IMPACT OF THE ORDER OF EAC AND SOS DURING INSTRUCTION ON RATIOS

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DEDICATION

I dedicate this thesis to my family, friends, and amazing teachers that I have worked with or been a student of. It is because of the endless amount of support and kindness that I have been shown throughout my life that I have been pushed to achieve my dreams and pursue my passions. I would not be where I am without the support and guidance of those in my life.

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ABSTRACT

The purpose of this study was twofold. First, a set of instructional materials surrounding proportional reasoning with ratios (particularly the understanding of the multiplicative relationship between the quantities within the ratio, referred to as functional reasoning throughout this thesis) were created using the free online tool, Desmos, with a goal of determining the impact of the lesson materials on student understanding. The second goal was to explore the impact of the order in which two instructional strategies, Explicit Attention to Concepts (EAC) and Students' Opportunity to Struggle (SOS), had on student understanding. The lesson materials consisted of 5 lessons. These 5 lessons had two forms: EAC then SOS or SOS then EAC. In each of these instructional groups, all EAC and SOS sections were identical in each of the five lessons, the difference between materials in each of these groups was the order in which the EAC and SOS sections occurred. Students' understanding was assessed anonymously, and answers were scored dichotomously (i.e. correct or incorrect). There was a total of 22 items on the full assessment with 8 items addressing functional reasoning specifically. The major findings of this study include that the lesson materials led to an increase in understanding for both overall understanding and the sub-area of functional reasoning, and the EAC then SOS instructional group's understanding of functional reasoning was higher than that of the SOS then EAC instructional group.

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LIST OF ABBREVIATIONS

EAC	Explicit Attention to Concepts
SOS	Students’ Opportunity to Struggle
HLT	Hypothetical Learning Trajectory
KDU	Key Developmental Understanding

CHAPTER ONE: INTRODUCTION

In this study, I explored the constructs of Explicit Attention to Concepts (EAC) and Students' Opportunity to Struggle (SOS) and the impact of the order in which these two constructs occur within instruction. To facilitate this exploration, I created a set of lessons using Desmos which focus on ratios and proportional reasoning. Desmos was chosen with the intent to create online materials that are effective and flexible in that they can be applied in a variety of teaching settings. In this chapter, I begin by discussing why this mathematical context is meaningful to explore, explain the motivation behind exploring the impact of the order in which EAC and SOS occurs in instruction, then provide an overview of the study. Finally, I present definitions of relevant key terms to be used throughout this thesis.

Motivations for Lesson Materials

Though ratios and proportional reasoning itself have been researched for decades, we have entered a time when developing effective online materials is more important than ever. Due to the current Covid-19 pandemic, we have an unprecedented need for online materials that can be used flexibly by teachers in a variety of contexts such as online synchronous learning, online asynchronous learning, and hybrid learning (in which some students attend class online and others attend class in person). With such need, I chose to develop lesson materials using the free online tool, Desmos.

The lessons created for this study focus on ratios and proportional reasoning. Specifically, the multiplicative relationship between two quantities in a ratio, which is sometimes referred to as the functional relationship (Simon & Placa, 2012; Carney et al., 2016). Developing students' understanding of and ability to reason with ratios has been described as important both for students' understanding of future mathematics and science concepts and also for use in the real world (Lamon, 1993; Akatufba & Wallace, 1999; Langrall & Swafford, 2000; Steinthorsdottir & Sriraman, 2009; Team, 2011; Lobato et al., 2014; Ramful & Narod, 2014). It has also been described as a challenging skill for students to develop (Tourniaire & Pulos, 1985; Lobato et al., 2014). The importance of and challenge of understanding this concept for students has been well documented, and there is a large breadth of literature available describing ratio problems and proportional reasoning tasks and how students interact with them. This meant I could use the literature as a support for selecting and creating meaningful tasks for students, ensuring that the lesson was mathematically sound to guide students towards my lesson goals.

Role of EAC and SOS

Previous research (Hiebert & Grouws, 2007; Stein et al., 2017) identified EAC and SOS as teaching practices which can lead to increased depth of conceptual understanding in students, and suggests instruction that contains both of these practices will have the greatest increase in depth of understanding. However, there is still more to be known about these practices and how they impact student understanding. One aspect of the implementation of EAC and SOS that has not yet been studied is the impact of the order in which these constructs occur in instruction. The question arises: Is there a difference in student understanding when EAC occurs before SOS or vice versa? This question is the central focus of the study at hand.

Purpose

The purpose of this study is to investigate if the lesson materials increase students' understanding of ratios and proportional reasoning, particularly the use of functional reasoning, and how the order in which opportunities for EAC and SOS are presented in the lessons impact the level of student understanding, if at all.

Research Questions

With the goals specified above, two primary research questions arise as the focus of this study:

- Do the lesson materials lead to an increased understanding of proportional and functional reasoning with ratios?
- Is there a difference between students' understanding of the functional relationship in ratios when instruction focuses first on EAC then on SOS compared to instruction that focuses first on SOS then on EAC?

Hypotheses

The null hypotheses for both research questions, respectively, are as follows:

1. There is no difference between student understanding of proportional reasoning with ratios prior to and after the implementation of the lesson materials.
2. There is no difference between student understanding of the functional reasoning with ratios prior to and after the implementation of the lesson materials.

3. Students' understanding of the functional reasoning with ratios does not differ based on which instructional group (EAC then SOS or SOS then EAC) they are in.

Because the lesson materials were created after a thorough review of the literature surrounding ratios and proportional reasoning, the alternative hypothesis for the first and second research questions are that there will be an increase in understanding of the proportional reasoning and the functional relationship, respectively, after students have worked with the lesson materials.

As mentioned previously, there is a gap in literature specifically surrounding the impact of the order in which instruction focuses on EAC and SOS. Thus, it is unclear as to whether or not there will be a difference in understanding as a result of alternating the order in which these constructs are presented. However, the studies by Schwartz et al. (2011) and Kapur (2014) address similar ideas. In these studies, the authors explore the impact of providing students with the opportunity to explore mathematical ideas before giving explicit instruction on them, and show that there is an increased level of understanding. Based on the results of these studies, a reasonable alternative hypothesis for the third research question would be that students whose instruction focuses on SOS prior to EAC will have higher levels of understanding than students whose instruction focuses on EAC prior to SOS. However, it is important to note that the EAC does not necessarily provide specific formulas or present a single way of solving ratio problems. It instead focuses on pressing connections between students' ideas or ideas presented to them. Thus, it is not identical to the explicit instruction described by Schwartz et al. (2011) and Kapur (2014).

Research Design

When constructing the lessons used in this research, I found that the Hypothetical Learning Trajectory (HLT) described by Simon (first introduced in 1995) fit well with my natural approach to developing lessons, making it a useful tool for structuring the development of the lesson materials. Along with this trajectory, research into ratio and proportional reasoning tasks, development of students' understanding of ratios, and EAC and SOS in instruction informed my creation of a set of 5 lessons on ratios and proportional reasoning with a goal of facilitating understanding of the functional relationship in ratios. There were two forms of the lessons: EAC then SOS and SOS then EAC. Each lesson was designed to take one class period (roughly 50 minutes), and had two distinct parts: one section focusing on EAC and the other focusing on SOS. In this way, both EAC then SOS and SOS then EAC focused lessons contained exactly the same content, with the only difference being the order in which those two sections were presented.

Sixth, seventh, and eighth grade teachers volunteered to implement these lesson materials in their classrooms and used a pre-/post-assessment designed with the support of my thesis chair (see Appendix A). Students were given the assessment prior to the implementation of the lessons and were given the same assessment after teachers implemented all five lessons.

The assessment was created using a Google form and understanding was gauged by scoring questions as a 1 if they were correct or 0 if they were incorrect. The entire assessment had a possibility of 22 points, with 8 of those specifically addressing functional reasoning. Paired t-Tests were used to compare students' pre- and post-assessment scores (matched via anonymous names) for both the full assessment and the functional reasoning specific portion of the assessment. This allowed me to look at growth in student understanding. ANCOVA was used to statistically compare the post-assessment means of the students whose instruction focused on EAC then SOS and those whose instruction focused on SOS then EAC with the pre-assessment as a covariate.

Assumptions and Limitations

This study relied on several assumptions. The first assumption is the use of radical constructivism. I ascribe to the idea that new knowledge is built upon previous knowledge and this shapes our perception of reality. This idea of radical constructivism is described further in the literature review, but it is important to note that this underlying idea of how we learn guides my own perspective. There are also a few assumptions central to the assessment process used in this study. I assumed that students would do their best on the pre- and post-assessments, even though they are anonymous, that their effort will match their understanding, and that students will complete this assessment without support from others. When scoring the data, I assumed that the number a student submitted was the number they intended to write. For example, if a student wrote "108" when the problem's answer is "180," I assume 108 is the solution the student actually got, rather than a typo. This could mean that some students' responses are considered erroneous due to mistyping rather than actual misunderstanding.

There are several notable limitations of this study. Firstly, because this assessment is relatively short and scored via an overall score of correct answers, it may not be very sensitive to changes in depth of understanding. Here, when I refer to "depth" of understanding, I mean the development of a conceptual understanding of proportional and functional reasoning with ratios rather than a procedural understanding. It is possible that students may answer questions correctly on the pre-assessment using a procedural understanding from previous instruction, and that they may again get the same questions correct on the post-assessment, which would not reflect any growth in conceptual understanding that may have occurred. Though descriptive answer questions were included on the assessment in hopes of being able to identify some of this type of growth, students' explanations varied widely in terms of detail and as such these questions were often not enough to pick up on changes in conceptual understanding.

Secondly, the freedom with which teachers had control over the implementation of the materials, including freedom in format (the materials could be used in face-to-face, online, or hybrid settings) is a limitation of this study. While teachers were asked to use the materials without changing any of their content, and to use strictly only one set of materials (the EAC first then SOS materials, or SOS then EAC materials, but not any mixture of the two), they had the freedom to make their own pedagogical calls when doing so. This freedom allows for the materials to be used in a natural way by the teachers, making the results easier to generalize to a wider population of teachers, but does cause some ambiguity in terms of interpreting whether the results of the study were due primarily to the differences in the lesson materials themselves or perhaps to differences of instructional choice by the teachers implementing the materials.

Thirdly, the structure of this study includes pre- and post-assessments that were given to students were given within one week prior to and following the implementation of the instructional materials, respectively. Due to time constraints, we were unable to administer an additional delayed post-assessment to consider differences in retention.

That said, it is unclear if any observed differences (or lack thereof) will be maintained over a longer period of time or perhaps that a difference in retention might appear between the two groups that shows a difference in understanding that was not indicated by the immediate post-assessment.

Finally, three teachers who participated in this study were also using these materials in an ongoing research grant, the ROOT project. For this project, they needed to collect data in a specified time frame, which occurred after they had completed the first three lessons. As a result, these teachers had to administer the post-assessment after completion of the third lesson as well as after completion of the fifth lesson. This means that their students were exposed to the pre-/post-assessment three times, instead of twice. Thus, there may be an increased testing effect with these students due to more exposure to the assessment.

Definitions of Key Terms

In this thesis, there are several terms of particular significance. These terms are described below:

- *Explicit Attention to Concepts (EAC)* - An instructional strategy that focuses on addressing mathematical concepts and connections between concepts or representations directly.
- *Students' Opportunity to Struggle (SOS)* - An instructional strategy that focuses on providing students with mathematical tasks that are within reach of understanding but whose solutions are not immediately apparent and/or multiple solution strategies can be used.
- *Rate* - A collection of infinitely many equivalent ratios (Lobato et al., 2010). A rate is distinct not because of the units (e.g. a ratio of quantities with

different units or a ratio where one of the quantities is a measure of time), but rather because of the way the student conceptualizes the quantities and is able to recognize *any* equivalent ratio (including non-integer ratios).

- *Composed Unit* - A joining of two quantities in a ratio into a single unit, used primarily for partitioning or iterating/scaling.
- *Scalar Reasoning* - Students conceptualize a ratio as a composed unit and can iterate or partition it. Students may iterate the unit using repeated addition or may move to more efficient methods such as multiplying both quantities in the composed unit by the same value. For example, given a paint mixture that is 4 parts blue and 8 parts red, students might add 4+4 and 8+8 to get an equivalent ratio of 8 parts blue and 16 parts red.
- *Functional Reasoning* - Students identify and use the multiplicative relationship between the two quantities in a ratio. For example, given a paint mixture that is 4 parts blue and 8 parts red, students recognize that the amount of red is 2 times the amount of blue.

CHAPTER TWO: REVIEW OF THE LITERATURE

In this literature review, I discuss the lesson design constructs that guided my creation of lesson materials, important understandings of ratios and proportional reasoning, characteristics of ratio and proportional reasoning tasks, and provide a more detailed explanation of Explicit Attention to Concepts (EAC) and Students' Opportunity to Struggle (SOS) and the research surrounding the identification of and impact of these practices.

Lesson Design Constructs

The primary underlying lesson design construct used in this study is the Hypothetical Learning Trajectory (HLT). The HLT was introduced by Martin A. Simon in his 1995 article, which has been cited over 2000 times. Simon (1995) describes how the HLT developed, what the HLT encapsulates, and how it has impacted his pedagogical decisions. A primary theoretical framework for the HLT is social constructivism, which Simon describes as the, "coordination of psychological and sociological analyses," (p. 117) of the constructivist perspective. So, Simon's social constructivist lens brings two perspectives on learning together. Namely, those focused on the cognitive individual and learning motivated by the social aspects of the classroom. Though Simon's framework is primarily social constructivist, I found myself considering the development of student understanding through a radical constructivist lens, focusing on the cognitive individual.

Constructivism

Piaget's work on cognitive development, particularly his ideas on assimilation and accommodation of knowledge, provides a well-described theoretical perspective related to the development of intelligence and knowledge in children. In his book, *The Origins of Intelligence in Children* (1952), he describes six stages of children's development of intelligence from their first sensorimotor reflexes to the use of intention, coordination of schema and their application, experimentation, and the use of deductive reasoning. An emphasis on adaptation, and specifically the roles of assimilation and accommodation is placed in each one of the stages described. In his introduction he connects adaptation of intelligence to evolutionary adaptation, describing, "The organism adapts itself by materially constructing new forms to fit them into those of the universe, whereas intelligence extends this creation and by constructing mentally structures which can be applied to those of the environment," (p. 4). This description highlights intelligence as an organization of ideas which can be applied to the world outside the individual. He goes on to describe assimilation as a method of incorporating new ideas or actions into an existing schema that successfully interacts with the environment in a way that fits with the current expectations. Accommodation on the other hand occurs when a change in the environment results in a new outcome that does not fit with the current schema or expectation, and as a result the child must modify their schema to allow this new outcome to fit.

Nowhere in this book does Piaget mention constructivism, and yet his ideas of assimilation and accommodation are so deeply connected to constructivism. Fox (2001) describes several claims of constructivism and highlights, "Learning is an active process," and "Knowledge is constructed, rather than innate, or passively absorbed," (p. 24) as two of the most central claims of constructivism, the second claim being a more expanded version of the first. In this way, a feature of the constructivist perspective is that students learn through interactions with the environment, which includes children's own active investigation (actions). This is contrary to other ideas of children as empty vessels waiting to be filled with knowledge (passively receiving knowledge) or that knowledge is attained through stimulus-response conditioning. The concepts of assimilation and accommodation rely on interaction between the environment and children's ideas, and continually involve looking at new experiences through the lens of previous experiences. Each new experience either fits well-enough with the students' existing knowledge and is assimilated to further define their current conceptions, or it creates disequilibrium and requires the modification of existing schema to incorporate this new knowledge. In both cases, new knowledge is built onto existing knowledge to create the child's reality.

Von Glasersfeld (1984) describes radical constructivism as the perspective that knowledge is created by the way we perceive experiences, and that knowledge is disconnected from an objective reality (or that, indeed, there is no perceivable objective reality and that our reality is instead defined by our unique perspective and experiences). He describes, "Radical constructivism, thus, is radical because it breaks with convention and develops a theory of knowledge in which knowledge does not reflect an "objective" ontological reality, but exclusively an ordering and organization of a world constituted by our experience," (p. 9). Von Glasersfeld himself described connections between Piaget's work and this aspect of the radical constructivist perspective. For example, he asserts:

> Whenever he [Piaget] says, for instance, that knowledge must not be thought of as a picture or copy of reality (and he says that often enough), it is easy to mistake it for a conventional admonition that a cognitive organism's picture of the world would necessarily be incomplete or somewhat distorted. Any realist will read it as such, rather than take it as an assertion that knowledge, of its nature, cannot have any iconic correspondence with ontological reality," (1982, p. 614).

In this way, he establishes a clear connection between the language of Piaget and the argument that knowledge cannot represent an objective reality.

Though the lack of objective reality may seem a radical idea, it's connection to the perspective that our knowledge is developed through our experiences is not radical, and von Glasersfeld describes how this view has been presented by those even as far back as pre-socratic philosophers. Further, he describes the resolution to the issue of whether or not there is an objective reality by redressing the issue of knowledge as not trying to understand an objective truth, but instead, "as a search for fitting ways of behaving and thinking," (p. 18). This extends the constructivist perspective of the building of knowledge through experiences and connects it to the pursuit of knowledge as an understanding of the environment. Piaget's (1963) connection of assimilation and accommodation to the adaptation of an organism to its environment fits snuggly within this perspective. We see an argument that a child's knowledge represents their current reality and only when their interaction with the environment does not fit with their reality are they prompted to adapt their knowledge.

In essence, von Glasersfeld's radical constructivism presents learning as identifying behaviors and ideas that are consistent throughout repeating events, and as such involves the identification of whether two events are a repetition or two separate events that can have different fitting behaviors. We, as learners, are tasked then with identifying concepts or behaviors that “work” in different experiences. These concepts represent our reality until we are presented with an experience in which they do not work, and thus we can adapt them (or accommodate, to use the language of Piaget) to create a better fit. In this way, our reality builds on our knowledge from previous experiences and we must be presented with the experiences required to adapt our concepts to best fit reality. In other words, we will look at new experiences through the lens of our earlier understanding, and only adapt that understanding when it no longer fits the reality. Simon (1995) describes this aspect of constructivism as well, stating that, “we construct our knowledge of our world from our perceptions and experiences, which are themselves mediated through our previous knowledge,” (p. 116).

From this, we see the perspective of constructivism that is central to the design of the lessons: our new knowledge is built upon our previous knowledge, and we only adapt our mental concepts when we are presented with experiences where our current concept is not the best fit. This radical constructivist perspective focuses on the cognitive perspective of understanding by focusing on the individual’s construction of knowledge, and allows the creator of lesson materials to consider opportunities in which students are pressed to further solidify or challenge their existing knowledge. The teacher can design opportunities that press students to a point of accommodation, leading to the students’ creation of beliefs and actions that are viable in the new reality we (teachers) have presented to them.

It is worth noting that while this perspective does not explicitly focus on the development of knowledge through social interaction, as a social constructivist perspective would, students are not developing knowledge in a vacuum and their interaction with the environment includes not only their interaction with lesson materials, but also interactions with peers and the teacher. Piaget has recognized this interpersonal interaction as well, "[t]he individual would not come to organize his operations in a coherent whole if he did not engage in thought exchanges and cooperation with others," (Piaget, 1947, p. 174 as cited in Lourenço, 2012). However, as these more interpersonal interactions can be harder to predict, I find the focus on the individual that is present in radical constructivism and Piaget's work to be a more fitting framework for the development of lesson materials that may be used by others and in a variety of learning environments.

Features of the Hypothetical Learning Trajectory

In Simon's 1995 article, he discussed how he developed the Hypothetical Learning Trajectory (HLT) and how it connects to a specific teaching experience in which he engaged. He explains a lesson in which he uses his previous teaching experience to predict the depth of understanding of a group of prospective elementary teachers related to units of measure and the area of a rectangle. He believed that the teachers would have a formulaic or rule-bound approach to finding area and wanted to generate a deeper understanding of the formula for area and the creation of a standard unit of measure. After setting his goal and predicting the incoming knowledge, he considered tasks that were available and the types of thinking and learning the tasks would provoke. In Simon and Tzur (2004), they summarize the HLT with these characteristics, stating, "An HLT consists of the goal for the students' learning, the mathematical tasks that will be used to promote student learning, and hypotheses about the process of the students' learning," (p. 91). This process was very similar to my own natural approach to designing lessons, which made it a natural choice for the underlying lesson design.

In the HLT, the teacher's learning goal provides the direction for the learning trajectory. As such, it is very influential on the overall structure of lesson(s) that will be used to reach said goal. Simon (2006) recommended Key Developmental Understandings as one way to choose an instructional goal. A Key Developmental Understanding (KDU) can be summarized as a conceptual advance without which students lack a particular mathematical ability. A KDU is often an essential step in understanding that students must make sense of in order to move to more advanced mathematical concepts. However, understanding of a KDU is not black and white; students may have a more complete understanding of some KDUs than others and may be in the process of learning multiple KDUs at the same time. I mentioned that a KDU is a conceptual advance, Simon (2006) describes a conceptual advance as, "a change in students' ability to think about and/or perceive particular mathematical relationships," (p. 362). A KDU is not a single piece of information but rather students' ability to think about and perceive mathematical ideas. For example, in the context of fractions a KDU would be, "Understanding that equal partitioning creates specific units of quantity," (p. 361). If a student lacks a KDU, this does not mean that they will not be able to move forward, but it does mean that they will find future concepts more challenging and may rely more on rote memorization rather than creating further conceptualization of what is happening.

With a learning goal in mind, the teacher must hypothesize students' understanding and consider tasks that will bridge their current understanding with the desired goal. How teachers form an hypothesis of student understanding can draw on a variety of sources, such as, "experience with the students in the conceptual area, experience with them in a related area, pretesting, experience with a similar group, and research data," (Simon, 1995, p. 132). Additionally, as the teacher begins to work with students on a particular understanding, their conceptualization of the students' understanding will develop further and likely become more accurate. As a result, when implementing a lesson, teachers will likely modify their hypothesis of student understanding.

The consideration of lesson tasks and the learning they may provoke is heavily influenced by the teachers' own beliefs. Simon's (1995) article provided little guidance on how one might think about the learning process, select a mathematical task, or conjecture the role of the mathematical tasks in the learning process. Simon and Tzur (2004) attempts to provide a framework for this process of considering mathematical tasks, the learning process, and the interaction between the two. They propose reflection on the activity-effect relationship as guidance for selection of mathematical tasks and a method of considering the learning that may be evoked. Simon and Tzur discuss Piaget's idea of assimilation where students' new knowledge is assimilated into their prior conceptions. The process of reflection on the activity-effect relationship begins with the learner setting a goal. This goal may not be directly related to the mathematical goal. For example, their goal may be to win a game, which does not relate to the mathematical goal explicitly. After setting their goal, students will choose activities in an attempt to reach that goal and continuously (though not necessarily consciously) reflect on the effect of their activity in regard to meeting their goal. To use this mechanism for selecting and analyzing mathematical tasks, teachers should consider:

- What will the students' goal be when they are presented with the task?
- Based on the hypothesis of student understanding, what activity/ies might students choose to do?
- What effect will that activity have in regard to the goal students chose?

These questions allow teachers to identify if students' engagement with a given mathematical task will lead to the intended understanding. The goal of asking themselves this question is described by Simon and Tzur when they state, "We next endeavor to design or select tasks that are likely to cause the students to set a goal, to call on the intended activity, and to reflectively abstract the intended concept," (p. 97). A teacher might first consider the activity-effect relationship they want students to go through, and then look at tasks and the (student) goals associated with them to consider if the activity-effect relationship that students will go through matches the one they intend.

Finally, I want to discuss the reason why the trajectory is hypothetical. The teacher cannot be sure of the students' knowledge on the subject (regardless of how much the teacher has worked with a student, they do not have any direct access to the knowledge of a student and thus must hypothesize about the students' knowledge). Based on the goal, the hypothesis of student understanding, and the learning that they believe will occur during the instruction, the teacher creates a plan for instruction. However, just like with planning a trip, as Simon (1995) analogized, no matter how detailed the plan, in the moment we must react to conditions and often have to make modifications. Thus, the HLT provides a structured way to plan lessons with specific goals and student understanding in mind, but this trajectory is not set in stone and will likely be modified continually once enacted.

Understanding Ratios and Proportional Reasoning

When assessing students' understanding of ratios and their ability to reason proportionally, it is important to understand the connection between what students "do" (i.e., how they approach solving problems) and what students "understand" (i.e., the mental connections and ideas they are attending to as they solve the problems). Students' solution strategies and their depth of understanding are naturally very intertwined, and it is important to consider both when assessing student understanding.

Common Reasoning Strategies and Errors

As students approach problems involving ratios and proportional reasoning, the literature has clearly identified common strategies (including erroneous strategies) that students use. The strategies commonly identified throughout the literature are:

- Random Calculations (erroneous) - students use operations randomly with the numbers given, rather than basing their arithmetic on the context of the situation (Langrall & Swafford, 2000; Steinthorsdottir & Sriraman, 2009)
- Ignoring Information (erroneous) - students solve without attending to both quantities, for example by comparing only the numerators of two ratios even though the denominators differ as well (Tourniaire & Pulos, 1985; Lobato et al., 2010).
- Incorrect Additive Reasoning (erroneous) - students try to use an additive relationship with the ratio rather than a multiplicative relationship by either adding the same amount to both quantities in the ratio or by maintaining a

constant difference between the two quantities in a ratio (Tourniaire & Pulos, 1985; Lamon, 1993; Steinthorsdottir & Sriraman, 2009; Lobato et al., 2010; Team, 2011).

- Scalar Reasoning - Students iterate a ratio or multiply both quantities in a ratio by the same value in order to scale it to the appropriate value. The language 'scalar' is used by several authors (e.g., Tourniaire & Pulos, 1985; Misailidou & Williams, 2003; Carney et al., 2016). This strategy is often referred to as 'building up' (Tourniaire & Pulos, 1985; Steinthorsdottir & Sriraman, 2009). Scalar reasoning can be further broken down into additive and multiplicative scalar reasoning. Additive scalar reasoning occurs when students iterate a ratio by repeatedly adding it to itself. Multiplicative scalar reasoning is a more sophisticated building up strategy (Tourniaire & Pulos, 1985) and occurs when students scale the ratio using multiplication rather than repeated addition. This may start initially with whole number multiples (or whole number division), but also applies to fractional multiples as a more sophisticated version of the strategy. This concept of multiplicative scalar reasoning has also been referred to as a 'between ratio' multiplicative strategy (Steinthorsdottir & Sriraman, 2009). Both categories of scalar reasoning are described as the "recursive relationship" by Simon & Placa (2012, p. 40), and Lobato et al. (2010) refers to this reasoning as 'composed unit'. The idea of a 'composed unit' and its connection to scalar reasoning is addressed in more detail in the Depth of Understanding section.

- Functional Reasoning - Students identify the multiplicative relationship from one quantity in a ratio to the other. The term ‘functional reasoning’ is used by Simon & Placa (2012) and Carney et al. (2016). This reasoning has also been referred to as a ‘within-ratio’ multiplication strategy (Steinthorsdottir & Sriraman, 2009), a ‘multiplicative comparison’ (Lobato et al., 2010) or, simply, ‘multiplicative’ reasoning (Tourniaire & Pulos, 1985). I choose to use the language ‘functional’ instead of other terms to keep this reasoning distinct from scalar reasoning since both involve multiplication (though in different ways).

A visualization of each of these strategies is shown in Figure 1a below.

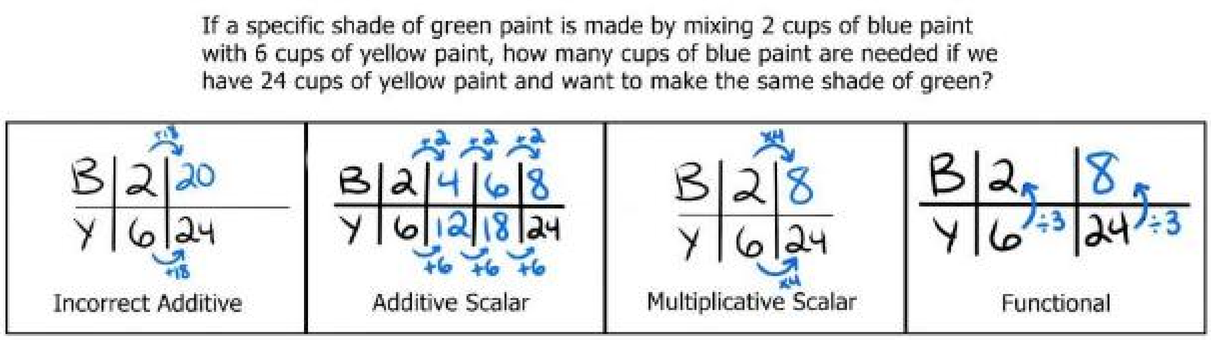

Figure 1a Examples of Different Reasoning for the Same Problem

Along with these different methods of reasoning, the literature also recognizes the use of the unit rate in student solution strategies. I did not state unit rate in the above list of strategies because, though it is certainly a unique and identifiable strategy, it is generally the result of either scaling a given ratio or recognizing the functional relationship between quantities in the ratio. The ‘unit rate’ refers to the identification of the amount of one quantity in the ratio that is required when the other quantity is one unit. This means that for any ratio there are two unit rates, depending on which quantity is the unit. Figure

1b shows the use of the unit rate used with multiplicative scalar reasoning to solve the same paint problem as in Figure 1a.

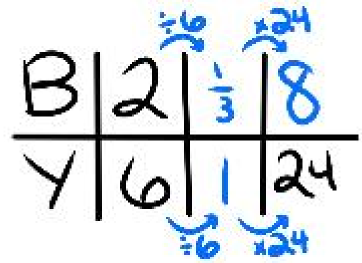

Figure 1b Use of the Unit Rate within a Solution that uses Multiplicative Scalar Reasoning

Though each of these strategies are presented distinctly here, it is very likely that students may use a mix of strategies in their work. This is true both across a variety of problems and within the same problem. For example, students may combine additive and multiplicative scalar strategies when solving problems, such as what is shown in Figure 1c below.

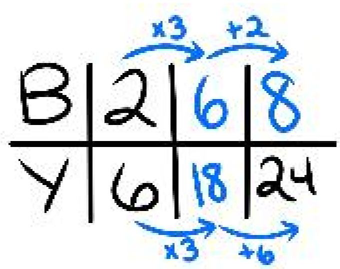

Figure 1c Solution Strategy Using Both Multiplicative and Additive Scalar Reasoning

Depth of Understanding

Along with the various strategies that students use when solving problems involving ratios and proportional reasoning, the literature describes several understandings that students encounter as they make sense of the mathematical concepts surrounding ratios.

From the literature, a picture of a general progression of ratio and proportional reasoning concepts can be ascertained as follows:

1. Identification of Ratio - When making sense of ratios and developing proportional reasoning, students must be able to identify contexts in which the use of ratio is appropriate and recognize that the use of ratio requires identification of a multiplicative (rather than additive) change between quantities (Langrall & Swafford, 2000; Lobato et al., 2010).
2. Composed Unit - Students can coordinate the quantities in a ratio by composing them into a single unit that can be iterated (Langrall & Swafford, 2000; Steinthorsdottir & Sriraman, 2009; Lobato et al., 2010).
3. Multiple Composed Units - Students recognize that there are multiple composed units that can represent a single ratio, such as by creating a new composed unit by partitioning or iterating the original (Langrall & Swafford, 2000; Steinthorsdottir & Sriraman, 2009). This idea is presented in Lobato et al. (2010) as the concept of a rate, specifically stating that, "A rate is a set of infinitely many equivalent ratios," (p.42). Students who have this understanding will be able to solve a wider variety of problems than those in the previous stage because they can work with a range of composed units to get an equivalent ratio that is not a whole number multiple or a whole number factor of the original ratio.
4. Unit Rate - Students recognize and use the unit rate to solve problems (Langrall & Swafford, 2000). The unit rate is highlighted by Lobato et al. (2010) as a method to connect scalar and functional reasoning.

5. Functional Reasoning - Students understand that the multiplicative relationship between the quantities within a ratio does not change even when the ratio is iterated or partitioned and can use this to solve problems. In other words, students are able to identify the functional relationship between quantities in the ratio, and can use functional reasoning to solve problems (Langrall & Swafford, 2000; Steinthorsdottir & Sriraman, 2009; Lobato et al., 2010).

This is by no means the only progression of understanding that has been presented, and it is not intended to argue that a student has to fully grasp one part of the progression before being able to grasp the next idea. This progression begins with the ideas that students most naturally develop first (Tourniaire & Pulos, 1985; Steinthorsdottir & Sriraman, 2009; Lobato et al., 2010), and transitions to more sophisticated concepts that are often developed later on. This does not mean that students can't be showing levels of reasoning that occur in different locations of the progression, and it also does not mean that students in a higher level of the progression will not use ideas from earlier levels (in fact, at higher levels of understanding, students should be able to apply any relevant strategies flexibly to solve problems).

Lobato et al. (2010) describes a progression similar to the one presented above (and in fact, many parts overlap, which can be seen in the citations above), but there are some key differences. Lobato et al. separates the identification of contexts in which proportional reasoning applies from the identification of the multiplicative relationship and places this contextual recognition at the end of their presented ideas. Lobato et. al. (2010) presents functional reasoning (which they describe as 'multiplicative comparison') along with scalar reasoning (which they describe as reasoning with a 'composed unit'), suggesting that the two ideas can be developed in tandem. However, this does not necessarily go against the progression above because the authors state agreement that the scalar reasoning is something that is less sophisticated, "Forming a ratio as a composed unit does not by itself mean that the student has attained the sophisticated understanding of proportionality… Forming a composed unit is a rudimentary, yet foundational concept…" (Lobato et al., 2010, p. 19). Beyond the progression described here, Lobato et al. (2010) describe more than the development of ratio and proportional reasoning in isolation, and instead also connect the idea of ratio to that of fractions and quotients.

The Progressions for the Common Core State Standards in Mathematics document (Team, 2011), and common core standards (on which the Idaho State Standards are currently based) aligns with the trajectory described above. The progression supports the idea that recognizing ratio in grade 6 is a key idea, and clearly describes students' understanding as beginning with scalar strategies and building up to unit rate and functional reasoning. This is explicitly described in the progressions document, and is further supported by the standards including the fact that function reasoning ideas are not explicitly required in sixth grade but are required in seventh grade. As well, the recognition of contexts in which proportional reasoning is applicable is specified in seventh grade, but not sixth, suggesting that these standards align with the ideas of Lobato et al. presented above.

Ratio and Rate

It is worth noting that the literature is inconsistent in regard to the terms ‘ratio’ and ‘rate’. I do not describe this distinction in detail, but if you would like to learn more about the distinction between the concepts and how they might be operationalized in the classroom, Thompson (1994) provides an excellent overview of the ambiguity in literature and an argument for what the distinction between the two should be. I ascribe to Thompson’s chosen definition of rate which is also the definition that Lobato et al. (2010) uses. This definition relies on how students conceptualize the situation rather than relying on characteristics of the problem setting and can be summarized as students conceptualizing “a set of infinitely many equivalent ratios,” (Lobato et al., 2010, p. 42). This means that students have conceptualized that all equivalent ratios have the same rate between them. For example, a 2:5 ratio of blue to yellow paint to make green is conceptualized as a rate not when students write ⅖ blue per unit yellow, but when they can identify *any* equivalent ratio using this rate.

Characteristics of Tasks

Ratio problems have been categorized in a vast variety of ways. Two main methods of categorizing ratio problems are what students are being asked to find (e.g., missing value problems, comparison problems, part-part-whole problems, etc.) (Ben-Chaim et al., 1998; Lobato et al., 2010), or categorizing by the type of information that we are providing (e.g., mixture problems, ‘rate’ problems, etc.) (Tourniaire & Pulos, 1985; de la Cruz, 2013). However, these categories are overlapping (for example, one could be comparing two mixtures, making that problem both a mixture problem and a comparison problem). In this way, these two methods of categorization didn’t seem sufficient on their own, and

I chose to look at ratio problems using the structure presented by Heller et al. (1989), which compares primarily 2 aspects of ratio problem contexts: problem setting and ratio type. Though not the primary goal of the study, this article also discussed the problem format. Along with these three characteristics, I have additionally included number set as an important characteristic.

- Problem Setting: The combination of the objects in the context, the variables used to describe the objects, and the units used to measure the variables. For example, consider the problem: a student runs 2 laps around the track in 7 minutes. If they keep up this pace, how long will it take for them to run 4 laps? In this problem, the object is a student, the variables are distance and time, and the measurements for those variables are laps and minutes, respectively. The more familiar that a student is with the problem setting, the more accessible the problem is for them. The inclusion of a visual model with the problem could be considered an aspect of the problem setting and can increase the accessibility of a problem (Misailidou & Williams, 2003). The choice of variable and measurement also impacts whether the quantities are going to be discrete or continuous. For example, if a variable is an amount of chocolate chips, this could be measured discretely with individual chocolate chips or with more continuous measurements, such as ounces.
- Ratio Type: The type of ratio is connected to the variables of the problem setting. Heller et. al. (1989) described 9 ratio types, some of which are: exchange (buying goods or services, money earned per week), mixture (mix two or more things into one whole, such as lemon juice and sugar to make

lemonade), and speed (how fast or slow an object moves). This category aligns with the "the type of information that we are providing" category that I described earlier. Several problems can have the same ratio type but different problem settings. For example, earlier I described a mixture problem involving lemonade, but another mixture problem could involve making a specific color of paint.

- Problem Format: The problem-format aligns with categories of ratio problems that are distinguished by what students are asked to find and includes missing value and comparison problems. Comparison problems are often considered more complex than missing value problems (Tourniaire & Pulos, 1985).
- Number Set: This could be considered part of the problem setting, but is distinct in that we can change the number set without changing the problem setting and impact the difficulty of the problem as a result. The number set in a problem consists of both the numbers that are presented to students as well as the number relationship between the given information and the solution, and the solution itself. Number choice can greatly impact the challenge of a task (Tourniaire & Pulos, 1985). de la Cruz (2013) described one aspect of the number set that refers to the change between quantities in the ratio. These can be described as four categories:
    a. the two ratios have a whole number scalar relationship, but not a whole number functional relationship
    b. the two ratios have a whole number functional relationship, but not a whole number scalar relationship

c. the two ratios have both a whole number scalar and whole number functional relationship

d. the two ratios have neither a whole number scalar nor whole number functional relationship

Of these four categories, de la Cruz described that type d was significantly more difficult than the other three.

Heller et al. (1989) looked to identify whether the ratio type or problem setting had a larger impact on the difficulty of ratio problems for two problem formats (missing value and comparison problems). They found that the ratio type has a larger impact on problem difficulty than the problem setting, but that familiarity with the problem setting (or lack thereof) became increasingly important as the ratio type became more challenging. The ratio types that they used were exchange, speed, and consumption. They describe consumption as, “how efficiently something is consumed (used up) or produced (made),” Their problem settings for each of these contexts were buying gum and buying records, running laps around a track and driving cars, and gas mileage of trucks and the oil consumption of furnaces, respectively. Of their ratio types, buying was the easiest and consumption was the most challenging for students.

Explicit Attention to Concepts (EAC) and Students’ Opportunity to Struggle (SOS)

<u>Definition and Characteristics</u>

In 2007, Hiebert & Grouws looked across empirical research to identify similarities in instruction that led to an increase in conceptual understanding. They were able to identify two features of instruction that appeared consistently in research that led to increased conceptual understanding. They described these characteristics as, “Teachers and

Students Attend Explicitly to Concepts," (p. 383), and "Students Struggle with Important Mathematics," (p. 387). I refer to these characteristics as Explicit Attention to Concepts (EAC), and Students' Opportunity to Struggle (SOS) using the language of Stein et al. (2017).

Hiebert & Grouws describe attending to concepts as, "treating mathematical concepts in an explicit and public way," (p. 383). They further describe:

> This could include discussing the mathematical meaning underlying procedures, asking questions about how different solution strategies are similar to and different from each other, considering the ways in which mathematical problems build on each other or are special (or general) cases of each other, attending to the relationships among mathematical ideas, and reminding students about the main point of the lesson and how this point fits within the current sequence of lessons and ideas (p.383).

On the other hand, students' opportunity to struggle involves students' opportunity to explore and wrestle with mathematical ideas and to make sense of mathematics. Specifically, they describe that 'struggle' occurs when students are asked to, "figure something out that is not immediately apparent," (p. 387). It is important to note the distinction they make about the term struggle:

> We do not use struggle to mean needless frustration or extreme levels of challenge created by nonsensical or overly difficult problems. We do not mean the feelings of despair that some students can experience when little of the material makes sense. The struggle we have in mind comes from solving problems that are within reach and grappling with key mathematical ideas that are comprehendible but not yet well formed (p. 387).

In this way, when students are given opportunities to 'struggle' they are not pushed to a place where they are overwhelmed or want to give up, but they are presented with problems that are within reach, but not fully formed, and have the opportunity to explore more deeply than in situations where students are asked to memorize or repeat a demonstrated process.

Impact

With these characteristics in mind, we can start to see what these constructs look like in instruction. Stein et al. (2017) built on the work of Hiebert & Grouws and looked at the impact of these two constructs, specifically the impact on students' understanding when instruction has different levels of both EAC and SOS. They considered four combinations of EAC and SOS in instruction: High EAC and high SOS, high EAC and low SOS, low EAC and high SOS, and low EAC and low SOS. These categories were represented in a 2 x 2 matrix and are henceforth referred to as 'quadrants'. The researchers were interested in the relationship between which quadrant teachers primarily fell in and their students' understanding as shown by standardized scores on the Tennessee Comprehensive Assessment Program (TCAP) and constructed response assessment (CRA). The TCAP test was a more procedural or skills-based assessment, and the CRA was a more conceptual assessment.

Teachers were categorized into one of the four quadrants based on a survey involving self-reported preference for instructional practices related to EAC and SOS as well as through video evidence and artifacts of student work. Then, they used the TCAP and CRA data to gauge students' understanding. Students whose teachers' instruction was high in both EAC and SOS had the highest scores on both the TCAP (skills-based) and CRA (conceptual) assessments. Students whose teachers focused primarily on EAC had the next highest scores on both TCAP and CRA. Those who focused primarily on SOS followed, and those whose teachers had rarely had either element scored lowest. Only instruction with a high focus on both EAC and SOS had CRA scores higher than the TCAP. However, the only statistically significant differences occurred between the CRA assessments with students whose teachers had high EAC and high SOS being statistically significantly higher than those whose teachers focus on only SOS, and from those who had both low EAC and SOS.

Summary

Building an understanding of ratios is a complex process, but the literature provides excellent guidance in this process. The hypothetical learning trajectory provides an underlying structure for designing lessons towards a set goal by having the instructor select an instructional goal, actively consider students' incoming knowledge, and reflect on the impact of different activities considering students' incoming knowledge and the goal for understanding. Though the Hypothetical Learning Trajectory provides an outline for unit design, the use of KDUs and the reflection on the activity-effect relationship provide a more defined structure for goal selection and mathematical task selection, respectively.

Students' understanding of ratios has been studied extensively and a meaningful progression of conceptualizations can be identified from this literature. Additionally, the literature describes characteristics of ratio and proportional reasoning tasks and the impacts of these characteristics students' ability to engage with material. Overall, students' understanding has been shown to progress from additive scalar reasoning to multiplicative scalar reasoning to identification of the unit rate, and finally to the use of functional reasoning.

Tasks can be characterized in a variety of ways, but the problem setting, problem format, ratio type, and number set provide key characteristics for anticipating students' ability to access, engage, and be challenged by the task. The problem setting includes surface level features including the objects, variables, and how they are measured in a task. The choice of objects, variables, and measures can impact students' engagement based on how familiar they are with that context. For example, when working with exchange ratio types, students are likely to be more familiar with buying cookies than they would be with buying stocks and bonds. Each of these characteristics impact the accessibility of a problem for students and when working with more challenging number sets or ratio types, it is increasingly beneficial to provide students with more familiar contexts.

Along with developing lesson materials and developing the understanding of ratios, the instructional constructs of EAC and SOS describe two aspects of instruction that have appeared frequently in empirical studies that show an increased conceptual understanding in students. SOS provides students with an opportunity to grapple with mathematics ideas and create connections between new ideas and their existing understanding. This plays nicely with the idea of constructivism that also underlies the hypothetical learning trajectory because they both include the feature of building on students’ existing understanding to create a new perception that is more accurate. Explicit Attention to Concepts serves as an opportunity to further solidify the connections that students are making or to encourage students to identify new connections that they may not have yet observed themselves. This further establishes connections between existing ideas or connections between new ideas and existing ideas. Having high levels of both EAC and SOS has been shown to encourage the highest conceptual understanding of students compared to instruction with lower levels of EAC and SOS. However, it is not yet clear if the order in which these instructional constructs occur impacts this level of understanding, and this question is the focus of the current study.

CHAPTER THREE: METHODOLOGY

The goal of this study was to investigate if the lesson materials increased students' understanding and if the order in which opportunities for EAC and SOS were presented in the lessons impact the level of student understanding. Data were collected using a pre-/post-assessment via a Google form. In this chapter, I describe the creation of the lesson materials, the research design, the participants, and the methods for data collection and analysis.

Development of Lesson Materials

Considering the progression of development of students' understanding, I decided to focus on the learning goal of developing students' ability to identify and use the functional relationship to solve ratio problems. This concept has been described as challenging for students to grasp (Simon & Placa, 2012), and is requisite for students to make the connection between ratios and proportional linear equations. Due to the importance of this understanding for future mathematics concepts, and a personal interest in functional understanding of students in general, I chose this as the focus of my lessons. The selection of a learning goal is the first step in a hypothetical learning trajectory. After making this selection, I began to consider task selection and reflected on the activity-effect relationship when doing so. Based on the literature, I anticipated that most students would initially begin by using scalar reasoning to solve problems, and that they would apply scalar reasoning with varying levels of confidence. In particular, I anticipated that most students, but certainly not all, would be comfortable with multiplicative scalar reasoning using integers, while others would be able to use this reasoning with non-integers, and that there would be again others who are only comfortable using additive scalar reasoning.

When first thinking about tasks, I wanted to limit confusion that could be caused by using several different problem settings. As a result, I chose to use primarily a single problem setting throughout the build of the materials (all but the EAC section of the final lesson, which focused on connecting students' previous solutions to other problem settings). The problem setting used throughout the 5 lessons was the context of creating soap. This context uses a mixture ratio type and is similar to mixing paint problems, but uses materials that I had readily available. This allowed me to create videos representing problems and solutions throughout the lessons. The variables of the problems were volume (of different colors) measured in teaspoons. Problems presented students with a ratio of colors measured in teaspoons (e.g., 4 teaspoons of blue and 8 teaspoons of white) to create a specific color or shade of soap. Out of 34 problems presented, all but 7 were missing value or comparison problems.

The missing value problems used in the lessons began with integer functional relationships and non-integer scalar relationships and ended with both non-integer functional and scalar relationships. For example, the first lesson's SOS section used the ratio of 4 teaspoons of blue to 8 teaspoons of white, and had students determine different ways to correct a mixture of 3 teaspoons of blue to 7 teaspoons of white so that it makes the same shade. While 8 is in an integer multiple of 4, neither 3 nor 7 are integer multiples of 4 or 8. However, this doesn't mean students will use the functional relationship to solve the problem. The problem required them to find two different solutions. One solution could be adding 1 teaspoon of blue and 1 teaspoon of white to the mixture to match the 4:8 given ratio. For the second solution, they might double the given 4:8 ratio to get 8:16 and thus add 5 teaspoons of blue and 9 teaspoons of white to the incorrect mixture to make it match. In doing this, students would not have had to use the functional relationship. However, additional questioning asking students to describe the relationship between blue and white was designed to encourage students to attend to the functional relationship, along with the EAC section, which included connections between both scalar and functional reasoning solutions.

Before designing the next lesson, I reflected on the activity-effect relationship to anticipate where student understanding would likely be as a result of the lesson. With the SOS problems, I intended students' goal to be to fix a given incorrect paint mixture, but, as previously stated, this doesn't guarantee that they will do so using functional reasoning. Regardless of if they identify this reasoning or not, however, they are presented with this relationship during the EAC section to make connections. As a result, I anticipated that by the end of the lesson they could identify an integer functional relationship, and see a potential benefit for it, even though it still may not be the relationship that is natural or most comfortable to them (they may still prefer using scalar reasoning). With this in mind, I used comparison problems in the second lesson, which allow for both scalar, functional, and use of unit rate to solve. Students can solve comparison problems by scaling up or down two or more ratios so one of the quantities in the ratio are the same, by finding the unit rate for each ratio, or by identifying the functional relationship (which involves similar reasoning to that of the unit rate). In this way, there were many opportunities to connect these three solution methods and allow students to identify places where each might be more efficient.

This process of reflecting on the activity and potential ideas that students could develop from the activity allowed me to continually build each lesson on the previous with the idea that students' new understanding should build on the ideas that they already have. The EAC portion of each lesson was crucial in being as sure as possible that students were able to make connections between functional reasoning and their current understanding if they were not yet using functional reasoning. As the problems progressed, students were asked to identify functional relationships more explicitly (such as with prompts like, "Complete this sentence: The amount of YELLOW soap is always _____ times the amount of BLUE soap."). The SOS section of the final lesson further reinforces the unit rate and function reasoning by explicitly asking for them to be identified, and the EAC section connects students' understanding of these mixture problems to additional ratio types. Namely, rate (miles per minute) and exchange (cupcakes to dollars). There is also additional reinforcement connecting visual and symbolic representations of functional reasoning in the EAC section of the final lesson. The intent here was to encourage transfer of the ideas to additional ratio types and further emphasize the connection between functional relationships symbolic representation and its visual representation, further solidifying the connection between these ideas for both integer and non-integer functional relationships.

Research Design

In this study I focused on the following research questions:

1. Do the lesson materials created lead to an increased understanding of proportional reasoning as a whole?
2. Do the lesson materials created lead to an increased understanding of the functional relationship in ratios?
3. Is there a difference between students' understanding of the functional relationship in ratios when instruction focuses first on EAC then on SOS compared to instruction that focuses first on SOS then on EAC?

To explore these questions, this research uses a quasi-experimental design in which there are two groups compared with (in this case) a pre-assessment, and post-assessment. This design does not include a control group but is instead looking for differences between students whose instruction focused on SOS then EAC compared to those whose instruction focused on EAC then SOS. There is no random assignment of students into these groups (it is instead determined by the selection of materials made by their teachers). A visualization of this research design is included in Figure 2. The lesson materials can be found in Desmos at the following links:

- SOS then EAC materials: https://teacher.desmos.com/collection/5fa83cc4ee1cac78b386d5b1
- EAC then SOS materials: https://teacher.desmos.com/collection/5fa83d7423d9f01b310d198b

- A hard-copy version of the EAC then SOS materials was also created to accommodate two students who did not have a device to access the materials online (see Appendix B).

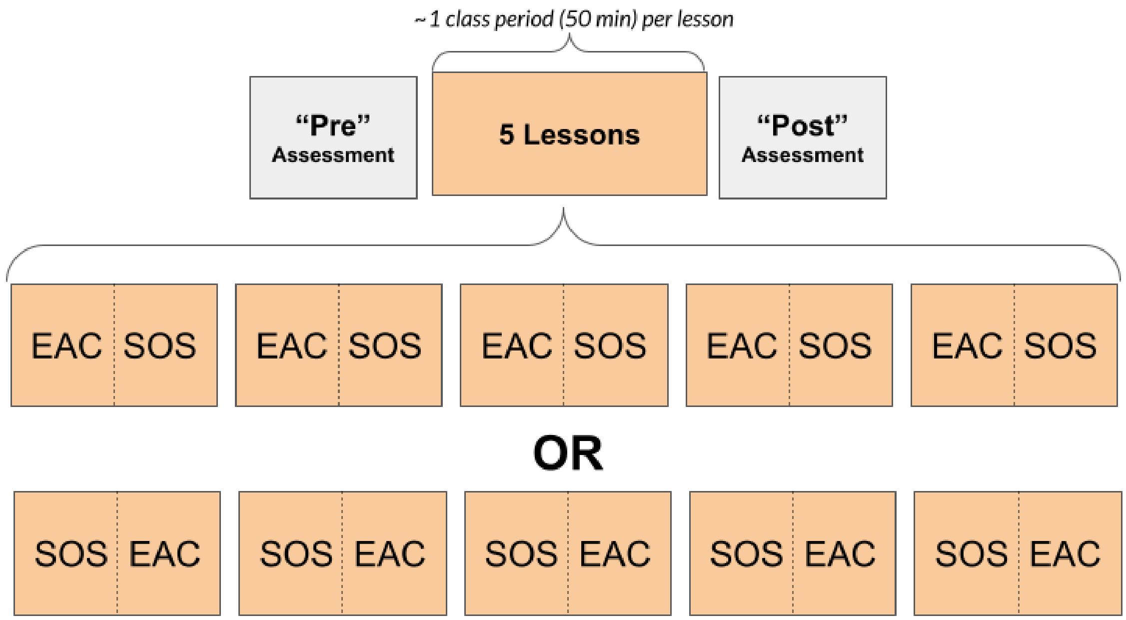

Figure 2 **Structure of Research Design**

Research Context

This study was conducted during Fall 2020 and Spring 2021 in the midst of a national pandemic caused by Covid-19. Due to these circumstances, most local educators faced the challenge of working flexibly in a variety of formats including online, hybrid, and socially distanced in person instruction. In response to this unprecedented time, the lesson materials created for this study were designed with flexibility in mind. The use of Desmos allowed for the lesson materials to be implemented in an online-only, hybrid, or in-person format, provided that students had access to both an internet-accessible device and the internet. Of the seven teachers, six were in a hybrid setting with some students working online synchronously and others working in person, and one teacher was fully in-person. All pre- and post- assessments were taken in the classroom, though the form itself required that the assessment be submitted online. A hard-copy version of the materials was also created upon request to accommodate two students who did not have access to the required technology to complete the materials online.

Independent Variable

The independent variable of this study was the lesson materials used during instruction. Teachers chose whether to use the set of lessons that focused on EAC then SOS or the set that focused on SOS then EAC during instruction. This structure was present in each of the 5 lessons that they implemented. This created two categories of students to be compared based on the type of instruction they received. The lesson materials were created with a section dedicated specifically to EAC and another section dedicated to SOS. The two sets of lessons differed only by the order in which these two sections were taught in each lesson. For example, in Lesson 1 of the EAC then SOS materials, slides 2-14 focused on EAC and slides 15-27 focused on SOS (slide 1 instructed students to get out pencil and paper), whereas these sections were switched in SOS then EAC materials, having slides 2-14 focus on SOS and slide 15-27 focus on EAC. It was not the case that every lesson had the same number of slides dedicated to each section, but the overall structure of only swapping the order of two sections for the different materials was consistent.

Dependent Variable

The dependent variable of this study was student understanding of proportional reasoning and functional reasoning with ratios as measured by a proportional reasoning assessment.

Students took the assessment via a Google form (see Appendix A). After creating an anonymous name to be used for matching pre- and post-assessments, students were presented with four item blocks, each with a total of 5 items plus a prompt at the end of each item block for students to explain how they solved the items. The first item block included a visual support, but no other item blocks included a visual with the context. Within the item blocks, parts “a” and “e” were missing value problems, parts “b” and “c” required students to identify the functional relationship, and part “d” asked students to identify the unit rate. The final section of the assessment was a two-part comparison problem with prompts to explain how they solved the problem.

Participant Selection

Participants were sixth, seventh, and eighth grade students whose teachers voluntarily chose to implement the lesson materials and pre-/post-assessment as part of their regular instruction. Seven teachers of grades 6-8 implemented the lesson materials for this study. Three of the teachers implemented the SOS then EAC materials, and four of the teachers implemented the EAC then SOS materials. In the Data Collection section below, Table 1 describes the number of pre- and post-assessments that were taken by students in each instructional group and grade level.

Data Collection

Data were collected anonymously via a Google form assessment. Teachers provided students with the link to the Google form that I created. The data was collected this way so that students could take the assessment at home or in-person depending on the school’s current teaching format, and all teachers who administered the materials were able to give the assessment to their students while they were in the classroom. In the assessment, students were prompted to create an anonymous name, which was used to match pre- and post-assessments. This anonymous name was generated by answering the questions:

- What is the first letter of your middle initial (if none, write X)?
- What day of the month is your birthday?
- Number of Older Brothers (half-brother, living, or deceased, if none write 0)?
- Number of Older Sisters (half-sister, living or deceased, if none write 0)?

Unfortunately, not all students consistently entered the identifier from the pre- to post-assessment. The number of pre-assessments, post-assessments, and the number of pre- and post- assessment that were able to be matched is described in Table 1 below.

Table 1 Number of Assessments Completed

	Grade	Pre-Assessment	Post-Assessment	Matched Assessments
SOS then EAC instruction	6	73	90	27
	7	12	12	10
	8	8	17	8
	Total	93	119	45
EAC then SOS instruction	6	90	96	47
	7	78	81	38
	Total	168	177	85

After the creation of their anonymous name, students also provided their teachers name and their grade. Then, students answered a total of 22 items (5 items each within item blocks 1-4 and two comparison items not included in the item blocks). The score for the assessment was the total correct numeric answers with a max score of 22. Before scoring, the data were cleaned so responses included only the number (e.g., “8 cookies” would be changed to “8”), and fractions and decimals were written in the form of a decimal rounded to two decimal places (e.g., “¼” would be changed to “0.25”).

The total correct score was used to analyze student growth on proportional reasoning understanding (research question 1). To look at students’ understanding of functional reasoning with ratios (research question 2), the total correct from parts “b” and “c” of the first four item blocks was analyzed (total functional reasoning correct). The score for functional reasoning (out of 8 possible) was compared between the two groups of students to identify differences between the understanding of the EAC then SOS and SOS then EAC instructional groups (research question 3). Throughout this paper, the phrase “proportional reasoning scores” will refer to the score (out of 22) for the entire assessment and “functional reasoning scores” will refer to the score (out of 8) of the specific functional reasoning questions within the assessment.

Analysis Approach

To address research questions 1 and 2, Paired Sample t-Tests were used to identify if there is a significant difference in the understanding of proportional reasoning and functional reasoning between the pre- and post-assessments. These Paired Sample t-Tests used the scores from the entire sample of students, without separating by instructional group. Analysis of Covariance (ANCOVA) was used to statistically compare the means of the post-assessment scores for paired data between the two groups of students to identify if there was a difference in the understanding of the students’ functional reasoning ideas. The ANCOVA was chosen to compare the means of these two groups with their scores on the pre-assessment as a covariant. By using the pre-assessment as a covariate, I hoped to equalize differences between the students making it more likely that any observed difference is due to the difference in instruction, rather than differences in students' initial understanding. During the analysis of data, I observed that one student's full assessment score decreased by 14 points from 17 to 3. This was the only student with such an extreme decrease, the next highest decrease being 9 points by another student. Due to this stark difference from the rest of the data, I chose to remove this student from my final data analysis, though I did run each of the statistical tests with this student as well and found the same levels of significance across each of the tests.

CHAPTER FOUR: RESULTS

Students' understanding of proportional reasoning and functional reasoning with ratios was measured by the proportional reasoning assessment administered via Google form. Here, I describe the statistical results addressing student growth in understanding of proportional reasoning with ratios as a whole group, growth of understanding of functional reasoning with ratios as a whole group, and differences in understanding of functional reasoning with ratios between the two instructional groups (SOS then EAC compared to EAC then SOS instruction). For each of the statistical analyses conducted, I used a significance level of $\alpha = 0.05$.

Understanding of Proportional Reasoning with Ratios

To address my first research question surrounding growth of understanding of proportional reasoning resulting from the lesson materials, I used a Paired Sample t-Test. Students' scores on the entire assessment (out of a possible 22 points) were paired by their anonymous name ($n = 129$), and then the Paired Sample t-Test was used to identify whether growth in understanding had occurred. My alternative hypothesis was that the mean score of the post-assessment would be higher than the mean score of the pre-assessment. So, I used a one-tailed t-test. The results of the t-test indicated that there was a statistically significant difference between the pre- and post-assessment means ($t = 6.238$, $df = 128$, $p < 0.001$). The gain score (mean of the differences) was 1.946 and the median of the gain scores was 2. This positive difference shows an increase in understanding, and is further highlighted in the histogram of gain scores shown in Figure

3. The Cohen's *d* effect size for Paired Sample t-Tests, which is based on the standard deviation of the differences, was 0.549.

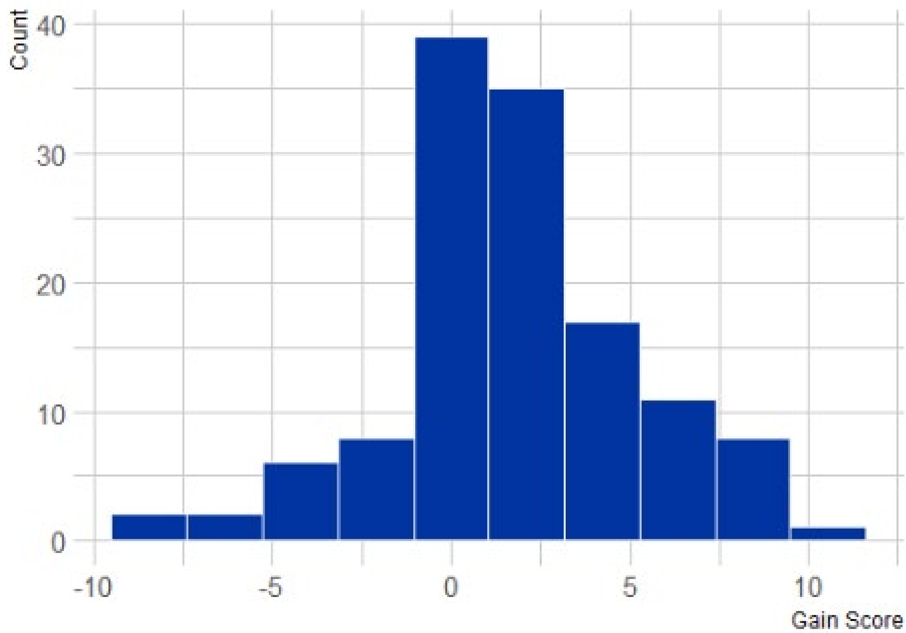

Figure 3 Histogram of Differences in Full Assessment Scores for the Whole Group

Table 2 Full Assessment Summary Results

Score	n	Mean	Std. Deviation
Pre-assessment	129	8.620	5.842
Post-assessment	129	10.566	6.049
Gain	129	1.946	3.543

Understanding of Functional Reasoning with Ratios

To address my second research question surrounding the growth of understanding of functional reasoning with ratios, I again used a Paired Sample t-Test. Questions specifically addressing functional reasoning (scored out of a possible 8 points) were paired by their anonymous name (n = 129), and then a Paired Sample t-Test was used to identify if a growth in understanding of functional reasoning with ratios had occurred. My alternative hypothesis was that the mean score of the post-assessment results would be higher than that of the pre-assessment results. So, I again used a one-tailed t-test. The results of the t-test indicated that the difference between the pre- and post-assessment results was statistically significant ($t = 4.911$, $df = 128$, $p < 0.001$). The mean difference in score was 0.729, though the median was 0. This positive difference represented by the mean score shows an increase in understanding, and this increase is further highlighted in the histogram of gain scores shown in Figure 4. The Cohen's *d* effect size for Paired Sample t-Tests was 0.432.

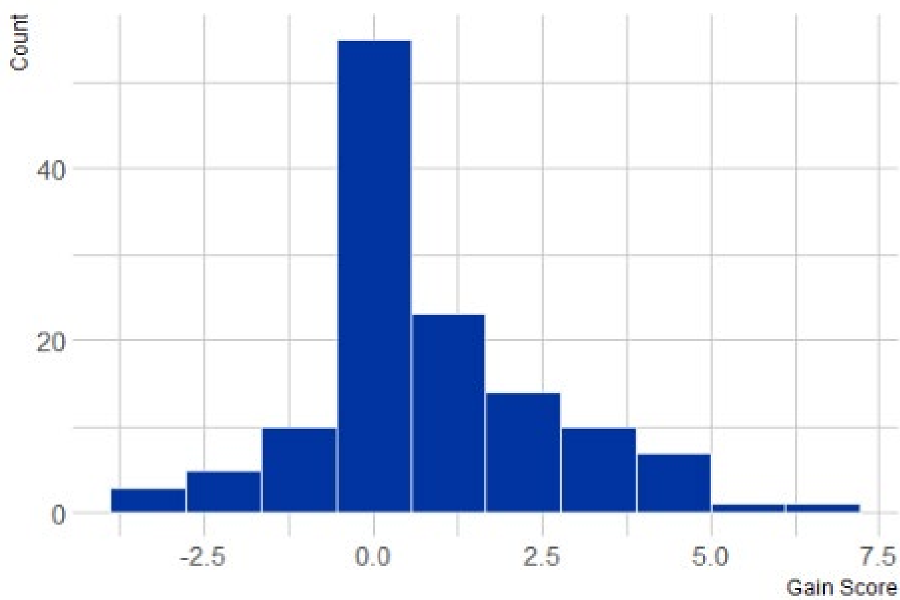

Figure 4 Histogram of Differences in Functional Assessment Scores for the Whole Group

Table 3 Functional Reasoning Assessment Summary Results

Score	n	Mean	Std. Deviation
Pre-assessment	129	1.736	2.064
Post-assessment	129	2.465	2.456
Gain	129	0.729	1.685

Comparison of Functional Reasoning Across Instructional Groups

To address my third and final research question surrounding differences in understanding, I ran an ANCOVA using the paired scores (n = 129) to determine if there was a difference between students' functional reasoning with ratios score (out of 8 points) using the pre-assessment as a covariate. Based on the literature, my alternative hypothesis was that the mean score of the SOS then EAC group would be higher than that of the EAC then SOS group. The ANCOVA F-statistic has an asymmetrical distribution, and detects only differences between the groups not where those differences are (e.g. which group has a higher mean). Therefore, though my alternative hypothesis was directional, I did not run a "one-tailed" test as it is not applicable in this context. The results of the ANCOVA indicate that there is a significant difference between the mean scores of the two instructional groups ($F_{(1, 126)} = 10.395$, $p = 0.002$).

Because we only have two groups being compared (EAC first and SOS first), we know that the significant difference observed is between these groups. For post-hoc analysis, I looked at the estimated marginal means. The Bonferroni correction for multiple comparisons was not necessary because only two groups were being compared (EAC first and SOS first), and thus there was only one comparison. I found estimated marginal means (controlling for the covariate of the pre-assessment score) of 2.81 +/- 0.347 (95% confidence) and 1.81 +/- 0.488 (95% confidence) for the EAC first and SOS first groups, respectively. This shows that when taking the pre-assessment into account the mean score for EAC then SOS instructional group was statistically significantly higher than that of the SOS then EAC instructional group. The box plots in Figure 5 shows the differences between the pre- and post-assessments for both instructional groups, which further highlights the increased growth in understanding in the EAC then SOS group compared to the SOS then EAC group. The effect size, partial eta squared, was 0.076. Table 4 summarizes the pre-assessment, post-assessment, and gain scores for the two groups. You may notice that the actual mean post-assessment score for the SOS first group is higher than that of the EAC first group, but keep in mind that this mean does not account for the pre-assessment as covariate as the previously reported estimated marginal mean does. The raw post-assessment scores lose the paired nature of the data when considered alone.

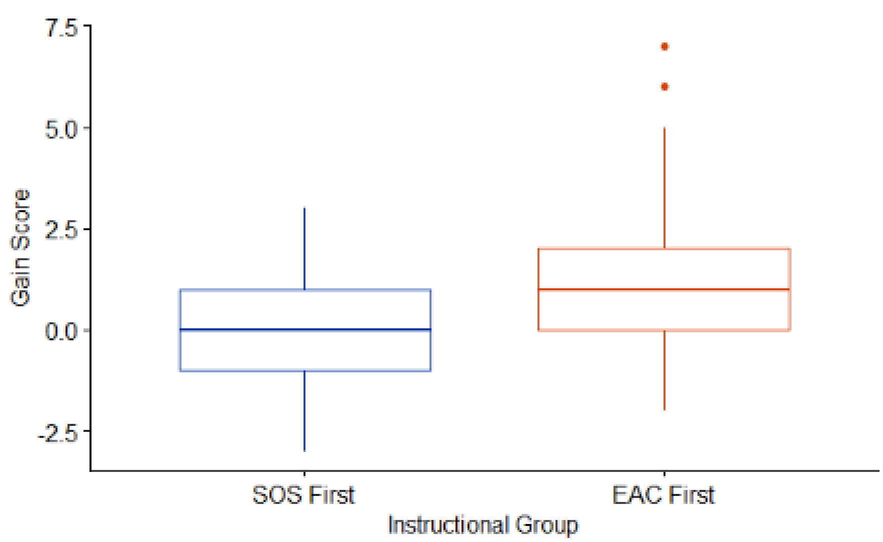

Figure 5 Box Plots of Gain Scores for Each Group using Matched Data

Table 4 Functional Reasoning Assessment Results Compared by Group

Group	Score	n	Mean	Std. Deviation
	Pre-assessment	44	2.455	2.556
SOS First	Post-assessment	44	2.477	2.706
	Gain	44	0.023	1.372
	Pre-assessment	85	1.364	1.654
EAC First	Post-assessment	85	2.459	2.333
	Gain	85	1.094	1.722

CHAPTER FIVE: DISCUSSION

In this study, I was interested in determining if the lesson materials I created led to increased understanding of proportional reasoning and functional reasoning with ratios, and in exploring differences in functional reasoning understanding that may have resulted from the order in which the instructional strategies of EAC and SOS were present. In this chapter, I interpret the results and discuss their implications and limitations.

Growth of Understanding

The data indicate that the lesson materials led to an increased understanding of proportional reasoning and functional reasoning with ratios that was likely due to instruction rather than random chance. For the full assessment, the median and mean of the differences were 2 and 1.946, respectively. For the functional reasoning questions, the median and mean differences were 0 and 0.729, respectively. Though the change that was observed is not likely to be due to random chance, the increase in understanding was not great as I had hoped to result from the lesson materials, particularly in regard to functional reasoning. Still, the fact that these materials were administered during a pandemic which meant that there were varied and difficult learning environments for students, and that the assessment was administered online (which often results in students trying to do calculations more mentally rather than doing their calculations on paper), this growth is notable. As well, there was a medium effect size for the full assessment and functional reasoning sub-section, respectively, which further supports the effectiveness of the lesson materials.

It is also worth noting that because many of the 6th grade students were being formally exposed to ratios for the first time, whereas higher grade level students would have been exposed to it in 6th grade, these materials may not have provided appropriate attention to the additive scalar and multiplicative scalar reasoning ideas that would have been more appropriate for students' initial understanding. Without having the time to explore these ideas in depth, it may have been even harder for students to grasp the more complex functional reasoning ideas.

Additionally, not only were the students learning when presented with the lessons, but the teachers may have been learning as well. One teacher reflected, "At the beginning I struggled on what I needed to say and what to expect from the students but as the lessons progressed, I was better at presenting the material!" It is possible that had teachers been more practiced with delivering instruction with Desmos activities (as well as delivering material in hybrid settings), there would have been a different amount of growth.

Along with the growth in understanding, it is also worth noting that one aspect of learning that was not assessed was students' engagement with lesson materials. Another teacher who implemented the materials commented, "One of my students emailed pictures of a yoga studio wall design she is painting and the ratio table she created to mix perfect paint combinations. You did a good job of making math authentic!" This comment demonstrates the engagement of one student with these materials. Another potentially interesting topic to explore surrounding SOS and EAC instructional strategies would be the impact of these strategies on student engagement. Was it the context of the problems alone that engaged this student (and hopefully others) or did the incorporation of SOS and EAC strategies on top of a real-world context lead to increased engagement?

Though engagement itself can be challenging to measure, this is an additional instructional characteristic that would be an interesting topic for future research and helpful for teachers to better understand.

Differences in Understanding

The data indicated a statistically significant difference in functional reasoning skills between the two instructional groups with the EAC then SOS group showing higher understanding than that of the SOS then EAC group. Not only was the understanding of the EAC then SOS instructional group statistically significantly higher than that of the SOS then EAC group (when accounting for initial differences in the pre-assessment), but there was a medium effect size for this difference. This indicates that the order in which the EAC and SOS instructional strategies occur may impact student understanding, specifically indicating this difference in a direction that contradicts earlier research.

My alternative hypothesis based on the work of Schwartz et al. (2011) and Kapur (2014) was that the SOS then EAC group would have a deeper understanding of functional reasoning. However, there are differences in these past studies compared to the study at hand. It is worth noting that the deeper understanding observed by Schwartz et al. (2011) and Kapur (2014) was that of conceptual (rather than procedural) understanding. Kapur (2014) found that both teaching concepts and procedures then practicing problems, and working on problems prior to being explicitly taught concepts and procedures led to equal procedural knowledge, but that there was a statistically significant difference in conceptual knowledge. It is possible that the assessment used was not sensitive enough to more subtle conceptual understanding differences due to the reliance on numeric answer questions rather than explanatory questions, and this may have impacted where the observed differences arose.

Additionally, Schwartz et al. (2011) and Kapur (2014) were not specifically using the instructional strategies of EAC and SOS in their studies. They were, instead, looking at explicit instruction prior to problem exploration and vice versa. Though these are similar, there may be differences in both direct instruction and exploratory opportunities in their studies compared to the strategies used here. One notable potential difference is the type of explicit instruction. Schwartz et al. (2011) describe explicit instruction as a lecture on the topic at hand and providing formulas and worked examples prior to instruction. These lectures and worked examples may not be strategies that would be categorized as EAC because EAC strategies focus on connections between solutions, representations (e.g. connecting a visual to a symbolic representation), and ideas (e.g. connecting the current lesson to a 'big picture'). It is unclear how many of these types of connections would have been made during the explicit instruction in the Schwartz et al. (2011) and Kapur (2014) articles. As well, some of these students in my study explored the lesson materials online at-home, which means it may be less likely that those students engaged as deeply in productive struggle without the support of teachers and peers that they would have in a classroom setting. So, the engagement with struggle in my study may differ from that of previous studies, though all students engaged in at least some of the lesson materials in the classroom (through hybrid and in-person settings).

Along with differences between the 'explicit' instruction in my lesson materials compared to that of Schwartz et al. (2011) and Kapur (2014), it is possible that differences in the teachers' instruction may have created differences in understanding for the two instructional groups. If some teachers had more experience teaching ratios and proportional reasoning or more experience teaching with Desmos materials, this experience could have impacted the quality of instruction that the students received, and thus impacted the resulting student understanding.

It is also important to note that, due to the lack of delayed post-assessment, it is unclear how these differences will be reflected in retention (if at all). Schwartz et al. (2011), observed that students' conceptual understanding (demonstrated by students' ability to transfer ratio problem structure to different physical applications) was statistically different both with the immediate transfer task and the delayed transfer task. There is no way of currently telling if the difference in understanding observed in my study would be retained. Future studies would benefit from including an additional delayed post-assessment to provide insight into differences in retention of understanding (if they exist).

CHAPTER SIX: CONCLUSIONS

The instructional materials themselves led to an increased understanding of both proportional reasoning with ratios as a whole and in the sub-area of functional reasoning with ratios. Though the increase in understanding may seem relatively small, there was medium or greater effect size, showing that this growth in understanding is meaningful. This suggests that the materials created are useful in increasing student understanding, even in a range of instructional formats (remote, hybrid, in-person, or a mix).

With its combination of learning gains and the low learning curve required to successfully implement these lesson materials into instruction, teachers, no matter their instructional formats or pedagogical habits, can easily integrate these materials into their current curricula to affect growth in their students' understanding. However, a teacher should reflect, of course, on their learning goals for their students. If the learning goal is to foster a conceptual understanding of the functional relationship between quantities in a ratio, then these materials may be a good fit. However, it may be helpful for teachers to help students formalize their additive and multiplicative scalar reasoning strategies and build a strong foundational understanding of ratios in general prior to working on more complex ideas like functional reasoning with ratios.

In this study, I worked under the assumption that Hiebert & Grouws (2007) and Stein et. al. (2017) were correct in concluding that the incorporation of EAC and SOS instructional strategies leads to increased understanding, particularly conceptual understanding. With that in mind, the results of this study provide some preliminary evidence that the order in which these two instructional strategies occur may impact students' understanding. Specifically, I found that students' whose instruction focused on EAC before SOS showed a statistically significant increase in understanding of functional reasoning with ratios compared to those who were exposed to SOS before EAC. This contradicts the work of Schwartz et al. (2011) and Kapur (2014), which suggested that minimally aided problem exploration before direct instruction would lead to increased understanding compared to students' who were exposed to direct instruction prior to exploring problems. However, additional research is necessary to identify if these differences are still present in long term retention. Further, particularly because this study contradicts earlier evidence, it will be important for future research to focus on the impact of the order in which EAC and SOS instructional strategies occur in order to identify if these results are replicable.

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APPENDIX A

Proportional Reasoning Assessment (Google Form)

Proportional Reasoning Assessment

It will be helpful for you to have a pencil and piece of paper available as you work through these problems.

* Required

Create an Anonymous name

To create your anonymous name, answer the following questions. Put all of your responses together with no spaces to create your anonymous name:

What is the first letter of your middle initial (in none, write X)?
What day of the month is your birthday?
Number of Older Brothers (half-brother, living, or deceased, if none write 0)?
Number of Older Sisters (half-sister, living or deceased, if none write 0)?

Example:
My middle name is Marlene, I was born on July 26th, I have 0 older brothers and 2 older sisters. My anonymous name is
M2602

Enter Your Anonymous Name *

Your answer

What is your teacher's name? *

Your answer

What grade are you in? *

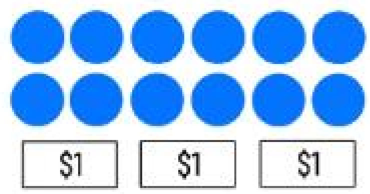
Your answer

Next

Proportional Reasoning Assessment

Ellie's Cookies

1a) Ellie bought 12 cookies for $3. How many cookies can Ellie buy with $2?



Your answer

If the relationship between cookies and cost ($) remains the same no matter how many cookies you buy:

12 cookies for $3

Complete the following statements about the relationship between cookies and cost ($).

1b) number of cookies = ____ • cost

• indicates multiplication

Your answer

1c) The cost is always ____ times the number of cookies

Your answer

1d) One cookie costs ____

Your answer

1e) You can buy _____ cookies for $45

Your answer

Describe in your own words how you solved the problems above.

Your answer

Back Next

Never submit passwords through Google Forms.

Proportional Reasoning Assessment

Jane's Hike

2a) Jane can hike 5 miles in 4 hours. How far can Jane hike in 16 hours?

Your answer

If the relationship between miles and hours remains the same no matter how many miles Jane hikes:

5 miles in 4 hours

Complete the following statements about the relationship between miles and hours.

2b) hours = _____ • number of miles

• indicates multiplication

Your answer

2c) The number of miles is always _____ times the hours

Your answer

2d) In one hour, Jane hikes _____ miles.

Your answer

2e) It would take _____ hours to hike 90 miles.

Your answer

Describe in your own words how you solved the problems above.

Your answer

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Never submit passwords through Google Forms.

Proportional Reasoning Assessment

Luisa's Paint

3a) Luisa mixed 5 ounces of red paint with 4 ounces of white paint to create pink paint. Luisa needs to finish painting. She has 20 ounces of red paint. How much white paint should she add to create the same color of pink she has before?

Your answer

The relationship between red and white paint must remain the same to create the correct color of pink paint.

5 ounces of red paint to 4 ounces of white paint

Complete the following statements about mixing the correct color of pink paint.

3b) ounces of white paint = _____ • ounces of red paint

• indicates multiplication

Your answer

3c) The ounces of red paint is always _____ times the ounces of white paint

Your answer

3d) For one ounce of white paint you need _____ ounces of red paint

Your answer

3e) For 75 ounces of red paint you need _____ ounces of white paint

Your answer

Describe in your own words how you solved the problems above.

Your answer

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Never submit passwords through Google Forms.

Proportional Reasoning Assessment

Alyssa's hike

Alyssa can hike 6 miles in 3 hours. How long will it take her to hike 8 miles?

Your answer

If the relationship between miles and hours remains the same no matter how many miles Alyssa hikes:

6 miles in 3 hours

Complete the following statements about the relationship between miles and hours.

4b) hours = _____ • number of miles

• indicates multiplication

Your answer

4c) The number of miles is always _____ times the hours

Your answer

4d) In one hour, Alyssa hikes _____ miles

Your answer

4e) It would take _____ hours to hike 108 miles

Your answer

Describe in your own words how you solved the problems above.

Your answer

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Proportional Reasoning Assessment

Mariana and David's Cookies

Mariana and David are making cookie's for a bake sale.

With every 2 cups of cookie dough, Mariana uses 50 chocolate chips. With every 3 cups of cookie dough, David uses 60 chocolate chips.

If both Mariana and David's chocolate chips are distributed evenly in their cookie dough, whose cookies will have more chocolate chips?

- ◯ Mariana's cookies will have more chocolate chips
- ◯ David's cookies will have more chocolate chips
- ◯ They will have the same amount of chocolate chips

Explain how you arrived at your answer for the problem above.

Your answer

Mariana is selling 8 cookies for $3 and David is selling 5 cookies for $2.

Who is giving their customers a better deal?

- ◯ Mariana's 8 cookies for $3 is the better deal
- ◯ David's 5 cookies per $2 is the better deal
- ◯ Neither, these deals have the same value

Explain how you arrived at your answer for the problem above.

Your answer

Back

Submit

APPENDIX B

EAC then SOS Lessons Worksheet Format

Lesson 1

Shades of Blue

	White	Blue
Original	2	6
New	5	?

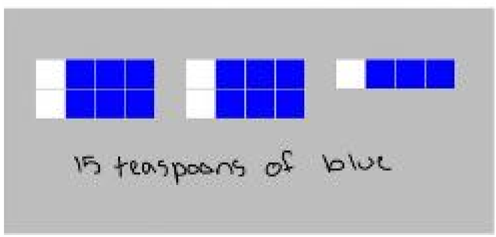
Josie is mixing white and blue to make the shade of blue pictured to the left.

This shade of blue is made with 2 parts white and 6 parts blue.

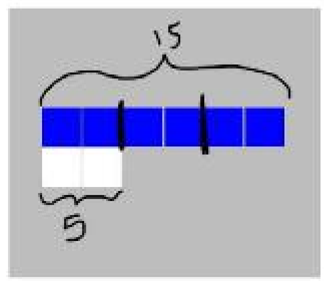
Josie wants to know how much blue she will need if she uses 5 teaspoons of white.

The next four solutions show the work that 4 students did to help Josie determine how much blue she needs to add to get the correct shade.

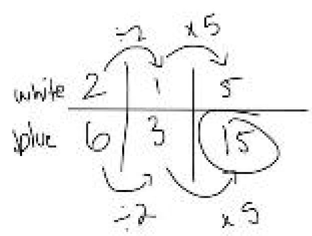
Student A



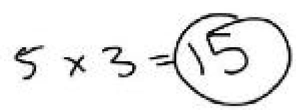
Student B



Student C

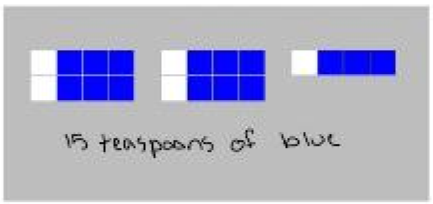
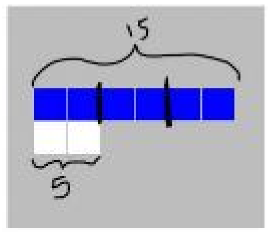


Student D

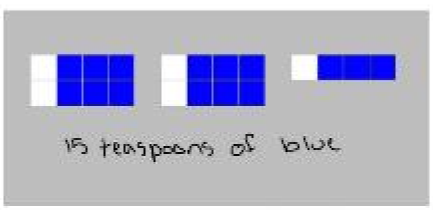
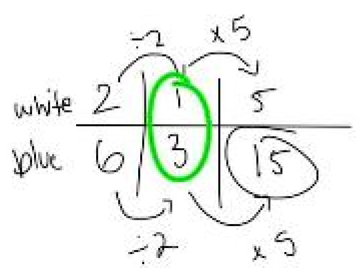


1. Which of these four solutions is the easiest to explain for a friend who might not understand how to find the correct answer? Explain what is happening in this solution and why it works.

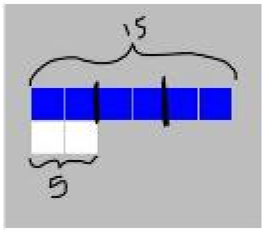
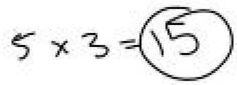
2. Compare and contrast these two student solutions. Name at least one similarity and one difference. Feel free to draw on them to highlight your thinking.

Student A	Student B
	

3. Look at the work of Student A and Student C pictured.
   Looking at the work of Student A, where do you see the quantity circled in green represented visually? Feel free to draw on them to highlight your thinking.

Student A	Student C
	

4. Compare and contrast these two student solutions. Name at least one similarity and one difference. Feel free to draw on them to highlight your thinking.

Student B	Student D
	

Problem 2 Explanation
Students A and B used a visual representation. Student A repeated the given ratio as much as possible then used half of the given ratio to get to the total of 5 tsp of white to 15 tsp of blue. Student B showed that the amount of Blue was 3 times the amount of white, and so multiplied the 5 tsp of white by 3 to get 15 tsp of blue.

Problem 3 Explanation
The circled quantity in Student C's work is the unit rate. We can see this unit rate in the last bar of Student A's work where there is 1 square of white and 3 squares of blue. We can also see it in the earlier work by looking at how the amount of blue is always 3 groups of the white.

Problem 4 Explanation
The difference between the work of Student B and D is that student B worked visually by drawing out the ratio, but Student D worked only symbolically. Their work is similar because they both noticed a multiplicative relationship between the amount of white and blue. They both noticed that the amount of blue is 3 times the amount of white.

Reflect

5. Did any of the explanations highlight something that you didn't notice originally? If so, what was new to you?

Josie is Making Blue Soap

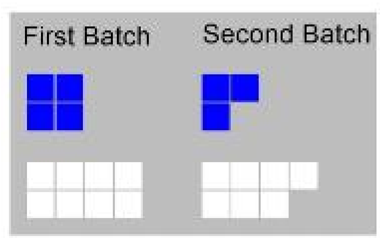
Today she wants to make some beach-themed shell soaps.

To make this shade of blue, she is going to mix some blue and white soap like what is shown in the image below. The cup in the center is where she is going to mix the soap.

Below shows two batches of soap that Josie mixed.

6. Looking at these pictures, what are some things you notice and wonder?

7. In the first batch of soap, Josie mixed 4 teaspoons of blue and 8 teaspoons of white. For the second batch she mixed 3 teaspoons of blue and 7 teaspoons of white.
   Will her second batch be the same shade of blue as the first? Explain your thinking.



8. Below is a picture of the actual soap colors. Does what you see match what you expected? (The soap on the left (batch 1) is a darker blue than the soap on the right (batch 2))

Correcting the Second Batch

Josie was upset when she poured the second batch into the soap mold. She had wanted the two colors to match.

Fill out the table below to show how much blue and/or white you would add to correct the mixture.

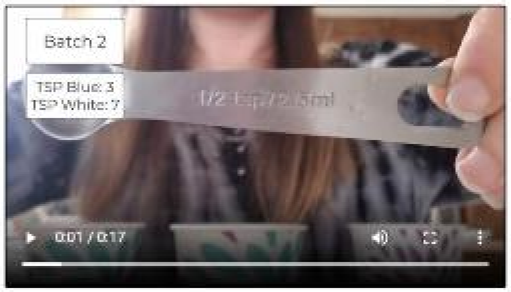
	TSP of Blue	TSP of White
First Batch	4	8
Second Batch	3	7
What You'd Add		
Resulting Mixture		

Thanks for helping Josie with her problem. What is another way Josie could solve her problem? Fill out the table below with a different solution.

	TSP of Blue	TSP of White
First Batch	4	8
Second Batch	3	7
What You'd Add		
Resulting Mixture		

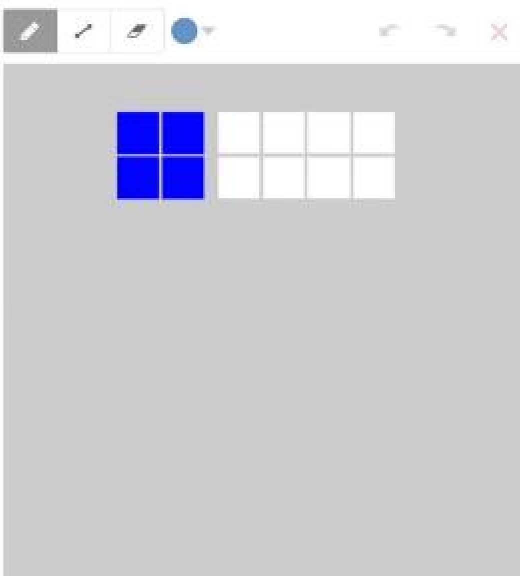
The Reveal

Josie fixed the soap color by adding 0.5 tsp to the blue to get a mixture of 3.5 tsp blue to 7 tsp white.



9.

Different Mixtures for the Same Shade



The original mixture that Josie used to make the correct shade of blue was 4 teaspoons of blue to 8 teaspoons of white.

Select the mixtures from the list below that WOULD make the same shade of blue.

You can use the drawing tool to help support your thinking.

(Select all that apply.)

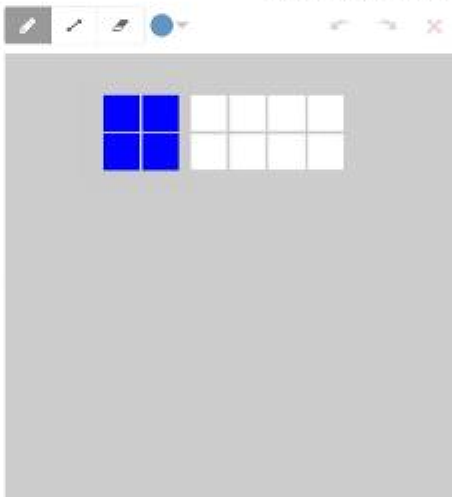
- A: 3 ounces of blue to 6 ounces of white
- B: 1 teaspoon of white and 2 teaspoons of blue
- C: 1 teaspoons of blue and 2 teaspoons of white
- D: 1 cup of white and 1/2 cup of blue

Submit

Problem 9 Feedback:
A is correct because the amount of white (6 ounces) is twice the amount of blue (3 ounces).
B is NOT correct because the amount of white (1 teaspoon) is not twice the amount of blue (2 teaspoons).
C is correct because the amount of white (2 teaspoons) is twice the amount of blue (1 teaspoon).
D is correct because the amount of white (1 cup) is twice the amount of blue (1/2 cup).

10.

Always Make the Correct Shade of Blue



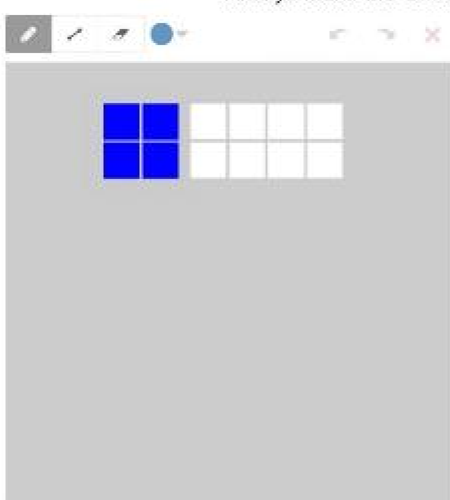
Describe a strategy for figuring out how much white soap to add to any amount of blue soap to make the same shade of blue that Josie likes.

That is, if I gave you any amount of blue soap (12 teaspoons, 1/2 cup, 327 ounces, etc.), **how could you figure out how much white soap you need?**

Share with Class

11.

Always Make the Correct Shade of Blue (Part 2)



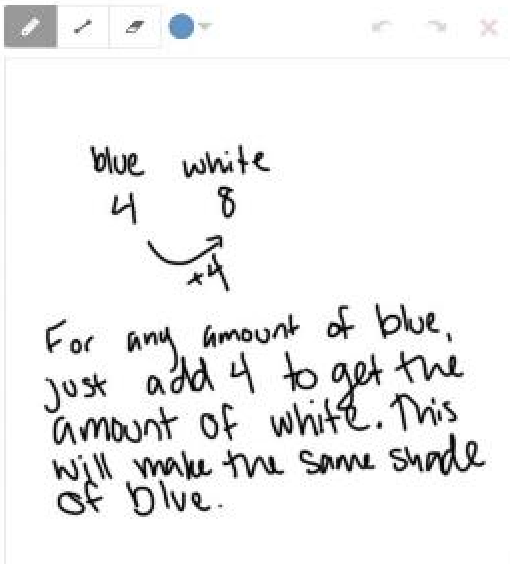
Describe a strategy for figuring out how much blue soap to add to any amount of white soap to make the same shade of blue that Josie likes.

That is, if I gave you any amount of white soap, **how could you figure out how much blue soap you need?**

Share with Class

12.

Will This Strategy Work?

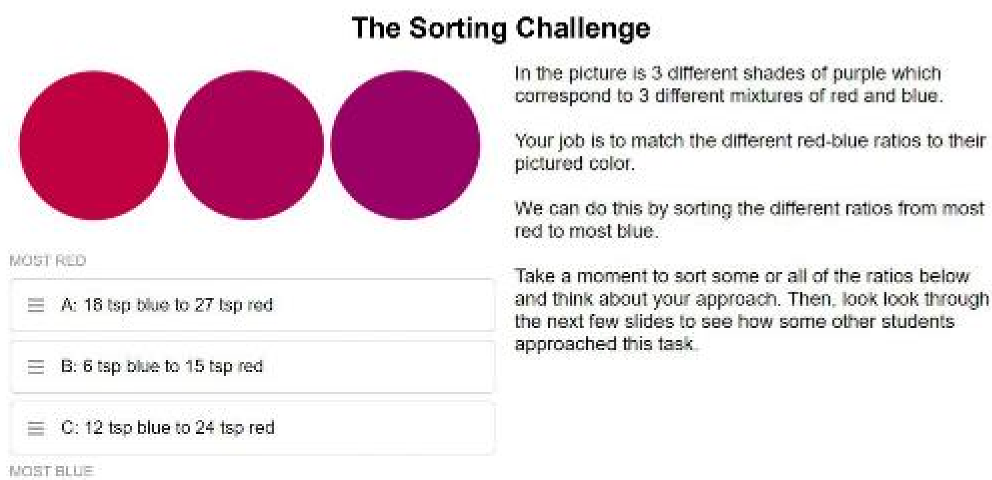


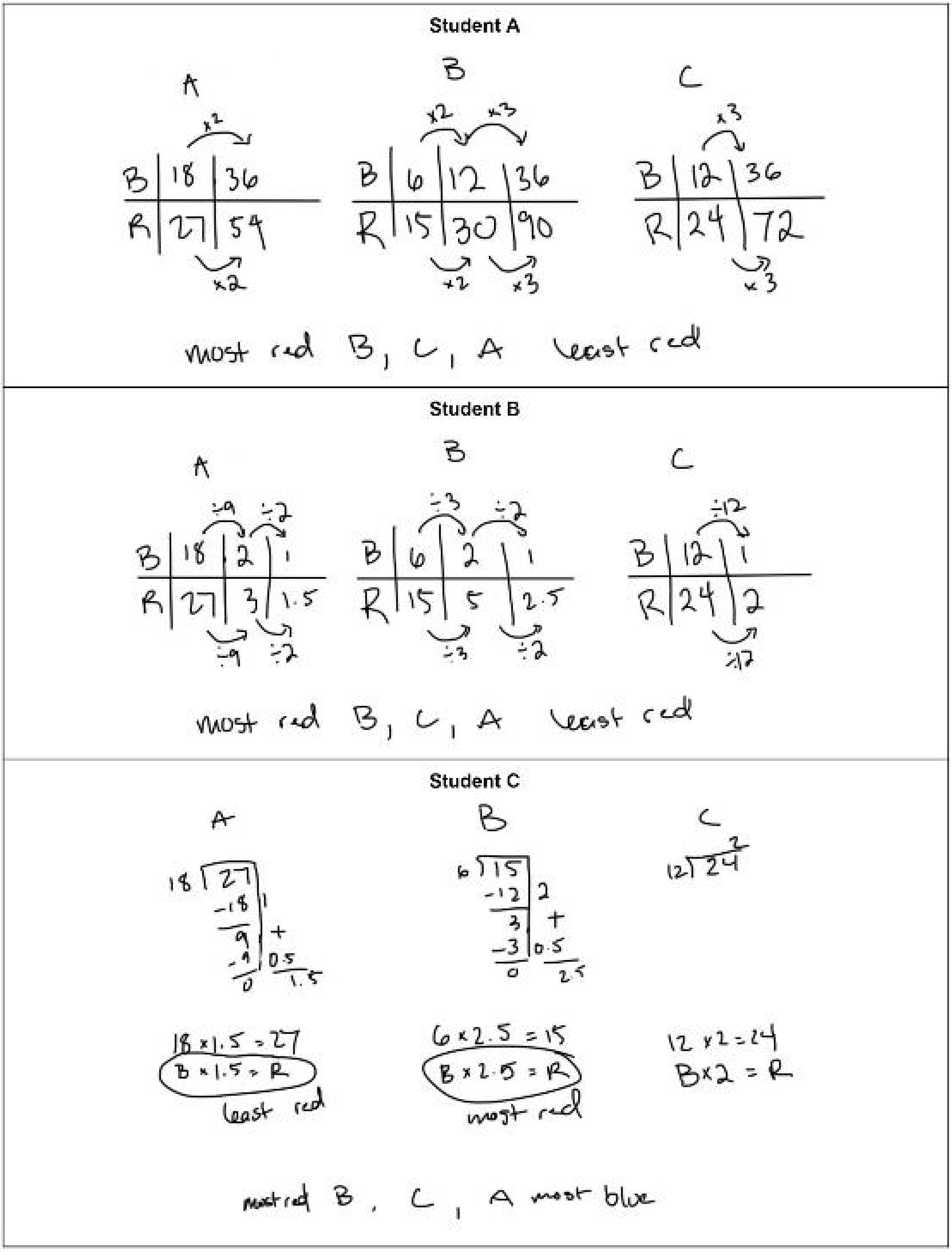
The work shown to the left describes one student's strategy for determining how much white soap they need for any amount of blue soap.

Do you agree with this response? Explain why or why not.

Share with Class

Lesson 2


The Sorting Challenge
In the picture is 3 different shades of purple which correspond to 3 different mixtures of red and blue.
Your job is to match the different red-blue ratios to their pictured color.
We can do this by sorting the different ratios from most red to most blue.
Take a moment to sort some or all of the ratios below and think about your approach. Then, look look through the next few slides to see how some other students approached this task.
MOST RED
A: 18 tsp blue to 27 tsp red
B: 6 tsp blue to 15 tsp red
C: 12 tsp blue to 24 tsp red
MOST BLUE


Student A
most red B, C, A least red
Student B
most red B, C, A least red
Student C
18 × 1.5 = 27
B × 1.5 = R
least red
6 × 2.5 = 15
B × 2.5 = R
most red
12 × 2 = 24
B × 2 = R
most red B, C, A most blue

1. Compare and contrast the work of Students A and B. Name at least one similarity and one difference. Feel free to draw on them to highlight your thinking.

2. Compare and contrast the work of Students B and C. Name at least one similarity and one difference. Feel free to draw on them to highlight your thinking.

Problem 1 Explanation
Both students (A and B) used a ratio table and scaled to get equal amounts of blue

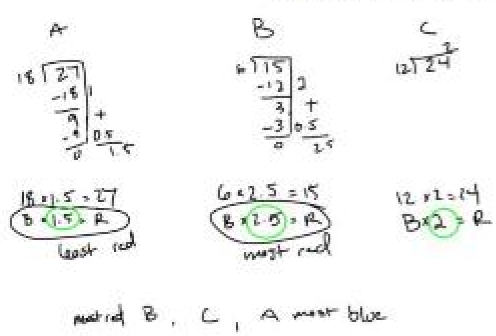
Student A scaled up so blue was 36 teaspoons, Student B scaled down to find a unit rate giving them the number of teaspoons of red for each teaspoon of blue.

Problem 2 Explanation
Both students (B and C) used division and ended up finding the numbers "1.5", "2", and "2.5" and determined that the higher the number the more red the color will be.

Student B scaled the entire ratio by dividing both the larger and smaller quantity by the same amount in order to get the unit rate.

Student C looked at the relationship between the larger and smaller quantities by dividing the larger quantity by the smaller. They thought about how much they need to multiply the amount of blue by to get the amount of red.

NEW VOCAB: Constant of Proportionality



Student C solved their problem by identifying the **constant of proportionality** for each of the three ratios.

The constant of proportionality is how much we multiply one quantity by to get the other in a ratio. For example, because in ratio A we multiply the amount of blue by 1.5 to get the amount of red, 1.5 is the constant of proportionality.

The constant of proportionality would be different if we wanted to get the amount of blue from the amount of red.

Reflect

3. Did any of the explanations highlight something that you didn't notice originally? If so, what was new to you?

Josie is Making Teal Soap

Today she wants to make some teal soap.

To make this shade of teal she mixed together a specific ratio of blue and green.

Notice/Wonder

Note: This video has no sound

Someone wants to pull a prank on Josie by messing with her recipe before she pours the mix into the soap molds.

Watch the short video to the left.

What do you notice and wonder?

I noticed that...

I wonder...

Share with Class

4. Even though you can't watch the video, what do you notice and wonder based on the information above?

5. Fixing the Sabotage - Strategy

	TSP Blue	TSP Green
Original Mix	4	2
Mix A	$4+1=5$	$2+1=3$
Mix B	$4+6=10$	$2+2=4$

Feeling guilty, the prankster confesses. They know that Josie can fix it because she has studied hard and understands how proportions work.

They reveal that they added 6 tsp of blue and 2 tsp of green to one of the soaps, and 1 tsp of blue and 1 tsp of green to the other.

Describe how Josie might be able to tell which soap is which. You don't need to solve anything here, just describe what you think she might need to look for in order to tell.

Share with Class

6. Fixing the Sabotage - Identifying the Mixtures. Circle your answer in the prompt below.

	TSP Blue	TSP Green
Original Mix (Left Soap)	4	2
Mix A	5	3
Mix B	10	4

The image to the left shows the soaps after the sabotage.

The one on the left didn't have anything added to it. The other two are clearly different colors!

Josie noticed that the middle one looks more green and the other looks more blue than the original.

Which new mix is going to be MORE BLUE than the original?

Mix A | Mix B

7. Explain how you made your selection in the previous problem.

Fixing the Sabotage - Solution - Mix B

Mix B is the mix that is too blue.

Since we know which color corresponds to which mix, we can add to it to correct the color.

In the table below, add blue, green, or both to fix the mixture.

.

	TSP Blue	TSP Green
Original Mix	4	2
Mix B	10	4
What You'd Add		
New Mix		

Fixing the Sabotage - Solution - Mix A

Mix A was the mix that was too green.

Since we know which color corresponds to which mix, we can add to it to correct the color.

In the table below, add blue, green, or both to fix the mixture.

.

	TSP Blue	TSP Green
Original Mix	4	2
Mix A	5	3
What You'd Add		
New Mix		

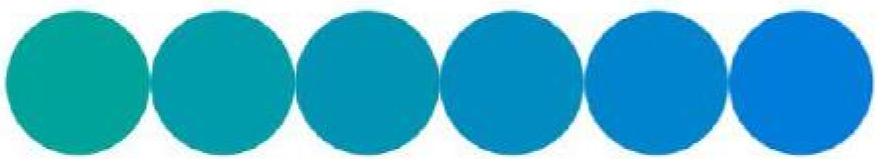
The Reveal

Mix A: Josie added one tsp of blue to Mix A to get 6 tsp blue to 6 tsp green.
Mix B: Josie added one tsp of green to Mix B to get 10 tsp blue to 5 tsp green.

A New Challenge

Josie's friend was impressed with how Josie fixed her soap. So, she made 5 new shades of blue-green and wanted to see if Josie could identify which color corresponds to which mixture.

The colors are shown below.



A New Challenge - Part 1

	TSP Blue	TSP Green
A	15	3
B	42	6

Two of the new mixtures are shown in the table. Which of these two will be MORE BLUE?

A B

8. Will mix A or mix B be more blue? Explain your thinking.

A New Challenge - Part 2

	TSP Blue	TSP Green
A	15	3
C	6	4

Two of the new mixtures are shown in the table. Which of these two will be MORE BLUE?

A | C

9. Will mix A or mix C be more blue? Explain your thinking.

A New Challenge - Part 3

	TSP Blue	TSP Green
D	4	$\frac{1}{2}$
E	27	6

Two of the new mixtures are shown in the table. Which of these two will be MORE BLUE?

D | E

10. Will mix D or mix E be more blue? Explain your thinking.

11. Sort all mixtures below from most green (1) to most blue (6)

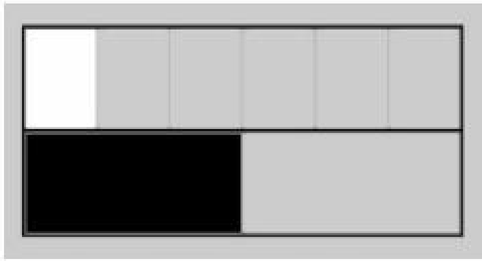
Rank	Mixture
	15 tsp blue to 3 tsp green
	4 tsp blue to 2 tsp green
	42 tsp blue to 6 tsp green
	6 tsp blue to 4 tsp green
	4 tsp blue to ½ tsp green
	27 tsp blue to 6 tsp gree

Problem 11 Solution

12. Did all of your solutions match this order? If not, pick one that was incorrect and explain how you can see from the ratios that it was out of place.

Lesson 3

Lump of Coal Soap



Josie is making soap that looks like lumps of coal as a joke gift to put in her family's stockings for Christmas.

To make this soap, she is going to mix black and white soap to make a dark grey.

To do this she needs to mix 1/6 of a teaspoon of white and 1/2 of a teaspoon of black.

There is nothing to do on this slide. Hit next when you are ready to continue.

Another Mix

	White	Black
Original Recipe	$\frac{1}{6}$	$\frac{1}{2}$
Big Batch	?	9

After mixing the 1/6 teaspoon of white and 1/2 teaspoon black Josie had created the grey pictured to the left.

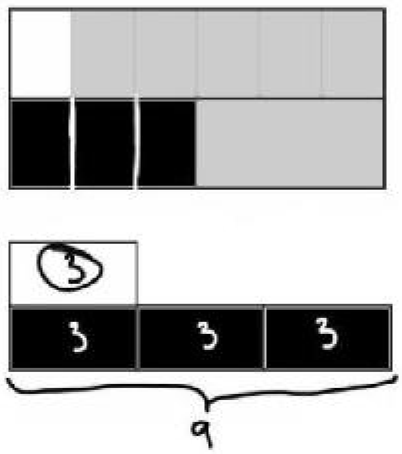
She decided that this grey was exactly what she wanted and now she was ready to make a bigger batch.

This time she started by adding 9 ounces of black. **How much white should she add if she wants to make the same shade of grey?**

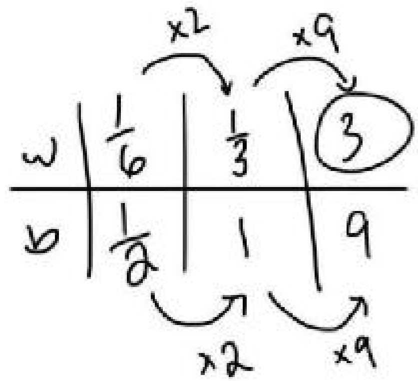
Think about how you would solve this problem. Then move on to the next slides to see how other students solved this problem.

You do not need to submit anything on this slide.

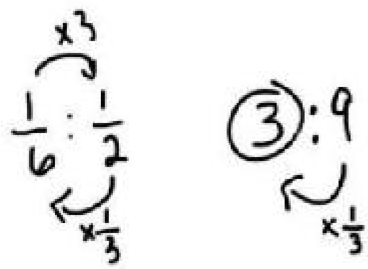
Student A



Student B



Student C



1. Compare and contrast the work of Students A and B. Name at least one similarity and one difference. Feel free to draw on them to highlight your thinking.

2. Compare and contrast the work of Students B and C. Name at least one similarity and one difference. Feel free to draw on them to highlight your thinking.

Problem 1 Explanation - Students A and B
Similarities: They both were able to show that the amount of white is 1/3 of the amount of black.

Differences: Student A used the fact that the white is 1/3 of the black directly, by dividing the amount of black into 3 pieces to find the amount of white. Student B scaled the unit rate of 1 part black to 1/3 parts white by multiplying it by 9. Of course, they also used different models (Student A used a bar model, and Student B used a ratio table).

Problem 2 Explanation - Students B and C
Though these solutions look very different, their strategies are actually very similar.
Student A used a visual model and Student B worked symbolically. However, that is really the only way they differ. Both students identified that the amount of white was 1/3 of the amount of black, and used this to find their answer by finding 1/3 of 9 to get 3 teaspoons of white.

Both students identified the constant of proportionality as being ⅓ when we want to identify the amount of white soap given the amount of black soap (because we multiply the amount of black by 1/3 to get the amount of white).

However, if we want to identify the amount of black given the amount of white, the constant of proportionality would be 3 because we multiply the amount of white by 3 to get the amount of black.

Reflect

3. Did any of the explanations highlight something that you didn't notice originally? If so, what was new to you?

Josie is Making Pumpkin Soap

Today she wants to make some orange pumpkin soap.

To make this shade of orange she mixed together a specific ratio of red and yellow.

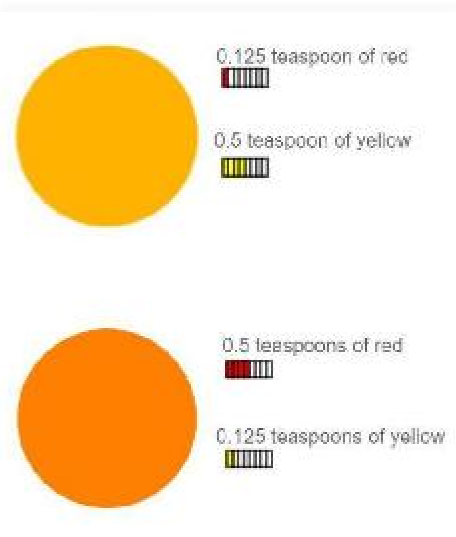
Consider the image below.

4. Looking at these pictures, describe what Josie did wrong.

5. What do you think the question is? In other words, what problem do you think we need to solve?

6. Without doing any calculations, do you think we can fix the soap color without throwing out the current mixture and starting over? Explain.

7. In the picture below, the top is the correct shade of orange, and the bottom is Josie's current shade of orange. In the table, add red, yellow, or both to Josie's mix so that the final mix makes the right shade of orange.



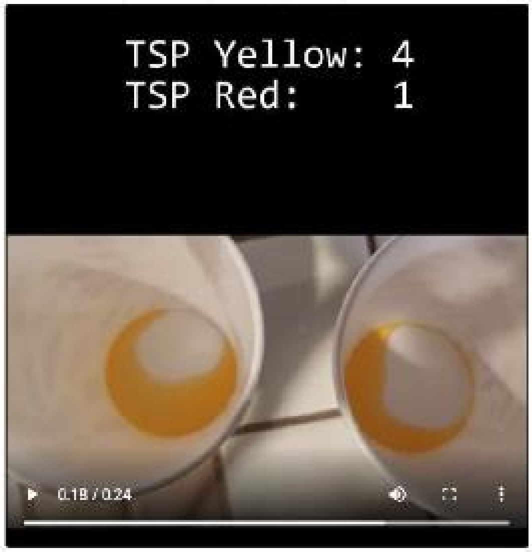
	TSP Red	TSP Yellow
Recipe	**1/8**	**1/2**
Josie's Mixture	**1/2**	**1/8**
What You'd Add		
The Final Mix		

8. What is another way Josie could solve her problem? Fill out the table below with a **different solution**.

	TSP Red	TSP Yellow
Recipe	**1/8**	**1/2**
Josie's Mixture	**1/2**	**1/8**
What You'd Add		
The Final Mix		

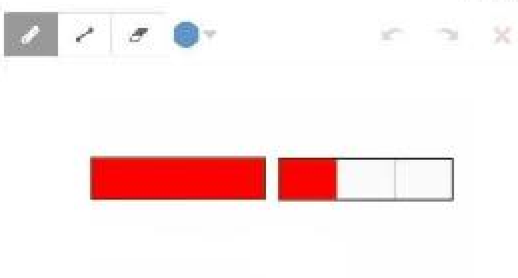
The Reveal

Josie fixed the soap color by adding ½ tsp red and 3+¼ tsp yellow.



9.

A Bigger Mix



Josie's original recipe was $\frac{1}{8}$ tsp of red to $\frac{1}{2}$ tsp of yellow.

Later, Josie wanted to make a lot of pumpkin soaps to give away as gifts. She added $1\frac{1}{3}$ cup of red soap.

How much **yellow** should she add to get the same shade of orange?

Feel free to use the sketch tool to help support your thinking.

Submit

10. Below is a table of mixes of this same shade of orange that we have used so far.

TSP Red	TSP Yellow
1/8	1/2
1/2	2
1	4
4/3 = 1 1/3	16/3 = 5 1/3

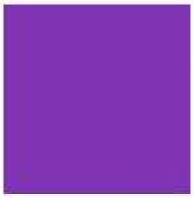
Complete the sentence:

The amount of yellow is always ____ times the amount of red.

(or in other words: What is the constant of proportionality?)

Lesson 4

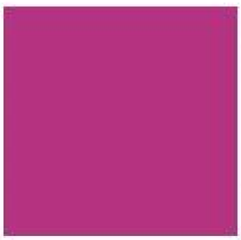
Purple Soap Recipe



Josie wants to make purple soap. In order to make the purple to the left, she would need to mix 3 tsp of red and 5 tsp of blue.

*note: tsp stands for teaspoon(s)

The Mistake



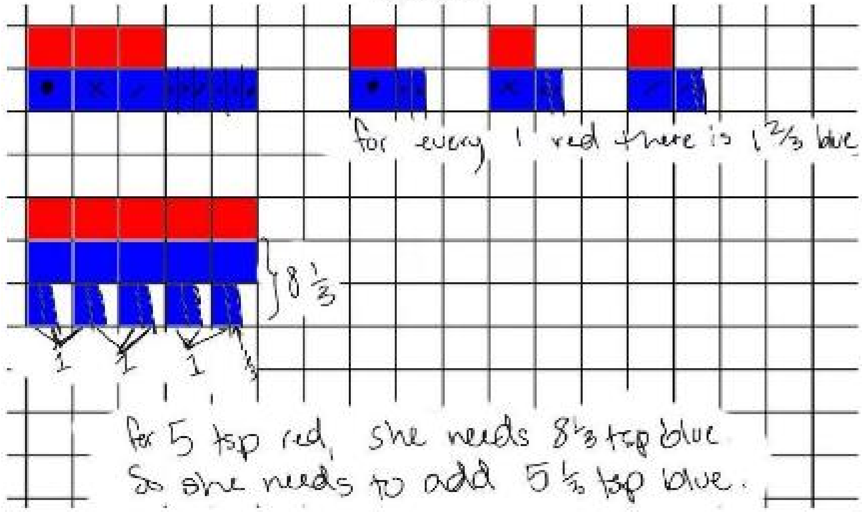
	TSP Red	TSP Blue
Recipe	3	5
Josie's Mixture	5	3
What You'll Add	0	
Final Mix	5	3 + ?

Instead of mixing 3 tsp of red to 5 tsp of blue like she was supposed to, she mixed up the recipe and accidentally mixed 5 tsp of red to 3 tsp of blue!

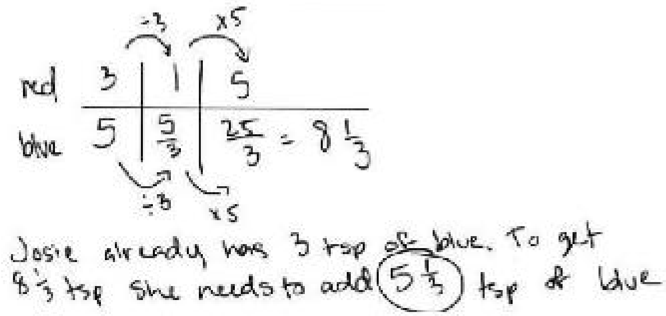
Josie wants to fix her mixture without adding any more red soap.

Think for a moment about how you would solve her problem, then continue and see some other students' solutions.

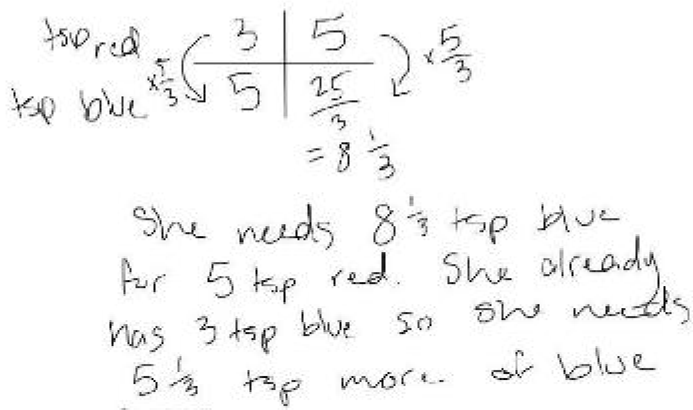
Student A



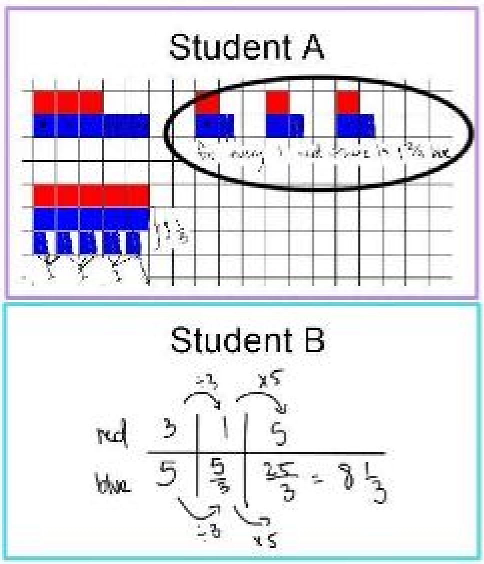
Student B



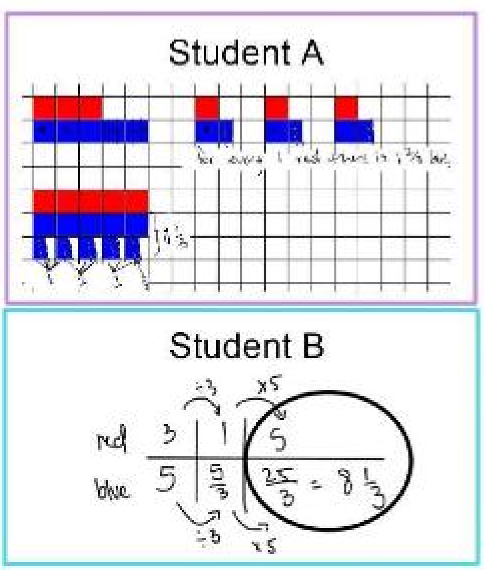
Student C



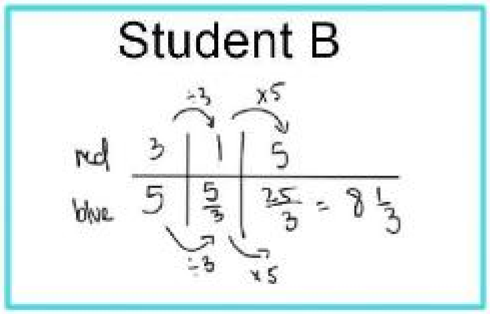
1. Describe at least one connection you see between the circled part of Student A's work (in the image below) and the work of Student B. Feel free to draw on them to highlight your thinking.

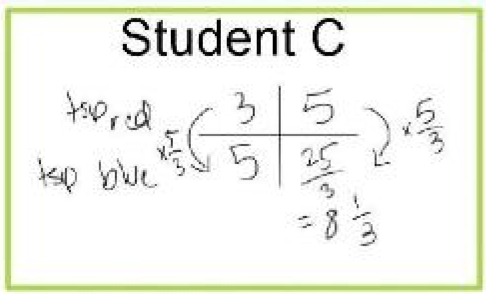


2. Describe at least one connection you see between Student A's work and the circled part of Student B's work (in the image below). Feel free to draw on them to highlight your thinking.



3. Compare and contrast the work of Student B and Student C. Describe at least one similarity and one difference. Feel free to draw on them to highlight your thinking.





Problem 1 Explanation
One connection you might have noticed between the circled part of Student A's work and the work of Student B is that both students started by dividing the 3:5 ratio of red to blue into 3 groups to get the unit rate.

Student A's work circled in black is a visual version of Student B's 1 to 5/3 unit rate in their ratio table.

Student A wrote '1 ⅔' instead of '5/3', but these two quantities are the same.

Problem 2 Explanation
Student B found '25/3' or '8 ⅓' by multiplying the unit rate by 5. Student A did the same thing, but visually. They drew 1 red square and 1 ⅔ blue squares 5 times and counted how many total blue squares there were, getting the answer of 8 ⅓.

Both Student A and Student B scaled their unit rate by repeating it 5 times to get how much total blue was needed if Josie used 5 teaspoons of red.

Problem 3 Explanation

Student B and Student C both chose to use a ratio table for their solution, but they got their answers in different ways.

Student B scaled the ratio of red and blue down to 1 teaspoon of red by dividing by 3. Then, they multiplied that new ratio (the unit rate) by 5 to find out how much blue Josie must use if she has 5 teaspoons of red.

Student C did not scale the given ratio. Instead, Student C found the constant of proportionality. **Student C answered the question, "Blue is how many times the amount of red?"**

Student C identified that the amount of blue was 5/3 times the amount of red. So, Student C multiplied 5 by 5/3 to find out how much blue Josie must use.

Reflect

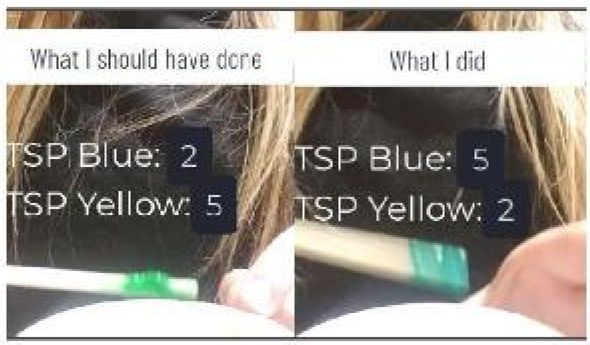
4. Did any of the explanations highlight something that you didn't notice originally? If so, what was new to you?

Josie is Making Green Soap

Today, Josie wants to make the soap pictured to the left.

To make this bar of soap, she is going to mix a specific ratio of blue and yellow soap.

Consider the image below.

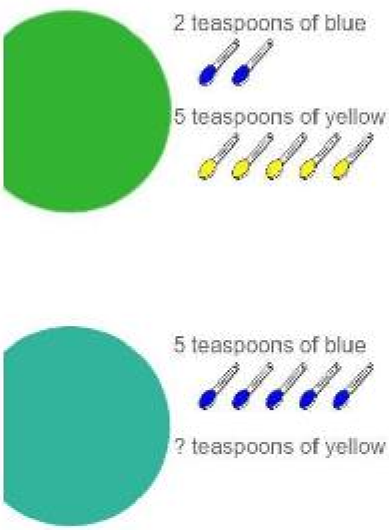


5. Looking at these pictures, describe what Josie did wrong.

6. What do you think the question is? In other words, what problem do you think we need to solve?

7. Without doing any calculations, do you think we can fix the soap color without throwing out the current mixture and starting over? Explain.

8. In the picture below, the top is the correct shade of green, and the bottom is Josie's current shade of green. In the table, add blue, yellow, or both to Josie's mix so that the final mix makes the right shade of green.



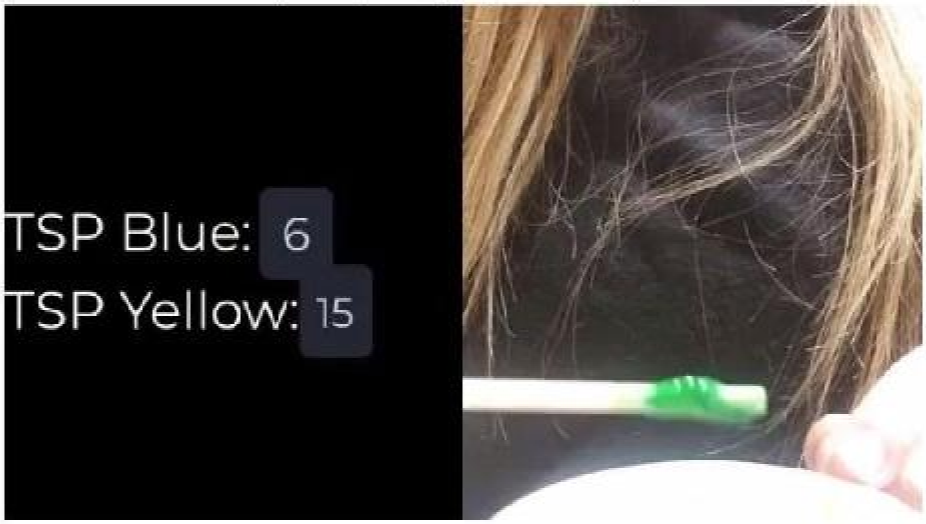
	TSP Blue	TSP Yellow
Recipe	2	5
Josie's Mixture	5	2
What You'll Add		
The Final Mix		

9. What is another way Josie could solve her problem? Fill out the table below with a **different solution**.

	TSP Blue	TSP Yellow
Recipe	2	5
Josie's Mixture	5	2
What You'll Add		
The Final Mix		

The Reveal

Josie fixed the soap color by adding 1 tsp blue and 13 yellow to her mixture



10. Even more solutions. If Josie wanted to fix the mixture without adding more blue soap...How many teaspoons of yellow soap would she need to add?
(note: decimals or fractions are okay)

	TSP of Blue	TSP of Yellow
Recipe	2	5
Josie's Mixture	5	2
What You'll Add	0	
Final Mixture		

11. Another mistake. Consider the problem picture below. Write down and explain your answer.

Another Mistake

	TSP of Blue	TSP of Yellow
Recipe	2	5
Josie's Mix	3	0 + ?

Later when Josie was trying to make the same shade of green, she messed up again!

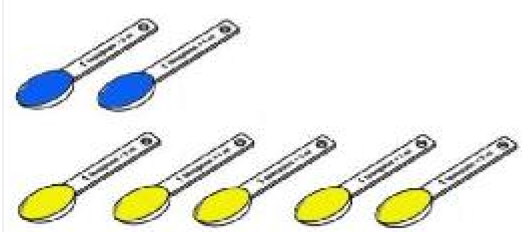
Josie added 3 teaspoons of blue, but realized her mistake before adding any yellow.

How much yellow should she add if she wants to make the correct shade of green?

Submit and Explain

12.

Constant of Proportionality



If we know that the recipe for the correct shade of green is

2 teaspoons of blue soap for every 5 teaspoons of yellow soap

Complete this sentence:
The amount of YELLOW soap is always _____ times the amount of BLUE soap.

Submit

Lesson 5

Connecting to Other Contexts

Over this week we have worked with ratios and thought about how to identify and use the constant of proportionality to solve problems. However, we only worked with making soap.

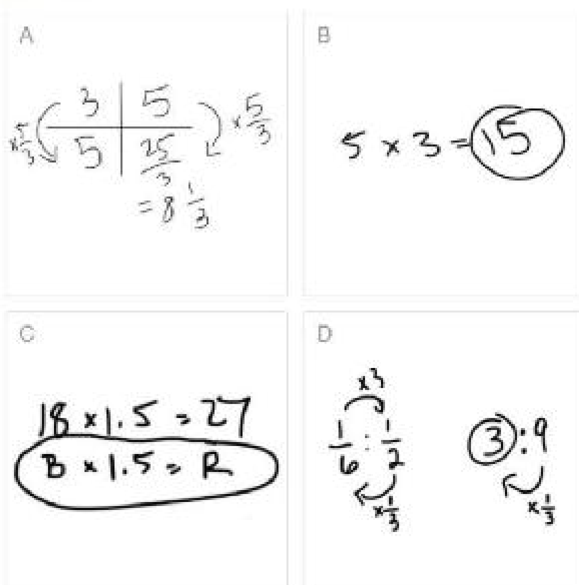
So, we are going to look at some other ratio situations and connect them to the ones we have already done.

1. Circle your choice below and explain your thinking in the space to the right.

Context 1

Josie drives 3 miles in 5 minutes. If Josie has driven 5 miles, how long has she driven for?

Which of the solutions below (which were used to solve previous problems) could be used to solve this problem?



Problem 1 Explanation

Solution A was used in an earlier problem about making purple soap with 3 teaspoons of red and 5 teaspoons of blue.

This solution used the fact that the amount of blue is 5/3 times the amount of red.

We can use the exact same logic in this new situation, where the amount of minutes is 5/3 times the amount of miles.

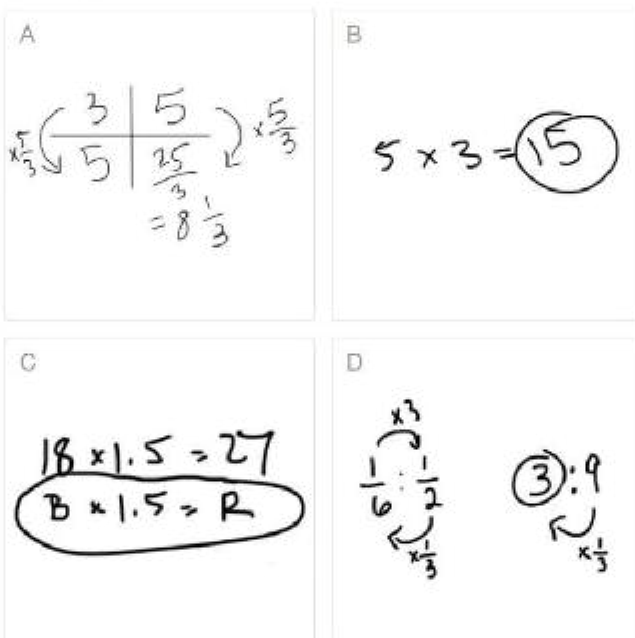
So, if Josie has driven 5 miles, then the amount of time she has been driving is 5/3 times that amount or 8 1/3 minutes.

2. Circle your choice below and explain your thinking.

Context 2

For a bake sale, Josie sold 4 cupcakes for S6. If she sold 18 cupcakes, how much money did she make?

Which of the solutions below (which were used to solve previous problems) could be used to solve this problem?



Problem 2 Explanation

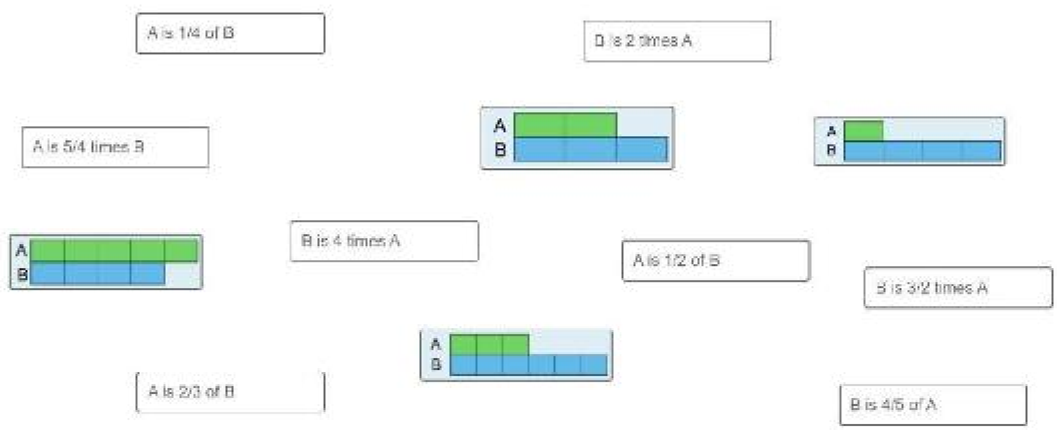
Solution C was used in an earlier problem where we sorted different shades of purple based on how red they were.

Reflect

3. Did any of the explanations highlight something that you didn't notice originally? If so, what was new to you?

4.

Match the Visual with the Multiplicative Relationships Described



One Last Soap Problem - Josie is Making Pink Soap

Today, she wants to make the soap pictured to the left.

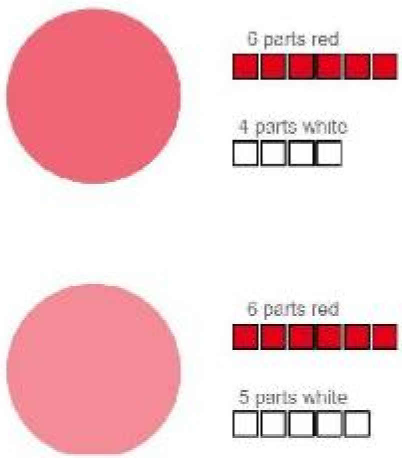
To make this bar of soap, she is going to mix a specific ratio of red and white soap.

1. Consider the images below. In the space to the right, describe what Josie did wrong.

2. What do you think the question is? In other words, what problem do you think we need to solve?

3.

Fix It



In the image to the left, the top is the correct shade of pink, and the bottom is Josie's current shade of pink.

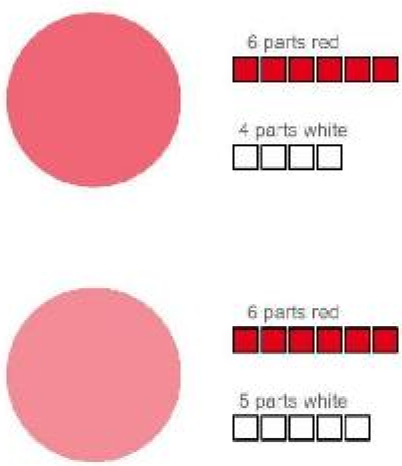
In the table below, add red, white, or both to Josie's mix so that the final mix makes the right shade of pink.

Note: Fractional or decimal parts are okay!

	Red	White
Recipe	6	4
Josie's Mix	6	5
What You'll Add		
Final Mix		

4.

Solution Using as Little Soap as Possible



In the image to the left, the top is the correct shade of pink, and the bottom is Josie's current shade of pink.

How much red soap should we add if we want to fix the color without adding any white soap?

Note: Fractional or decimal parts are okay!

	Red	White
Recipe	6	4
Josie's Mix	6	5
What You'll Add		0
Final Mix		

The Reveal

Josie fixed the mixture by adding 1.5 tsp of red.

5.

Unit Rate

	Red	White
Recipe	6	4
Josie's Solution	7.5	5
Unit Rate (amount of red per unit white)		1
Unit Rate (amount of white per unit red)	1	

The unit rate can be helpful for identifying the constant of proportionality (how much we multiply one quantity by to get the other).

Fill out the table to the left to identify the unit rate.

Note: Fractional and decimal parts are okay!

6.

Constant of Proportionality 1

	Red	White
Recipe	6	4
Josie's Solution	7.5	5
Unit Rate (amount of red per unit white)	$\frac{3}{2}$	1
Unit Rate (amount of white per unit red)	1	$\frac{2}{3}$

Complete the sentence:

The amount of **white** is always ___ times the amount of **red**

Submit and Explain

7.

Constant of Proportionality 2

	Red	White
Recipe	6	4
Josie's Solution	7.5	5
Unit Rate (amount of red per unit white)	$\frac{3}{2}$	1
Unit Rate (amount of white per unit red)	1	$\frac{2}{3}$

Complete the sentence:

The amount of **red** is always ___ times the amount of **white**

Submit and Explain