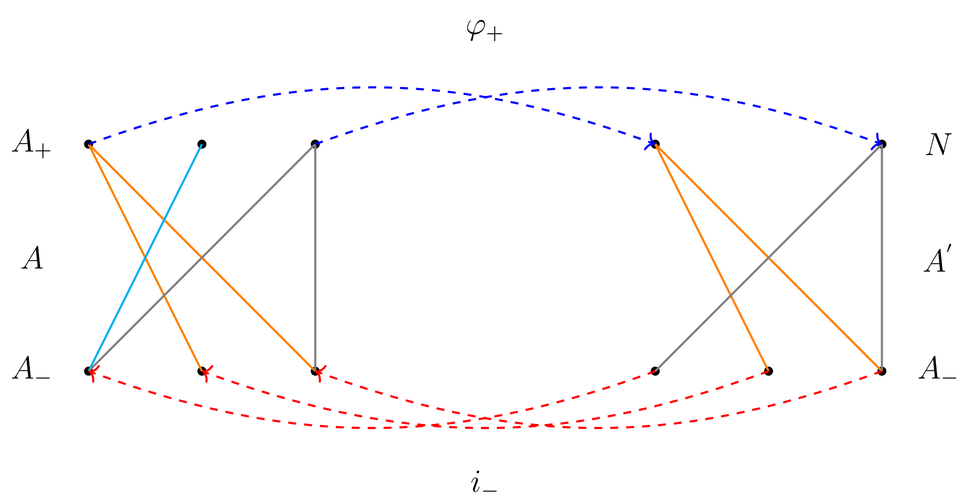


Figure 3.6: Illustration of Lemma 3.2.6. Here we delete non-maximal points from A_+ to obtain $\mathbf{A}'$ and find a morphism from $\mathbf{A}$ to $\mathbf{A}'$

Twin points can be seen as duplicate points in the relation, and as such can be removed to create a relation that is bimorphic with the original. This is illustrated in Figure 3.7.

Lemma 3.2.7. *Let* $\mathbf{A}$ *be a relation and let* $R \subseteq A_-$ *and* $S \subseteq A_+$ *be sets of points such that each point in* $A_- \setminus R$ *has an* $\mathbf{A}$*-twin in* R *and each point in* $A_+ \setminus S$ *has an* $\mathbf{A}$*-twin in* S*. Let* $\mathbf{A}' = \mathbf{A}[R, S]$*. Then there is a morphism from* $\mathbf{A}$ *to* $\mathbf{A}'$ *and from* $\mathbf{A}'$ *to* $\mathbf{A}$*.*

Proof. Let $i_- : R \to A_-$ be the inclusion function and define $\varphi_+ : A_+ \to S$ as

$$\varphi_+(x) = \begin{cases} x & \text{when } x \in S \\ y & \text{when } x \in A_+ \setminus S \text{ for } y \in S \text{ such that } y \text{ is an } \mathbf{A}\text{-twin of } x \end{cases}$$

Then we see that (i_-, φ_+) is a morphism from $\mathbf{A}$ to $\mathbf{A}'$: Suppose $i_-(b)Aa$ for some $b \in R$ and $a \in A_+$. Then bAa. Either $a \in S$ or $a \in A_+ \setminus S$. If $a \in S$, then $bA'a$ and $bA'\varphi_+(a)$. If $a \in A_+ \setminus S$, then it has an $\mathbf{A}$-twin $c \in S$ and bAc. This implies that $bA'c$ and $bA'\varphi_+(a)$.

Now, let $i_+ : S \to A_+$ be the inclusion function and define $\varphi_- : A_- \to R$ as

$$\varphi_-(x) = \begin{cases} x & \text{when } x \in R \\ y & \text{when } x \in A_- \setminus R \text{ for } y \in R \text{ such that } y \text{ is an } \mathbf{A}\text{-twin of } x \end{cases}$$

Then we see that (φ_-, i_+) is a morphism from $\mathbf{A}'$ to $\mathbf{A}$: Suppose $\varphi_-(b)A'a$ for some $b \in A_-$ and $a \in S$. Either $b \in R$ or $b \in A_- \setminus R$. If $b \in R$, then $bA'a$. Then bAa and $bAi_+(a)$. If $b \in A_- \setminus R$, then it has an $\mathbf{A}$-twin $c \in R$ and cAa. This implies that bAa and $bAi_+(a)$. $\square$

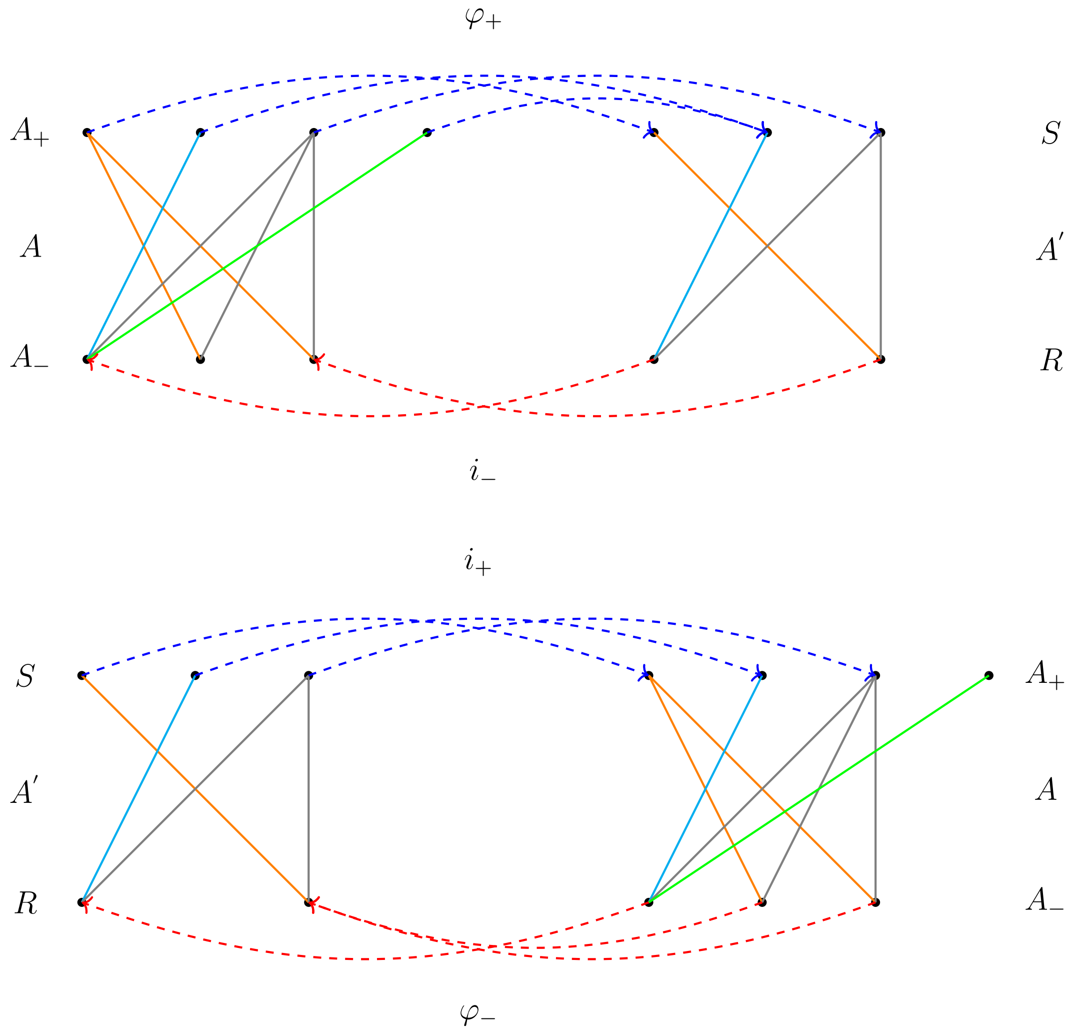


Figure 3.7: Illustration of Lemma 3.2.7. Here we delete twin points from A to obtain A′ and find morphisms from A to A′ and from A′ to A.

3.3 Skeleton Bimorphic Form

The operations of deleting twins, deleting non-minimal points from A_-, and deleting non-maximal points from A_+ allow us to transform a relation into one which is less complex, but still bimorphic with the original. We can then repeat these operations until they are no longer applicable to obtain a bimorphic form of "minimal" size.

The question arises whether the order of these operations matters. The status of the minimality or maximality of a point can change if other points are deleted first. However, the next two lemmas state that if a non-minimal or non-maximal point becomes minimal or maximal through the deletion of other points, it will become the twin of an existing point. In particular, once a point becomes "deletable," it stays deletable.

Lemma 3.3.1. *Let $\mathbf{A} = (A_-, A_+, A)$ be a relation and let $x \in A_-$ be a non $\mathbf{A}$-minimal point. Let $\mathbf{A}' = \mathbf{A}[A_-, A_+ \setminus \{y\}]$, for some $y \in A_+$. Then x is either non $\mathbf{A}'$-minimal or is an $\mathbf{A}'$-twin of an $\mathbf{A}$-minimal point.*

Proof. Because x is non $\mathbf{A}$-minimal, there exists $z \in A_-$ such that $N_{\mathbf{A}}(z) \subsetneq N_{\mathbf{A}}(x)$. Without loss of generality, suppose z is $\mathbf{A}$-minimal. Suppose $x \neg Ay$. Then $z \neg Ay$. Then $N_{\mathbf{A}'}(x) = N_{\mathbf{A}}(x)$ and $N_{\mathbf{A}'}(z) = N_{\mathbf{A}}(z)$. Then $N_{\mathbf{A}'}(z) \subsetneq N_{\mathbf{A}'}(x)$ and x is non $\mathbf{A}'$-minimal. Now suppose that xAy. Then either zAy or $z \neg Ay$.

First, suppose zAy. Then $N_{\mathbf{A}'}(x) = N_{\mathbf{A}}(x) \setminus \{y\}$ and $N_{\mathbf{A}'}(z) = N_{\mathbf{A}}(z) \setminus \{y\}$. Then $N_{\mathbf{A}'}(z) \subsetneq N_{\mathbf{A}'}(x)$ and x is non $\mathbf{A}'$-minimal.

Second, suppose $z \neg Ay$. Then $N_{\mathbf{A}'}(x) = N_{\mathbf{A}}(x) \setminus \{y\}$ and $N_{\mathbf{A}'}(z) = N_{\mathbf{A}}(z)$. Then $N_{\mathbf{A}'}(z) \subseteq N_{\mathbf{A}'}(x)$. Then either $N_{\mathbf{A}'}(z) \subsetneq N_{\mathbf{A}'}(x)$ and x is non $\mathbf{A}'$-minimal or $N_{\mathbf{A}'}(z) = N_{\mathbf{A}'}(x)$ and x is an $\mathbf{A}'$-twin of an z. ◻

Lemma 3.3.2. *Let $\mathbf{A} = (A_-, A_+, A)$ be a relation and let $x \in A_+$ be a non $\mathbf{A}$-maximal point. Let $\mathbf{A}' = \mathbf{A}[A_- \setminus \{y\}, A_+]$, for some $y \in A_-$. Then x is either non $\mathbf{A}'$-maximal or is an $\mathbf{A}'$-twin of an $\mathbf{A}$-maximal point.*

Proof. Because x is non $\mathbf{A}$-maximal, there exists $z \in A_+$ such that $N_{\mathbf{A}}(z) \supsetneq N_{\mathbf{A}}(x)$. Without loss of generality, suppose z is $\mathbf{A}$-maximal. Suppose $y \neg Az$. Then $y \neg Ax$.

Then $N_{\mathbf{A}'}(x) = N_{\mathbf{A}}(x)$ and $N_{\mathbf{A}'}(z) = N_{\mathbf{A}}(z)$. Then $N_{\mathbf{A}'}(z) \supsetneq N_{\mathbf{A}'}(x)$ and x is non $\mathbf{A}'$-maximal. Now suppose that yAz. Then either yAx or $y\neg Ax$.

First, suppose yAx. Then $N_{\mathbf{A}'}(x) = N_{\mathbf{A}}(x) \setminus \{y\}$ and $N_{\mathbf{A}'}(z) = N_{\mathbf{A}}(z) \setminus \{y\}$. Then $N_{\mathbf{A}'}(z) \supsetneq N_{\mathbf{A}'}(x)$ and x is non $\mathbf{A}'$-maximal.

Second, suppose $y\neg Ax$. Then $N_{\mathbf{A}'}(x) = N_{\mathbf{A}}(x)$ and $N_{\mathbf{A}'}(z) = N_{\mathbf{A}}(z) \setminus \{y\}$. Then $N_{\mathbf{A}'}(z) \supseteq N_{\mathbf{A}'}(x)$. Then either $N_{\mathbf{A}'}(z) \supsetneq N_{\mathbf{A}'}(x)$ and x is non $\mathbf{A}'$-maximal or $N_{\mathbf{A}'}(z) = N_{\mathbf{A}'}(x)$ and x is an $\mathbf{A}'$-twin of an z. $\square$

Putting this all together, we obtain a simple algorithm for finding a less complex bimorphic form of a relation.

Definition 3.3.3. The **skeleton bimorphic form** of a relation $\mathbf{A}$ is the relation resulting from the following algorithm:

1. Create an induced subrelation by retaining all the $\mathbf{A}$-maximal points in A_+ and all the $\mathbf{A}$-minimal points in A_-.

2. Repeat Step 1 until every point in A_+ is $\mathbf{A}$-maximal and every point in A_- is $\mathbf{A}$-minimal.

3. Create an induced subrelation by retaining a single representative for each set of twins.

The simplifying power of this algorithm from Definition 3.3.3 can be seen in Figure 3.8. The algorithm's usefulness lies in the fact that each operation yields a relation that is bimorphic with the original, justifying the name.

Lemma 3.3.4. *A relation* $\mathbf{A}$ *is bimorphic with its skeleton bimorphic form.*

Proof. By Lemmas 3.2.3 and 3.2.6, there is a morphism from $\mathbf{A}$ to the relation resulting from Step 1 of Definition 3.3.3. By Lemmas 3.2.5 and 3.2.4, there is a morphism from the relation resulting from Step 1 of Definition 3.3.3 to $\mathbf{A}$. Then they are bimorphic. Then by induction, $\mathbf{A}$ is bimorphic with the relation resulting after Step 2. Finally, by Lemma 3.2.7, $\mathbf{A}$ is bimorphic with the relation resulting after Step 3. □

3.4 Special Cases

This section will explore the morphism decision problem for special cases of $\mathbf{A}$ and $\mathbf{B}$. The case when $\delta(\mathbf{B}) = 1$ turns out to be very simple.

Lemma 3.4.1. *Let* $\mathbf{A}$*,* $\mathbf{B}$ *be finite relations. If* $\delta(\mathbf{B}) = 1$ *then there exists a morphism from* $\mathbf{A}$ *to* $\mathbf{B}$*.*

Proof. Let φ_+ map all points in A_+ to the element of a minimal $\mathbf{B}$-dominating family. Then $bB\varphi_+(a)$ is true for all $b \in B_-$ and $a \in A_+$. Then the implication $\varphi_-(b)Aa \implies bB\varphi_+(a)$ is true for all $\varphi_-(b) \in A_-$, $a \in A_+$, and $b \in B_-$. Then for any $\varphi_- : B_- \to A_-$, $\varphi = (\varphi_-, \varphi_+)$ is a morphism from $\mathbf{A}$ to $\mathbf{B}$. □

Figure 3.9 example illustrates Lemma 3.4.1. Note that in this example, $\delta(\mathbf{A}) = 2$ and $\delta(\mathbf{B}) = 1$. If φ_+ maps all values of A_+ to the single dominating element in B_+, then the morphism is guaranteed to work because the consequent $bB\varphi_+(a)$ of the implication $\varphi_-(b)Aa \implies bB\varphi_+(a)$ is always true. This can also be seen by examining the dual, in which case the antecedent $\varphi_+(a)\neg\check{B}b$ of the implication is always false, guaranteeing that the implication $\varphi_+(a)\neg\check{B}b \implies a\neg\check{A}\varphi_-(b)$ will be true.

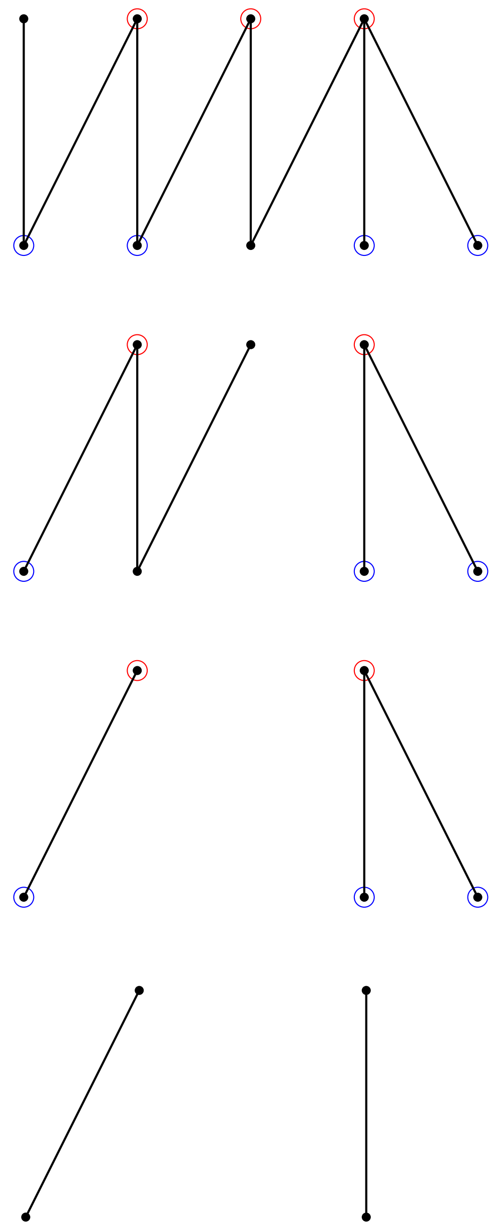


Figure 3.8: Finding the Skeleton Bimorphic Form of a Relation. Maximal points in A_+ are circled in red. Minimal point in A_- are circled in blue.

3.4.1 Ladder Relations

For a given dominating number, there is a certain class of special skeleton bimorphic forms whose bipartite graphs are shaped like a ladder, motivating the Definition

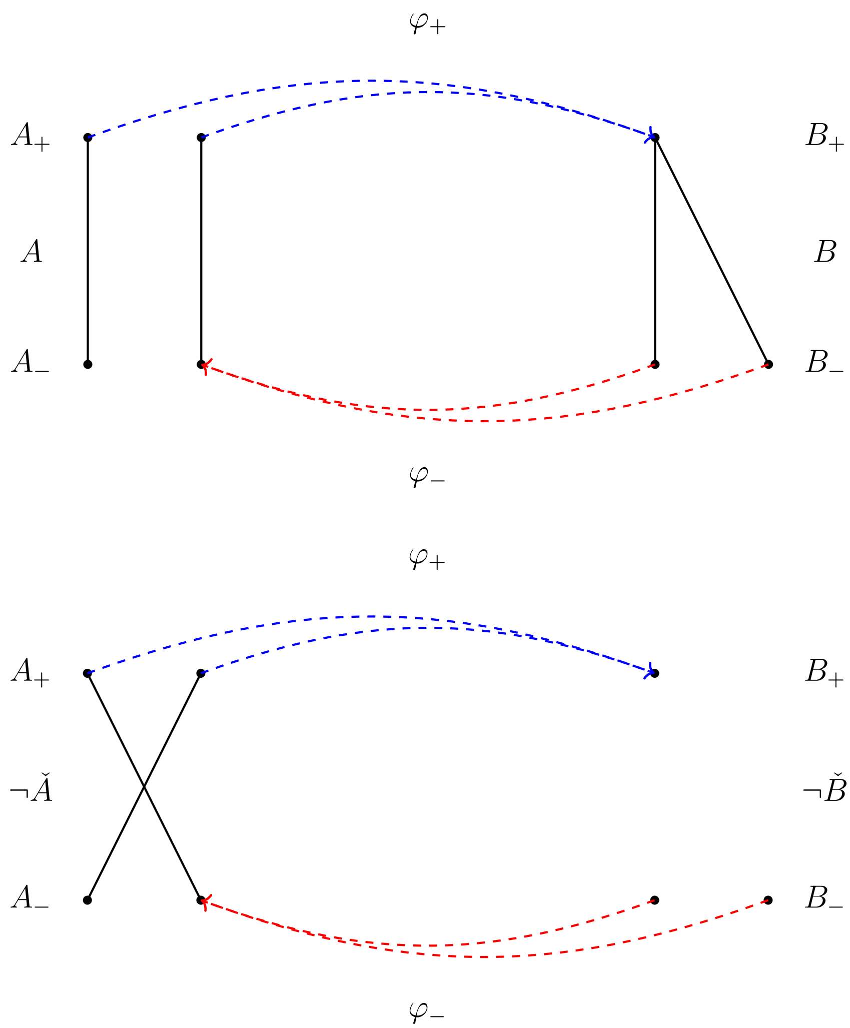


Figure 3.9: When $\delta(\mathbf{B}) = 1$, there is a morphism from A to B.

3.4.2.

Definition 3.4.2. A **ladder relation L** is a finite relation such that for each $x \in L_-$ there exists a unique $y \in L_+$ such that xLy and for each $y \in L_+$ there exists a unique $x \in L_-$ such that xLy. A ladder relation with dominating number n is referred to as an **n-ladder**.

The relation **A** in Figure 3.9 is an example of a 2-ladder.

Ladder relations are special in the sense that they can morphism onto any relation that shares their dominating number. In this way they are also the least complex relation with regards to the morphism relation for a given dominating number. Figure 3.10 provides an illustration.

Lemma 3.4.3. *Let* $\mathbf{A}$ *be a ladder relation and* $\mathbf{B}$ *be a relation such that* $\delta(\mathbf{A}) = \delta(\mathbf{B})$*. There is a morphism from* $\mathbf{A}$ *to* $\mathbf{B}$*.*

Proof. Let $D_{\mathbf{B}}$ be a minimal $\mathbf{B}$-dominating family and let $\varphi_+ : A_+ \to D_{\mathbf{B}}$ be a bijection. For each $b \in B_-$, $bB\beta$ for some $\beta \in D_{\mathbf{B}}$. Define $\varphi_-(b) = a$ for $a \in A_-$ such that $aA(\varphi_+)^{-1}(\beta)$.

Suppose $\varphi_-(b)A\alpha$ for some $b \in B_-$ and $\alpha \in A_+$. Then $\varphi_-(b)A(\varphi_+)^{-1}(\beta)$ for some $\beta \in D_{\mathbf{B}}$. By definition of φ_-, $bB\beta$. Then $bB\varphi_+(\alpha)$, which implies that (φ_-, φ_+) is a morphism from $\mathbf{A}$ to $\mathbf{B}$. □

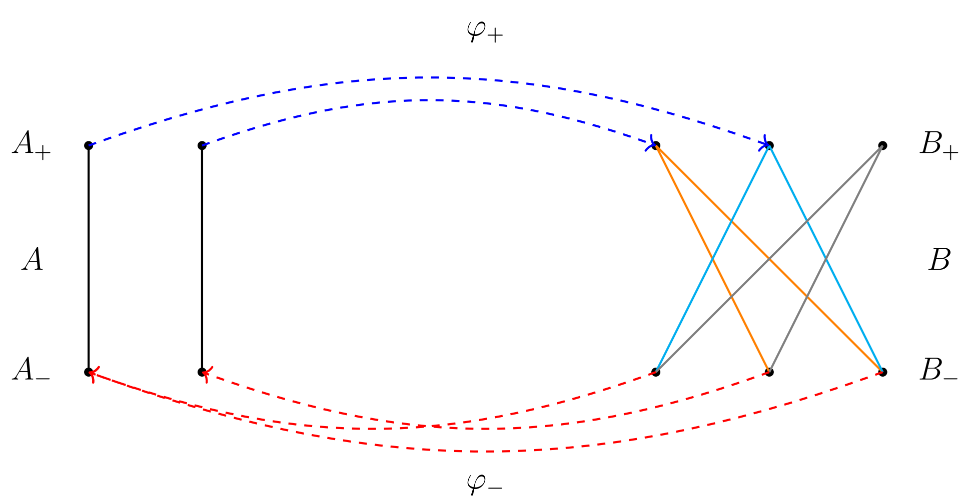


Figure 3.10: Example of Lemma 3.4.3. There is a morphism from the 2-ladder to any relation with dominating number 2.

Ladder relations also morphism onto other ladder relations of lesser size. An example can be seen in Figure 3.11. Then by transitivity, ladders morphism onto any relation with a dominating number less than or equal to their own.

Lemma 3.4.4. *If* $\mathbf{A}$ *and* $\mathbf{B}$ *are ladder relations such that* $\delta(\mathbf{A}) \geq \delta(\mathbf{B})$*, then there is a morphism from* $\mathbf{A}$ *to* $\mathbf{B}$*.*

Proof. Let $\varphi_- : B_- \to A_-$ be an injective function. For each $\alpha \in A_+$, define $\varphi_+(\alpha)$ as follows: For the unique $a \in A_-$ such that $aA\alpha$, if $a = \varphi_-(b)$ for some $b \in B_-$, then define $\varphi_+(\alpha) = \beta$, where $bB\beta$. Otherwise, α can be mapped to any point in B_+.

Suppose $\varphi_-(b)A\alpha$ for some $b \in B_-$ and $\alpha \in A_+$. Then by definition of φ_+, $\varphi_+(\alpha) = \beta$, where $bB\beta$. Then $bB\varphi_+(\alpha)$, which implies that (φ_-, φ_+) is a morphism from $\mathbf{A}$ to $\mathbf{B}$. $\square$

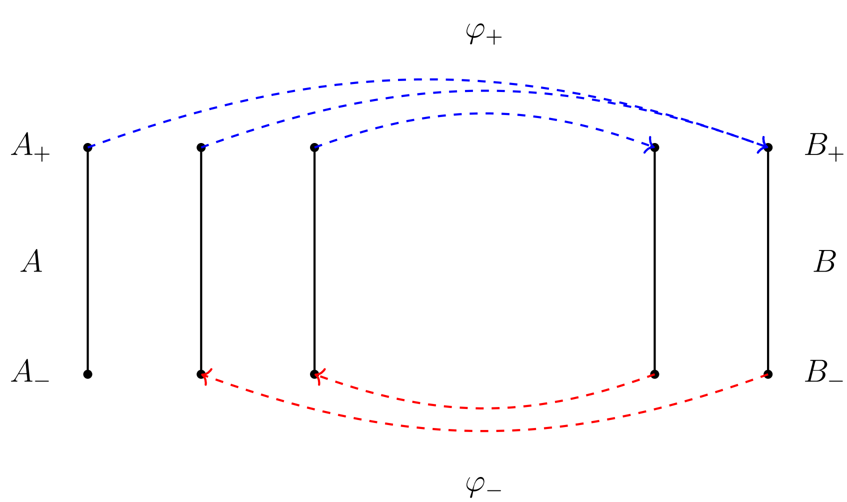


Figure 3.11: Example of Lemma 3.4.4. There is a morphism from the 3-ladder to the 2-ladder because $3 \geq 2$.

2-Ladders

The 2-ladder is particularly interesting because it is isomorphic to its dual. This leads to some interesting results that suggest the centrality of the 2-ladder in finite relations.

Lemma 3.4.5. *Let* $\mathbf{A}$ *be a relation. If* $\delta(\mathbf{A}^{\perp}) = 2$*, then there is a morphism from* $\mathbf{A}$ *to the* 2*-ladder.*

Proof. Let $\mathbf{B}$ be a 2-ladder. Note that $\mathbf{B}^{\perp}$ is also a 2-ladder. Then by Lemma 3.4.3, there is a morphism from $\mathbf{B}^{\perp}$ to $\mathbf{A}^{\perp}$. Then there exists a morphism from $\mathbf{A}$ to $\mathbf{B}$. □

In particular, this implies that any relation with a dominating number of 2 that has a dual relation with dominating number of 2 is bimorphic with the 2-ladder.

Corollary 3.4.6. *Let* $\mathbf{A}$ *be a relation. If* $\delta(\mathbf{A}) = \delta(\mathbf{A}^{\perp}) = 2$*, then* $\mathbf{A}$ *is bimorphic with the* 2*-ladder.*

Proof. Let $\mathbf{B}$ be a 2-ladder. By Lemma 3.4.3, there is a morphism from $\mathbf{B}$ to $\mathbf{A}$. By Lemma 3.4.5, there is a morphism from $\mathbf{A}$ to $\mathbf{B}$. □

Another implication of Lemma 3.4.5 is that $\mathbf{A}$ is comparable via morphism to $\mathbf{A}^{\perp}$ as long as at least one of them have a dominating number of 2.

Corollary 3.4.7. *For a relation* $\mathbf{A}$*, if* $\delta(\mathbf{A}^{\perp}) = 2$*, then there is a morphism from* $\mathbf{A}$ *to* $\mathbf{A}^{\perp}$*.*

Proof. By Lemma 3.4.5, there is a morphism from $\mathbf{A}$ to the 2-ladder. By Lemma 3.4.3 there is a morphism from a 2-ladder to $\mathbf{A}^{\perp}$. Then by transitivity, there is a morphism from $\mathbf{A}$ to $\mathbf{A}^{\perp}$. □

3.4.2 Further Special Results on Morphism Existence

Informally, the idea behind Lemma 3.4.8 is that if φ_- has to be a non-surjective function, then the points that aren't included in the image of φ_- can be thought of as having been deleted.

Lemma 3.4.8. *Let* $\mathbf{A}$ *and* $\mathbf{B}$ *be relations such that* $|A_-| \geq |B_-|$. *There exists a morphism from* $\mathbf{A}$ *to* $\mathbf{B}$ *if and only if there exists* $\mathbf{A}' = \mathbf{A}[C, A_+]$, *where* $C \subseteq A_-$ *and* $|C| = |B_-|$, *such that there exists a morphism from* $\mathbf{A}'$ *to* $\mathbf{B}$.

Proof. Suppose $\varphi = (\varphi_-, \varphi_+)$ is a morphism from $\mathbf{A}$ to $\mathbf{B}$. Note that $|\varphi_-(B_-)| \leq |B_-|$, so $\varphi_-(B_-) \subseteq C$ for some $C \subseteq A_-$ and $|C| = |B_-|$. Then let $\mathbf{A}' = \mathbf{A}[C, A_+]$ and let $\varphi_-' : B_- \to C$ be defined as $\varphi_-'(x) = \varphi_-(x)$ for all $x \in B_-$. Then (φ_-', φ_+) is obviously a morphism from $\mathbf{A}'$ to $\mathbf{B}$.

Now suppose that there exists $\mathbf{A}' = \mathbf{A}[C, A_+]$, where $C \subseteq A_-$ and $|C| = |B_-|$, such that there exists a morphism from $\mathbf{A}'$ to $\mathbf{B}$. By Lemma 3.2.3, there is a morphism from $\mathbf{A}$ to $\mathbf{A}'$. Then by transitivity, there is a morphism from $\mathbf{A}$ to $\mathbf{B}$. $\square$

We can now prove that the relations in Figure 2.5 provide a counter example to the conjecture that $\delta(\mathbf{A}) \geq \delta(\mathbf{B}) \wedge \delta(\mathbf{A}^\perp) \leq \delta(\mathbf{B}^\perp) \implies \mathbf{A} \to \mathbf{B}$. As can been seen in Figure 3.12, each subrelation of $\mathbf{A}$ that retains 3 points in A_- has a dominating number < 3. Then by Corollary 2.1.8 there does not exist a morphism from any of these subrelations to $\mathbf{B}$. Then by Lemma 3.4.8, there cannot exist a morphism from $\mathbf{A}$ to $\mathbf{B}$.

Lemma 3.4.9 provides a sufficient condition for a morphism to exist. Informally, the idea behind this lemmas is that if two larger relations are made up of smaller, disjoint relations, then the larger relations will have a morphism if the smaller ones do.

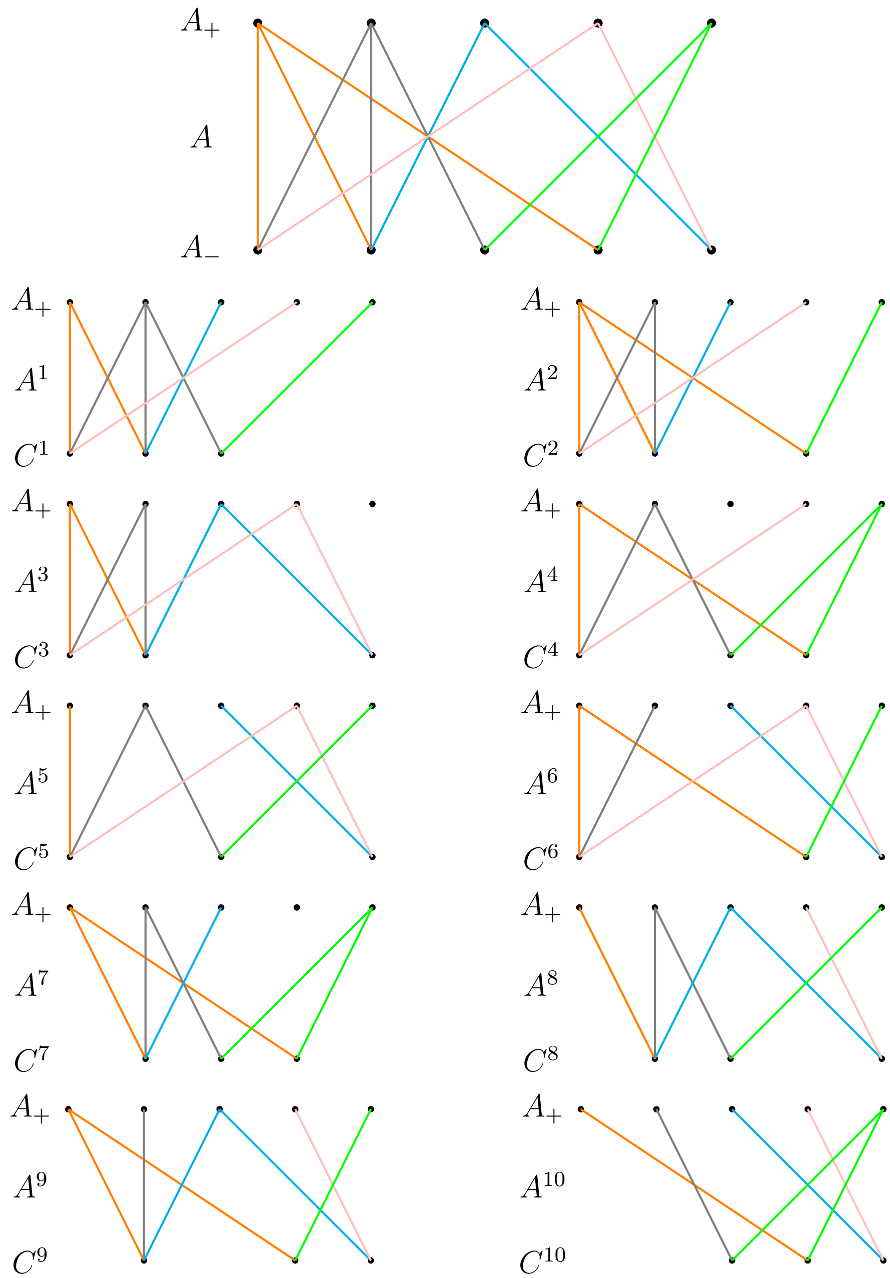


Figure 3.12: Counter Example from Figure 2.5 continued. Each subrelation has a dominating number < 3.

Lemma 3.4.9. *Let* $\mathbf{A}$, $\mathbf{B}$, $\mathbf{C}$, *and* $\mathbf{D}$ *be disjoint relations, and let* $\mathbf{X}$ *and* $\mathbf{Y}$ *be relations such that* $X_- = A_- \cup C_-$, $X_+ = A_+ \cup C_+$, $X = A \cup C$, $Y_- = B_- \cup D_-$, $Y_+ = B_+ \cup D_+$,

$Y = B \cup D$. *If there exists a morphism from* $\mathbf{A}$ *to* $\mathbf{B}$ *and from* $\mathbf{C}$ *to* $\mathbf{D}$*, then there exists a morphism from* $\mathbf{X}$ *to* $\mathbf{Y}$.

Proof. Suppose $\varphi = (\varphi_-, \varphi_+)$ is a morphism from $\mathbf{A}$ to $\mathbf{B}$ and $\vartheta = (\vartheta_-, \vartheta_+)$ is a morphism from $\mathbf{C}$ to $\mathbf{D}$. We will show that $\varsigma = (\varsigma_-, \varsigma_+)$, where $\varsigma_- = \varphi_- \cup \vartheta_-$ and $\varsigma_+ = \varphi_+ \cup \vartheta_+$, is a morphism from $\mathbf{X}$ to $\mathbf{Y}$: Suppose for some $y \in Y_-$ and $x \in X_+$ that $\varsigma_-(y)Xx$. Then either $y \in B_-$, $x \in A_+$ and $\varphi_-(x)Ay$ **or** $y \in D_-$, $x \in C_+$ and $\vartheta_-(x)Cy$. This implies either $xB\varphi_+(y)$ or $xD\vartheta_+(y)$. Either way, $xY\varsigma_+(y)$. $\square$

CHAPTER 4

CLASSIFICATION OF SMALL FINITE RELATIONS

Definition 4.0.1. The set of relations with m elements in the domain and n elements in the codomain will be denoted as $\mathbf{R}_{m,n}$. i.e.

$$\mathbf{R}_{m,n} = \{(A_-, A_+, A) : |A_-| = m \wedge |A_+| = n\}$$

We observe that if $m^{'} \leq m$ and $n^{'} \leq n$, for all $\mathbf{A} \in \mathbf{R}_{m^{'},n^{'}}$, there exists $\mathbf{A}^{'} \in \mathbf{R}_{m,n}$ such that $\mathbf{A}$ and $\mathbf{A}^{'}$ are bimorphic. This is because a smaller relation can be "padded" with twins to create a morphism that is bimorphic with the original (see Lemma 3.2.7). Rather than study morphisms between all relations in $\mathbf{R}_{m^{'},n^{'}}$ such that $m^{'} \leq m$ and $n^{'} \leq n$, it suffices to study the relations just in $\mathbf{R}_{m,n}$.

4.1 $\mathbf{R}_{5,5}$ and Computation

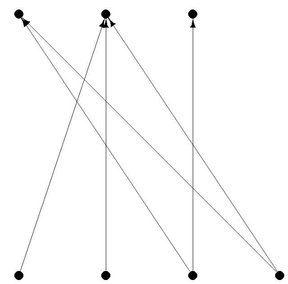
This section focuses on classifying the morphism problem for all relations in $\mathbf{R}_{m,n}$ such that $m \leq 5$ and $n \leq 5$. Even though we could just consider the relations in $\mathbf{R}_{5,5}$, we include the smaller relations due to the ease of computation. The computation is done using the R programming language. The full code is included in Appendix A.

In the code, we represent the relations primarily as *incidence matrices*. For a given $\mathbf{A} = (A_-, A_+, A)$, the elements of A_- are represented as rows and the elements of A_+ as columns of the matrix. For each element x_{ij} of the matrix, $x_{ij} = 1$ if the corresponding elements of A_- and A_+ relate to each other. Otherwise, $x_{ij} = 0$. For example, the code generates the following 4×3 matrix, representing a relation with 4 elements in A_- and 3 elements in A_+:

```
> matrices_1to5_x_1to5$Size_4x3[[101]]
     [,1] [,2] [,3]
[1,]    0    1    0
[2,]    0    1    0
[3,]    1    0    1
[4,]    1    1    0
```

We can also convert these matrix objects to directed graph objects. This is helpful for visualization and for checking for isomorphisms between relations. Using the same matrix from the previous example:

```
> graph_from_matrix(matrices_1to5_x_1to5$Size_4x3[[101]])
IGRAPH 59f9fa7 D--B 7 6 --
+ attr: type (v/l)
+ edges from 59f9fa7:
[1] 1->6 2->6 3->5 3->7 4->5 4->6
```



Using this framework, we generated all finite relations in $\mathbf{R}_{m,n}$ such that $m \leq 5$ and $n \leq 5$. We then ran all of these relations through the algorithm defined in Definition 3.3.3 to get their corresponding skeleton bimorphic form. This resulted in 32 unique (up to isomorphism) relations. The graphical output for each of these can be seen in Table 4.1.

This set of 32 is closed with respect to the dual operation, that is, for each $\mathbf{A}$ in the set, $\mathbf{A}^{\perp}$ is also in the set. This is explained by Lemma 2.2.7 and Corollary 2.2.8, which state that points in A_- that are $\mathbf{A}$-minimal are $\mathbf{A}^{\perp}$-maximal and points in A_+ that are $\mathbf{A}$-maximal are $\mathbf{A}^{\perp}$-minimal (and vice versa).

For each relation in Table 4.1, the dual and the associated dominating numbers are included in Table 4.2. This is output from the code.

To check for morphisms between each of the 32 relations, we ran the following algorithm for each pair of relations. The corresponding function in R code is called *morphism_from_AtoB* and can be seen in Appendix A.

1. Input two adjacency matrices, $\mathbf{A}$ and $\mathbf{B}$, representing the relations. (Rows of $\mathbf{A}$ represent the elements of A_-, Columns of $\mathbf{A}$ represent the elements of A_+.)

2. Find all the possible functions $\varphi_- : B_- \to A_-$ by getting all permutations (with repetition) of the rows of $\mathbf{A}$ of length $|B_-|$.

3. Find all the possible functions $\varphi_+ : A_+ \to B_+$ by getting all permutations (with repetition) of the columns of $\mathbf{B}$ of length $|A_+|$.

4. For each combination of φ_- and φ_+, do the following:

   (a) Create a $|B_-| \times |A_+|$ matrix representing the relation $\varphi_-(b_i)Aa_j$. Call this matrix $\mathbf{M}_1$.

   (b) Create a $|B_-| \times |A_+|$ matrix representing the relation $b_iB\varphi_+(a_j)$. Call this matrix $\mathbf{M}_2$.

   (c) Compare $\mathbf{M}_1$ and $\mathbf{M}_2$ coordinate-wise. If each element of $\mathbf{M}_2$ is greater than or equal to its corresponding element of $\mathbf{M}_1$, then (φ_-, φ_+) is a morphism from $\mathbf{A}$ to $\mathbf{B}$.

Each pair of $\mathbf{A}$ and $\mathbf{B}$ such that there is a morphism from $\mathbf{A}$ to $\mathbf{B}$ is output from the code and can be seen in Appendix B.

The Hasse Diagram in Figure 4.1 is created by using the output in Appendix B. A line that goes *downward* from a relation $\mathbf{A}$ to a relation $\mathbf{B}$ signifies that there exists a morphism from $\mathbf{A}$ to $\mathbf{B}$. Relations that are bimorphic are included on the same line and are separated by comma. The dominating number information from Table 4.2 is also included.

4.2 Discussion of Hasse Diagram

The Hasse Diagram allows us to observe experimentally many of the behaviors that were described in Chapter 3.

As stated in Lemma 3.4.4, ladders of larger size morphism onto ladders of smaller size. As stated in Lemma 3.4.3, ladders morphism onto relations that have the same dominating number.

As stated in Lemma 3.4.5, if $\delta(\mathbf{A}^{\perp}) = 2$ then $\mathbf{A}$ morphisms onto the 2-ladder and, as stated in Corollary 3.4.7, $\mathbf{A}$ morphisms onto $\mathbf{A}^{\perp}$. As stated in Corollary 3.4.6, all relations $\mathbf{A}$ such that $\delta(\mathbf{A}) = \delta(\mathbf{A}^{\perp}) = 2$ are bimorphic with the 2-ladder. These results, along with duality, provide explanation for the symmetry of the diagram.

The 3-ladder (4) is bimorphic with 16. This was anticipated by Lemma 3.4.9 since $4 = 3 \cup 2$ and $16 = 9 \cup 2$ (note that 3 and 9 are bimorphic).

Question 4.2.1. *Can we find general lemmas that justify the existence (and non-existence) of all the morphisms in the Hasse diagram?*

Question 4.2.2. *What does the classification look like for larger relations?*

Question 4.2.3. *What structural properties does the morphism order satisfy? (infinite chains, antichains, etc.)*

Question 4.2.4. *How does the number of skeleton bimorphic relations grow in $\mathbf{R}_{k,k}$ for $k > 5$?*

Question 4.2.5. *Can we find a "simpler" bimorphic form? i.e. one that has the property that if $\mathbf{A}$ and $\mathbf{B}$ are bimorphic then they are isomorphic? What is the algorithm to arrive at such a form?*

Question 4.2.6. *What is the distribution of dominating numbers for finite relations up to a certain size? What is the bivariate distribution of $\delta(\mathbf{A})$ and $\delta(\mathbf{A}^{\perp})$?*

Table 4.1: $\mathbf{R}_{5,5}$ Skeleton Bimorphic Forms

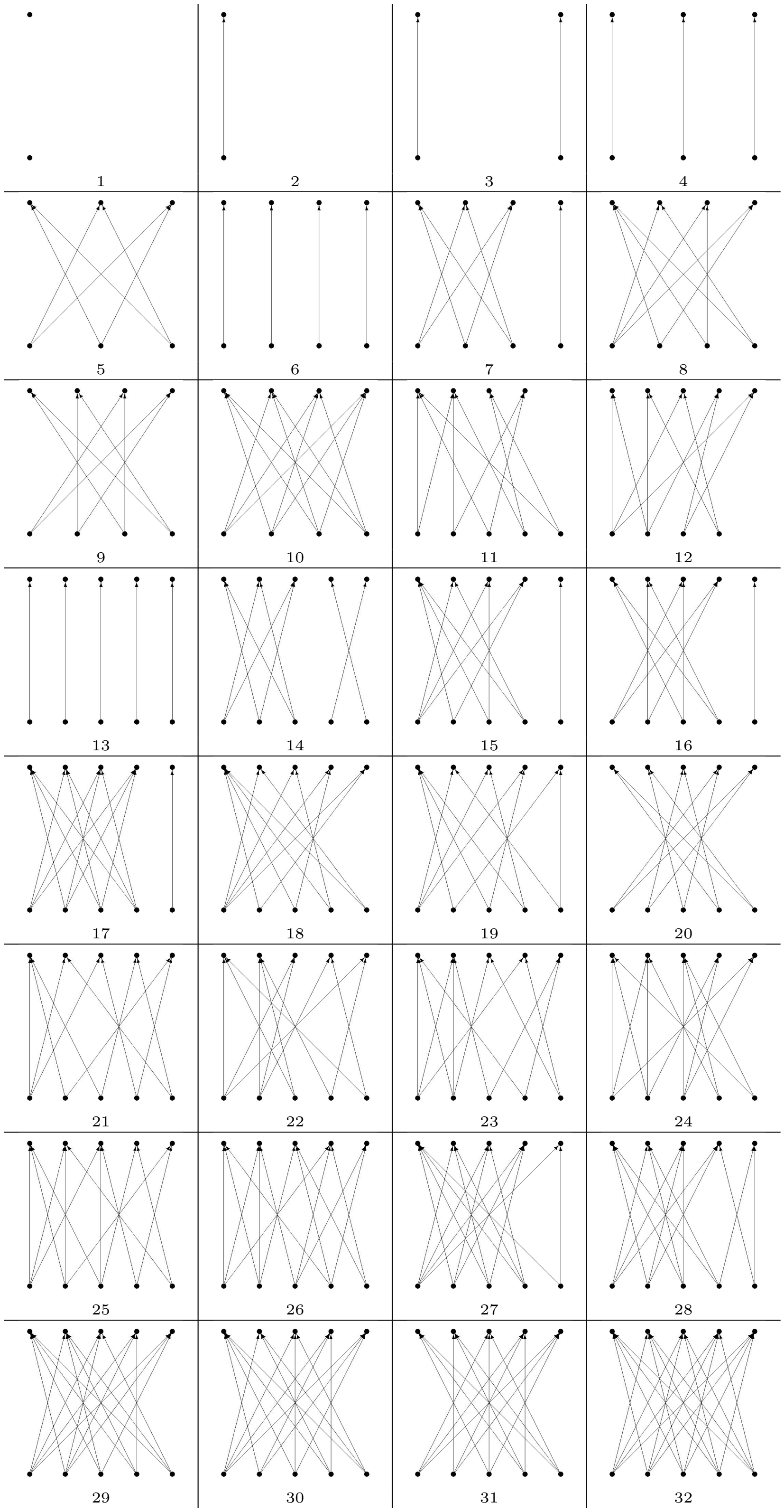


Table 4.2: Duals and Dominating Numbers for $R_{5,5}$ Skeleton Bimorphic Forms

$\mathbf{A}$	$\mathbf{A}^{\perp}$	$\delta(\mathbf{A})$	$\delta(\mathbf{A}^{\perp})$
1	2	∞	1
2	1	1	∞
3	3	2	2
4	5	3	2
5	4	2	3
6	10	4	2
7	8	3	2
8	7	2	3
9	9	2	2
10	6	2	4
11	12	2	2
12	11	2	2
13	32	5	2
14	29	4	2
15	27	3	2
16	30	3	2
17	18	3	2
18	17	2	3
19	28	2	2
20	31	3	2
21	26	3	2
22	25	2	2
23	24	3	2
24	23	2	3
25	22	2	2
26	21	2	3
27	15	2	3
28	19	2	2
29	14	2	4
30	16	2	3
31	20	2	3
32	13	2	5

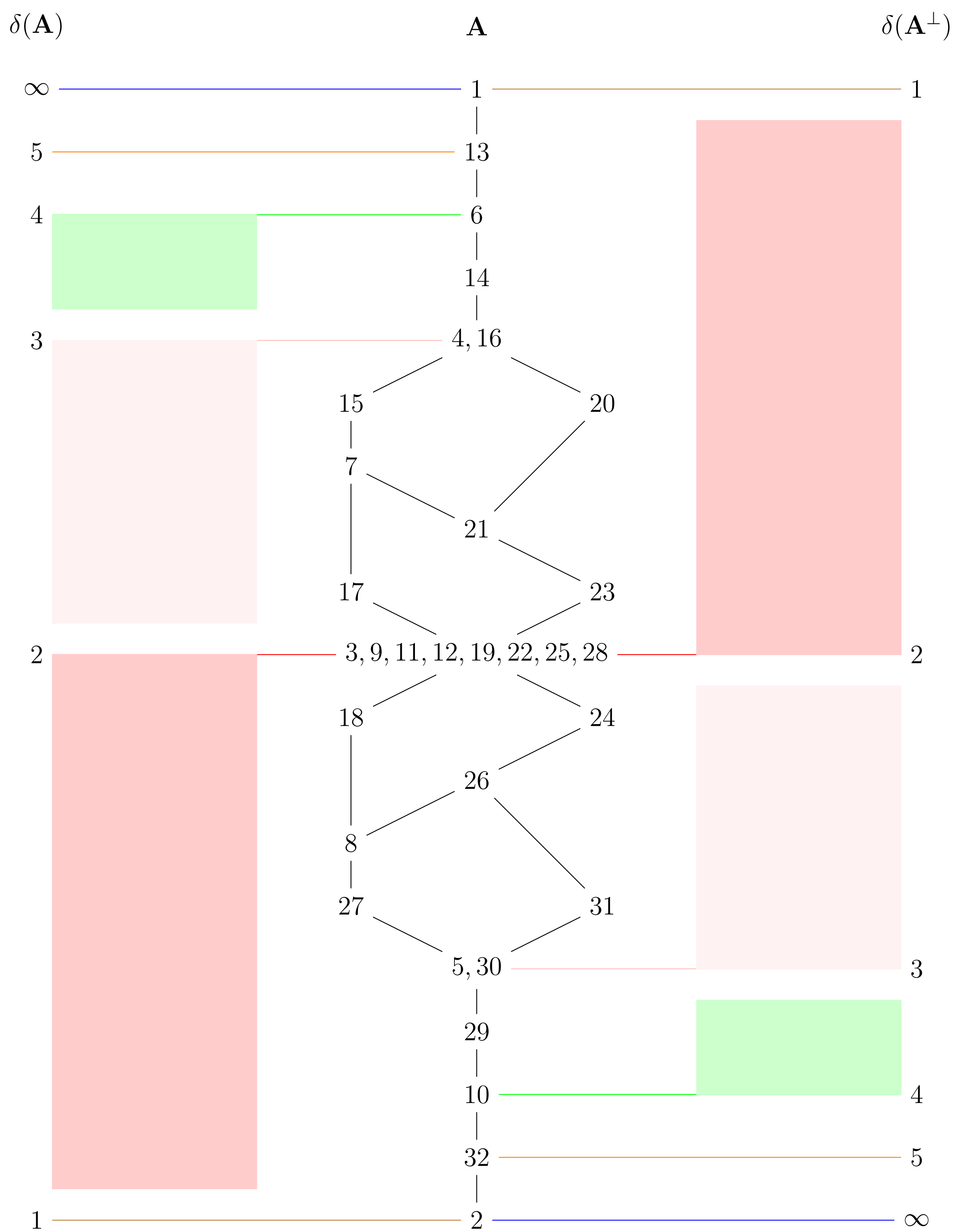


Figure 4.1: Hasse Diagram of $\mathbf{R}_{5,5}$ Skeleton Bimorphic Forms

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APPENDIX A

R CODE

```
install.packages(c("igraph","R.utils","gtools","data.table","RcppGreedySetCover"))

#Define Functions

graph_from_matrix <- function(M,plotGraph = T,graphName = "",saveGraph = F) {
  #Convert incidence matrices to graphs
  library(igraph)
  graphs_temp <- as.directed(graph.incidence(M), mode = "arbitrary")
  if(plotGraph == T) {
    if(length(M) == 1) {
      i <- 1
      j <- 1
    } else {
      i <- nrow(M)
      j <- ncol(M)
    }
    layout <- matrix(c(seq(0,i-1),seq(0,j-1),rep(0,i),rep(1,j)),byrow = F,nrow = i+j, ncol = 2)
    par(mar = c(0.5, 0.5, 0.5, 0.5))
    plot(graphs_temp,layout = layout, vertex.color = 'black',
         vertex.label = NA, vertex.size = 7,
         edge.arrow.size=1, edge.color = 'black', main = paste(graphName,sep=''))
  }
  if(saveGraph == T) {
    assign(graphName,graphs_temp,envir = .GlobalEnv)
  }
  return(graphs_temp)
}
```

```
generate_finite_relations <- function(Usizes,Vsizes,
                                      saveMatrices = T,saveGraphs = T,
                                      matricesListName = 'matrices',graphsListName = 'graphs'
) {
  library(R.utils)
  #Generate finite relations of a given size
  library(igraph)
  library(gtools)
  grid <- expand.grid(Usizes,Vsizes)
  #Create all possible binary vectors of length j
  #2^j possibilities at each step
  perm <- lapply(X = Vsizes,function(k) permutations(n= 2, r = k, v = 0:1,
                                                     repeats.allowed = TRUE))
  #Get "all" ixj matrices by combining (with replacement)
  #the vectors from the previous step i times
  #The zero vectors can be left out, since we are only interested in relations
  #that have dominating families
  #((2^j-1)+i-1)!/(i!((2^j-1)-1)!) possibilities at each step
  combi <- lapply(X = 1:nrow(grid),
                  function(k) combinations(n = nrow(perm[[match(grid[k,2],Vsizes)]])-1,
                                           r = grid[k,1],
                                           v = 2:nrow(perm[[match(grid[k,2],Vsizes)]]),
                                           repeats.allowed = TRUE))
  matrix_temp <- lapply(X = 1:length(combi),
                        function(k) lapply(X = 1:nrow(combi[[k]]),
                                           function(n) matrix(perm[[match(grid[k,2],Vsizes)]]
                                                                  [combi[[k]][n,],],
                                                              byrow = F, nrow = grid[k,1],
                                                              ncol = grid[k,2])))
  if(saveMatrices == T) {
    names(matrix_temp) <- lapply(X = 1:nrow(grid),
                                 function(k) paste0("Size_",grid[k,1],"x",grid[k,2]))
    matrix_name <- paste(matricesListName)
    assign(matrix_name,matrix_temp, envir = .GlobalEnv)
  }
  if(saveGraphs == T) {
    graphs_temp <- lapply(X = matrix_temp,
                          function(M) as.directed(graph.incidence(M), mode = "arbitrary"))
```

```
    graphs_name <- paste(graphsListName)
    assign(graphs_name,graphs_temp, envir = .GlobalEnv)
  }
}

count_of_graph_generation <- function(Usizes,Vsizes) {
  n <- 0
  for(j in Vsizes){
    #2^j possibilities at each step
    for(i in Usizes) {
      #Get "all" ixj matrices by combining (with replacement) the vectors from
      #the previous step i times
      #The zero vectors can be left out, since we are only interested
      #in relations that have dominating families
      #((2^j-1)+i-1)!/(i!((2^j-1)-1)!) possibilities at each step
      n <- n + factorial((2^j-1)+i-1)/(factorial(i)*factorial((2^j-1)-1))
    }
  }
  return(n)
}

dual_relation <- function(A) {
  #Find the dual of a finite relation
  Adual <- t(1- A)
  return(Adual)
}

setsystem_from_matrix <- function(M){
  library(data.table)
  #Convert incidence matrix to two column data frame (set system):
  j = ncol(M)
  S1 <- c()
  S2 <- c()
  for (k in 1:j) {
    # - 1st column is points in A+ (columns in the incidence matrix)
    S1 <- c(S1,rep(k,length(which(M[,k] == 1))))
    # - 2nd column contain the points in A- that they relate to
    S2 <- c(S2,which(M[,k] == 1))
```

```
  }
  S <- data.table("set" = S1, "element" = S2)
  return(S)
}

dominating_number <- function(M) {
  #Find the (approximate) dominating number of a finite relation
  #Test for if a dominating number exists
  if(0 %in% rowSums(M)) {return(NA)}
  #Test for dominating number = 1
  if(nrow(M) %in% colSums(M)) {return(1)}
  #Test for dominating number = 2
  combs <- t(combn(x = 1:ncol(M),m = 2))
  comp_value <- apply(X = combs, MARGIN = 1,
                      function(ind) pmax(M[,ind[1]],M[,ind[2]]))
  nrow(M) %in% colSums(comp_value)
  if(nrow(M) %in% colSums(comp_value)) {return(2)}
  #Test for dominating number = 3
  combs <- t(combn(x = 1:ncol(M),m = 3))
  comp_value <- apply(X = combs, MARGIN = 1,
                      function(ind) pmax(M[,ind[1]],M[,ind[2]],M[,ind[3]]))
  nrow(M) %in% colSums(comp_value)
  if(nrow(M) %in% colSums(comp_value)) {return(3)}
  #Test for dominating number = 4
  combs <- t(combn(x = 1:ncol(M),m = 4))
  comp_value <- apply(X = combs, MARGIN = 1,
                      function(ind) pmax(M[,ind[1]],M[,ind[2]],M[,ind[3]],M[,ind[4]]))
  nrow(M) %in% colSums(comp_value)
  if(nrow(M) %in% colSums(comp_value)) {return(4)}
  #Otherwise use a greedy set cover algorithm
  library(RcppGreedySetCover)
  S <- setsystem_from_matrix(M)
  invisible(capture.output(res <- greedySetCover(S,TRUE)))
  D <- uniqueN(res[,1])
  return(D)
}

maximal_points_from_matrix <- function(M,RowsOrCols = "Cols"){
```

```
  #Identify maximal points from a matrix
  if(RowsOrCols == "Cols") {
    z <- unlist(lapply(X = 1:ncol(M),
                       function(k) all(unlist(lapply(X = 1:ncol(M),
                           function(n) any(M[,k,drop = FALSE] > M[,n,drop = FALSE]) |
                                           all(M[,k,drop = FALSE] == M[,n,drop = FALSE]))))))
  }
  if(RowsOrCols == "Rows") {
    z <- unlist(lapply(X = 1:nrow(M),
                       function(k) all(unlist(lapply(X = 1:nrow(M),
                           function(n) any(M[k,,drop = FALSE] > M[n,,drop = FALSE]) |
                                           all(M[k,,drop = FALSE] == M[n,,drop = FALSE]))))))
  }
  return(z)
}

minimal_points_from_matrix <- function(M,RowsOrCols = "Rows"){
  #Identify maximal points from a matrix
  if(RowsOrCols == "Cols") {
    z <- unlist(lapply(X = 1:ncol(M),
                       function(k) all(unlist(lapply(X = 1:ncol(M),
                           function(n) any(M[,k,drop = FALSE] < M[,n,drop = FALSE]) |
                                           all(M[,k,drop = FALSE] == M[,n,drop = FALSE]))))))
    return(z)
  }
  if(RowsOrCols == "Rows") {
    z <- unlist(lapply(X = 1:nrow(M),
                       function(k) all(unlist(lapply(X = 1:nrow(M),
                           function(n) any(M[k,,drop = FALSE] < M[n,,drop = FALSE]) |
                                           all(M[k,,drop = FALSE] == M[n,,drop = FALSE]))))))
    return(z)
  }
}

remove_twins <- function(M){
  #Remove twin points from a matrix
  A <- M
  row_dups <- duplicated(A)
```

```
  col_dups <- duplicated(t(A))
  A <- A[!row_dups,!col_dups, drop = FALSE]
  return(A)
}

skeleton_bimorphic_form <- function(M){
  A <- M
  maximal_points <- maximal_points_from_matrix(A,RowsOrCols = "Cols")
  minimal_points <- minimal_points_from_matrix(A,RowsOrCols = "Rows")
  while(sum(maximal_points) < length(maximal_points) ||
        sum(minimal_points) < length(minimal_points)) {
    A <- A[which(minimal_points),which(maximal_points),drop = FALSE]
    maximal_points <- maximal_points_from_matrix(A,RowsOrCols = "Cols")
    minimal_points <- minimal_points_from_matrix(A,RowsOrCols = "Rows")
  }
  A <- remove_twins(A)
  return(A)
}

canonical_form <- function(M) {
  i <- nrow(M)
  j <- ncol(M)
  I <- rowSums(M)
  J <- colSums(M)
  if(identical(I,rep(1,i)) & identical(J,rep(1,j))) {
    return(diag(i))
  }
  else if (identical(I,rep(j-1,i)) & identical(J,rep(i-1,j))) {
    return(1-diag(i))
  }
  else {
    return(M[order(rowSums(-M)),order(colSums(-M)),drop = FALSE])
  }
}

morphism_from_AtoB <- function(A,B,printMorphisms = F) {
  library(R.utils)
  #Explicitly Test for Morphism
```

```
  Amin <- 1:nrow(A)
  Bmin <- 1:nrow(B)
  Apl <- 1:ncol(A)
  Bpl <- 1:ncol(B)

  phimin <- permutations(n = length(Amin),r = length(Bmin), v = Amin, repeats.allowed = TRUE)
  phipl  <- permutations(n = length(Bpl),r = length(Apl), v = Bpl, repeats.allowed = TRUE)

  rowsandcols <- unlist(lapply(X = 1:nrow(phimin),
                              function(k) lapply(X = 1:nrow(phipl),
                                  function(n) list(phimin[k,],phipl[n,]))),recursive = FALSE)

  M1 <- lapply(X = 1:length(rowsandcols), function(k) A[rowsandcols[[k]][[1]],])
  M2 <- lapply(X = 1:length(rowsandcols), function(k) B[,rowsandcols[[k]][[2]]])
  morphism <- unlist(lapply(X = 1:length(rowsandcols), function(k) all(M2[[k]] >= M1[[k]])))

  if(printMorphisms == T) {
    z <- rowsandcols[which(morphism)]
  }
  else{z <- length(which(morphism)) > 0}

  return(z)
}




#Generate all finite relations graphs of size 5x5 or less
generate_finite_relations(Usizes = 1:5,Vsizes = 1:5,
                          saveMatrices = T,saveGraphs = F,
                          matricesListName = "matrices_1to5_x_1to5",
                          graphsListName = "")

#Create Skeleton Relations
#Get the canonical form (order by degree) to make the duplicates more obvious
skeleton_matrices_1to5_x_1to5 <- lapply(X = 1:length(matrices_1to5_x_1to5),
                                       function(k) lapply(X = matrices_1to5_x_1to5[[k]],
                                        function(M) canonical_form(skeleton_bimorphic_form(M))))
```

```
#Find Unique (up to isomorphism) Skeleton Relations
#Remove exact duplicates
unique_skeleton_matrices_1to5_x_1to5_stage <- c(list(matrix(0)),
                                                unique(unlist(skeleton_matrices_1to5_x_1to5,
                                                recursive = FALSE)))
unique_skeleton_graphs_1to5_x_1to5_stage <- lapply(X = unique_skeleton_matrices_1to5_x_1to5_stage,
                                                   function(M) graph_from_matrix(M,plotGraph = F))

#Find all graphs that are isomorphic
isomorphic_list <- lapply(X = unique_skeleton_graphs_1to5_x_1to5_stage,
                          function(G1) unlist(lapply(X = unique_skeleton_graphs_1to5_x_1to5_stage,
                                                     function(G2) isomorphic(G1,G2,method = 'vf2'))))
isomorphism_classes <- unique(lapply(X = isomorphic_list, function(L) which(L)))

#Find all matrices that are symmetric
symmetric_list <- unlist(lapply(X = unique_skeleton_matrices_1to5_x_1to5_stage, function(M) t(isSymmetric(M))))
symmetric_indices <- which(symmetric_list)

#Choose a representative for each isomorphic class. Choose a symmetric relation if one exists.
unique_skeleton_relations_final_indices <- unlist(lapply(X = isomorphism_classes,
                                                        function(C){if(length(intersect(C,symmetric_indices)) == 0)
                                                        {C[1]} else{intersect(C,symmetric_indices)[1]}}))


unique_skeleton_matrices_1to5_x_1to5 <-
    unique_skeleton_matrices_1to5_x_1to5_stage[unique_skeleton_relations_final_indices]
unique_skeleton_graphs_1to5_x_1to5 <-
    unique_skeleton_graphs_1to5_x_1to5_stage[unique_skeleton_relations_final_indices]

#Check that skeleton relations are closed under dual
unique_skeleton_matrices_1to5_x_1to5_duals <- lapply(X = unique_skeleton_matrices_1to5_x_1to5,
                                                     function(M) dual_relation(M))
unique_skeleton_graphs_1to5_x_1to5_duals <- lapply(X = unique_skeleton_matrices_1to5_x_1to5_duals,
                                                   function(M) graph_from_matrix(M,plotGraph = F))
isomorphic_list_duals <- lapply(X = unique_skeleton_graphs_1to5_x_1to5_duals,
                                function(G1) unlist(lapply(X = unique_skeleton_graphs_1to5_x_1to5,
                                                           function(G2) isomorphic(G1,G2,method = 'vf2'))))
Aindex <- 1:length(unique_skeleton_matrices_1to5_x_1to5)
dualIndex <- unlist(lapply(X = isomorphic_list_duals, function(L) if(length(which(L)) == 0){NA}else{which(L)}))
```

```
identical(sort(dualIndex),Aindex)#returns TRUE

#Caclulate the Dominating Numbers, Dual Dominating Numbers, Size of A- and A+
Aminus <- unlist(lapply(X = unique_skeleton_matrices_1to5_x_1to5,function(M) nrow(M)))
Aplus  <- unlist(lapply(X = unique_skeleton_matrices_1to5_x_1to5,function(M) ncol(M)))
D      <- unlist(lapply(X = unique_skeleton_matrices_1to5_x_1to5,function(M) dominating_number(M)))
Ddual  <- unlist(lapply(X = unique_skeleton_matrices_1to5_x_1to5,function(M) dominating_number(dual_relation(M))))
skeleton_characteristics <- cbind(Aindex,Aminus,Aplus,D,Ddual,dualIndex)
colnames(skeleton_characteristics) <- c("Aindex","Aminus","Aplus","d(A)","d(Adual)","dualIndex")
skeleton_characteristics

#Print out all skeleton graphs
lapply(X = 1:length(unique_skeleton_matrices_1to5_x_1to5),
       function(k) {
         M <- unique_skeleton_matrices_1to5_x_1to5[[k]]
         mypath <- file.path("C:/Users/Jared/Desktop/Rhett Thesis",
                   paste0("[",k,"].jpg", sep = ""))
         jpeg(file=mypath)
         graph_from_matrix(M, plotGraph = T, saveGraph = F)
         dev.off()
       })

#Classification of morphisms for graphs up to 5x5
#Create a data frame with all combinations of skeleton relations up to 5x5
morphisms_1to5_x_1to5 <- merge(data.frame(skeleton_characteristics = skeleton_characteristics),
                               data.frame(skeleton_characteristics = skeleton_characteristics),
                               by = NULL)
names(morphisms_1to5_x_1to5) <- c("Aindex","Aminus","Aplus","dA","dA_dual","A_dualIndex",
                                  "Bindex","Bminus","Bplus","dB","dB_dual","B_dualIndex")

#Explict morphism checking
MorphismsFromAtoB <-
    lapply(X = 1:nrow(morphisms_1to5_x_1to5),
           function(k) morphism_from_AtoB(unique_skeleton_matrices_1to5_x_1to5[[morphisms_1to5_x_1to5[k,1]]],
           unique_skeleton_matrices_1to5_x_1to5[[morphisms_1to5_x_1to5[k,7]]],
                                                                  printMorphisms = T))

MorphismFromAtoB <- unlist(lapply(1:length(MorphismsFromAtoB),
```

```
                                  function(k) if(length(MorphismsFromAtoB[[k]]) == 0){FALSE} else{TRUE}))

morphisms_1to5_x_1to5 <- cbind(morphisms_1to5_x_1to5,MorphismFromAtoB)
morphisms_1to5_x_1to5[which(morphisms_1to5_x_1to5$MorphismFromAtoB == TRUE),c("Aindex","Bindex")]
```

APPENDIX B

MORPHISMS BETWEEN SKELETON RELATIONS

```
> morphisms_1to5_x_1to5[which(morphisms_1to5_x_1to5$MorphismFromAtoB == TRUE),c("Aindex","Bindex")]

    Aindex Bindex
1        1      1
33       1      2
34       2      2
35       3      2
36       4      2
37       5      2
38       6      2
39       7      2
40       8      2
41       9      2
42      10      2
43      11      2
44      12      2
45      13      2
46      14      2
47      15      2
48      16      2
49      17      2
50      18      2
51      19      2
52      20      2
53      21      2
54      22      2
55      23      2
56      24      2
57      25      2
58      26      2
59      27      2
60      28      2
61      29      2
62      30      2
63      31      2
64      32      2
65       1      3
67       3      3
68       4      3
70       6      3
71       7      3
73       9      3
75      11      3
76      12      3
77      13      3
78      14      3
79      15      3
80      16      3
81      17      3
83      19      3
84      20      3
85      21      3
86      22      3
87      23      3
89      25      3
92      28      3
97       1      4
100      4      4
102      6      4
109     13      4
110     14      4
112     16      4
129      1      5
131      3      5
132      4      5
133      5      5
134      6      5
135      7      5
136      8      5
137      9      5
139     11      5
140     12      5
141     13      5
142     14      5
143     15      5
144     16      5
145     17      5
146     18      5
147     19      5
148     20      5
149     21      5
150     22      5
151     23      5
```

152	24	5
153	25	5
154	26	5
155	27	5
156	28	5
158	30	5
159	31	5
161	1	6
166	6	6
173	13	6
193	1	7
196	4	7
198	6	7
199	7	7
205	13	7
206	14	7
207	15	7
208	16	7
225	1	8
227	3	8
228	4	8
230	6	8
231	7	8
232	8	8
233	9	8
235	11	8
236	12	8
237	13	8
238	14	8
239	15	8
240	16	8
241	17	8
242	18	8
243	19	8
244	20	8
245	21	8
246	22	8
247	23	8
248	24	8
249	25	8
250	26	8
252	28	8
257	1	9
259	3	9
260	4	9
262	6	9
263	7	9
265	9	9
267	11	9
268	12	9
269	13	9
270	14	9
271	15	9
272	16	9
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275	19	9
276	20	9
277	21	9
278	22	9
279	23	9
281	25	9
284	28	9
289	1	10
291	3	10
292	4	10
293	5	10
294	6	10
295	7	10
296	8	10
297	9	10
298	10	10
299	11	10
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313	25	10
314	26	10
315	27	10
316	28	10
317	29	10
318	30	10
319	31	10
321	1	11
323	3	11
324	4	11
326	6	11
327	7	11
329	9	11
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332	12	11
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335	15	11
336	16	11
337	17	11
339	19	11
340	20	11
341	21	11
342	22	11
343	23	11
345	25	11
348	28	11
353	1	12

355	3	12
356	4	12
358	6	12
359	7	12
361	9	12
363	11	12
364	12	12
365	13	12
366	14	12
367	15	12
368	16	12
369	17	12
371	19	12
372	20	12
373	21	12
374	22	12
375	23	12
377	25	12
380	28	12
385	1	13
397	13	13
417	1	14
422	6	14
429	13	14
430	14	14
449	1	15
452	4	15
454	6	15
461	13	15
462	14	15
463	15	15
464	16	15
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484	4	16
486	6	16
493	13	16
494	14	16
496	16	16
513	1	17
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593	17	19
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601	25	19
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614	6	20
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624	16	20
628	20	20
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644	4	21
646	6	21
647	7	21
653	13	21
654	14	21
655	15	21
656	16	21
660	20	21
661	21	21
673	1	22
675	3	22
676	4	22
678	6	22
679	7	22
681	9	22
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694	22	22
695	23	22
697	25	22
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705	1	23
708	4	23
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717	13	23
718	14	23
719	15	23
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760	24	24
761	25	24
764	28	24
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775	7	25
777	9	25
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839	7	27
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868	4	28
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1001	9	32
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1003	11	32
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1013	21	32
1014	22	32
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1016	24	32
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1018	26	32
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