EXPLORING THE BEGINNINGS OF ALGEBRAIC K-THEORY

by

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ABSTRACT

According to Atiyah, K-theory is that part of linear algebra that studies additive or abelian properties (e.g. the determinant). Because linear algebra, and its extensions to linear analysis, is ubiquitous in mathematics, K-theory has turned out to be useful and relevant in most branches of mathematics. Let R be a ring. One defines $K_0(R)$ as the free abelian group whose basis are the finitely generated projective R-modules with the added relation $P \oplus Q = P + Q$. The purpose of this thesis is to study simple settings of the K-theory for rings and to provide a sequence of examples of rings where the associated K-groups $K_0(R)$ get progressively more complicated. We start with R being a field or a principle ideal domain and end with R being a polynomial ring on two variables over a non-commutative division ring.

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CHAPTER 1

INTRODUCTION

We quote from Atiyah's K-Theory:"K-theory is that part of linear algebra that studies additive or abelian properties (e.g. the determinant). Because linear algebra, and its extensions to linear analysis, is ubiquitous in mathematics, K-theory has turned out to be useful and relevant in most branches of mathematics."

A vector bundle over a topological space X is a space E together with a continuous map $p\colon E \to X$ so that X can be covered by open sets $X = \bigcup U_j$ with the property $p^{-1}(U_j) \cong U_j \times \mathbb{R}^n$ (homeomorphic) and transitions maps are linear. Vector bundles can be added (Whitney sum) and it turns out that vector bundles are "projective": If X is compact and $p\colon E \to X$ is a vector bundle then there exists another vector bundle $p\colon E' \to X$ so that $E \oplus E' \cong X \times \mathbb{R}^m$. Thus $p\colon E \to X$ is a projection of an obvious vector bundle $X \times \mathbb{R}^m \to X$. Let $K_0(X)$ be the free abelian group with basis the set of vector bundles with the added relation $E \oplus E' = E + E'$. $K_0(X)$ is an abelian group, accessible via linear algebra and analysis, that measures how tangled the space X is. For example, if X is contractible, then every vector bundle is of the form $X \times \mathbb{R}^n$ for some n, and so $K_0(X) = \mathbb{Z}$. The circle S^1 already admits vector bundles that are not products, the line bundle coming from the Möbius band for example. So $K_0(S^1) \neq \mathbb{Z}$.

Now let R be a ring. An R-module M is called projective if it is a summand of a free R-module R^m. So M is a projection of an obvious R-module. Let $K_0(R)$ be the free abelian group with basis the set of projective R-modules with the added relation $M \oplus M' = M + M'$. $K_0(R)$ is an abelian group that reflects the complexity of the ring R. For example if R is a field, then every projective R-module is of the form R^n for some n, and so $K_0(R) = \mathbb{Z}$.

The purpose of this thesis is to study simple settings of the K-theory for rings and to provide a sequence of examples of rings where $K_0(R)$ gets progressively more complicated. A unimodular row is a surjective linear map $\alpha\colon R^m \to R$. The kernel $\ker \alpha$ is a projective R-module that may or may not be projective. We show:

1. The kernel $\ker \alpha$ of an unimodular row $\alpha\colon R^m \to R$ is free if and only if it can be extended to an invertible $m \times m$-matrix.

2. If R is a PID, then every unimodular row can be extended to an invertible matrix. In fact, every finitely generated R-module is free.

3. If R is a local ring, then every unimodular row can be extended to an invertible matrix. In fact, every projective R-module is free. However, there do exist local rings where finitely generated modules are not always free.

4. If R is a commutative ring and the unimodular row has length 2, then it can be extended to an invertible 2×2-matrix. However, there do exist commutative rings so that a unimodular row of length 3 can not be extended to an invertible 3×3-matrix. In particular there exists projective R-modules that are not free.

5. For every non-commutative division ring D there does exists a length 2 unimodular row over the polynomial ring $R = D[x, y]$ that can not be extended to an invertible 2×2-matrix.

The main sources for this thesis are the first chapter in Weibel's K-Book [7] Rotman's "Advanced Modern Algebra" [5], Daniel Chan's video on vector bundles [1], Wikipedia's article on Algebraic K-theory [8], Jason Polak's blog post on projective modules over local rings [4], and Nicholson's book "Introduction to Abstract Algebra" [3].

CHAPTER 2

MODULES OVER PIDS

The main source for this chapter is Rotman, Advanced Modern Algebra [5]

An R-module has the same definition as a vector space, except scalars are in a ring R instead of a field.

Definition 2.0.1. Let R be ring with an identity element 1. A ***left*** $R-$***module*** is an (additive) abelian group M equipped with a scalar multiplication such that the following axioms hold for all $m, m' \in M$ and all $r, r', 1 \in R$:
(i) $r(m + m') = rm + rm'$
(ii) $(r + r')m = rm + r'm$
(iii) $(rr')m = r(r'm)$
(iv) $1m = m$

Definition 2.0.2. A ***PID*** (Principal Ideal Domain) is an integral domain where every proper ideal can be generated by a single element.

Examples of PIDs:
(i) The ring of integers
(ii) Any field
(ii) Euclidean rings

Examples of Rings which are not PIDs:

(i) $\mathbb{Z}[x]$

(ii) $\mathbb{Q}[x, y]$

Definition 2.0.3. An R-module F is called a ***free R-module*** if there exists a linearly independent generating set $B = \{b_i : i \in I\}$, called a basis. Note that in that case F is isomorphic to a direct sum of copies of R: The isomorphism

$$F \to \bigoplus_{i \in I} R_i$$

send a linear combination $\sum_{i \in I} r_i b_i$ to the tupel $(r_i)_{i \in I}$.

Lemma 2.0.4. *Suppose we have an epimorphism $f \colon M \to F$ where M is an R-module and F is a free R-module. Then there exists a splitting $g \colon F \to M$ ($f \circ g = id$) and the homomorphism $h \colon M \to F \oplus \ker(f)$ defined by $h(m) = (f(m), m - g \circ f(m))$ is an isomorphism.*

Proof. Since F is free, it has a basis: $b_1, ..., b_k$. We know that f is onto so choose $m_1, ..., m_k$ so that $f(m_i) = b_i$. Thus, we can define a homomorphism $g \colon F \to M$ by $g(b_i) = m_i$ and it follows that $f(g(b_i)) = f(m_i) = b_i$. Now, h is a homomorphism. Let us consider the coordinates in the image of h. f is an epimorphism, so it is a homomorphism. Further, the composition of two homomorphisms is a homomorphism, so $m \mapsto m - g \circ f(m)$ is also a homomorphism. We now show that the homomorphism h is an isomorophism. First, suppose $h(m) = (0, 0)$. Then clearly $f(m) = 0$ and so we have $m - g(0) = 0$. Since g is a homomorphism we have $g(0) = 0$ and can conclude that $m = 0$, and therefore h is injective. To show that h is onto, we must show that given $(p, q) \in F \oplus$ ker (f) there exists an m so that $h(m) = (p, q)$. Let $m := g(p) + q$. Then it follows:

$$\begin{aligned} f(m) &= f(g(p)) \\ &= p \end{aligned}$$

From this we see that

$$\begin{aligned} m - g(f(m)) &= g(p) + q - g(p) \\ &= q \end{aligned}$$

So we have

$$\begin{aligned} h(m) &= (f(m), m - g(f(m))) \\ &= (p, q) \end{aligned}$$

□

Definition 2.0.5. Let M be an $R-$module. The ***annihilator*** of $m \in M$ is defined as the following: ann $m = \{r \in R : rm = 0\}$.

Definition 2.0.6. An element $m \neq 0$ in an R-module M is called a torsion element if ann $m \neq 0$. An R-module M is ***torsion-free*** if it does not contain torsion elements. M is a ***torsion module*** if every nonzero element in M is a torsion element.

Definition 2.0.7. If N is a submodule of an $R-$module M, then the ***quotient module*** is the quotient group M/N equipped with the scalar multiplication

$$r(m + N) = rm + N.$$

Theorem 2.0.8. *(Rotman [5])If R is a PID, then every finitely generated torsion-free R-module M is free.*

Proof. The theorem is proven by induction on the number of generators of M. So suppose M is a torsion-free R-module generated by the set of non-zero elements $\{v_1, ..., v_n\}$. Each $m \in M$ has the form $m = r_1v_1 + ... + r_nv_n$ where the r_i's are elements of R. For the base case, assume $M = \langle v_1 \rangle$, which means that if $m \in M$, then $m = rv_1$ for some $r \in R$. Since M was assumed to be torsion-free and $v_1 \neq 0$, $rv_1 = 0$ implies $r = 0$. This shows independence, hence $\{v_1\}$ is a basis for M.

For the inductive step, let $M = \langle v_1, ..., v_{n+1} \rangle$ and define

$$S = \{m \in M : \text{there is } r \in R, r \neq 0, \text{ with } rm \in \langle v_{n+1} \rangle\}$$

First, we check that S is a submodule of M. To show closure under addition, suppose we have two elements of $S : m_1, m_2$. Then we have the following:

$$r_1m_1 = r_1'v_{n+1}$$

$$r_1r_2m_1 = r_2r_1'v_{n+1}$$

and we have:

$$r_2m_2 = r_2'v_{n+1}$$

$$r_1r_2m_2 = r_1r_2'v_{n+1}$$

Adding the equations together, we see that $r_1r_2(m_1+m_2) = (r_2r_1'+r_1r_2')v_{n+1}$. Thus, S is closed under addition. Furthermore, if $m \in S$, then $r'm = r''v_{n+1}$ where $r', r'' \in R$. Multiplying both sides of the equation by r we have: $r'rm = rr''v_{n+1}$. Since $rr' \in R$, we can conclude that S is closed under scalar multiplication.

Now, to show that M/S is torsion-free, we assume otherwise and arrive at a contradiction. So suppose $x \in M, x \notin S$, and $r(x+S) = 0 + S = S$, with $r \neq 0$. Then we have the following:

$$r(x+S) = S$$
$$rx + S = S$$

Thus, $rx \in S$. Since this is the case, there exists an $r' \in R$ with $r' \neq 0$ and $rr'x \in \langle v_{n+1} \rangle$. Since $rr' \neq 0$, we have $x \in S$, a contradiction. So M/S is torsion-free. Now we have the following:

$$\begin{aligned} m + S &= (r_1 v_1 + ... + r_n v_n) + S \\ &= r_1 v_1 + S + ... + r_n v_n + S \\ &= r_1(v_1 + S) + ... + r_n(v_n + S) \end{aligned}$$

So M/S is generated by n elements. Thus, M/S is free by the inductive hypothesis. Note that we have a projection map $M \to M/S$ with kernel S. So it follows from Lemma 2.0.4 that

$$M \cong S \oplus (M/S)$$

Since the direct sum of free modules is free, all we have left to show is that $S \cong R$. If $x \in S$, then there is some nonzero $r \in R$ with $rx \in \langle v_{n+1} \rangle$; that is, there exists $a \in R$ with $rx = av_{n+1}$. Define $\varphi : S \to Q =$Frac(R) by $\varphi : x \mapsto \frac{a}{r}$. Let $r_1 x = a_1 v_{n+1}$ and $r_2 x = a_2 v_{n+1}$. Then $x = \frac{a_1}{r_1} v_{n+1}$ and $x = \frac{a_2}{r_2} v_{n+1}$, so it follows that $\frac{a_1}{r_1} = \frac{a_2}{r_2}$. Thus, φ is well-defined. Suppose $x_1, x_2 \in S$ and $\varphi(x_1) = \varphi(x_2)$. Then we have

$$r_1 x_1 = a_1 v_{n+1}$$

$$r_2 x_2 = a_2 v_{n+1}$$

$$\frac{a_1}{r_1} = \frac{a_2}{r_2}$$

If $r = r_1 r_2, a = a_1 r_2 = a_2 r_1$, then it follows that $rx_1 = av_{n+1}$ and $rx_2 = av_{n+1}$. We have

$$rx_1 = rx_2$$

$$r(x_1 - x_2) = 0$$

$$x_1 - x_2 = 0$$

$$x_1 = x_2$$

Thus, φ is injective. If $D = \mathrm{im}\varphi$, then D is a finitely generated submodule of Q. (D is finitely generated because it is the image of S, and S is finitely generated because S is a direct summand of the finitely generated module M) We know that D is a submodule of Q because if $\frac{a}{b}, \frac{c}{d} \in D$, then clearly their sum is in D as well. It is also clear that if $\frac{a}{b} \in D$, and $r \in R$, $r \cdot \frac{a}{b}$ is in D. Now suppose

$$D = \langle \tfrac{b_1}{c_1}, \ldots \tfrac{b_m}{c_m} \rangle$$

where $b_i, c_i \in R$. Let $c = \prod_i c_i$ and define $f : D \to R$ by $f : d \mapsto cd$ for all $d \in D$. It is clear that f has values in R because for each $d \in D$ we have $d = r_1 \frac{b_1}{c_1} + \ldots r_m \frac{b_m}{c_m}$ and $c = c_1 \cdot c_2 \cdot \ldots \cdot c_m \cdot \ldots$, so multiplying d by c clears all denominators. Since D is torsion-free, f is an injective R-map, and so D is isomorphic to an ideal of R. Since R is a PID, every nonzero ideal in R is isomorphic to R; hence, $S \cong \mathrm{im} f = D \cong R$. Since S is isomorphic to R, it follows that S is free and since M/S is free, $M \cong S \oplus (M/S)$ is free as well.

CHAPTER 3

PROJECTIVE MODULES OVER LOCAL RINGS

The main result in this chapter is due to Kaplansky [2]. Our discussion follows the blog entry of Jason Polak [4].

Definition 3.0.1. A ***local ring*** is a ring in which the set of nonunits forms an ideal.

Note that this ideal of nonunits is the unique maximal ideal in the ring. In fact, local rings are exactly the rings with unique maximal ideals. A local ring does not have to be commutative.

Examples of Local Rings:

(i) All fields are local rings since the only non-unit, $\{0\}$, forms an ideal.

(ii) The ring of rational numbers with odd denominator is local. The set of non-units consist of the fractions with an even numerator and an odd denominator.

Definition 3.0.2. An R-module is projective it is a direct summand of a free module.

It is clear that a direct summand of a projective module is projective.

An Example of a Projective Module Which is Not Free:

It is well known that projective modules need not be free. As an example, consider the abelian group: $\mathbb{Z}_6 = \mathbb{Z}_3 \oplus \mathbb{Z}_2$. Since $\mathbb{Z}_6$ is a free $\mathbb{Z}_6$-module, (basis $\{\bar{1}\}$ or $\{\bar{5}\}$, it follows that $\mathbb{Z}_3$ is a projective $\mathbb{Z}_6$-module. Now $\mathbb{Z}_3 \oplus \{0\} = \{(0,0),(1,0),(2,0)\}$ has three elements. Let's say that each free $\mathbb{Z}_6$ module has a basis with n elements.

A finitely-generated free $\mathbb{Z}_6$-module is a direct sum of n copies of $\mathbb{Z}_6$, so any free $\mathbb{Z}_6$-module has 6^n elements. Hence, $\mathbb{Z}_3 \oplus \{0\}$ is a projective $\mathbb{Z}_6$-module that is not free.

Corollary 3.0.3. *Let S be a finite ring. Then $S \oplus \{0\}$ is a projective $S \oplus S-$module that is not free.*

Proof. Suppose that $S = \{r_1, ..., r_n\}$. Then $S \oplus S$ has n^2 elements. Also suppose that $\{x_1, ..., x_k\}$ is a basis for $S \oplus \{0\}$. Then $S \oplus \{0\} = (S \oplus S)x_1 \oplus ... \oplus (S \oplus S)x_k$ and is a direct sum of k copies of $S \oplus S$. So any free $S \oplus S$-module has $(n^2)^k$ elements. Hence, $S \oplus \{0\}$ is not a free $S \oplus S$-module. □

Lemma 3.0.4. *Let R be a ring and M a finitely generated R-module. Suppose that any direct summand N of M has the following property: for any element $x \in N$, there exists a free direct summand F of N such that $x \in F$. Then M is free.*

Proof. . Let $\{x_1, \ldots, x_n\}$ be a generating set for M. Since M is a direct summand of M thus there exists a decomposition $M = F_1 \oplus M_1$ with $x_1 \in F_1$. Let y_i be the projection of x_i to M_1. Then $\{y_2, \ldots, y_n\}$ is a generating set for M_1. Some of the y_i might be zero, throw them out. If they are all zero then $M = F_1$ and we are done. Assume without loss of generality that y_2 is not zero. Then there exists a decomposition $M_1 = F_2 \oplus M_2$ so that $y_2 \in F_2$. We proceed in this fashion and in the end arive at $M = F_1 \oplus F_2 \oplus \cdots \oplus F_k$. So M is free. □

This Lemma has the following consequence.

Lemma 3.0.5. *Let R be a ring and suppose that every finitely generated projective R-module P has the following property: for any element $x \in P$ there exists a free summand of P that contains x. Then all finitely generated projectives are free.*

Proof. Let P be a finitely generated projective. Let N be a summand and $x \in N$. Then N is a finitely generated projective and hence N contains a free summand that contains x. Thus P satisfies Lemma 3.0.4. So P is free. □

Theorem 3.0.6. *(Kaplansky [2]) A projective module over a local ring is free.*

Proof. We show the theorem only for finitely generated projectives P. Let $x \in P$. By Lemma 3.0.5 it suffices to construct a free summand S of P that contains x. Because P is projective, we can write $F = P \oplus Q$ where F is free. Choose a basis $\{u_i\}$ of F so that the number of generators required to express x is minimal, and write

$$x = a_1u_1 + ... + a_nu_n$$

Because of minimality no a_i in this sum can be expressed in terms of the other a_j. For suppose the simplest case, $x = a_1u_1 + a_2u_2$ and $a_2 = ra_1$. Then $x = a_1(u_1 + ru_2)$. We can now switch to the basis $u_1' = u_1 + ru_2, u_2, u_3, \ldots$, and in that basis $x = a_1u_1'$, which contradicts minimality. Let y_i be the image of u_i under the projection $F \to P$. We have

$$x = a_1u_1 + ... + a_nu_n = a_1y_1 + ... + a_ny_n$$

Now write $y_i = (\sum_{j=1}^{n} c_{ij}u_j) + t_i$ where t_i is a linear combination of the remaining basis elements u_j not in $\{u_1, ..., u_n\}$. We have $y_i - t_i = \sum_{j=1}^{n} c_{ij}u_j$. If we can show that the $n \times n$-matrix (c_{ij}) is invertible, then the $y_i - t_i$ together with the u_j, $j \neq 1, \ldots, n$ is a basis for F. And therefore the y_i together with the u_j, $j \neq 1, \ldots, n$ is a basis for F. Then the y_i are a basis for a free summand S of F (and hence of P) that contains x and we are done.

If we input the y_i into the equation $a_1u_1 + ... + a_nu_n = a_1y_1 + ... + a_ny_n$ and it is easy to see that

$$a_j = a_1c_{1j} + ... + a_nc_{nj}$$

Rearranging the above equation, we see that $(1 - c_{jj})$ is a non-unit. Indeed, we have

$$(1 - c_{jj})a_j = a_j - c_{jj}a_{jj} = a_1c_{1j} + \dots a_{j-1}c_{j-1j} + a_{j+1}c_{j+1j} + \dots + a_nc_{nj}.$$

So if $(1 - c_{jj})$ were a unit, we could divide and express a_j in terms of the other a's, a contradiction to minimality (see the first paragraph of the proof). It follows that since $(1 - c_{jj})$ is a non-unit, c_{jj} must be a unit. Here we use that our ring R is local. In a similar way, we conclude that c_{ij} is a non-unit. Thus, we have a matrix with units on the diagonal and non-units off the diagonal. Such a matrix over a local ring is invertible. In order to see this decompose the C into $C = D + OD$, where D is diagonal part of C and OD is the off-diagonal part. To see that C is invertible, we use the Jacobson ideal and its properties. See Rotman [5]. The Jacobson radical $J(R)$ of a ring is the intersection of all maximal ideals. So if R is local then $J(R)$ are the nonunits. In general we have that $J(M(n, R)) = M(n, J(R))$. Now look at $C = D + OD$. We have $OD \in M(n, J(R))$ and so $OD \in J(M(n, R))$. Thus $C + J(M(n, R)) = D + J(M(n, R))$. Since D is a unit, C has to be one as well (see Rotman [5] page 544, Proposition 8.3). $\square$

CHAPTER 4

UNIMODULAR ROWS AND STABLY FREE MODULES

The main source for this chapter is Weibel's K-book [7].

Let R be a ring with identity that satisfies the rank invariance property. R-modules are right R-modules. Let $\alpha\colon R^m \to R^n$ be an onto map $(m \geq n)$. We say $\hat{\alpha}\colon R^m \to R^n \oplus R^{m-n}$ is an isomorphic extension of α if $\hat{\alpha}$ is an isomorphism and $p \circ \hat{\alpha} = \alpha$, where p is the projection on the first n coordinates.

Theorem 4.0.1. *Let $\alpha\colon R^m \to R^n$ be an onto module homomorphism $(m \geq n)$. Then α has an extension $\hat{\alpha}$ if and only if $\ker\alpha$ is free of rank $m-n$*

Proof. Let us assume first that α has an isomorphic extension $\hat{\alpha}$. Note that $\ker\alpha = \ker(p \circ \hat{\alpha}) = \hat{\alpha}^{-1}(R^{m-n})$. So $\ker\alpha$ is free of rank $m-n$.

Next suppose that $M = \ker\alpha$ is free of rank $m-n$. Choose an isomorphism $\beta\colon M \to R^{m-n}$ and a splitting $s\colon R^n \to R^m$ of α. We have $\alpha \circ s = 1$. Note that

$$\begin{aligned} \alpha(\vec{r} - s(\alpha(\vec{r}))) &= \alpha(\vec{r}) - \alpha(s(\alpha(\vec{r}))) \\ &= \alpha(\vec{r}) - \alpha(\vec{r}) \\ &= 0 \end{aligned}$$

so $\vec{r} - s(\alpha(\vec{r})) \in \ker\alpha$. It follows that every element in R^m can be uniquely written as $\vec{r} = s \circ \alpha(\vec{r}) + (\vec{r} - s \circ \alpha(\vec{r}))$, so $R^m = s(R^n) \oplus M$. Now define $\hat{\alpha}\colon R^m \to R^n \oplus M \to$

$R^n \oplus R^{m-n}$ by $\hat{\alpha}(\vec{r}) = \alpha(\vec{r}) + \beta(\vec{r} - s \circ \alpha(\vec{r}))$. This is an isomorphism with the desired property. □

Here is the matrix version of the above theorem.

Theorem 4.0.2. *Let α be a $n \times m$ matrix (n rows and m columns), $m \geq n$, that defines a surjection $\alpha : R^m \to R^n$ with kernel M. Then M is isomorphic to R^{m-n} if and only if α can be extended to an invertible $m \times m$ matrix $\hat{\alpha} \in GL(m, R)$.*

The $(m-n) \times m$ matrix $\alpha' \colon R^m \to R^{m-n}$ which, together with the matrix α gives $\hat{\alpha}$ (in case $\ker \alpha$ is free) is given by $\alpha'(\vec{r}) = \beta(\vec{r} - s \circ \alpha(\vec{r}))$. Note that $\hat{\alpha}$ is obtained by stacking α and α' on top of each other.

Corollary 4.0.3. *If R is a field then an $n \times m$-matrix α, $m \geq n$, of rank n can be extended to an invertible $m \times m$-matrix $\hat{\alpha}$. The $(m-n) \times m$ matrix α' that extends α to $\hat{\alpha}$ can be chosen so that every row of α' is orthogonal to every row of α.*

Proof. We give an independent elementary proof. Let $\vec{a_1}, \ldots, \vec{a_n}$ be the rows of α. Since the rank of α is n we know that these rows are linearly independent. Note that $R^m = (\ker \alpha)^\perp \oplus \ker \alpha$ and that $(\ker \alpha)^\perp$ is the row space of α with basis $\vec{a_1}, \ldots, \vec{a_n}$. Let $\vec{a'_{m-n+1}}, \ldots, \vec{a'_m}$ be a basis for $\ker \alpha$. Then the rows $\vec{a_1}, \ldots, \vec{a_n}$ together with the rows $\vec{a'_{m-n+1}}, \ldots, \vec{a'_m}$ form a set of m linearly independent vectors. So they make up an invertible $m \times m$-matrix. □

A *unimodular row* is a matrix $[a_1, \ldots, a_m]$ consisting of a single row that defines a surjection $R^m \to R$.

Theorem 4.0.4. *Let $[a\ b]$ be a unimodular row over a commutative ring R. Then it can be extended to an invertible 2×2 matrix.*

Proof. If $\begin{bmatrix} a & b \end{bmatrix}$ is a unimodular row then there exists $c, d \in R$ such that $ac + bd = 1$. Thus, we can extend $\begin{bmatrix} a & b \end{bmatrix}$ to $\begin{bmatrix} a & b \\ -d & c \end{bmatrix}$, which is invertible because its determinant is 1. □

This is not true for longer unimodular rows: there does exist a unimodular row of length 3 over a commutative ring that does not extend to an invertible 3×3 matrix. Before we give an example we recall some algebraic topology.

A tangent vector field on S^n is a continuous map $v\colon S^n \to \mathbb{R}^{n+1}$ so that the dot product $x \cdot v(x) = 0$, i.e. x and $v(x)$ are orthogonal.

Theorem 4.0.5. *S^n admits a non-zero tangent vector field if and only if n is odd.*

For a proof see Hatcher, Theorem 2.28, page 135. We are now ready to give our example. Let $R = \mathbb{R}[x, y, z]/(x^2 + y^2 + z^2 = 1)$, a quotient of a polynomial ring. Note that $\alpha = [x\ y\ z]$ is a unimodular row because

$$xx + yy + zz = 1.$$

Suppose this row could be extended to a invertible 3×3 matrix

$$\begin{bmatrix} x & y & z \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

where the f_{ij} are polynomials in x, y, z, considered modulo the given ideal. Let

$$V : S^2 \to \mathbb{R}^3$$

defined by $V(a,b,c) = \pi(f_{21}(a,b,c), f_{22}(a,b,c), f_{23}(a,b,c))$, where $\pi: \mathbb{R}^3 \to \mathbb{R}^3$ is the projection onto the orthogonal complement of (a,b,c). Since the vectors (a,b,c) and $(f_{21}(a,b,c), f_{22}(a,b,c), f_{23}(a,b,c))$ are linearly independent for every (a,b,c), this is a non-zero tangent vector field on S^2, contradicting Theorem 4.0.5

Remark. The tangent bundle $\mathrm{T}(\mathrm{S}^2)$ on the 2-sphere S^2 is a non-trivial 2-dimensional real vector bundle. If we add the normal line bundle we get the trivial bundle. The connection between $T(S^2)$ and the non-free projective module we considered (the kernel of the unimodular row $[x\ y\ z]$) is made explicit in Swan [6], Example 1.

CHAPTER 5

UNIMODULAR ROWS FOR NON-COMMUTATIVE RINGS

The next example illustrates that there are unimodular rows of length 2 over non-commutative rings whose kernels are not free. See Exercise 1.6 in Weibel's K-book [7].

Consider the following: Let D be a division ring which is not a field. Choose $\alpha, \beta \in D$ such that $\alpha\beta - \beta\alpha \neq 0$. Note first that $\sigma = [x + \alpha \;\; y + \beta]$ is a unimodular row over $R = D[x, y]$: We must show that there exists $d_1, d_2 \in D[x, y]$ such that $x + \alpha)d_1 + (y + \beta)d_2 = 1$. Consider the following combination:

$$\begin{aligned}(x+\alpha)(y+\beta) - (y+\beta)(x+\alpha) &= xy + \beta x + \alpha\beta + \alpha y - yx - \alpha y - \beta x - \beta\alpha \\ &= \alpha\beta - \beta\alpha\end{aligned}$$

Since we assumed $\alpha\beta - \beta\alpha \neq 0$, we can divide both sides of the equation by $\alpha\beta - \beta\alpha$ to arrive at our intended result. Therefore, there exists $d_1, d_2 \in D[x, y]$ such that $(x + \alpha)d_1 + (y + \beta)d_2 = 1$: $d_1 = \frac{(y+\beta)}{\alpha\beta-\beta\alpha}$ and $d_2 = \frac{-(x+\alpha)}{\alpha\beta-\beta\alpha}$.

Let P =ker(σ). We have $P \oplus R \cong R^2$ by Lemma 2.0.4. So P is a rank 1 stably free projective.

Theorem 5.0.1. *P is not a free R-module.*

Lemma 5.0.2. *P does not contain a vector $\begin{pmatrix} f \\ g \end{pmatrix}$ where both f and g are constant or both are linear.*

Proof. Let $f = c_1 + c_2x + c_3y$ and $g = d_1 + d_2x + d_3y$. Set up the equation:

$$(x + \alpha)(c_1 + c_2x + c_3y) + (y + \beta)(d_1 + d_2x + d_3y) = 0$$

Solving for the coefficients yields $c_2 = d_3 = 0$ and so we have the following equations:

$$\begin{aligned} c_1 + \beta d_2 &= 0 \\ c_3 + d_2 &= 0 \\ \alpha c_3 + d_1 &= 0 \\ \alpha c_1 + \beta d_1 &= 0 \end{aligned}$$

Doing the appropriate substitutions we obtain the following:

$$\begin{aligned} c_1 &= -\beta d_2 \\ c_1 &= -\alpha^{-1}\beta d_1 \\ c_3 &= -d_2 \\ c_3 &= -\alpha^{-1} d_1 \end{aligned}$$

Setting the $c_1's$ equal we see that $d_1 = \beta^{-1}\alpha\beta d_2$. Substituting d_1 into the c_3 equation, we see that $c_3 = -\alpha^{-1}(\beta^{-1}\alpha\beta)d_2$. Finally, we set this c_3 equal to $-d_2$ and obtain a contradiction: $\alpha\beta = \beta\alpha$. Thus, f and g cannot both be linear. □

Lemma 5.0.3. *P does contain a vector $\begin{pmatrix} f \\ g \end{pmatrix}$ where both f and g are quadratic without constant term: $f = c_1x + c_2y + c_3xy + c_4y^2$ and $g = d_1x + d_2y + d_3xy + d_4x^2, (c_i, d_i \in D)$.*

Proof. Start by solving the following equation:

$$(x + \alpha)(c_1x + c_2y + c_3xy + c_4y^2) + (y + \beta)(d_1x + d_2y + d_3xy + d_4x^2) = 0$$

You will then get the following equations from the coefficients:

$$\begin{aligned} c_1 + \beta d_4 &= 0 \\ c_2 + \alpha c_3 + d_1 + \beta d_3 &= 0 \\ c_3 + d_4 &= 0 \\ c_4 + d_3 &= 0 \\ \alpha c_1 + \beta d_1 &= 0 \\ \alpha c_2 + \beta d_2 &= 0 \\ \alpha c_4 + d_2 &= 0 \end{aligned}$$

Making some substitutions and letting $d_2 = u$ we obtain the following solutions:

$$c_1 = -\beta d_4$$

$$c_2 = -\alpha^{-1}\beta u$$

$$c_3 = -d_4$$

$$c_4 = -\alpha^{-1}u$$

$$d_1 = \beta^{-1}\alpha\beta d_4$$

$$d_2 = (\alpha^{-1}\beta - \beta\alpha^{-1})^{-1}(-\alpha + \beta^{-1}\alpha\beta)d_4$$

$$d_3 = \alpha^{-1}u$$

$$d_4 = d_4$$

□

Lemma 5.0.4. *P does contain a vector $\begin{pmatrix} f \\ g \end{pmatrix}$ where both f and g contains a constant term. In fact, f and g can be chosen to have the special form*

$$f = \gamma_0 + \gamma_1 y + y^2, g = \delta_0 + \delta_1 x - \alpha y - xy$$

and both $\gamma_0 = \beta u^{-1}\beta u$ and δ_0 are nonzero.

Proof. We first show that P contains a vector of the form given. First, we solve the equation:

$$(x + \alpha)(\gamma_0 + \gamma_1 y + y^2) + (y + \beta)(\delta_0 + \delta_1 x - \alpha y - xy) = 0$$

We obtain the following equations:

$$\gamma_0 + \beta\delta_1 = 0$$

$$\gamma_1 + \delta_1 = \beta$$

$$\alpha\gamma_1 + \delta_0 = \beta\alpha$$

$$\alpha\gamma_0 + \beta\delta_0 = 0$$

By substitution we have the following solution for γ_0:

$$\gamma_0 = \beta(-(\beta\alpha - \alpha\beta)^{-1} \cdot \beta(-(\beta\alpha - \alpha\beta))$$

So indeed γ_0 is a constant. Completing some substitutions in the above equations, we see that $\delta_0 = \beta\alpha - \alpha(\beta - (-\beta^{-1}\gamma_0))$, which again is a constant. □

We can now prove Theorem 5.0.1. Assume P is free, say $P = R^n$. Since $P \oplus R \cong R^2$ it follows that $R^n \oplus R \cong R^2$ and hence $n = 1$ (R has the rank invariance property). So P is cyclic. Say it is generated by $\begin{pmatrix} r \\ s \end{pmatrix}$. Let $\begin{pmatrix} f \\ g \end{pmatrix}$ be a vector where f and g are both quadratics without constant terms as in Lemma 5.0.3. We have

$$\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} r \\ s \end{pmatrix} h$$

for some $h \in R$. So $2 = \deg f = \deg r + \deg h$ and it follows that $\deg r = 2 - \deg h$. Similarly $\deg s = 2 - \deg h$. From Lemma 5.0.2 it follows that $\deg h = 0$ and so $\deg r = \deg s = 2$. So $0 \neq h \in D$ and it follows that $\begin{pmatrix} r \\ s \end{pmatrix}$ is a vector of quadratic polynomials without constant term. But then it follows that every vector in P has polynomials without constant term. This contradicts Lemma 5.0.4. □

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