ON THE FUNDAMENTAL GROUP OF PLANE CURVE COMPLEMENTS

by

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A thesis

submitted in partial fulfillment

of the requirements for the degree of

Master of Science in Mathematics

Boise State University

May 2019

BOISE STATE UNIVERSITY GRADUATE COLLEGE

DEFENSE COMMITTEE AND FINAL READING APPROVALS

of the thesis submitted by

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Thesis Title: On the Fundamental Group of Plane Curve Complements

Date of Final Oral Examination: 01 March 2019

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The final reading approval of the thesis was granted by Jens Harlander, Ph.D., Chair of the Supervisory Committee. The thesis was approved by the Graduate College.

ACKNOWLEDGMENTS

This was made possible by the generous Graduate Teaching Assistantship from Boise State University, without which I would not be here. I would first and foremost like to acknowledge my parents, both of whom instilled a desire to be the best I can be, in whatever endeavor I choose. I thank my fiancée for always being proud and certain of my abilities. Next I acknowledge Jens Harlander, a great mathematician, professor and friend who always encouraged me when feeling unable to grasp difficult material. I want to extend thanks to Zach Teitler, another great mathematician, for always being willing and available for a quick technical question, and for building my intuition in previous courses. I would like to acknowledge Uwe Kaiser, a third great mathematician, for building on and making solid a foundation in topology. Moreover I would like to thank all other staff I came in contact with at Boise State University who undoubtedly made a lasting impression on me. Finally I would like to acknowledge and thank my fellow graduate students, who were always willing to talk mathematics whether it be for work or for fun, especially my office mates who were always interested in bouncing questions off one another and providing support.

ABSTRACT

Given a polynomial $f(x, y)$ monic in y of degree d, we study the complement $\mathbb{C}^2 - C$, where C is the curve defined by the equation $f(x, y) = 0$. The Zariski-Van Kampen theorem gives a presentation of the fundamental group of the complement $\mathbb{C}^2 - C$. Let NT be be the set of complex numbers x for which $f(x, y)$ has multiple roots (as a polynomial in y). Let $\tilde{f} : \mathbb{C} - NT \to \mathbb{C}^d - \Delta$ be the map that sends x to the d-tuple of distinct roots (Δ is the diagonal in $\mathbb{C}^d$). It induces a map $\nabla : F_r \to B_d$ on the fundamental group level, where F_r is the free group on r letters and B_d is the braid group on d strands. In order to write down the Zariski-Van Kampen presentation one needs an explicit understanding of ∇. This is hard to come by in general. It turns out that under special circumstances ∇ can be computed directly from combinatorial and visual (real) information on the curve C. The method in these special situations is similar to the computation of the presentation of the fundamental group of a knot complement in $\mathbb{R}^3$.

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CHAPTER 1

INTRODUCTION

Given a curve $C \subset \mathbb{C}^2$, defined by the polynomial $f(x,y) = 0$, we study the complement $\mathbb{C}^2 - C$. In particular we are interested in the fundamental group

$$\pi_1(\mathbb{C}^2 - C).$$

The simplest possible example is $f(x,y) = y$, so the curve C is just the x-axis in $\mathbb{C}^2$. The projection map $p\colon \mathbb{C}^2 \to \mathbb{C}$, $p(x,y) = x$, induces a map $\mathbb{C}^2 - C \to \mathbb{C}$ with fiber $L_0 = p^{-1}(0)$ the complex y-axis with $(0,0)$ removed. It is easy to see that this fibration is trivial, that is $\mathbb{C}^2 - C$ is the product of the base space with the fiber, so $\mathbb{C}^2 - C = \mathbb{C} \times L_0$. Thus we have

$$\pi_1(\mathbb{C}^2 - C) = \pi_1(\mathbb{C}) \times \pi_1(L_0) = \mathbb{Z}.$$

The next simplest example would be something like $f(x,y) = (y-1)(y-2)(y-3) = 0$, in which case C consists of three horizontal lines. The reasoning is exactly as above, the only difference is that the fiber L_0 is $\mathbb{C} - \{1,2,3\}$. So we obtain

$$\pi_1(\mathbb{C}^2 - C) = F_3,$$

the free group of rank 3.

This argument breaks down for the next simplest case $f(x, y) = y(y - x)$. The curve C now consists of the complex x-axis $y = 0$ together with the complex line $y = x$, and these lines cross at $(0, 0)$. The projection $\mathbb{C}^2 - C \to \mathbb{C}$ is not a fibration, $p^{-1}(0) = \mathbb{C} - \{0\}$, whereas $p^{-1}(1) = \mathbb{C} - \{0, 1\}$. However, if we remove the vertical line as well as C, things look better. So let C^+ be $C \cup L_0$. Then $\mathbb{C}^2 - C^+ \to \mathbb{C} - \{0\}$ is a (locally trivial, or even trivial) fibration. If we can compute $\pi_1(\mathbb{C}^2 - C^+)$, using fibration theory, and then control what happens when we stick the vertical line L_0 back in, perhaps making use of Van Kampen's theorem, we have a chance. The case at hand can be settled easier. Note that $\mathbb{C}^2 - C$ is homeomorphic to $\mathbb{C}^2$ with the x and the y-axis removed, which is homeomorphic to the product $\mathbb{C} - \{0\} \times \mathbb{C} - \{0\}$. Thus

$$\pi_1(\mathbb{C}^2 - C) = \mathbb{Z} \times \mathbb{Z}.$$

We got lucky.

As our final example consider the curve C defined by $x^2 + y^2 = 1$. First we show that C is homeomorphic to a cylinder: $C \to S^1 \times (-\frac{\pi}{2}, \frac{\pi}{2})$. Let $x = a(1+i) + b(1-i)$ and $y = c(1 + i) + d(1 - i)$ with $a, b, c, d \in \mathbb{R}$. Thus the real and imaginary parts of the equation $x^2 + y^2 = 1$ are $4ab + 4cd = 1$ and $a^2 - b^2 = c^2 - d^2$ respectively. Note that this can be seen by expanding $x^2 + y^2 = 1$ with our assumed x and y, and then separating the terms by real and imaginary components. We cannot have $a^2 + d^2 = 0$ for that would imply that the real part of the equation would read $0 = 1$. So with $r = \sqrt{a^2 + d^2}$, we have that $a = r\cos(\theta)$, $b = r\cos(\phi)$, $c = r\sin(\phi)$, and $d = r\sin(\theta)$ for angles θ and ϕ. We substitute into the real equation to get

$$4r\cos(\theta)r\cos(\phi) + 4r\sin(\phi)r\sin(\theta) = 1$$
$$(2r)^2\cos(\theta - \phi) = 1. \qquad \text{(by sum identity)}$$

Thus we must have that $\cos(\theta - \phi) > 0$. Take as parameters $u = \theta - \phi$ for $u \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $v = \phi$ for $v \in [0, 2\pi)$, where 0 and 2π are identified. Thus we see that

$$r = \frac{1}{2\sqrt{\cos(u)}},$$

and we therefore have the following:

$$a = \frac{\cos(u+v)}{2\sqrt{\cos(u)}} \quad b = \frac{\cos(v)}{2\sqrt{\cos(u)}} \quad c = \frac{\sin(v)}{2\sqrt{\cos(u)}} \quad d = \frac{\sin(u+v)}{2\sqrt{\cos(u)}}.$$

This gives us a homeomorphism from the solution set of $x^2 + y^2 = 1$ as a subset of $\mathbb{C}^2$ to the cylinder $(-\frac{\pi}{2}, \frac{\pi}{2}) \times (\mathbb{R} \text{ modulo } 2\pi)$. Now, what we do not have is a good idea of how this cylinder sits in complex 2-space. It is even more difficult to have an understanding of what the complement of this cylinder looks like, and is thus an unlikely candidate for finding the fundamental group of the complement easily or from first principles. Our luck has run out.

In Chapter 2 we see that the Zariski-Van Kampen theorem gives a presentation of the fundamental group of the complement $\mathbb{C}^2 - C$. In the example where C is the curve defined by $x^2 + y^2 = 1$ Zariski-Van Kampen gives

$$\pi_1(\mathbb{C}^2 - C) = \mathbb{Z}.$$

Let NT be be the set of complex numbers x for which $f(x, y)$ has multiple roots (as a polynomial in y). Let $\tilde{f} : \mathbb{C} - NT \to \mathbb{C}^d - \Delta$ be the map that sends x to the d-tuple of distinct roots (Δ is the diagonal in $\mathbb{C}^d$). It induces a map $\nabla : F_r \to B_d$ on the fundamental group level, where F_r is the free group on r letters and B_d is the braid group on d strands. In order to write down the Zariski-Van Kampen presentation one needs an explicit understanding of ∇. This is hard to come by in general. The purpose of this thesis is to explore settings where ∇ can be computed directly from combinatorial and visual (real) information on the curve C. The method in these special situations is similar to the computation of the presentation of the fundamental group of a knot complement in $\mathbb{R}^3$.

The following references [3, 5, 6, 7, 9, 11, 12, 13, 16, 17, 19] are not cited explicitly in this paper, but did serve a roll in the process of creating this thesis. It is recommended that the reader consult any and all sources in this document for further reading.

CHAPTER 2

ZARISKI-VAN KAMPEN

2.1 Fundamental Group and Presentations of Groups

The fundamental group is a group associated to any given pointed topological space. It provides information about the basic shape, or "number of holes" of the topological space in question. It should be noted that the fundamental group π_1 is the first in a hierarchy of homotopy groups π_i, $i = 1, 2, 3, \ldots$. It is a topological invariant: homeomorphic topological spaces have isomorphic fundamental groups. The intuition behind π_1 is the following: Start with a space, and some point in it, call it a base point, and then consider all the loops both starting and ending at that base point. Two loops may be combined together by traveling along the first loop, and then along the second. Any two loops are considered equivalent if one can be continuously deformed into the other. The set of such loops, up to deformation, is the fundamental group for that particular space.

A more mathematically precise definition (that can be found in greater detail in [8]) now follows. Let X be a topological space, and let x_0 be a point of X. The set of continuous functions

$$\{f : [0, 1] \to X : f(0) = x_0 = f(1)\},$$

is called the set of loops based at x_0. Two loops f and g can be deformed into each other if there is a continuous function

$$H\colon [0,1] \times [0,1] \to X$$

such that $H(s,0) = f(s)$, $H(s,1) = g(s)$, $H(0,t) = x_0$, and $H(1,t) = x_0$. The map H is called a homotopy (or deformation) from f to g. Homotopy defines an equivalence relation on the set of loops. We denote the equivalence class containing f by $[f]$. Concatenation of loops

$$(f * g)(t) = \begin{cases} f(2t) & 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \frac{1}{2} \leq t \leq 1 \end{cases}$$

defines a multiplication on the set of homotopy classes: $[f][g] = [f * g]$. This makes the set

$$\pi_1(X, x_0) = \{[f] \mid f : [0,1] \to X : f(0) = x_0 = f(1)\}$$

into a group. The identity element is the class containing the constant path at the base point, and the inverse of a loop $f(t)$ is defined by $f(1-t)$, i.e. travelling along the loop backwards.

As a simple example, consider the 2-disk X with a hole in it. Let x_0 be a point on the boundary of the disc. Any loop that does not travel around the hole is homotopy equivalent to the constant loop. Furthermore, any loop that does find itself traveling around the hole in the 2-disk is notably *not* homotopy equivalent to the constant loop. The hole provides an obstruction for deformations. Thus there are at least two distinct homotopy classes. It turns out that a loop that travels around the hole once can not be deformed to a loop that travels around twice, and so on. Homotopy classes of loops are completely determined by how many times the loop travels around the hole, so

$$\pi_1(X, x_0) = \mathbb{Z}.$$

A presentation of a group is a set of generators of some group G that satisfy "relations". More precisely: Let S be a set of letters and R be a set of words in these letters and their inverses. We denote by $R = 1$ the set of equations $\{r = 1 \mid r \in R\}$. Then

$$P = < S \mid R = 1 >$$

is called a presentation. It defines a group $G(P)$ in the following way. Elements are words in S which we are allowed to rewrite using the equations $r = 1$. Multiplication is just concatenation of words. More formally,

$$G(P) = F(S)/N(R)$$

where $F(S)$ is the free group on S and $N(R)$ is the normal closure of R in $F(S)$. Recall that the free group over a set S is defined to be all the possible words that can be built from the letters from S and their inverses. Multiplication in $F(S)$ is concatenation, and we are allowed to cancel a pairs of the form aa^{-1} or $a^{-1}a$. The empty word is the identity in $F(S)$ generally expressed as 1 (or e in line with notation for groups). An example now follows:

$$P = < a_1, a_2, a_3, a_4 \mid a_1a_2a_3 = a_3a_2a_1, a_1^3 = a_4^2 >$$

where we have generators a_1 through a_4 and two relations which, in standard form could be written as $a_1a_2a_3(a_3a_2a_1)^{-1} = 1$ and $a_1^3a_4^{-2} = 1$. It is common to write these relations such that they are equal to the empty word. The structure of the group $G(P)$ is typically hard to understand just from its presentation alone on a general basis. However, sometimes we get lucky. Consider

$$P = < a \mid a^n = 1 >$$

where the relation tells us that if we repeat the letter a, n times, we get the identity of the group. This can be recognized as the cyclic group of order n. Thus $G(P) \cong \mathbb{Z}_n$. Consider

$$P = < a, b \mid a^2 = 1, b^2 = 1, (ab)^3 = 1 > .$$

This is a little more difficult. We can list elements of $G(P)$ according to word length. Note that since $a = a^{-1}$ and $b = b^{-1}$ we only have to list group elements represented by positive words. 1 is the only element of length 0. a and b are the only elements of length 1, ab and ba are the only elements of length two, and aba is the only element of length 3 because $aba = bab$. Indeed, $ababab = 1$, so $aba = b^{-1}a^{-1}b^{-1} = bab$. We claim that this completes the list of elements of $G(P)$. Suppose by induction that all elements of $G(P)$ that can be represented by words w of length $\leq n-1$ are already listed. Let w be a word of length n. If $w = abab...$ then replacing aba by bab gives a word $w \cong babb... \cong ba....$ Thus the element of $G(P)$ that w represents can be represented by a word of shorter length. Hence it is on the list. If $w = baba...$ the argument is similar. This serves to show that $|G(P)| \leq 6$. Now consider D_3, the group of symmetries of an equilateral triangle. This group is generated by two reflections g and h. The relations $g^2 = 1$, $h^2 = 1$, $(gh)^3 = 1$ hold in D_3. Thus we have a group epimorphism $G(P) \to D_3$ sending a to g and b to h. Since $G(P)$ has at most 6 elements and D_3 has exactly 6 elements, it follows that the epimorphism is an isomorphism.

Theorem (Van Kampen [8]). Let X be the union of path-connected open sets A_1 and A_2. Assume that the intersection $A_1 \cap A_2$ is path connected and contains the point x_0. Let $j_1\colon A_1 \cap A_2 \to A_1$ and $j_2\colon A_1 \cap A_2 \to A_2$ be the inclusion maps. Then

$$\pi_1(X, x_0) = \pi_1(A_1, x_0) * \pi_1(A_2, x_0)/N,$$

where N is normally generated by elements of the form $j_{1_\sharp}(\omega)j_{2_\sharp}(\omega)^{-1}$, where $\omega \in \pi_1(A_1 \cap A_2, x_0)$.

A CW-complex (or what is commonly just referred to as a complex) is a topological space that is built from building blocks called cells. We can construct these complexes inductively by attaching cells. 0-cells are vertices, 1-cells are edges, 2-cells are faces, etc. The n-skeleton of a complex is the union of cells whose dimension is at most n. As an example, the standard CW structure on the real numbers has the integers as 0-cells, and the intervals $\{[n, n+1] | n \in \mathbb{Z}\}$ as 1-cells. A presentation

$$P = \langle x_1, \ldots, x_n \mid r_1 = 1, \ldots, r_m = 1\rangle$$

not only defines a group $G(P)$ but also a 2-complex $K(P)$. Its 1-skeleton is a wedge of circles $S^1_{x_1} \vee ... \vee S^1_{x_n}$. For each relator r_i (the equations $r_i = 1$ are called relations among the generators, and the r_i are referred to as relators) we attach a 2-cell D^2_i along the path that spells the word r_i.

Corollary (2-complex). Let P be a presentation and $K(P)$ the 2-complex defined by P. Then $\pi_1(K(P))$ is isomorphic to $G(P)$.

Proof. Let $P = \langle x_1, ..., x_n | r_1 = 1, ..., r_m = 1\rangle$. Then $K(P)$ is a wedge of circles in one-to-one correspondence to the generators to which 2-cells D_j^2 are attached using r_j as attaching path. So

$$K(P) = K(P)^{(1)} \cup D_1^2 \cup ... \cup D_m^2.$$

Let $A_1 = K(P)^{(1)}$ and $A_2 = D_1^2$. Note that the A_i are not open but we could put a collar on it to make it so. Then $A_1 \cap A_2$ is a circle S^1. Let ω be the generator of $\pi_1(A_1 \cap A_2)$. Now $j_{1_\sharp}(\omega) = r_1$ and $j_{2_\sharp}(\omega) = 1$ (note that $\pi_1(A_2) = 1$). Thus Van Kampen tells us that

$$\pi_1(K(P)^{(1)} \cup D_1^2) = F(x_1, ..., x_n) * 1/N$$

where N is normally generated by $r_1 \cdot 1 = r_1$. We iterate this process to obtain

$$\pi_1(K(P)) = F(x_1, ..., x_n)/N$$

where N is normally generated by $r_1, ..., r_m$. By definition $G(P) = F(x_1, ..., x_n)/N$. This completes the proof. □

As we proceed, we will present methods for finding presentations of the fundamental group of the complement of particular curves. To do so we first need to describe the action of the braid group on the free group.

2.2 The Braid Group and its Action on the Free Group

The braid group on n strands, given by B_n, is the group whose elements are the equivalence classes of n-braids, and whose group operation is that of compositions of braids. This operation can be thought of analogously to the operation for concatenation of paths. The braid group on n strands is generated by "half-twists" σ_i for $1 \leq i \leq n-1$. An arbitrary half-twist with the inverse half twist is seen below:

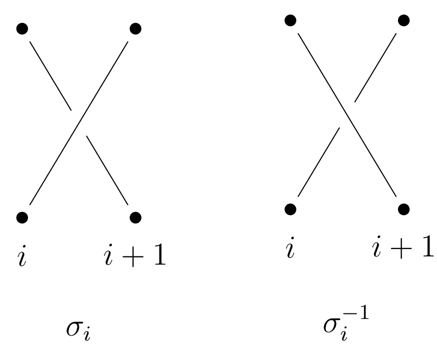


Figure 2.1: Half twists in the braid group

where these braids are read from bottom to top. Note the distinction between the half twists, (over-under). Here, we are defining braid groups as fundamental groups of a configuration space, and defining free groups as the fundamental groups of a plane with punctured holes. This will be made more precise shortly. Below is what we define to be the left action of the braid group B_n on the free group F_n. Also shown is the inverse action for completeness.

$$^{\sigma_i}a_j := \begin{cases} a_j & \text{if } j \neq i, i+1 \\ a_i a_{i+1} a_i^{-1} & \text{if } j = i \\ a_i & \text{if } j = i+1. \end{cases} \qquad ^{\sigma_i^{-1}}a_j := \begin{cases} a_j & \text{if } j \neq i, i+1 \\ a_{i+1} & \text{if } j = i \\ a_{i+1}^{-1} a_i a_{i+1} & \text{if } j = i+1. \end{cases}$$

The following figure which serves to show the geometric action of the braid group on the free group visually (from a top-down viewpoint) has been recreated from an image in [14]. Note that the tilde notation is used to talk about the "same loop" but in its new location after the action.

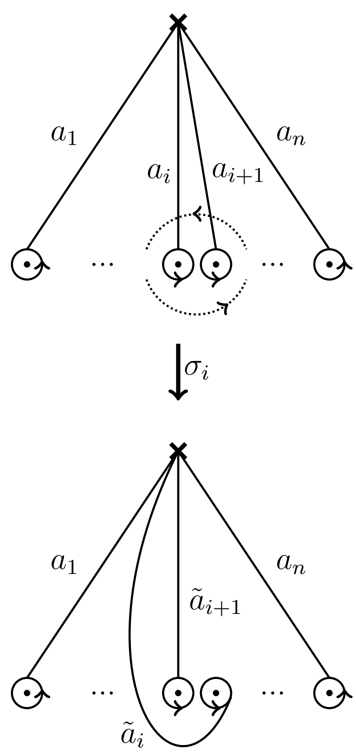


Figure 2.2: Braid action on free group

We can see that in fact under the action of some σ_i in the braid group, that the generators of the free group, namely $a_1, \ldots, a_n$ in this case, are permuted (up to conjugation). Notice that the class of loops $[\tilde{a}_{i+1}] = [a_i]$, and that the class of loops $[a_{i+1}]$ is conjugated, and we see that $[\tilde{a}_i] = [a_i][a_{i+1}][a_i]^{-1}$. Note that only two loops next to each other are affected by the action at any generator σ_i (as is realized by the half-twists in the braid group). This is precisely the action described above. To add some more intuition, below is another figure taken from the point of view of the braid itself, inspiration for which came from [15]:

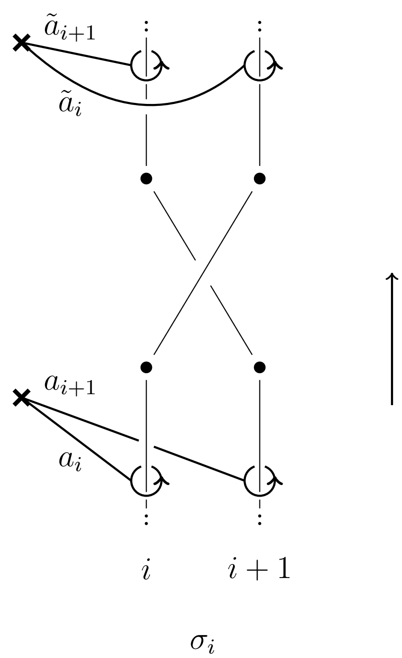


Figure 2.3: Braid action via half twist

where we view the basepoint as being in the background, and note that there are not two basepoints, but it is drawn in this way to make clear how these loops change as they move across the braid. Note again that $[\tilde{a}_{i+1}] = [a_i]$, and $[\tilde{a}_i] = [a_i][a_{i+1}][a_i]^{-1}$.

After the braiding $\tilde{a}_i$ has to "go around" the i-th strand to loop the $i+1$-st strand, then come back around the i-th strand again. This is the geometric realization of the conjugation we see in the action.

2.3 The Zariski-Van Kampen Theorem

The machinery of Zariski-Van Kampen allows us a way to compute the fundamental group of complements to plane curves, and it does so in terms of generators and relations. In other words it reveals this as a presentation. This classical method can be interpreted in the following way: Consider a plane algebraic curve $C \subset \mathbb{C}^2$ defined by $C = \{(x, y) \in \mathbb{C}^2 \mid f(x, y) = 0\}$ where f is a polynomial. Without loss of generality we assume $f(x, y) = y^d + g(x, y)$, where $g(x, y)$ is a polynomial of y-degree less than d (this form can be achieved by a change of coordinates and division by a nonzero constant). Define

$$NT = \{x_0 \in \mathbb{C} \mid f(x_0, y) = 0 \text{ has roots with multiplicity as a polynomial in } y\}.$$

NT stands for "not transversal". Note that this set is finite. Let r be the number of elements in NT. For every $x_0 \in \mathbb{C}$ we have a vertical complex line $L_{x_0} = \{(x_0, y) \mid y \in \mathbb{C}\}$. We say the line is generic if $x_0 \notin NT$. Geometrically this means that it intersects the curve transversally, or in other words that it stays away from cusps, is not a vertical tangent to the curve, or does not intersect a node or some other irregular point. Given a generic vertical line L, we see that $L \cap C$ consists of d points (the degree of f in y). That means $L - C$ is a real plane that has been punctured at d points. Thus this fundamental group is the free group of rank d, denoted F_d.

One can show that a basis of loops in the group $\pi_1(L-C)$ also generates $\pi_1(\mathbb{C}^2-C)$ (that is, the inclusion $L-C \to \mathbb{C}^2 - C$ is π_1-surjective.)

The fat diagonal in $\mathbb{C}^d$ is defined as $\Delta = \{(z_1, \ldots, z_d) \in \mathbb{C}^d \mid z_i = z_j \text{ for some } i \neq j\}$. Then there is a continuous mapping $\tilde{f} : \mathbb{C} - NT \to \mathbb{C}^d - \Delta$ defined by $\tilde{f}(x) = (z_1, ..., z_d)$, where the z_i are the d distinct roots of $f(x, y)$, viewed as a polynomial of degree d with single variable y. This is continuous since $x \notin NT$, and continuity of roots applies. Now the space we have mapped into can be naturally identified with the configuration space of d different points in the complex plane $\mathbb{C}$. The fundamental group of this space is known to be the braid group on d strands, denoted B_d. So $\tilde{f}$ induces a mapping at the level of fundamental groups $\tilde{f}_\# = \nabla : F_r \to B_d$. This precise wording of the theorem comes from [1], and can be originally found in [18, 10].

Theorem (Zariski-Van Kampen). The fundamental group of $\mathbb{C}^2 - C$ is the quotient of F_d by the subgroup normally generated by ${}^{\tau}ww^{-1}$, $w \in F_d$, $\tau \in B_d$. If $b_1, \ldots, b_r$ is a free generating system of F_r, then it admits the following presentation:

$$\left\langle a_1, \ldots, a_d \;\middle|\; a_j = {}^{\nabla(b_i)}a_j,\ j = 1, \ldots d,\ i = 1, \ldots, r \right\rangle.$$

We end this section by giving a proof sketch of the Zariski-Van Kampen theorem. Let $C^+ = C \cup \bigcup_{i-1}^{r} L_{x_i}$, where the x_i are the numbers in NT. The projection $p\colon \mathbb{C}^2 \to \mathbb{C}$ onto the first coordinate gives a locally trivial fibration

$$p\colon E = \mathbb{C}^2 - C^+ \to B = \mathbb{C} - NT$$

with fiber $F = \mathbb{C} - \{d \text{ points}\}$. Associated with this fibration comes a long exact homotopy sequence

$$\cdots \to \pi_2(B) \to \pi_1(F) \to \pi_1(E) \to \pi_1(B) \to 1$$

(the map on the right is onto because both the fiber and the total space are connected). Since B is topologically a punctured plane we have $\pi_2(B) = 0$. Thus we have

$$1 \to F_d = F(a_1, ..., a_d) \to \pi_1(E) \to F_r = F(b_1, ..., b_r) \to 1.$$

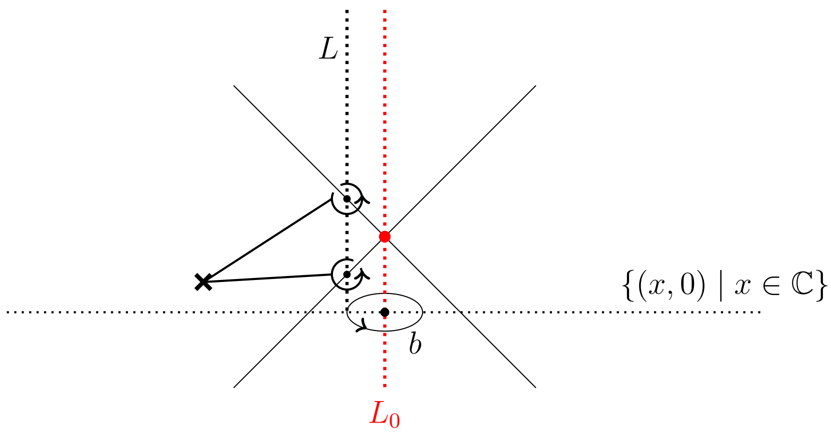
Conjugation action of F_r on the normal subgroup F_d is the monodromy action on the fiber. Thus we obtain a presentation

$$\langle a_1, ..., a_d, b_1, ..., b_r \mid b_i a_j b_i^{-1} = {}^{\nabla(b_i)}a_j \rangle.$$

Inserting the vertical lines L_{x_i} back into E trivializes the b_i and we obtain the desired presentation for $\pi_1(\mathbb{C}^2 - C)$.

2.4 Some Examples

Example 1. Let $f(x, y) = (y - (x + 1))(y - (-x + 1))$. Then $NT = \{0\}$, so $\mathbb{C} - NT$ is the plane punctured at 0. Let there be a map $b : [0, 1] \to \mathbb{C} - NT$ defined by $b(t) = e^{2\pi it}$, where $b(t)$ is the loop around the puncture. Note that $[\tilde{f} \circ b] = \nabla[b]$. Then $\nabla(b) = (-e^{2\pi it} + 1, e^{2\pi it} + 1)$. The two coordinates should be viewed as two moving points p_1 and p_2 in the plane. At $t = 0$ we have $p_1 = 0$ and $p_2 = 2$. Over time the points move on a circle of radius 1 and center 1, in the counterclockwise direction. At time $t = \frac{1}{2}$ the points have changed position, and at time $t = 1$, both points are back where they started. Visual interpretation of this is seen below



where the red line is the center of rotation, L is a generic vertical line, and these loops get wrapped around each other by the action of the braid group permuting these punctures as they move around the singularity. Note that the base point is exaggerated for intuition; this base point lives on the plane L. If σ is the braid for the transposition that interchanges p_1 and p_2, then $\nabla(b) = \sigma^2$. Call the two loops around the punctures a_1 and a_2 respectively. As these punctures move, the loops surrounding them are twisted. Since the degree of y is 2 is this example, we are mapping into $\pi_1(\mathbb{C}^2 - \Delta) \cong B_2$. σ_1 is the only half twist in B_2 and so we must have that $\sigma = \sigma_1$. Now, applying the action to our generators a_1 and a_2 we see

$$a_1 = {}^{\sigma_1^2}a_1 = {}^{\sigma_1}(a_1a_2a_1^{-1}) = a_1a_2a_1^{-1}a_1a_1a_2^{-1}a_1^{-1} = a_1a_2a_1a_2^{-1}a_1^{-1}.$$

$$a_2 = {}^{\sigma_1^2}a_2 = {}^{\sigma_1}a_1 = a_1a_2a_1^{-1}.$$

These relations both imply that $a_2a_1a_2^{-1}a_1^{-1} = 1$. This is the commutator between a_1 and a_2. Our presentation is now given by

$$P = < a_1, a_2 \mid a_1a_2 = a_2a_1 > .$$

Thus we have two generators that commute and so we see that $G(P) \cong \mathbb{Z} \times \mathbb{Z}$.

We now verify this answer using methods closer to first principles. Let the complex numbers z_1 and z_2 represent $(y - (x+1))$ and $(y - (-x+1))$ respectively. Then our curve C is given by $z_1 z_2$. Notice that $C = \{(z_1, z_2) : z_1 = 0 \text{ or } z_2 = 0\}$. This now implies logically that

$$\mathbb{C}^2 - C = \{(z_1, z_2) : z_1 \neq 0 \text{ and } z_2 \neq 0\}.$$

Now, $(\mathbb{C} - z_1) \times (\mathbb{C} - z_2) = \{(z_1, z_2) : z_1 \neq 0 \text{ and } z_2 \neq 0\}$. So we may come to the conclusion that $\mathbb{C}^2 - z_1 z_2 = (\mathbb{C} - z_1) \times (\mathbb{C} - z_2)$, i.e. we have that

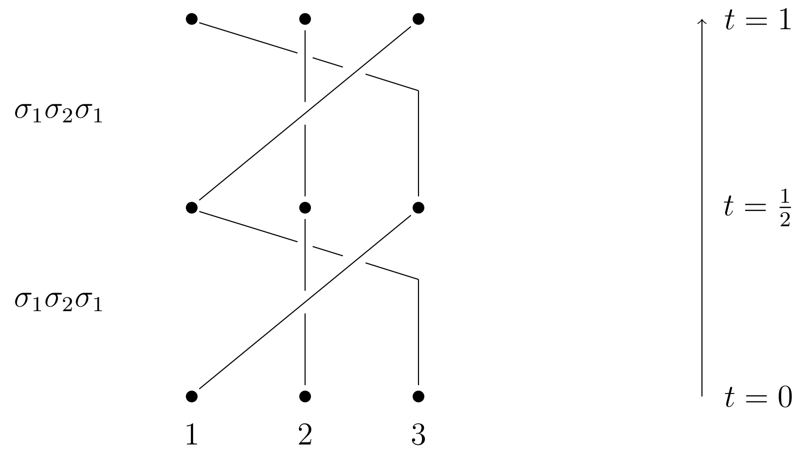
$$\begin{aligned} \pi_1(\mathbb{C}^2 - z_1 z_2) &= \pi_1((\mathbb{C} - z_1) \times (\mathbb{C} - z_2)) \\ &= \pi_1(\mathbb{C} - z_1) \times \pi_1(\mathbb{C} - z_2) \\ &= \mathbb{Z} \times \mathbb{Z} \end{aligned}$$

where the second equality comes from standard properties of the fundamental group, and the third coming from the fact that each piece is the fundamental group of the complex plane minus a point, which is infinite cyclic. Thus we see that this agrees with Zariski-Van Kampen. This works as z_1 and z_2 are both straight lines, which by a coordinate change could be made to be the x and y axis respectively.

Example 2. As another example let our curve C be defined by $f(x,y) = (y - 1)(y - (x+1))(y - (-x+1))$. Once more $NT = \{0\}$, so again $\mathbb{C} - NT$ is the plane punctured at 0. Let b be the loop around the puncture, with $b(t) = e^{2\pi i t}$. Notably there is a distinction from the last example, the singularity is a point of multiplicity three rather than two. The degree of y is 3 here and so we are mapping into $\pi_1(\mathbb{C}^3 - \Delta) \cong B_3$, in fact, $\nabla(b) = (-e^{2\pi it} + 1, 1, e^{2\pi it} + 1)$. The respective points p_1, p_2 and p_3 are moving in the plane (along with elements of the free group surrounding these punctures a_1, a_2, a_3), and just like the previous example, at $t = 0$ we have $(0, 1, 2)$, at $t = \frac{1}{2}$ these points have changed position (notably p_2 hasn't changed positions with p_1 or p_3), and at $t = 1$ these points are back where they started. The main distinction in this example is that B_3 has two half-twists as generators, namely σ_1 and σ_2. What we require to "get back to where we started" is to perform a full twist on the number of strands we are dealing with in the braid group. This means all strands return to their starting position at the end of the composition of braids. A full twist on n braids can be given by $(\sigma_1 \cdots \sigma_{n-1})^n$. In our case $n = 3$, and so this can be given by $(\sigma_1\sigma_2)^3$. Using relations in the braid group we can replace this composition of braids with a square of a different composition, i.e.

$$\begin{aligned}(\sigma_1\sigma_2)^3 &= \sigma_1\sigma_2\sigma_1\sigma_2\sigma_1\sigma_2 \\ &= \sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1 && \text{(by braid relation } \sigma_2\sigma_1\sigma_2 = \sigma_1\sigma_2\sigma_1\text{)} \\ &= (\sigma_1\sigma_2\sigma_1)^2.\end{aligned}$$

The point of this is to make intuitively clear what $t = \frac{1}{2}$ is taken to mean. By making this composition of braids a square, we see that if we do the braid once we are halfway around our singularity, and doing it again brings us all the way back. The described braid is seen below



So, $\nabla(b) = (\sigma_1\sigma_2\sigma_1)^2$ is the braid relation that will act on our free group elements a_1, a_2, a_3. We compute these now to see the presentation of the group.

$$\begin{aligned}
a_1 &= {}^{\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2\sigma_1}a_1 \\
&= {}^{\sigma_1\sigma_2\sigma_1\sigma_1\sigma_2}(a_1a_2a_1^{-1}) \\
&= {}^{\sigma_1\sigma_2\sigma_1\sigma_1}(a_1a_2a_3a_2^{-1}a_1^{-1}) \\
&= {}^{\sigma_1\sigma_2\sigma_1}(a_1a_2a_1^{-1}a_1a_3a_1^{-1}a_1a_2^{-1}a_1^{-1}) \\
&= {}^{\sigma_1\sigma_2\sigma_1}(a_1a_2a_3a_2^{-1}a_1^{-1}) \\
&= {}^{\sigma_1\sigma_2}(a_1a_2a_1^{-1}a_1a_3a_1^{-1}a_1a_2^{-1}a_1^{-1}) && \text{(since } (a_1a_2a_1^{-1})^{-1} = a_1a_2^{-1}a_1^{-1}\text{)} \\
&= {}^{\sigma_1\sigma_2}(a_1a_2a_3a_2^{-1}a_1^{-1}) \\
&= {}^{\sigma_1}(a_1a_2a_3a_2^{-1}a_2a_2a_3^{-1}a_2^{-1}a_1^{-1}) \\
&= {}^{\sigma_1}(a_1a_2a_3a_2a_3^{-1}a_2^{-1}a_1^{-1}) \\
&= a_1a_2a_1^{-1}a_1a_3a_1a_3^{-1}a_1^{-1}a_1a_2^{-1}a_1^{-1} \\
&= a_1a_2a_3a_1a_3^{-1}a_2^{-1}a_1^{-1}
\end{aligned}$$

This implies $a_1a_1a_2a_3 = a_1a_2a_3a_1$ which can be written as $[a_1a_2a_3, a_1] = 1$, and similarly for a_2 and a_3 we get $[a_1a_2a_3, a_2] = 1$ and $[a_1a_2a_3, a_3] = 1$ respectively. Then a presentation of $\pi_1(\mathbb{C}^2 - C)$ is given by

$$P = < a_1, a_2, a_3 \mid [a_1a_2a_3, a_1] = 1, [a_1a_2a_3, a_2] = 1, [a_1a_2a_3, a_3] = 1 > .$$

If we let $a_1a_2a_3 = c$ then $a_3 = a_2^{-1}a_1^{-1}c$. Our relations become $[c, a_1] = 1, [c, a_2] = 1, [c, a_2^{-1}a_1^{-1}c] = 1$. But c clearly commutes with itself, as well as a_1 and a_2 given by previous relations, so the last relation is redundant. Therefore we may write

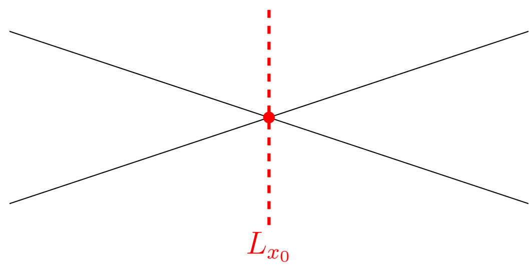
$$P = < a_1, a_2, c \mid [c, a_1] = 1, [c, a_2] = 1 >$$

and so $G(P) \cong F_2 \times \mathbb{Z}$.

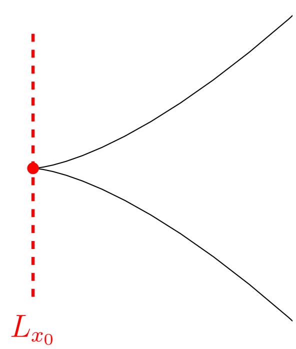
2.5 Relations

There are particular relations given by the type of non-transversal intersection we have with our vertical line. These are described in [1], and follow below for reference. Let us assume that b_1 corresponds to a small loop surrounding a point x_0 such that L_{x_0} is an ordinary tangent line. For suitable choices of loops and paths, $\nabla(b_1) = \sigma_1$ and the only nontrivial relation is given by $a_1 = a_2$. Analogously, for other non-transversal vertical lines L_{x_0}, one obtains the following braids and relations:

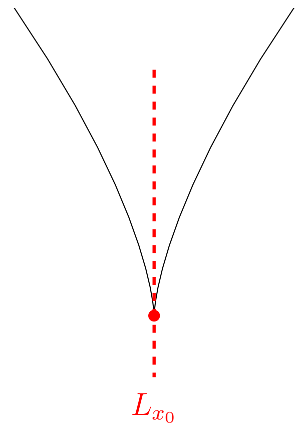
(a) If L_{x_0} passes transversally through a node, then $\nabla(b_1) = \sigma_1^2$ and the only nontrivial relation is given by $[a_1, a_2] = 1$.



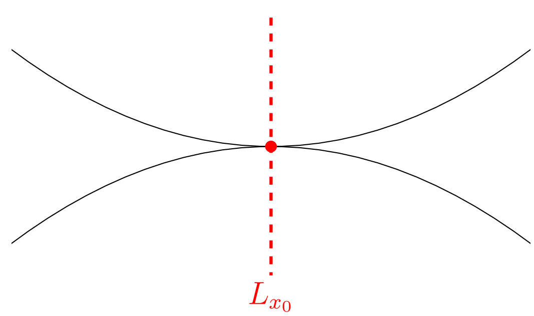
(b) If L_{x_0} passes through an ordinary cusp (specifically $y^2 = x^3$) in a direction different from its tangent cone, then $\nabla(b_1) = \sigma_1^3$ and the only nontrivial relation is given by $a_1a_2a_1 = a_2a_1a_2$.



(c) If L_{x_0} passes through an ordinary cusp (in a direction consistent with its tangent cone), then $\nabla(b_1) = (\sigma_2\sigma_1)^2$ and the only nontrivial relations are given by $a_1 = a_3$ and $a_2 = a_3a_2a_1a_2^{-1}a_1^{-1}$.



(d) If L_{x_0} passes transversally through an m-tacnode (two smooth branches with intersection number m and the same tangent direction), then $\nabla(b_1) = \sigma_1^{2m}$ and the only nontrivial relation is given by $(a_1a_2)^m = (a_2a_1)^m$.



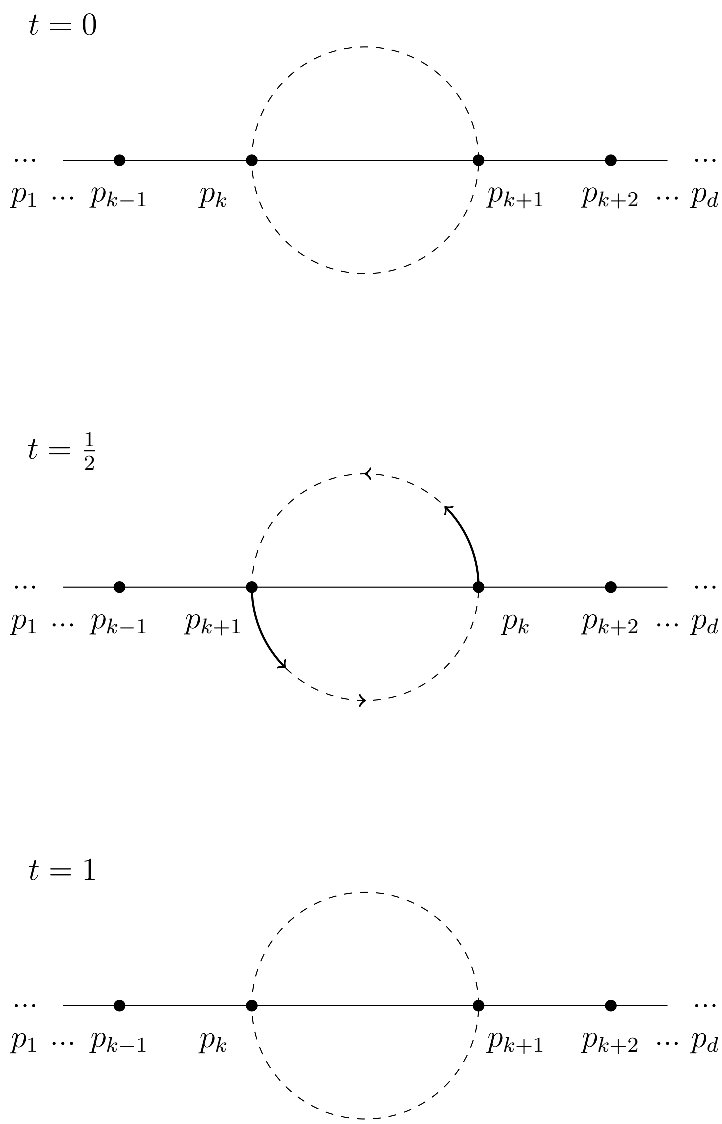
Now let us examine (a) above more closely. In the general case we have a curve C given by $f(x, y) = 0$ that has degree d in y. Let $NT = \{x_1, \dots, x_r\}$. Consider a vertical line L_{x_i} passing transversally through a node, and for $1 \leq i \leq r$ let there be maps $b_i : [0, 1] \to \mathbb{C} - \{x_i\}$ where $b_i(t)$ is a small loop surrounding the point

x_i. Now for $\epsilon > 0$, $[\tilde{f} \circ b_i] = \nabla[b_i] : \pi_1(\mathbb{C} - NT) \to \pi_1(\mathbb{C}^d - \Delta)$ maps the loop $b_i(t) = x_i + \epsilon \cdot e^{2\pi\sqrt{-1}t}$ in the following way:

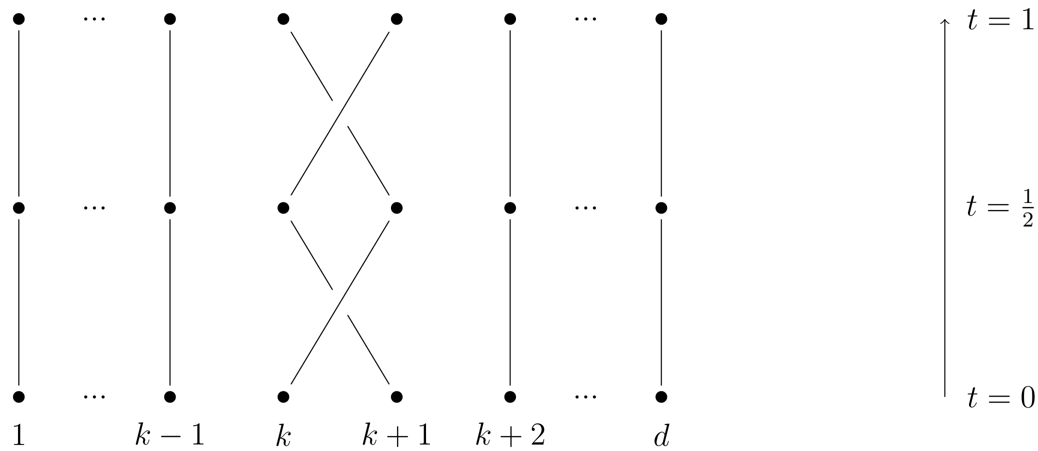
$$\nabla(b_i(t)) = (z_1(x_i + \epsilon \cdot e^{2\pi\sqrt{-1}t}), \ldots, z_i(x_i + \epsilon \cdot e^{2\pi\sqrt{-1}t}), \ldots, z_d(x_i + \epsilon \cdot e^{2\pi\sqrt{-1}t}))$$

where the z's are the distinct roots in the configuration space. The braid monodromy about x_i (for a sufficiently small loop b_i) then permutes the points of $L_{x_i - \epsilon} \cap C$ as it "runs around" the singular point x_i. Note that d many points are being permuted, but as there are no other singularities within this circle of radius ϵ (by construction), nothing interesting happens to the points that are not meeting at the node x_i.

Let $a_1, \ldots a_d$ be generators of the free group, in particular the loops around the d punctures in $L_{x_i-\epsilon} - C$. Let a_k and a_{k+1} be the loops around the punctures p_k and p_{k+1} that will meet the node at x_i. One can see that at time $t = 0$, p_k and p_{k+1} have yet to move, at time $t = \frac{1}{2}$, p_k and p_{k+1} have swapped places, and at $t = 1$ they have returned to where they started. What is important is this permutation has memory. This memory comes from the loops being attached to a base point (in fact the starting configuration). So as was noted previously, if σ is the braid that swapped their positions, then σ^2 is the braid that describes the entire journey around x_i. Accordingly this braid acts on the elements a_k and a_{k+1} (it will consist of half twists on strands k and $k+1$) and notably *does not* act on any other a_j, witnessed by the action. Consequently this means all other a_j's remain unchanged by this process. This has the visual interpretation seen below



where say, in the interval of one second (given by t) these paths complete their journeys. It is not displayed in the figure, but note that all other points in this configuration space are also moving around in this interval of one second, however they do not tangle with one another, and return to where they started, unchanged. This permutation can be identified by the following braid



where this is a concatenation of two braids, both σ_k, as this is the only half-twist in B_d that permutes these two strands. Thus we see that in fact $\nabla(b_i) = \sigma_k^2$, and given the action of the braid group on the free group, we see that the only relation that comes out of this (for the two loops around the curve a_k and a_{k+1} that meet at this node) is $[a_k, a_{k+1}] = 1$, which is seen below:

$$a_k = {}^{\sigma_k^2}a_k = {}^{\sigma_k}(a_k a_{k+1} a_k^{-1}) = a_k a_{k+1} a_k^{-1} a_k a_k a_{k+1}^{-1} a_k^{-1} = a_k a_{k+1} a_k a_{k+1}^{-1} a_k^{-1}$$

which implies $1 = a_{k+1} a_k a_{k+1}^{-1} a_k^{-1}$ i.e. $1 = [a_k, a_{k+1}]$, and furthermore

$$a_{k+1} = {}^{\sigma_k^2}a_{k+1} = {}^{\sigma_k}a_k = a_k a_{k+1} a_k^{-1}$$

which implies $1 = a_k a_{k+1} a_k^{-1} a_{k+1}^{-1}$ and again we see $1 = [a_k, a_{k+1}]$.

The other cases (b), (c) and (d), are worked out in a similar way (albeit more difficult). For example, (b) can be realized by noticing that again there are two points p_1 and p_2 (with loops a_1 and a_2 surrounding them) moving in space. The distinction here is that these points will round the circle to return to where they started, but then permute one more time. In other words there is a half-twist on two strands, three times, i.e. σ_1^3 is the braid that will act on our generators a_1, a_2. This triple twist is witnessed by the fact that a fractional power that is obtained at distinct roots of the ordinary cusp ($y^2 - x^3 = 0$) applied to our loop $b(t)$ gives a loop that traverses 3π rather than just 2π, which gives the extra permutation. In fact for $m = 2k$, if the cusp singularity is of type $\mathbb{A}_m$ ($y^2 - x^{m+1} = 0$), then the braid can be given by σ_1^{m+1}, i.e. if $m = 4$ ($k = 2$) then our loop $b(t)$ gives a loop that traverses 5π rather than 2π (2 more half-twists), generally this becomes $(m+1)\pi$ rather than 2π. It should be noted that in case (c) we are considering curves of the form $y^{m+1} - x^2 = 0$ for $m = 2k$, and in case (d) we are dealing with curves of the form $y^2 - x^{m+1} = 0$ for $m = 2k + 1$.

2.6 Zariski-Van Kampen Presentations are LOG Presentations

A labeled oriented graph Γ, LOG for short, is an oriented graph whose edges are labeled by words in $V^{\pm 1}$, where V is the set of vertices. The associated LOG presentation is defined as

$$\mathcal{P}(\Gamma) = \langle V \mid r_e = 1, e \in E \rangle,$$

where E is the set of edges. If e is an edge from a to b, labeled by the word w, then $r_e = aw(wb)^{-1}$. The group $G(\Gamma)$ defined by $\mathcal{P}(\Gamma)$ is called a LOG group. Knot groups, higher dimensional knot groups, and virtual knot groups are all LOG groups. In this section we show that the Zariski-Van Kampen presentation is a LOG presentation.

A fundamental open question about LOG presentations is the following: If the underlying graph Γ is a tree, is the standard 2-complex $K(\Gamma)$ built from the LOG presentation aspherical, that is $\pi_i(K(\Gamma)) = 0$ for $i \geq 2$? One motivation for this thesis came from the idea of using algebraic geometry to approach this question.

Lemma 1. Given a braid σ, for every $j = 1, \ldots, d$ there exists a word $w_j \in F_d$ so that ${}^{\sigma}a_j = w_j a_{\bar{\sigma}(j)} w_j^{-1}$.

Proof. Recall the action of the braid group (along with the inverse action):

$$ {}^{\sigma_i}a_j := \begin{cases} a_j & \text{if } j \neq i, i+1 \\ a_i a_{i+1} a_i^{-1} & \text{if } j = i \\ a_i & \text{if } j = i+1. \end{cases} \qquad {}^{\sigma_i^{-1}}a_j := \begin{cases} a_j & \text{if } j \neq i, i+1 \\ a_{i+1} & \text{if } j = i \\ a_{i+1}^{-1} a_i a_{i+1} & \text{if } j = i+1. \end{cases} $$

where $a_1, \ldots, a_d$ are generators of F_d and $\sigma_1, \ldots, \sigma_{d-1}$ are generators of B_d. Let $\sigma = \sigma_{k_1}^{\epsilon_1} \cdots \sigma_{k_n}^{\epsilon_n}$ for $\epsilon_i = \pm 1$. We will now do induction on n to show the desired result. First let $n = 1$ and $\epsilon_i = 1$ (the case $\epsilon_i = -1$ is handled similarly). Then we have $\sigma = \sigma_{k_1}$, which gives

$$ {}^{\sigma_{k_1}}a_j := \begin{cases} a_j & \text{if } j \neq k_1, k_1+1 \\ a_i a_{i+1} a_i^{-1} & \text{if } j = k_1 \\ a_i & \text{if } j = k_1+1. \end{cases} $$

Now suppose the claim holds for all $|\zeta| \leq n-1$, for $\zeta \in B_d$. Consider $|\sigma| = n$. Where we put $\sigma = \sigma_{k_1}\zeta$. Thus we have

$$ {}^{\sigma_{k_1}\zeta}a_j = {}^{\sigma_{k_1}}({}^{\zeta}a_j) = {}^{\sigma_{k_1}}(u_j a_{\bar{\zeta}(j)} u_j^{-1}) $$

for $u \in F_d$, where the first equality holds since our action is a homomorphism, and the second holds by the induction hypothesis. Again by the homomorphism we see that the resulting word is equal to

$$({}^{\sigma_{k_1}}u_j)({}^{\sigma_{k_1}}a_{\bar{\zeta}(j)})({}^{\sigma_{k_1}}u_j)^{-1}.$$

Now, u_j is a word in the free group and thus can be acted on by σ_{k_1}. Let ${}^{\sigma_{k_1}}u_j = v_j \in F_d$. Now notice the following

$$({}^{\sigma_{k_1}}u_j)({}^{\sigma_{k_1}}a_{\bar{\zeta}(j)})({}^{\sigma_{k_1}}u_j)^{-1} = v_j({}^{\sigma_{k_1}}a_{\bar{\zeta}(j)})v_j^{-1}$$

where

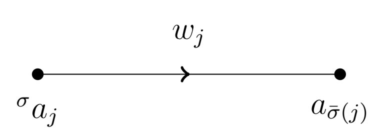
$${}^{\sigma_{k_1}}a_{\bar{\zeta}(j)} = s_j a_{\sigma_{k_1}\zeta(j)} s_j^{-1} = s_j a_{\bar{\sigma}(j)} s_j^{-1}.$$

for some s_j. Putting this together and letting $v_j s_j = w_j \in F_d$, we have the desired result

$${}^{\sigma}a_j = w_j a_{\bar{\sigma}(j)} w_j^{-1}.$$

□

Indeed we see that this produces a label oriented graph which has each a_j in the free group generating set as a vertex, with connecting edges given by these w_j in the free group. For example



$${}^{\sigma}a_j w_j = w_j a_{\bar{\sigma}(j)}.$$

CHAPTER 3

WIRTINGER PRESENTATIONS AND WIRTINGER CURVES

3.1 The Wirtinger Presentation of a Knot Complement

A knot k is an embedded circle in $\mathbb{R}^3$. A knot can be represented by a knot diagram, which is a planar 4-regular graph with over and under crossing information at the nodes. See Figure 3.1. The knot diagram gives a planar projection of the knot. Up to Reidemeister moves the knot diagram uniquely represents the knot. The space $\mathbb{R}^3 - k$ is the knot complement of k. Wilhelm Wirtinger observed that a presentation of the fundamental group $\pi_1(\mathbb{R}^3 - k)$ can be read off a knot diagram. It is generated by loops winding around each of the arcs, while each crossing gives rise to relations among the generators corresponding to the arcs meeting at the crossings. As an example, consider the trefoil knot. See Figure 3.1. We give it an orientation by picking a point on the knot, give it an arrow in the direction it is to travel, and then continue moving around the knot until you are back to the chosen point, we can "read off" a Wirtinger presentation. One reads off a presentation for $\pi_1(\mathbb{R}^3 - k)$ in the following way, where we have taken inspiration from the description of this in [15]:

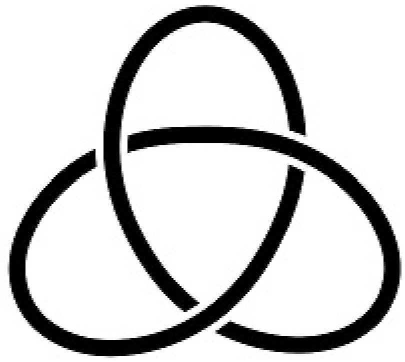
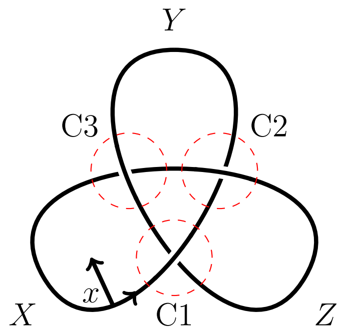
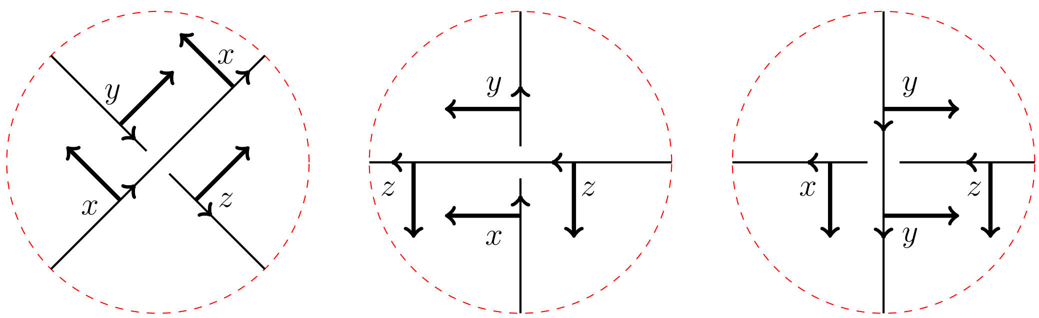


Figure 3.1: A knot diagram for the trefoil knot

Picture a small arrow pointing up from the bottom left loop on the knot (perpendicular to the knot). Label this arrow x. Fix a direction, say, move to the right along the knot. As we move, we will pass over the crossings. If it is an over crossing, nothing changes and we proceed (however we make note of the arrow's direction and label on either side of the crossing). If it is an under crossing, our label changes to y, again making note of the arrow's direction and label, and we proceed. We continue this process until we have returned to our starting point. Note that since there are three crossings, there are three arcs that exist in this figure. At each "circuit" around a crossing, we have a relation that is given by simply traveling this circuit clockwise. Crossings at C1, C2, and C3 are shown below where individual arcs are labeled X, Y and Z:





C1   C2   C3

We now read off the relations of this knot from the crossings by winding clockwise around the crossings (starting with the bottom most labeling) and writing down the label along with its orientation. i.e. we obtain the relationships for each crossing below

$$\text{C1: } xyx^{-1}z^{-1} = 1 \qquad \text{C2: } xz^{-1}y^{-1}z = 1 \qquad \text{C3: } y^{-1}x^{-1}yz = 1.$$

Notice that if we solve the second relation for x, namely $x = z^{-1}yz$, we now see some cancellations as we substitute it into the other two relations:

$$\begin{aligned} xyx^{-1}z^{-1} &= 1 & \qquad y^{-1}x^{-1}yz &= 1 \\ z^{-1}yzyz^{-1}y^{-1}zz^{-1} &= 1 & \qquad y^{-1}z^{-1}y^{-1}zyz &= 1 \\ z^{-1}yzyz^{-1}y^{-1} &= 1 & \qquad zyz &= yzy \\ yzy &= zyz & \qquad yzy &= zyz \end{aligned}$$

where we have removed a generator and reduced three relations down to one, giving the presentation

$$\pi_1(\mathbb{R}^3 - \text{trefoil}) = < y, z \mid yzy = zyz > \cong B_3.$$

Theorem (Wirtinger Presentation). Let k be a knot and W be the Wirtinger presentation read from a knot diagram for K. Then W is a presentation of the fundamental group of $\mathbb{R}^3 - k$.

3.2 The Wirtinger Presentation of a Real Curve

The hope is that reading off a presentation from (the real part of) a curve is done in an analogous way i.e. that we label edges, and consider relations that are obtained from running past a point of singularity (a vertex). This point of singularity will have different relations associated with it, such as if this point is a node, a cusp, etc. This process does not work in general, but it does work when the curve is a so called *Wirtinger curve*. We will define what it means for a curve to be of *Wirtinger type* first. Five conditions are to be met in order for $C \subset \mathbb{C}^2$ to qualify as a curve of Wirtinger type. These are listed below:

(W1) Ramification points of C are real: that is, if $(x, y) \in C$ and $x \in NT$, then $(x, y) \in \mathbb{R}^2$.

(W2) The local branches of C at ramification points are all real.

(W3) If $x \in \mathbb{R}$ and not in NT then the vertical (complex) line L_x intersects the real part of C in exactly d points, where d is the y-degree of the polynomial $f(x, y)$ that defines C.

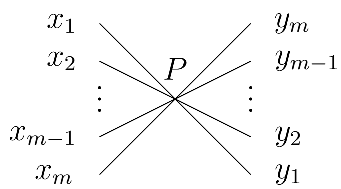
(W4) The curve C contains no vertical asymptotes and no vertical lines, and simple tangencies at smooth points are the only vertical lines in the tangent cone of C at any point.

(W5) The only singularities of C are either double, type $\mathbb{A}_m$, or ordinary (i.e. smooth branches with pairwise distinct tangent cones).

Factors in $f(x, y)$ of the form $(y^2 - x^2) = (y - x)(y + x)$ would give rise to an ordinary singularity, whereas factors of the form $(y^2 - x^{m+1})$, $m \geq 2$, would give rise to a type $\mathbb{A}_m$ singularity. We will exhibit examples of curves of Wirtinger type below.

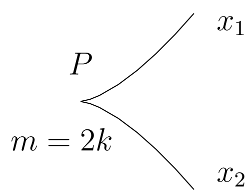
Given a curve C of Wirtinger type, we consider its diagram, $C_{\mathbb{R}} = C \cap \mathbb{R}^2$. Any singularities of this diagram will be referred to as the vertices of the curve, and the edges of C are the closures of the connected components of $C_{\mathbb{R}}$ with the vertices removed. A Wirtinger presentation $\mathcal{P}_C$ is given by a generating system parametrized by the edges of the curve. In addition, for each singular point P, the following relations are associated:

(R1) If P is an ordinary real singular point of multiplicity m, then the edges associated with P can be sorted out in two groups $\{x_1, \ldots, x_m\}$ and $\{y_1, \ldots, y_m\}$. Define $\bar{x}_k = x_k x_{k-1} \cdots x_1$ and $\bar{y}_k = y_1 y_2 \cdots y_k$. Then we have

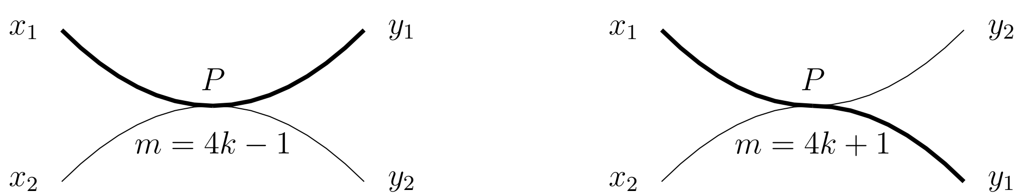


$$\begin{cases} [\bar{x}_m, x_j] = 1 \\ y_j = \bar{x}_j^{-1} \cdot x_j \cdot \bar{x}_j \end{cases}$$

(R2) If P is of type $\mathbb{A}_m$, then the following relations are added:



$$x_1(x_2x_1)^k = (x_2x_1)^k x_2$$

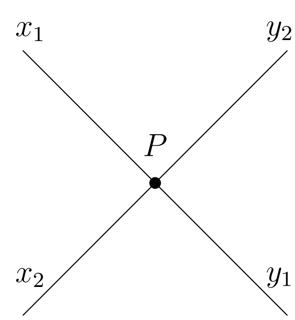


$$\begin{cases} (x_1x_2)^{\frac{m+1}{2}} = (x_2x_1)^{\frac{m+1}{2}}, \\ y_i = (x_2x_1)^{-k}x_i(x_1x_2)^k, \ i = 1, 2. \end{cases}$$

The relations given above come precisely from generalized relations when using the ∇ map from Chapter 2. This was alluded to in section 2.5 where an explanation of generalized relations from cusps was discussed. The discussion in that section is completely analogous to the ways in which we can generalize other $\mathbb{A}_m$ singularities.

Moreover one can see how the generalized relations of ordinary singularities come about by extrapolating from the example seen at the end of section 2.4 (i.e. full twists on more strands give rise to braid words that continue to add conjugations to the relation).

Example 1. First we look at a curve previously computed in section 2.4 using these ideas. Namely the curve C defined by $(y-(x+1))(y-(-x+1))=0$. Thus we have

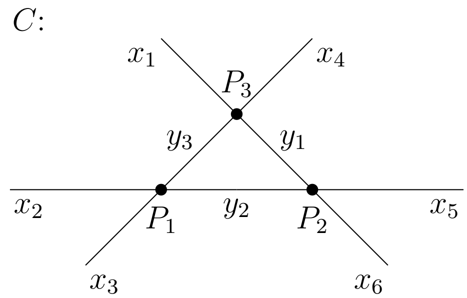


where edges are given by $\{x_1, x_2, y_1, y_2\}$ and relations are given by $[x_2, x_1] = 1$, $y_1 = x_1$, and $y_2 = x_2$. This reveals the presentation

$$\mathcal{P}_C = < x_1, x_2 \mid [x_2, x_1] = 1 >$$

which gives a group $G_C \cong \mathbb{Z} \times \mathbb{Z}$. This agrees with Zariski-Van Kampen.

Example 2. As a slightly more difficult example, consider a real arrangement of lines in $\mathbb{R}[x, y]$, i.e. the curve C defined by $y(y-(x+1))(y-(-x+1))=0$. In the figure below, notice that each relation we see is of the form (R1).



Now, we consider the "crossings" in an analogous way for when we computing the Wirtinger presentation of a knot complement. These crossings occur precisely at the nodes P_1, P_2, and P_3. We have to be cautious here, by noticing that (R1) gives us relations for a node of multiplicity m, but here $m = 2$, and these relations occur disjoint from one another at each point. In other words P_1: gives $\bar{x}_k = x_3x_2$, whereas P_3: gives $\bar{x}_k = x_1x_4$. We will now compute the relations given at each node.

$$P_1\colon \quad [\bar{x}_k, x_i] = [x_3x_2, x_i] = 1 \Rightarrow \begin{cases} 1 = [x_3x_2, x_2] & = x_3x_2x_2x_2^{-1}x_3^{-1}x_2^{-1} \\ & = x_3x_2x_3^{-1}x_2^{-1} = [x_3, x_2] = 1. \\ 1 = [x_3x_2, x_3] & = x_3x_2x_3x_2^{-1}x_3^{-1}x_3^{-1} \\ & = x_2x_3x_2^{-1}x_3^{-1} = [x_3, x_2] = 1. \end{cases}$$

$$P_2\colon \quad [\bar{x}_k, x_i] = [x_6x_5, x_i] = 1 \Rightarrow \begin{cases} 1 = [x_6x_5, x_5] & = x_6x_5x_5x_5^{-1}x_6^{-1}x_5^{-1} \\ & = x_6x_5x_6^{-1}x_5^{-1} = [x_6, x_5] = 1. \\ 1 = [x_6x_5, x_6] & = x_6x_5x_6x_5^{-1}x_6^{-1}x_6^{-1} \\ & = x_5x_6x_5^{-1}x_6^{-1} = [x_6, x_5] = 1. \end{cases}$$

$$P_3:\quad [\bar{x}_k, x_i] = [x_1x_4, x_i] = 1 \Rightarrow \begin{cases} 1 = [x_1x_4, x_1] & = x_1x_4x_1x_4^{-1}x_1^{-1}x_1^{-1} \\ & = x_4x_1x_4^{-1}x_1^{-1} = [x_1, x_4] = 1. \\ 1 = [x_1x_4, x_4] & = x_1x_4x_4x_4^{-1}x_1^{-1}x_4^{-1} \\ & = x_1x_4x_1^{-1}x_4^{-1} = [x_1, x_4] = 1. \end{cases}$$

Now notice that we also have $y_1 = x_1^{-1}x_1x_1 = x_1$, but moreover we have $y_1 = x_5^{-1}x_6^{-1}x_6x_6x_5 = x_5^{-1}x_6x_5$, but we have already seen that x_5 and x_6 commute, so in fact we see that $y_1 = x_1 = x_6$. Similarly $y_2 = x_2 = x_5$, and $y_3 = x_3 = x_4$. Putting this all together, the Wirtinger presentation of this curve then becomes

$$\mathcal{P}_C = < x_1, x_2, x_3 \mid [x_3, x_2] = 1, [x_1, x_2] = 1, [x_1, x_3] = 1 > .$$

The presentation says that we have 3 generators, all of which commute with each other, so in fact we have that the group G_C defined by $\mathcal{P}_C$ is isomorphic to $\mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$ (it should be noted that this agrees with the Zariski-Van Kampen method).

Given a curve C of Wirtinger type it is not too hard to compute ∇ and verify that the listed relations (R1) and (R2) do hold in $\pi_1(\mathbb{C}^2 - C)$. So if G_C is the group defined by the Wirtinger presentation $\mathcal{P}_C$, we have an epimorphism

$$G_C \to \pi_1(\mathbb{C}^2 - C).$$

It turns out that this is not always an isomorphism. However it is in case the curve C is of Wirtinger type and satisfies additional technical properties. We will discuss this in the next section.

3.3 Wirtinger Curves

Let C be a curve of Wirtinger type. We first define what it means for real branches to “face” a vertical line. Consider a vertical real line $L_{\mathbb{R}} \subseteq \mathbb{R}^2$ and a ramification point P of C not in $L_{\mathbb{R}}$. The vertical line through P cuts the real plane into two half-planes, one of them, call it H^+, containing $L_{\mathbb{R}}$. If H^+ contains real branches at P, then these branches are said to face $L_{\mathbb{R}}$. A curve C of Wirtinger type is called a *Wirtinger curve* if the following conditions hold:

(WC1) The real part of each irreducible component is connected.

(WC2) There exists a vertical complex line L satisfying (W3) and a closed topological disk (with piecewise smooth boundary) $B \subset \mathbb{R}^2$ such that:

(1) $B \cup C_{\mathbb{R}} \cup L_{\mathbb{R}}$ is simply connected.

(2) There is a parallel real plane $H_\epsilon = \mathbb{R} \times (\mathbb{R} + i\epsilon)$ to $\mathbb{R}^2$ with $\epsilon \neq 0$ such that

$$B_\epsilon \cap C = \emptyset, \text{ where } B_\epsilon = \{(x, y + i\epsilon) \in H_\epsilon \mid (x, y) \in B\} \subset H_\epsilon.$$

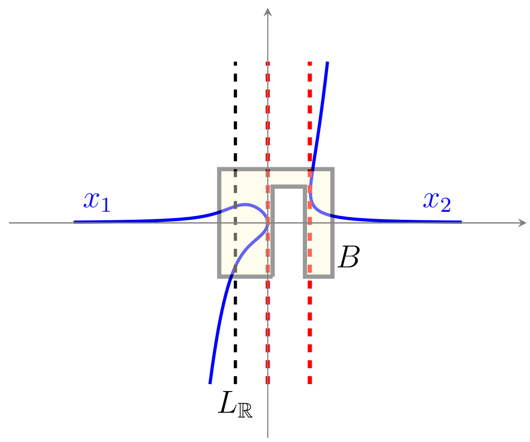
(3) All singularities of C face $L_{\mathbb{R}}$.

The following theorem was proven in [2]:

Theorem (Wirtinger Curve). Let C be a Wirtinger curve. Then $\mathcal{P}_C$ is a presentation of $\pi_1(\mathbb{C}^2 - C)$.

3.4 Examples

Example 1. Consider the curve $C : y^2 - x(yx^2 - 1) = 0$. The real part of C is shown below.

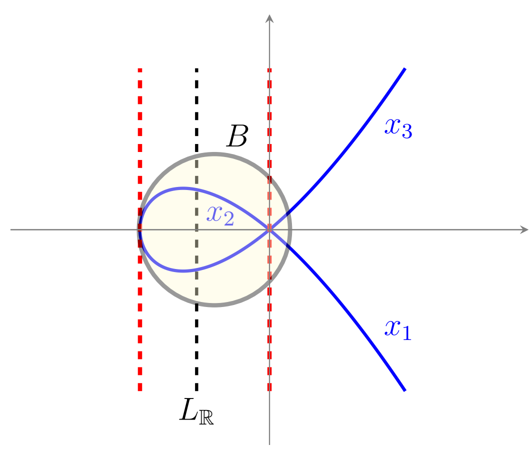


where the red dashed lines represent the vertical tangents to the curve, the other dashed line represents a generic vertical line, and B represents a topological disk satisfying (WC2). Notice that there are two connected components in this curve. C satisfies the conditions to be a curve of Wirtinger type, and the set of ramification points is $\{0, \sqrt[5]{4}\}$. Now, since the presentation $\mathcal{P}_C$ has generators in correspondence to our edges x_1 and x_2, and there are no relations given by vertices (since there are no vertices via a node or cusp, etc.), we find the presentation

$$\mathcal{P}_C = < x_1, x_2 \mid \emptyset > .$$

So we have that G_C gives the free group of rank 2 (corresponding to the number of connected components), but in fact we have by Zariski-Van Kampen that $\pi_1(\mathbb{C}^2 - C) = < x_1, x_2 \mid x_1 = x_2 > \cong \mathbb{Z}$. This example illustrates the need for the conditions put on a curve of Wirtinger type to make it a Wirtinger curve. Observe that when considering the left most red dashed line along with our generic vertical line $L_{\mathbb{R}}$, we see that the real local branches (of the red line) do not "face" $L_{\mathbb{R}}$ in the sense described. Thus this is a curve of Wirtinger type but we have a problem with "facing", and morover (WC1) also fails and so the conclusion of theorem does not hold. These extra conditions are all in an effort to ensure that the presentation given by $\mathcal{P}_C$ does indeed define $\pi_1(\mathbb{C}^2 - C)$.

Example 2. Next we observe an example that does satisfy all new conditions, namely $C : y^2 - x^2(x+1) = 0$ which can be seen below

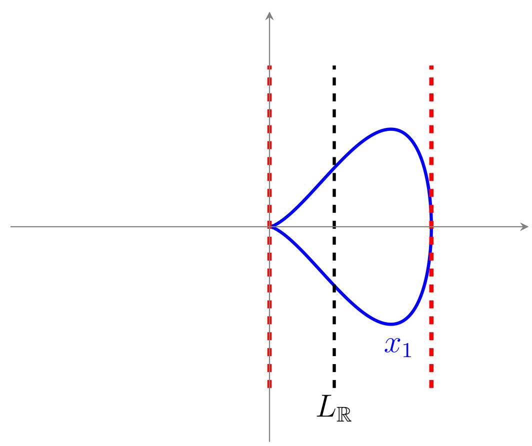


where again the red dashed lines on the curve represent a vertical tangent to the curve, and a node on the curve. The other dashed line represents a generic vertical line, and B is a topological disk that satisfies (WC2). Moreover, all singularities face $L_{\mathbb{R}}$, and for a parallel real plane H_ϵ for $\epsilon > 0$ we have that $B_\epsilon \cap C = \emptyset$ since the ramification points are real, and the local branches near these points are real. The set of ramification points is $\{-1, 0\}$. Now given the real diagram of the curve, we find the following relations

$$[x_2^2, x_2] = 1 \qquad x_3 = x_2, \qquad x_1 = x_2$$

which gives us $\mathcal{P}_C = < x_1 \mid \emptyset >$ which defines a group $G_C \cong \mathbb{Z}$. It should be noted that this agrees with Zariski-Van Kampen, where in that method we get a commutator from the node in the center, but it is trivialized by the vertical tangent to the curve.

Example 3. Now consider the curve C defined by $y^2 + x^4 - x^3 = 0$. This curve is known as the pear-shaped quartic, and its real diagram can be seen below.

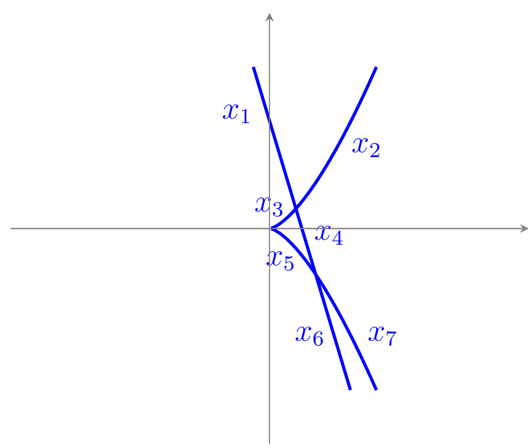


It can be checked that this curve satisfies the conditions to be a curve of Wirtinger type. Ramification points at the cusp and the vertical tangent are real, and moreover the local branches at the cusp, namely $y - \sqrt{x^3 - x^4} = 0$ and $y + \sqrt{x^3 - x^4} = 0$ define real equations. The black dotted line above is a line satisfying (WC2), and a topological disk could be chosen big enough to cover both the cusp and the vertical tangency. The ramification points face each other, so condition (3) of (WC2) is also satisfied. In [2], the second condition of (WC2) is generalized for these types of singularities, and so this is also satisfied (so long as our topological disk is only as large as it must be to satisfy (1) of (WC2)). Given this, a presentation of the fundamental group of the complement to this curve can be given by

$$\mathcal{P}_C = < x_1 \mid \emptyset >$$

which as we have seen before, defines a group $G_C \cong \mathbb{Z}$. Notice that we could have called each branch of the cusp an edge, and had generating edges x_1 and x_2. This would have led to relations given by the cusp $x_1x_2x_1 = x_2x_1x_2$, but edges x_1 and x_2 do meet each other on the curve (not at a vertex) and so we see the relation $x_1 = x_2$ as well.

Example 4. As another example, consider the curve C defined by the following polynomial $(y^2 - x^3)(y + 4x - 4) = 0$ seen below



It can be quickly checked that this is a Wirtinger curve by using precisely the same arguments as in the previous examples. This curve has a cusp, two nodes that are visible, and a third node. When choosing a topological disk B we take care to include the node on the curve that is not shown in the graph (where edges x_6 and x_7 meet again). A line $L_{\mathbb{R}}$ can be chosen between the cusp and the first node. Now using both (R1) and (R2) we see that relations given by generators $x_1, \ldots, x_7$ are

$$[x_1, x_3] = 1 \qquad x_1 = x_4 = x_6$$

$$x_3x_5x_3 = x_5x_3x_5 \qquad [x_4, x_5] = 1 \qquad x_2 = x_3$$

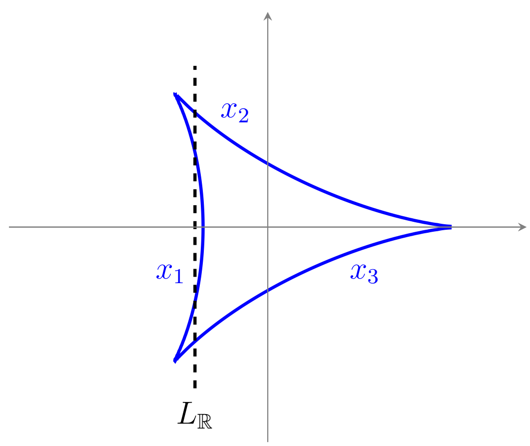
$$[x_6, x_7] = 1 \qquad x_5 = x_7.$$

Putting all this together, removing unnecessary generators and superfluous relations, we see the following presentation

$$\mathcal{P}_C = < x_1, x_2, x_5 \mid x_2x_5x_2 = x_5x_2x_5, [x_1, x_2] = 1, [x_1, x_5] = 1 > .$$

This gives rise to a group $G_C \cong B_3 \times \mathbb{Z}$.

Example 5. As a final example, consider the deltoid, which has a real equation defined by $C = \{(x, y) \in \mathbb{C}^2 \mid (x^2 + y^2)^2 + 18a^2(x^2 + y^2) - 27a^4 - 8a(x^3 - 3xy^2)\}$ with $a = 2$:



where the diagram of this is precisely the figure we see. A topological disk that covers all cusps (and the interior) on the diagram will suffice. We have three distinct edges x_1, x_2, x_3, and three vertices located at the cusps in the diagram. Thus by the relations (R2) we find the following:

$$x_1x_2x_1 = x_2x_1x_2, \qquad x_2x_3x_2 = x_3x_2x_3, \qquad x_3x_1x_3 = x_1x_3x_1$$

and so our presentation is given by

$$G_C = < x_1, x_2, x_3 \mid x_1x_2x_1 = x_2x_1x_2, x_2x_3x_2 = x_3x_2x_3, x_3x_1x_3 = x_1x_3x_1 >$$

which is known to be the Artin group of the triangle and coincides with $\pi_1(\mathbb{C}^2 - C)$.

CHAPTER 4

REAL ARRANGEMENTS OF LINES AND ASPHERICITY

Let $f(x,y) = (y - a_1(x)) \cdots (y - a_r(x))$, where each $a_i(x) \in \mathbb{R}[x]$ has degree 1. Thus, the complex curve $C \subseteq \mathbb{C}^2$ is a collection of complex lines ℓ_i defined by $y = a_i(x)$: $C = \bigcup_{i=1}^{r} \ell_i$. We assume that there are no vertical lines in this collection. In the following we will refer to the curve C as a *real line arrangement*. First we check that these arrangements are indeed a Wirtinger curve.

(W1) It will suffice to show that any pair of lines that intersect, do so at a real point $(x,y) \in \mathbb{R}^2$. Let ℓ_j and ℓ_k be two distinct lines in our arrangement C. These lines are defined by $y = a_j(x)$ and $y = a_k(x)$ respectively. These lines intersect precisely when $a_j(x) = a_k(x)$. Then we have that

$$\begin{aligned} m_j x + b_j &= m_k x + b_k \\ (m_j - m_k)x &= b_k - b_j \\ x &= \frac{b_k - b_j}{m_j - m_k} \end{aligned}$$

but $m_j, m_k, b_{,j}, b_k \in \mathbb{R}$ which implies that $x \in \mathbb{R}$ which implies that $y \in \mathbb{R}$. Thus a point of intersection (x,y) between a pair of arbitrary lines in this arragement is in $\mathbb{R}^2$. Note that if $m_j = m_k$ these lines would be parallel.

(W2) Local branches around any point P given by an intersection of a pair of lines are certainly real as each branch is given precisely by our irreducible factors, namely $y - a_j(x)$ and $y - a_k(x)$ for an intersection between these two factors. Each of these branches is defined over the reals. Notice that we only consider pairs, but allow points of higher multiplicity, i.e. > 2 lines intersecting at a point. We can check all pairs of lines in this setting using this process and come to the same conclusions.

(W3) Any vertical fiber $p^{-1}(x_0)$ for $x_0 \in \mathbb{R}$ that intersects C transversally (i.e. not a fiber of a ramification point) does so with maximal cardinality, in other words $p^{-1}(x_0)$ intersects r-many lines in C.

(W4) The line arrangement is stipulated to not contain vertical lines, so this is trivially satisfied.

(W5) All singularities of C are points of multiplicity and so they are ordinary.

This curve is of Wirtinger type. We now ensure it satisfies the further conditions as well so we can apply the Wirtinger curve theorem.

(WC1) The real part of each irreducible componenent, which is each $y - a_i(x)$, is connected since these lines are defined over the reals.

(WC2) Let there be a fiber satisfying (W3) and a closed topological disk $B \subset \mathbb{R}^2$. Then

(1) Given $\epsilon > 0$, some sufficiently large $r > 0$, and a point $(a, b) \in \mathbb{R}^2$, we may let $B = \{(x, y) \in \mathbb{R}^2 \mid (x-a)^2 + (y-b)^2 \leq R\}$ which is a topological disk

centered at (a, b) with radius $R = r + \epsilon$ sufficiently large. By sufficiently large we mean that it covers all points of multiplicity of C, as well as an ϵ worth of room past the ramification points to allow for a fiber satisfying (W3). Given this B, we then have that $B \cup C_{\mathbb{R}} \cup L_{\mathbb{R}}$ is simply connected, or in other words that it has trivial fundamental group.

(2) Let there be a real parallel plane $H_\epsilon = \mathbb{R} \times (\mathbb{R} + i\epsilon)$ to $\mathbb{R}^2$ for $\epsilon \neq 0$. Let $B_\epsilon = \{(x, y + i) \in H_\epsilon \mid ((x, y) \in B\} \subset H_\epsilon$. Now each irreducible component $y - a_k(x)$ gives rise to points that look like $(x, a_k(x))$, which notably do not look like $(x, a_k(x) + i\epsilon)$. In other words $y = a_k(x) + i\epsilon$ has no solution for $a_k(x) \in \mathbb{R}$. Thus $B_\epsilon \cap C = \emptyset$ as required.

(3) Since there are real branches on either side of a node created by intersecting lines, trivially all singularites "face" $L_{\mathbb{R}}$.

Thus an arrangement of real lines satisfies the conditions to be a Wirtinger Curve. Therefore $\mathcal{P}_C$ is a presentation of $\pi_1(\mathbb{C}^2 - C)$. Now, as was seen in the second example in section 3.2, we found that $G_C = \mathbb{Z}^3$, the free abelian group of rank 3. Consider a maximal intersection arrangement, which is taken to mean that any pair of lines intersect, and no three lines have an intersection. Note that for each line that is added, you pick up as many vertices as lines you had previously, i.e. a maximal intersection arrangement of three lines has three vertices, adding a line nets three more vertices, and so now the arrangment has four lines, and six vertices. By induction we see that if n is the number of lines, then $\frac{n(n-1)}{2}$ is the number of vertices.

Something important to note here is the following: around each node, which is the only kind of singularity that appears in this arrangement, there are four edges. In this setting, given the relations (R1), each edge is equal to the edge that is on the other side of the node (which is the same line). Notice that this was demonstrated in the example mentioned in the previous paragraph. Since each node in this arrangment is an isolated occurance, we may say that a line which has many nodes occurring on it has edges that are all equal. This is made visually apparent below:

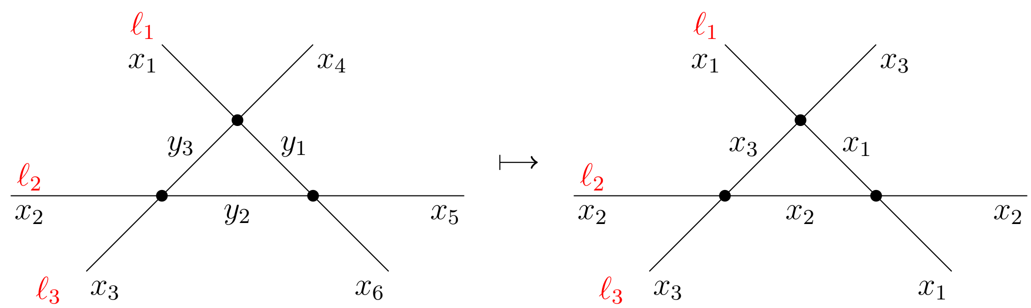


Figure 4.1: Redundant generators

So, if we have a general arrangment with r lines, after relabeling we have r distinct edges, and $\frac{r(r-1)}{2}$ vertices. Now, each of these vertices give a commutator between a pair of two lines, and since this is a general arrangement, commutators between all lines are realized. So a presentation is given by the following

$$\mathcal{P}_C = < x_1, \ldots, x_r \mid [x_i, x_j] = 1 \text{ for } x_i \neq x_j >$$

where $x_1, \ldots, x_r$ are given by r-many lines (as edges), and $\frac{r(r-1)}{2}$-many commutators are realized as relations. Since C is a Wirtinger curve we have

$$\mathbb{Z}^r = G_C = \pi_1(\mathbb{C}^2 - C).$$

Next consider a curve $C = \bigcup_{i=1}^{r} \ell_i$ that forms a central arrangement, i.e. all lines intersect at a point. Again we avoid vertical lines in this arrangment. This curve contains a single point of multiplicity r, and so it suffices to appeal to (R1) only one time. If we label the edges given by the lines on either side of the singularity, then we see the following figure

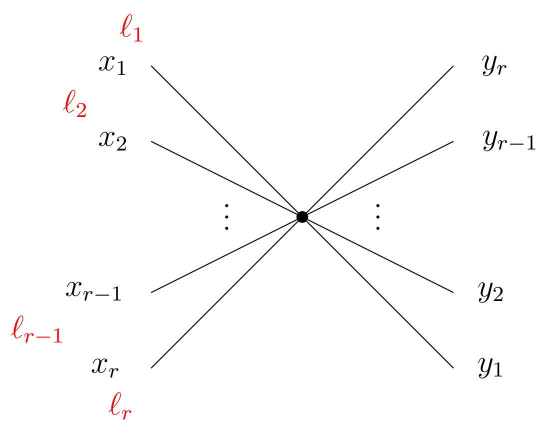


Figure 4.2: A centered arrangement

Note that in terms of the braid monodromy, we consider a full twist on r strands. If r is odd then the middle strand $\lceil \frac{r}{2} \rceil$ stays fixed up to conjugation of other strands (halfway through the braid word). If r is even no strand stays fixed. Viewed as a Wirtinger curve, this should be thought of as edges $y_i = x_i$ up to conjugation of the surrounding strands (as the relations (R1) alludes to). These relations would give a presentation of the form

$$\mathcal{P}_C = < x_1, \ldots, x_r, y_1, \ldots y_r \mid [\bar{x}_r, x_i] = 1, y_i = \bar{x}_i^{-1} x_i \bar{x}_i \text{ for } 1 \leq i \leq r >$$

however the y_i's are all words in terms of the x's and so they may be removed, in other words we may rewrite this presentation as

$$\mathcal{P}_C = < x_1, \ldots, x_r \mid [\bar{x}_r, x_i] = 1 \text{ for } 1 \leq i \leq r > .$$

Now as was done in the example right after the statement of the Zariski-Van Kampen theorem, we may make another adjustment: we can let $x_r x_{r-1} \cdots x_1 = c$. But what is c? We can think of this "full word" in the generators as a based loop that is the concatenation of r many loops surrounding r punctures in the plane. Then c is just the single "big loop" that surrounds all punctures at once. This gives that $x_r = c x_1^{-1} \cdots x_{r-1}^{-1}$, thus our presentation becomes

$$\mathcal{P}_C = < x_1, \ldots, x_{r-1}, c \mid [c, x_i] = 1 \text{ for } 1 \leq i \leq r-1 > .$$

Now the presentation makes it clear that this group G_C is isomorphic to $F_{r-1} \times \mathbb{Z}$, as c commutes with all generators, while $x_1, \ldots x_{r-1}$ have no relations imposed on them. Thus we have

$$F_{r-1} \times \mathbb{Z} = G_C = \pi_1(\mathbb{C}^2 - C).$$

There is a theorem first asserted by Zariski which says the following:

Theorem (Zariski). If C is a plane curve with only ordinary node singularities, then $\pi_1(\mathbb{C}^2 - C)$ is abelian.

Unfortunately Zariski's original proof of this fact relied on someone else's theorem that turned out to be incomplete. This was later proved by the work of Fulton and Deligne. This relates to the example of a maximal intersection arrangement, which indeed we found to be abelian.

4.1 Conclusions and Further Work

A space X is aspherical if $\pi_i(X) = 0$ for $i \geq 2$. The homotopy type of an aspherical space is determined by its fundamental group [4]. The construction of a suitable aspherical space with pre-described fundamental group is a basic problem in the study of infinite groups. It is known that knot complements $S^3 - k$ are aspherical, and one may wonder if this extends to curve complements $\mathbb{C}^2 - C$. In general the answer is "no". We have seen that if C is a real line arrangement of r lines, no two being parallel, and at every intersection point only 2 lines intersect, then $\pi_1(\mathbb{C}^2 - C) = \mathbb{Z}^r$; if $r \geq 5$ this group can not occur as the fundamental group of an aspherical space of dimension 4 or less, since $H_5(\mathbb{Z}^5) = \mathbb{Z}$ (i.e. not trivial) [4]. Thus $\mathbb{C}^2 - C$ is not aspherical.

A classification of aspherical real line arrangements (or even Wirtinger curves) does not seem to be out of reach and could be pursued in future work. Here are some ideas. Given a graph Γ, the right angled Artin Group (RAAG) $G(\Gamma)$ is defined in the following way: generators are the vertices, and two vertices commute if they are connected by an edge.

Question 1. Given a real line arrangement is $\pi_1(\mathbb{C}^2 - C)$ a RAAG?

This is true if at each intersection point only 2 lines intersect. However if more lines are involved at intersection points the relations are not just commutators and come from a more complicated braiding. One group we encountered in the case of a central arrangement of r lines is $F_{r-1} \times \mathbb{Z}$. But note that this group is a RAAG. The graph Γ for this group is a tree with one central vertex of valency $r-1$, and $r-1$ edges.

Question 2. Given a real line arrangement C so that at intersection points only two lines intersect and C does not contain triangles. Is $\pi_1(\mathbb{C}^2 - C)$ a RAAG (most likely "yes") and is $\mathbb{C}^2 - C$ is aspherical (most likely also "yes")?

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