EXAMINING THE INFLUENCE OF NUMBER LINE FRACTION TASK CHARACTERISTICS ON STUDENT WORK

by

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DEDICATION

To JaiseAnn Maie, you rewrote my dreams and made them come true all at the same time.

To Zach, thank you for your unending support. This accomplishment is ours. Because of you, I have truly been able to “have it all,” in the very way that I want. LYARL!!!

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ABSTRACT

Research supports the claim that classrooms with teachers who respond to student thinking in the mathematics classroom will see greater student gains and student success (Lamon, 1996; Sleep & Boerst, 2012). The topic of fractions is both difficult to teach and learn, but has important implications on future success in mathematics and in life. This study set out to explore the ways in which student work is influenced by characteristics of number line fraction tasks. By examining task type, number line structure, and number choice this study shares the way these task characteristics influenced student strategies. The relationship between task characteristic and student work is examined qualitatively; in addition to how well each task characteristic uncovers three key conceptual understanding fraction ideas: partitioning, iterating, and unitizing. Additionally, this study looks at which task characteristics better highlight informal or intuitive understanding of these key ideas and in what ways. The findings can be used to inform the selection of fraction tasks for the classroom.

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LIST OF ABBREVIATIONS

CCSS	Common Core State Standards
DOK	Depth of Knowledge

CHAPTER ONE: BACKGROUND AND PURPOSE

Comprehension of and competence with fractions is a crucial mathematical understanding (Siegler, Thompson, & Schneider, 2011; Wilkerson et al., 2015). Student performance with fractions is a predictor for future success in other areas of mathematics (Bailey, Hoard, Nugent, & Geary, 2012; Fuchs et al., 2013) and even success in adulthood (Siegler & Pyke, 2012a; Vukovic et al., 2014). Yang, Reys, & Wu (2010) described fraction knowledge as foundational mathematical knowledge.

Fraction knowledge is difficult for students to acquire (Behr, Lesh, Post, & Silver, 1983; Lamon, 1996) as it requires a shift from whole number thinking to rational number understanding (Ni & Zhou, 2005). Additionally, there are different contexts through which fractions are taught and most common textbooks only focus on one or two areas of rational number (Boesen, Lithner, & Palm, 2010), leaving other contexts that may help students develop more thorough understandings of rational number behind (Behr et al., 1983; Behr & Post, 1988). Despite students having informal understandings of rational number before kindergarten (Brizuela, 2006), many students struggle with concepts of rational number throughout their elementary and secondary years (Yang et al., 2010).

Fraction knowledge is difficult to teach (Behr, Wachsmuth, & Post, 1985; Ni & Zhou, 2005; Pitta-Pantazi, Gray, & Christou, 2004; Siegler & Pyke, 2012a; Zaslavsky, 2005) and is an area where many teachers struggle (Behr et al., 1985; Vukovic et al., 2014; Wong, 2013). Most teachers have seen there are a number of areas where students can become confused when learning about fractions due to different misconceptions and varying levels of prior knowledge (Ni & Zhou, 2005). The understanding of each student differs from one to the other. How can teachers inform their pedagogical approaches when teaching fractions to meet the individual needs of each student?

Mathematical tasks are defined in this study as a paper and pencil assessment given to students to solve. Mathematical tasks are one influencing factor used for informing teachers' instructional practices. Teachers use tasks to identify student informal and formal understanding of mathematical concepts (Boesen et al., 2010). The use of mathematical tasks in the classroom is important and the selection of tasks even more so (Mitchell & Clarke, 2010).

Mitchell and Clarke (2010) called for more information regarding student responses to fraction tasks, "In order to continue to refine tasks that may prove useful in establishing fraction growth points in later studies, close attention must be paid to the children's explanations of their answers to tasks." (p.372) Understanding student work and strategies around rational number will give more information regarding building and using fraction tasks in the most useful way in the classroom.

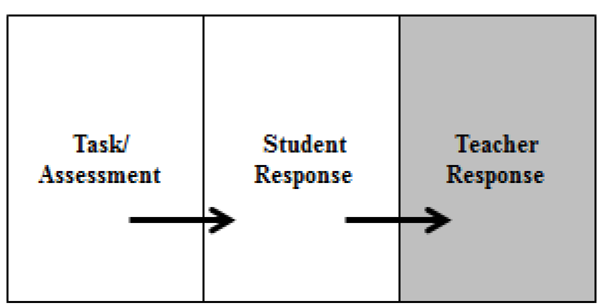


Figure 1.1 Theoretical Frameworks for Study

With the understanding of the influential role of tasks in the mathematics classroom, the purpose of this study is to uncover more information about fraction tasks and student responses to different tasks. The theoretical framework for this study (as shown in figure 1.1) displays the parts of the relationships between task, student response, and teacher response. This study will be studying the relationship between task and student response. This study will not study teacher responses.

Responding to Student Thinking

Mathematical tasks are one way teachers can address or respond to student thinking. In order to teach effectively, it is imperative that teachers engage students' prior knowledge and use their informal and formal understanding to help them make sense of new learning (Lamon, 1996; Sleep & Boerst, 2012). Tasks can be used to help teachers uncover and address student thinking.

Making formative assessment part of regular teaching practices is one way teachers can acquire this knowledge about their students' thinking. The implementation of formative assessment in the classroom will enhance student mathematical understanding and give teachers more insight into the best ways to address each student's mathematical understanding (Heritage & Niemi, 2006; Wong, 2013).

The Role of Tasks in Teaching and Learning Fractions

One way teachers gain insight into student understandings is through mathematical tasks. As Sidevenall, Lithner, and Jader (2015) stated, "Tasks are a cornerstone of students' work with mathematics." (p.533) Not all mathematical tasks will highlight student thinking or understanding well (Wing & Beal, 2004). For example, students may be given a task that asks them to name the amount of fourths it takes to make a unit of one. Students may be able to answer the question as 4 $^{1}/_{4}$ths, but may not be able to demonstrate how to create a fourth correctly or how to model 4 $^{1}/_{4}$ths. Tasks that uncover key understandings of fractions would be of greater use in the mathematics classroom (Simon, 2006).

Mathematical task characteristics influence student work or strategies in different ways. Boesen et al., (2010) found students confronted with procedural mathematical tasks (similar to those found in common textbooks) did not use new reasoning to solve problems. Similarly, Cwikla (2014) found most students have informal understandings of fractions and are unable to connect with the formal language and symbols used when learning and exploring fraction ideas. Fraction tasks that rely heavily on language or symbols may display a lack of understanding. Teachers may be left without a clear understanding of what their students know due to the informal understanding that may be hindered by the symbols and language used in a task.

Different kinds of tasks will promote different levels of student reasoning (Boesen et al., 2010; Pitta-Pantazi et al., 2004; Sullivan, Clarke, & Clarke, 2009; Sullivan, Warren, & White, 2000). Depending on the type of problem, students will use different strategies or reasoning to solve it (Hunting, 1999; Sidenvall et al., 2015). A study conducted by Brinker (1997) found students seemed to have more success in solving problems that prompted them to rely on informal strategies. Where one strategy a student uses may highlight conceptual or procedural knowledge, another strategy may not (Lamon, 1996).

Furthermore, different models appear to appeal to different ways of student thinking-each model with its own strengths and weaknesses (Bruner, 1966; Heron, 2014).

A model is a representation used to help students make sense of conceptual knowledge and can be displayed through hands-on methods as well as through pictorial or symbolic representations (Bruner, 1966). Hannula (2003) found when students completed two fraction tasks with two linear models (a bar model and a number line) their results were not consistent. Lamon (1996) found that student partitioning strategies were situationally specific. That is, not all tasks reveal what students know about fractions in the same way or to the same level or degree.

Statement

The purpose of this study is to identify fraction task characteristics that will help teachers better understand specific conceptual understandings students have about fractions.

Research Questions

What task characteristics influence student work or thinking around fractions and in what ways?

What task characteristics highlight informal or intuitive understanding?

Significance

"Mathematical tasks are important for teaching, and the nature of student learning is determined by the type of task and the way it is used." (Sullivan et al., 2009) Teachers rely heavily on textbooks for mathematical tasks (Hodges, Cady, & Collins, 2008). Many mainstream textbooks contain procedural tasks, many with algorithms and procedures explained within the task itself. These tasks reveal very little to teachers about what students know about fractions (Cady, Hodges, & Collins, 2015; Niemi, 1996).

With such importance placed on fraction understanding, tasks that help teachers address student misconceptions and build on informal knowledge will help teachers address student needs and help students succeed in their acquisition of fraction knowledge. Boesen, Lithner, and Palm (2010) declared a need for more information about the task characteristics used in mathematical tasks and the student reasoning required.

Some tasks show students arriving at correct or incorrect answers, but do not uncover what students do and do not know. This could be detrimental in a classroom where a teacher relies heavily (or solely) on these tasks and the corresponding responses. Well-chosen tasks can prompt explanations from students that lend insight into different aspects of their fraction knowledge (Mitchell & Clarke, 2010).

Some fraction tasks are easily solved by students because they do not challenge the way students think about fractions compared to the way they think about whole numbers. Some tasks (particularly part-whole tasks) allow students to still work with fractions using whole number thinking (Freeman & Jorgensen, 2015; Niemi, 1996).

Long-Term Impact

Stafylidou and Vosnidou (2004) found student interpretations of fractions shown in models revealed their misconceptions. Tasks that highlight student misconceptions will provide the students with an opportunity to have those misconceptions addressed by their teachers. Addressing student misconceptions and strengthening their conceptual understanding will impact their procedural understanding in a positive way (Niemi, 1996). Students with both strong procedural and conceptual understanding will be more likely to observe long-term success in mathematics (Siegler et al., 2011).

A better understanding of how student thinking and student work around fractions is influenced by task characteristics may also lead to the adoption of better tasks among teachers and curriculum developers. The use of tasks in the classroom that uncover conceptual student knowledge, would lead to the following claims:

Improving the Classroom Environment

Students with limited fraction understanding may require interventions or differentiated instruction. The intervention that is given to a student needs to be appropriate in terms of what part of the conceptual understanding they are lacking (Vukovic et al., 2014). Each student will have different misconceptions that need to be addressed as well as differing informal or formal understanding which teachers can build on. One way to increase students' likelihood for success is to foster a classroom where student needs are being addressed individually.

Tasks highlighting student understanding will provide students and teachers more opportunities to discuss the structure of the mathematics (Niemi, 1996). Having discussions regarding the tasks will improve the quality of discourse in the classroom surrounding the concept of rational number (Cramer, Post, & delMas, 2015). This improved discourse will assist student learning as they begin to understand the unique language of fractions.

Discovering tasks that highlight student thinking regarding fractions can and should also serve as formative assessment. The use of formative assessment may lead to greater success in the classroom (Heritage & Niemi, 2006). The National Council for Teaching Mathematics (2000) emphasized the importance of teacher analysis of student representation as a means of gaining insight into student thinking.

This section covered the role tasks play in the mathematics classroom—specifically in teaching fractions. Tasks play an important role and, as Simon (2006) declared, help determine student key understanding through careful observation of student work. The careful selection of tasks can positively impact student learning in the classroom. More information regarding student strategies on fraction tasks is needed (Mitchell & Clarke, 2010) in order to make more informed decisions when selecting fraction tasks.

Limitations

There are limitations to this study. The research regarding fraction tasks, student understanding of fractions, and the instruction of fractions highlight two very important influences of fraction understanding that extend beyond the type of fraction task being used and outside of the scope of this research. This study will not attempt to control for the following:

Student Background and Prior Knowledge

Students that do not have sufficient background knowledge have limited solution strategies (Hodges et al., 2008). Lamon (1996) found student thinking about rational number is heavily based on social practices and norms. Student performance with fractions can be predicted by student competence with the four operations of whole numbers as well as measurement (Behr et al., 1985; Pearn, 2007). Students with higher representational knowledge have been found to do better on measures of fraction understanding (Heritage & Niemi, 2006). There are many influences that impact the background knowledge that students bring to a classroom prior to learning about rational numbers.

Instructional Style and Teacher Content Knowledge

Teacher content knowledge impacts the classroom environment (Mueller & Maher, 2009; Zaslavsky, 2005). Students in a classroom with a heavy focus on procedural mathematical tasks may have difficulty or lack confidence when encountering tasks that press them conceptually (Hecht & Vagi, 2011). The structure of the classroom may impact a student's ability to complete a task or to comfortably attempt solving tasks without formal mathematical language or use of symbols.

Additionally, teacher content knowledge is necessary if conceptual understanding tasks are to be used in a way that impacts student understanding in a positive way. Teachers should understand the various student models that may arise in student work prior to presenting a task to students (Lamon, 2007). Teachers must first have a conceptual understanding of the task in order to determine what a student knows when attempting to solve a task (Sullivan et al., 2009). Teacher content knowledge will impact the classroom prior to this study and will also determine the value of the mathematical tasks this study uncovers.

Assumptions

This study assumes the tasks given to students are designed to elicit conceptual understanding in students. This study assumes the students completing these tasks are putting forth their best effort. This study also assumes the influencing factors on student work pulled from the sample will also influence the student work of other students of a similar population.

Definitions

a. Unitizing

Unitizing is the ability to name and work with an object or group of objects as a unit or whole ("unit of one"). Lamon (1996) describes unitizing as a "Cognitive process for conceptualizing the amount of a given commodity or share before, during, and after the process." (p. 171). It requires the separation of the unit or whole into equal parts. Unitizing is the ability for a student to answer the question of "how much" after partitioning or iterating an object.

b. Partitioning

Partitioning is an operation that generates quantity (Lamon, 1996). It requires the separation of the unit or whole into equal parts.

c. Iterating

Iterating is repeating a value or typically a unit-fraction in order to produce identical copies of it. Iterating a value can also look like repeating the same distance on a number line with a fractional value—repeated line segments with a value of $^1/_4$ for example. Iterating has a different meaning in mathematics when used regarding fractions compared to other places in math.

d. Model

Bruner (1966) discussed 3 different modes of representation. He theorized that each mode of representation was a way to store and keep new knowledge in memory. The three modes are briefly described below:

- Enactive-This describes hands-on modeling of mathematical concepts or understanding. Students actually act or work out the problems with manipulatives. With fractions this may include fraction bars or fraction strips.

- Iconic-This describes using visual drawings or images to convey understanding. This can be referred to as the "pictorial" stage. With fractions this may include bar models, number line models, and different part-whole or set models drawn out.
- Symbolic-This describes the "abstract" stage where students are able to use sophisticated or formal mathematical symbols or language to describe what is happening. With fractions this may include written equations, comparisons, and the use of mathematical language when describing fractional relationships such as the relationship between the numerator and denominator.

For this study, iconic representations are primarily used and discussed when referring to "models."

e. Unit Fraction

f. The Common Core State Standards (2010) require students to understand the concept of unit fractions by the end of 3rd grade. A unit fraction is a fraction with a numerator of one, which becomes the unit students can then count by or iterate to develop a new fraction (Strother, Brendefur, Thiede, & Appleton, 2016). Students can also see that the denominator indicates how many unit fractions are required in order to create or make a unit of one. $^1/_3$ is an example of a unit fraction. As students develop an understanding of the unit $^1/_3$ they can then begin to see $^2/_3$ as two copies (or iterations) of $^1/_3$.

CHAPTER TWO: REVIEW OF LITERATURE

In order to design and administer the fraction tasks for this study, research regarding student thinking around fractions and fraction tasks characteristics were examined. The research regarding student thinking around fractions was used to build a task framework and also to contribute to the methodology of categorizing student work. The research regarding task characteristics was used to narrow the task framework and to then create the assessment framework from which the tasks for this study were created.

Foundational Fraction Understandings

The following are three key understandings that have emerged from this review of literature: the student ability to unitize and recognize the unit (Clarke & Roche, 2009; Cramer et al., 2015; Gabriel et al., 2012) and partitioning, and iterating (Brizuela, 2006; Hunting, 1999; Lamon, 1996). Unitizing, partitioning, and iterating are foundational to student understanding of fractions. These understandings may manifest in informal or formal strategies, but are the building blocks for a solid understanding of fraction and rational number. Research also suggests that students demonstrate these key ideas in different ways (Lamon, 1996). Recognizing the way students approach tasks that highlight this foundational knowledge is central to creating fraction tasks that highlight important conceptual understandings of fractions.

Of course there are many more concepts that could be deemed foundational than the three reviewed in this chapter. The concepts of equivalence (Taube, 1997; Vance,

1993) and measurement understanding (Fuchs et al., 2013; Geary et al., 2008) are just a few additional foundational concepts that emerged in this review of literature. For the purpose of this study the foundational components of unitizing, partitioning, and iterating are the primary focus in building a task framework.

Unitizing

Unit is the basis for the construction of fraction ideas (Taube, 1997) In order for students to work with, model, and truly understand rational number, they must be able to identify the unit of one (Heron, 2014; Stafylidou & Vosniadou, 2004; Strother et al., 2016). When a child is able to identify the unit instead of look for the number of parts, it demonstrates potential conceptual understanding (Lamon, 1999).

In order to press students conceptually, providing them with opportunities to reunitize will highlight deeper levels of thinking and understanding (Lamon, 1996). An ability to reunitize would manifest in a student who is able to identify the unit as the new unit of one change. For example, a student may look at a bar model split into ten equivalent pieces. Those pieces could each represent a value of one if the bar is equal to ten, seen as ten miles for example. A teacher may then ask students to rename the value of each piece if the bar is now equivalent to one—now seen as one mile. Each equally partitioned piece is now equivalent to $^{1}/_{10}$. Student understanding of unit is a foundation of fractions. Understanding the way students work with units and the ability to recognize the unit will tell a teacher much about what a student knows or needs—specifically if this foundational knowledge even exists.

Partitioning and Iterating

Partitioning by young children was identified by Hunting (1999) as pre-fraction knowledge. Partitioning and iterating are strategies students likely have a great deal of informal understanding about due to socialized ideas about fair-sharing (Brizuela, 2006). Partitioning relies on intuitive knowledge (Charles & Nason, 2001) Observing the way that students partition or identifying their ability to partition, can divulge a lot about their mathematical understanding.

Simply put, partitioning is taking any amount and splitting or decomposing that amount. Equipartitioning is partitioning the whole into equivalent pieces (Brizuela, 2006). A student may partition a number line to show $^3/_4$ with or without equi-partitioning. A student who does not equipartition may make the correct number of partitions needed—cutting a number line into 4 parts, for example. Each partition in the number line, however, would not be equivalent in size even though each partition is meant to represent $^1/_4$.

Economy in partitioning, as Lamon (1996) calls it, is also telling. A student may partition the line into halves and then only partition the second half in half again to make fourths. The way that student decomposed the whole would be as $^1/_2 + ^1/_4 + ^1/_4$. A student may also partition the line into fourths and count each fourth to appropriately represent fourths. The way that student decomposed the whole would be as $^1/_4 + ^1/_4 + ^1/_4 + ^1/_4$. Each idea is accurate, but the approaches are different. The approach a student uses uncovers a great deal about what informal and/or formal understandings that student may hold. Partitioning is another foundational mathematical understanding—requiring mathematical tasks that will allow students to display their ability to partition as well as their approaches to partitioning.

Students partition in different ways. Some students might use what Davis and Pitkethly (1990) termed a “dealing procedure” where students distribute pieces equally among people. This strategy is often employed in students working with discrete objects (Hunting, 1999). When working with continuous items, students might be more likely to use what Confrey (1987) called “splitting” where students partition items into equal shares using a systematic strategies.

Even with the use of these two strategies, teachers can learn a lot from the way a student might “deal” when partitioning. Hunting (1999) discussed the different ways students partition and distribute: one to one, many to one, combinations of the previous two strategies, non-systematically, or by trial and error.

Student partitioning strategies tend to follow a developmental path where they start with halving, and then double those to find powers of two. From there students become more comfortable partitioning into even numbered pieces still relying on the strategy of halving. Eventually students overcome the need to halve and embrace new strategies that allow them to partition into odd numbers. By the end, students use multiplicative reasoning to partition—for example, trisecting into thirds to create ninths (Lamon, 1996). Additionally, looking at student strategies in regard to economy is telling as well. Students who partition using economical strategies demonstrate mastery of key understandings of rational number (Lamon, 1996).

The idea of building or creating a unit of one is supported by the Rational Number Project (Behr & Post, 1988; Behr et al., 1985). Building an amount through repetition of a unit is essentially the concept of iterating in mathematics. Iterating requires taking a unit and repeating, or copying, that unit in order to build an amount. Understanding how many iterations are required to make a unit of one ties directly into unitizing. An example of iterating might be taking a line segment that is given a distance of $^1/_3$ and copying that same distance to create a new distance. A student could make three copies of $^1/_3$ to make one or four copies of $^1/_3$ to make $^4/_3$.

Iterating is another key understanding needed for developing competence with fractions. The body of research surrounding iterating is not as through or rich as that which covers partitioning. The research on iterating is convincing (McCloskey & Norton, 2009; Norton et al., 2014). If partitioning is such a key understanding it would make sense that the inverse of partitioning, iterating, would hold a valuable place in fraction knowledge as well. In fact, Norton et al. (2014) states that iterating tasks also require partitive reasoning and designed partitive reasoning tasks that required students to both iterate and partition.

Student Understanding of Fractions

Students develop fraction knowledge through a number of different trajectories (Vukovic et al., 2014). In order to create tasks that will help identify the way students are developing fraction understanding, it is important to clearly research and discuss both the ways in which students demonstrate fraction knowledge and the common misconceptions that occur when learning about fractions.

Conceptual Versus Procedural Understanding

There are two overarching kinds of mathematical tasks and two kinds of mathematical understanding: conceptual and procedural. Conceptual knowledge describes student understanding of mathematical concepts. Gabriel et al.(2012) defined procedural knowledge as, “sequences of actions that can be put to play to solve specific problems.” (p.138). The researchers defined conceptual understanding as, “the knowledge of central concepts and principles and their interrelations in a particular domain.” (p.137). Procedural knowledge describes student ability to compute or calculate mathematical problems through procedures. With procedural knowledge, the solutions given by students are usually deemed correct or incorrect with little insight into student understandings or misconceptions.

The research supporting the assessment and instruction focused on conceptual knowledge is convincing. Most of the research defends the claim that students can demonstrate procedural understanding without conceptual understanding (Hecht & Vagi, 2011; Niemi, 1996; Pitsolantis & Osana, 2013; Sidenvall et al., 2015; Siegler et al., 2011; Wilkerson et al., 2015). Assessing conceptual knowledge focuses both on the correctness of the answer on the approach or reasoning a student used in order to arrive at an answer. A student can use sensible reasoning, but still arrive at an incorrect answer (Lithner, 2008). The student’s approach tells us more than the answer in many cases. The Standards for Mathematical Practice put forth by the Common Core State Standards (Officers, 2010) encourage the development of conceptual knowledge (Heron, 2014).

Simon (2006) said, “One of the most common uses of understanding is knowing why something is true or appropriate.” (p.360). Conceptual knowledge involves intuitive knowledge (Pitsolantis & Osana, 2013), reasoning (Wilkerson et al., 2015), and connecting relationships (Gabriel et al., 2012) which assist students in arriving at the answer to “why.” Tasks that highlight student conceptual understanding about knowledge will tell teachers which students understand the “why” as well. This study explored the task characteristics that influence student work and highlight both formal and informal understandings as well as common misconceptions through the use of tasks designed to assess conceptual knowledge. With the knowledge that understanding and assessing conceptual knowledge will allow teachers insight into student misconceptions (Niemi, 1996) this study focused on conceptual knowledge as that is the type of knowledge that highlights student understanding for teachers.

Models

Student mathematical models uncover what a student understands and has learned, both conceptually and procedurally. Of models, Heritage and Niemi (2006) stated, “The value of representations as a source of information about the students’ mathematical thinking has been widely recognized.” (p.267). Conceptual knowledge is tied closely to representations or models (Bruner, 1966; Niemi, 1996). There are physical models that are represented with paper and pencil, manipulatives, or using digital tools. There are also mental models, those are models that people have in their minds to help make sense of different mathematical concepts. Getting the mental models represented physically is one way to access and assess student thinking as Heritage and Niemi (2006) suggested.

A number line is a line that represents numbers from negative infinity to positive infinity (see Figure 2.1). Students and adults possess what researchers term a “mental number line” (Ebersbach, Luwel, & Verschaffel, 2015) which is a mental model. The mental number line is explained as a number line individuals visualize when working with fractions. Each mental number line varies from person to person, but some consistencies regarding factors like number choice have been discovered through research. Mental representations (including the mental number line) are heavily discussed in mathematical research regarding fractions. The intuitive fraction knowledge students possess is often also held as a mental representation (Cramer et al., 2015). Representing mental representations externally is the essence of and will highlight deeply-rooted mathematical ideas that need to be addressed (Goldin & Kaput, 1996)

Understanding about the mental number line that students possess and the way they operate with it can indicate different levels of thinking and understanding about student thinking (Pitta-Pantazi et al., 2004). For example students often use reference points when working with number line estimation (Ebersbach et al., 2015). Identifying their reference points will help teachers determine the level of sophistication of understanding of rational number. Half as a reference point tends to reveal informal and immature rational thinking as half is often the first fraction that students understand and work with (Hunting, 1999). Students who scale their reference point to greater or less than ½ indicate greater sophistication and understanding. In addition, students who are capable of estimating on a number line tell a teacher that the student has a strong understanding of number (Ebersbach et al., 2015), which leads to a greater potential for understanding rational number (Hannula, 2003) .

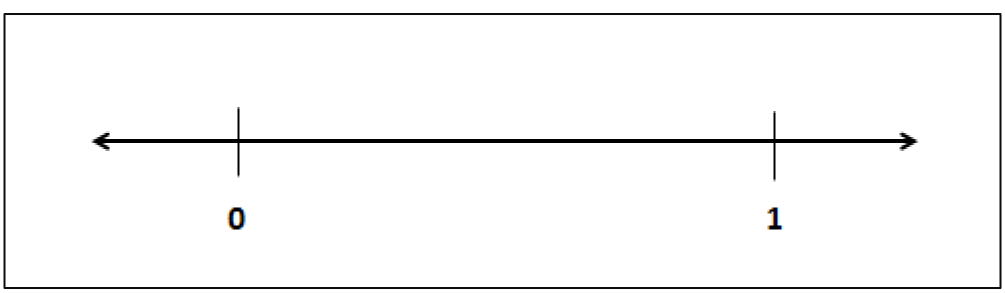


Figure 2.1 Example of a Number Line

<u>Misconceptions</u>

Thinking of fractions as numbers is one of the most cited misconceptions about fractions (Hannula, 2003; Siegler & Pyke, 2012a). Students often overlook or do not grasp the fact that fractions are numbers that fit on a number line and denote a distance from zero. Bodies of research support what Ni and Zhou (2005) found regarding whole number bias (Booth & Newton, 2012; Meert, Gregoire, & Noel, 2010; Pearn, 2007). Students working with rational number tend to address fraction ideas with the rules that apply to whole numbers and do not apply to fractions. Some examples of whole number bias include understanding that there are no whole numbers between two consecutive whole numbers (i.e. there are no whole numbers between 1 and 2). Unlike whole numbers, there is an infinite number of rational numbers between $1/_3$ and $1/_2$. Students often apply the whole number understanding that an increase in value occurs when the increase of digits occurs or simply see one digit as greater than another. This is displayed in student work or student reasoning when a student thinks that $1/_{10}$ is greater than $1/_5$ because 10 is greater than 5. Another way whole number bias may reveal itself is when students work with and handle numerators and denominators of two separate entities (Clarke & Roche, 2009; Cramer & Wyberg, 2009; Meert et al., 2010). This whole number approach to dealing with fractions is detrimental to a students' ability to unitize fractions. This whole number bias plays a role in student acquisition of fraction knowledge and heavily influences student misconceptions and informal thinking about fractions (Ni & Zhou, 2005).

Not understanding the unit is another misconception shared among students. Specification of the unit is a key understanding. Students may see a fraction like $3/_4$ and see it as “3 out of 4” but they are unable to answer the question “$^3/_4$ of which unit?” (Hannula, 2003). This becomes problematic when a student may be working with $^3/_4$ of a whole partitioned into 12 pieces to represent a 12-pack of soda.

While an understanding of half is a foundational and early piece of fractional understanding, there is another common misconception regarding half. Wong (2013) found that many students treated half as an action rather than a location. This may be because students develop a knowledge of half beginning as a qualitative unit and progressing to a quantitative unit as Hunting (1999) found.

Learning Progressions of Fractions

Student acquisition of fraction knowledge can follow different progressions, or trajectories. That is, there are certain foundational understandings that students must first acquire before moving forward. There are a number of different task or model progressions presented in research for the pathways in which students develop an understanding of rational number. Each progression may be specific to the type of fraction understanding acquired. Lamon (1996) and Pothier and Sawada (1983) explored the developmental progression of student partitioning strategies. Cramer and Wyberg (2009) explored the way students learn and understand different part-whole models. Hannula (2003) explored the progression of students understanding of the number line from 5^{th} to 7^{th} grade. Throughout this review of literature, some generalizable trajectories have been discovered. These trajectories or ideas about trajectories can be found in varying types of tasks and among different levels of student thinking.

Students tend to enter school with an idea of fair sharing as the most commonly found informal understanding of rational numbers (Brizuela, 2006; Cwikla, 2014;

Wilkerson et al., 2015). From there, student understanding of fraction often proceeds with students developing a partitive understanding of unit fractions (Norton et al., 2014). Thompson and Saldanha (2004) found that the progression of fraction understanding moves from additive reasoning to multiplicative reasoning. Students as early as kindergarten are asked to split food items equally among a certain number of people (Cwikla, 2014). In this early stage, students understand the need to split into the same number of pieces, but the students do not necessarily understand sharing equally *sized* pieces. For example, students may split a granola bar into 6 pieces to share among six people, but the pieces may not all be the same size. This budding understanding of rational number concepts begins with partitioning.

From the early stages of partitioning, student understanding of fractions often continues with students developing both a partitive and iterative understanding of unit fraction which Norton et al. (2014) explains as "both partitioning and iterating." (p.354). That is students understand more than just how many *equal* pieces or parts to divide the unit of one into, but also understand the base unit that can be iterated to make that unit of one. Here students see a number such as $^4/_6$ as an amount that can be made with 4 iterations of $^1/_6$ (Strother et al., 2016).

Thompson and Saldanha (2004) found the progression of fraction understanding moves from additive reasoning to multiplicative reasoning. This move is extremely complex, but at the earliest stages might show in a student that begins to understand that in order to create 9^{ths} out of 3^{rds}, each 3^{rd} should be partitioned into three. Students with this understanding would also understand the inverse and recognize that 9^{ths} can also be composed into 3^{rds} as well. Students who understand how the numerator changes in these situations are on their way to mastering multiplicative thinking at this level. For example, a student who understands that the more time the total unit is split, the smaller each piece becomes is demonstrating an understanding the relationship between the numerator and denominator with multiplicative reasoning.

Building a Task Framework

There are many different types of fraction tasks and even more influences on task outcomes within the task (Mueller & Maher, 2009; Sidenvall et al., 2015). Lesh, Post, and Behr (1988) identify five representation modes for mathematical tasks: real-world contexts, pictures, written language, manipulatives, and symbols. Some tasks may overlap and contain more than one representation. Cramer, Post, and delMas (2015) stated that fraction tasks *should* include physical objects, diagrams, and real-world situations. There are many ways to build a task framework regarding these influential factors with a large breadth of research supporting ideas about what makes a mathematically rich task. In order to create a reasonable task framework for this study, it was necessary to narrow and investigate certain task characteristics.

The tasks for this study are designed to assess the third grade standards, but will explore the impact the understanding of those third grade standards has on the fourth grade standard selected. The third grade Common Core State Standards (2010) used to help develop the task framework for this study are located in Appendix A.

Cognitive Demand

Hodges, Cady, and Collins (2008) discussed the way student understanding is enhanced through tasks that require student writing or explaining of representations through oral or written language. Koyama (1998) presented tasks to students in a problematic situation and asked the students to justify their solutions in order to see their ideas and internal representations. In Niemi's (1996) study students were asked to solve problems and then draw representations and use writing and drawing to demonstrate how they knew their strategies were correct. Tasks that ask students to draw, justify, explain, and reason not only help students acquire conceptual knowledge, but they also uncover that knowledge for teachers.

"Mathematical reasoning can be found at all levels of mathematical understanding." (Sidenvall et al., 2015). If mathematical reasoning is found at all levels, tasks need to be designed in a way that uncovers both informal and formal understanding. Words and written representations allow teachers to make inferences about students' thinking (Pearn, 2007; Pitsolantis & Osana, 2013; Pitta-Pantazi et al., 2004). Not all tasks are designed to uncover the mathematical reasoning. Tasks that assess conceptual understanding and ask students to justify or explain their thinking are more likely to help find the reasoning within students. There are many ways of looking at categorizing the rigor or cognitive demand of mathematical tasks. This study uses Webb's (2002) depth of knowledge (DOK) for four content areas to assess the cognitive demand of the mathematical tasks.

The DOK level 1 assesses procedural knowledge as it measures students' ability to recall and reproduce. These tasks require students to calculate or apply procedures to solve problems. The level of cognitive demand is low and, therefore, the student work would likely lack in rich reasoning. Tasks at a DOK level 1 will not provide much, if any, insight into student conceptual understanding. Asking students to demonstrate an understanding of fractions by labeling the location of a fraction on a number line would be an example of a DOK 1 task.

The DOK levels 2 and 3 address conceptual knowledge as they assess skills and concepts (level 2) and strategic thinking and reasoning (level 3). All of these categories of assessment would demand conceptual knowledge from students, though the level of conceptual knowledge will vary. These tasks would therefore be more likely to demand student reasoning and uncover student conceptual knowledge for teachers (Hess, 2013). Asking students to construct a number line to complete the unit of one and then reason about their strategy would be an example of a DOK 2 or 3 task.

The DOK level 4 looks at extended thinking. This combines both procedural and conceptual knowledge, but is meant to be used after material has been assessed and understood at other levels. Tasks that demand too much rigor from students with weak or unsophisticated understanding often result in students shutting down without sharing what they know or understand (Boesen et al., 2010; Sullivan et al., 2000). Asking students to critique a sample of student work with a mislabeled number line would be an example of a DOK 4 task.

For the purpose of this study, which is to create tasks that evaluate all levels of student conceptual understanding around fractions, the tasks created will be at DOK level 2 or 3. Tasks at a DOK level 2 or 3 are more likely to be used for understanding conceptual knowledge at a level that is more likely to uncover students' conceptual understanding.

<u>Context</u>

Behr, Harel, Post, & Silver (1983) narrowed Kieren's (1976) **7** subconstructs of fraction understanding to 5: part-whole, measure, operator, ratio, and quotient. These subconstructs essentially give 5 different contexts (or constructs) in which students need to understand fractions. Students' actions regarding the modeling of a fraction are influenced by the context given to the solution (Hannula, 2003). Therefore, without a clear context, students modeling $^{1}/_{4}$ may represent that amount as a distance from zero or as a part-whole relationship. Identifying a clear context for fraction tasks is essential in designing the assessment framework for this study.

Measure is one of the 5 subconstructs identified by Behr, Harel, Post, & Silver (1983) Tasks in the context of fractions as measure often have an inferior role in the classroom (Fuchs et al., 2013). Meaning measurement tasks are uncommon among curricular materials and textbooks, which makes them unfamiliar to students (Siegler et al., 2011; Sullivan et al., 2009). This has been shown in research studies where students will indicate adequate fraction knowledge in the completion of part-whole tasks and be unable to locate the same fraction they used in the part-whole task on a number line (Hannula, 2003).

Understanding fractions as measure is considered crucial (Gabriel et al., 2012) but is also considered perhaps one of the least intuitive (Hannula, 2003). In a study conducted by Freeman and Jorgensen (2015), the problem types created were all real-world linear context to support the measurement understanding of fractions. When students began, they started using part-whole representations, but as they noticed the linear context, began to shift toward linear models. Hannula (2003) remarked that students may solve part-whole fraction problems with greater ease, but that the understanding of fractions as measure is what supports students' understanding of operations with fractions. Student achievement with fractions as magnitude is linked to greater overall mathematics achievement from 5th to 8th grade (Siegler & Pyke, 2012a; Siegler et al., 2011). Gabriel et al. (2012) discovered that asking students to represent fractions as magnitudes helped children to connect to the foundational idea of unity in fractions. One could then argue that using fractions as measure in tasks to inform instruction might highlight student understanding of the unit or unitizing.

Fractions in the context of measure hold a great deal of significance and influence. Fraction misconceptions around whole number bias may be addressed by understanding fractions as measure. As Siegler and Pyke (2012a) pointed out, "The only property that all real numbers have in common is that they have magnitudes that can be located and ordered on number lines." (p.1994).

The context of fractions as measure will be the context for the tasks for this study. Fractions as measure allow exploration of student responses and influencing characteristics beyond the commonly-used and easily accessed part-whole tasks (Ni & Zhou, 2005). Additionally, this study is exploring tasks at a 3rd grade level that are designed to meet standards aimed at understanding fractions as measure.

Models

Bruner's (1966) work explores both the necessity for modeling in order to construct understanding and the progression of modeling students go through as they work through the construction of understanding. Student modeling progression begins with enactive representations which include physical objects that can be manipulated. In students developing an understanding of fraction, enactive modeling may involve the use of fraction strips. Iconic representations are the next level of progression and might include a student drawing a bar model on paper-often the transition from enactive to iconic takes place as students are asked to draw their enactive representations on paper. Eventually, students become comfortable modeling their thinking and conceptual understanding through pictures and drawings. Finally, students will model their understanding through symbolic notation which includes the sophistication of using formal mathematical language through an equation or formula.

Using models in fraction tasks is essential to creating strong tasks that elicit strong understanding (Son, 2011). When it comes to representing fractions: area, length, and set are the most commonly used models (Cady et al., 2015) and would be considered iconic modeling of fraction concepts. Each model has its own sets of strengths and weaknesses. Additionally, some tasks are more likely to fit particular contexts, standards, or learning goals more effectively. A set model, for example, is not designed to work with fractions as measure. Models designed to look at fractions as measure are the number line and other linear models like the bar model.

Researchers tend to agree that the number line is a necessary mathematical representation, but that the abstraction of this representation often results in misconceptions or erroneous thinking. Heron (2014) identified problems with the number line being as simple as not understanding its conventions—such as reading from left to right or counting the spaces between numbers rather than the tick marks or numbers themselves. A study conducted by Tunk-Pekkan (2015) found that students performed significantly lower on number line tasks than other fraction tasks, but that those tasks required more advanced fractional thinking.

Number line models can be challenging and useful for identifying a student's ability to understand unit because they often contain more than one whole (Heron, 2014). A student's ability to identify the unit on a number line may be a strong indicator of an ability to unitize or understand the unit. Niemi (1996) recognized this about the use of a number line, "Successful use of the number line requires at least two types of knowledge not implied by other representations: coordination of multiple units simultaneously and understanding that fractions are numbers representing relations between other numbers." (p. 353)

The number line is considered a continuous model. There is conflicting research regarding the use of continuous versus discrete models (Lamon, 1996). Discrete models are those consisting of singular objects that can be counted, whereas a continuous model is made of a unit that must be divided (Hunting, 1999). Studies conducted by Wilkerson et al.(2015) and Wing and Beal (2004) found students performed better on tasks using discrete models in comparison to continuous models. On the other hand, the suggestion of the Rational Number Project (Behr et al., 1985) is that students work with continuous models first and progress to the use of discrete models later—as they apply the knowledge they gained with continuous models to discrete models. With the consideration of this research and the standards being addressed and assessed by the mathematical tasks, the number line was the model used.

Task Characteristic Comparison Framework

Each task type and characteristic plays a role in uncovering student knowledge and misconceptions about fraction. Several influencing factors are discussed in an attempt to determine the task framework best fit for this study.

Influencing Task Characteristics

Within the narrowed framework for creating tasks for this study, considering the remaining characteristics opens the framework up again. It is with the intent of discovering how the remaining task characteristics influence student work and what that student work uncovers that this study hopes to compare both number choice and the number line design used within these partitioning and iterating tasks.

Number Choice

Number choice research around fractions focuses on both numerator and denominator and how each of those impacts computation and mental representations of number lines. Student informal understanding of fractions begins with halves and progresses to more sophisticated and economical thinking (Brizuela, 2006; Hunting, 1999; Lamon, 1996). According to Pothier and Sawada’s (1983) 5 levels of partitioning, number choice matters. Students use their most intuitive and informal understanding of half as a starting point and progress from even to odd denominators when completing partitioning tasks. Student partitioning strategies will reflect student understanding by the way students partition a model to represent the denominator.

In Niemi’s (1996) study, he chose 5 different fractions to assess student understanding: $^1/_2$, $^2/_4$, $^2/_3$, $^4/_6$, and $^3/_2$ based on the level of difficulty present in the representation of each fraction. Siegler et al. (2011) found that the two most common fractions that were correctly represented on number line estimation tasks were $1/2$ and $1/4$ and those fractions also elicited the most accurate estimates. This aligns with the research by Pothier and Sawada (1983) regarding the intuitive nature and ease of working with half and then half of half.

Number choice has also been shown to impact the comparison of fractions. Distance effects, as they have been named, have been found when studying response times when comparing pairs of fractions. The further apart the two fractions are on the number line, the shorter the response time. The closer together the two fractions are on the number line, the longer it takes both children and adults to compare them (Meert et al., 2010).

Research about number choice can help teachers draw conclusions about student thinking. With half being a foundational understanding, one can conclude that students using half as a starting point when partitioning, are at the beginning of their understanding of fractions. Furthermore, in the study completed by Pothier and Sawada (1983), they found students go through five stages of partitioning strategies. It is not until the fourth stage that students are able to successfully partition odd-numbered denominators. Whether or not students are successfully partitioning into odd-numbered denominators can help teachers more accurately place them on the learning trajectories for foundational concepts of fraction understanding.

This study compared student work with even and odd-numbered denominators. While the research suggests that students working with odd-numbered denominators may struggle more without strong formal backgrounds (Niemi, 1996; Pothier & Sawada, 1983; Siegler & Pyke, 2012b), this study is not simply looking at student solutions, but also student approaches. How do student partitioning strategies differ with even-numbered denominators compared to odd-numbered denominators? Does number choice impact student ability to unitize? What does a task with an odd-numbered denominator offer teachers that a task with an even-numbered denominator does not?

Number Line Models

There are two, more commonly used, types of number lines used in fraction tasks. Filled number lines which are lines marked into proportional segments and open (or empty) number lines are blank lines where the partitioning is left to the students (Diezmann, Lowrie, & Sugars, 2010). Some empty number lines have a start and end point, but there are no partitions between those two points. Other empty number lines are left completely blank. In addition, some empty number lines have an end point, others are continuous indicating that the numbers continue to in infinitely in both directions.

Steffe and Olive (2009) found that when students are doing the actual physical partitioning of regions and lengths, that new understanding is being developed or demonstrated rather than when presented with pre-partitioned models. Allowing students to do the partitioning themselves taps into intuitive knowledge (Cwikla, 2014; Lamon, 1999). Gaining insight into that intuitive knowledge is a primary goal of this study.

There is research behind each model and how its implementation in the classroom can lead to success, but there is little research surrounding the solution strategies students use with both number line models. What kind of student thinking might an empty number line elicit versus a structured number line and vice versa.

Through the literature review of this study, was narrowed to the context, level of cognitive demand, and the model that will be used. The framework for tasks was first narrowed down through the context of measure. The tasks were at a Depth of Knowledge level 2 or 3 and used a number line model with zero marked. From there, this study examined how the influencing characteristics of number choice and number line model impact the student work that results from these tasks. Using this assessment framework, the different types of student understanding that those characteristics uncover in student work are examined and discussed.

Purpose of Study

The purpose of this study was to identify the way students respond to different types of fraction tasks and what conceptual student knowledge is uncovered or influenced by these tasks in relation their characteristics.

Goals of Study

This study was designed to assist both teachers in teaching and students in learning about fractions. With limited resources for fraction materials that encourage conceptual understanding, such as textbooks and other common curricular materials (Hodges et al., 2008), teachers are often left to address the difficult concept of fraction without materials better equipped to assess student understanding (Behr et al., 1983). This study identifies fraction task characteristics that best highlight student thinking and understanding so teachers can select tasks for their students to more clearly understand and address student thinking and understanding.

This literature review led to the creation of a task framework that assesses conceptual understanding of fractions and explores the way characteristics within that framework influence student work. Ultimately this study not only provides information about how task characteristic influence student thinking, but also contributes to a needed body of research regarding the use of valuable and informative tasks in the mathematics classroom.

CHAPTER THREE: RESEARCH DESIGN AND METHODOLOGY

The research design for this study is non-experimental. This study was designed to capture and analyze student thinking through examination of work samples on fraction tasks designed using the framework developed and discussed in Chapter 2 of this study. Evaluating student work from a variety of student mathematic ability levels and socio-economic status of students was a priority in this study, allowing for the examination of student thinking from a variety of backgrounds, in order to capture intuitive and informal strategies as well as more formal strategies. Another goal of this study was to highlight an intuitive understanding as well misconceptions through the analysis of the student work.

Participants

This study looked at the work of 100 third grade student and 100 fourth grade students. The students in this study came from 4 different schools and 9 different classrooms. These schools represent varying levels of socio-economic status. Three of the schools are Title 1 schools. The schools represent varying levels of student prior knowledge. See Table 3.1 for a summarization of the schools used in this study.

Table 3.1 Student Sample Summary

School	% Free and Reduced Lunch	% 3rd Graders Proficient or Advanced During 2014-2015 School Year	% 4th Graders Proficient or Advanced During 2014-2015 School Year	# of Students Sampled
1	75%	39.2%	N/A	46
2	15%	71.4%	35.1%	62
3	57%	37.9%	27.6%	48
4	68%	N/A	N/A	49

Setting

The tasks for this study were administered during spring semester after students had completed their unit on fractions. The tasks were administered during normal school hours in their normal classroom setting. The tasks took approximately 20 minutes to complete. The process was briefly discussed with the students prior to them receiving the tasks. The students were told (by their teacher) that the tasks would be looked at in order to find better ways to write fraction tasks but that they would not be assessed or graded on their work and were asked to do their best.

The Instrument

A review of the research on influential task characteristics was completed during the literature review and the assessment framework was created in order to narrow the tasks for this study. Using assessment framework, tasks were created. Each task was designed to prompt students to either iterate or partition and each task shared the same level of cognitive demand, mathematical model, and context—as discussed in Chapter 2 of this study. The two instruments used in this study were developed from the set of fraction tasks found in Appendix B. Two worksheets (Form H and Form J) containing 6 fraction task. The worksheets were created from a set of 12 tasks created based on the framework from Chapter 2. One iterating task was created for each category and two partitioning tasks were created. The tasks were divided into 2 worksheets. Each worksheet contained a task with one of the task characteristics shown in Table 3.2. The instruments were evenly distributed in each classroom--that is, in each classroom half of the students received Form H and half of the students received Form J.

Administration of Tasks

Administration of the tasks took place in general education classrooms to third and fourth grade students. The students were given time in class to work on the tasks with paper and pencil. No teacher assistance was given to help solve the problems, but teachers could reread instructions if needed. Tasks were collected after they were completed on the same day.

Analysis, Evaluation, and Categorization of Tasks

The tasks were analyzed first quantitatively and then qualitatively. The task outcomes were analyzed first by relative frequencies. The percentage correct was determined for each task category from the task matrix. The purpose of the quantitative analysis was to provide insight as to which, if any, tasks are easier. This was taken into consideration as the qualitative analysis was explored. One of the goals of this analysis was to determine whether or not tasks that may be easier to complete correctly offer the same quality of information accompanying student work. Additionally, the analysis of percentage correct was used to determine whether there are any significant differences in student performance on tasks with a certain type of task characteristic compared to the other.

Table 3.2 Matrix for Task Framework

	Even-Numbered Denominators	Odd-Numbered Denominators
Filled Number Lines	Iterating	Iterating
	Partitioning	Partitioning
	Even-Numbered Denominators	Odd-Numbered Denominators
Empty Number Lines	Iterating	Iterating
	Partitioning	Partitioning

The qualitative analysis for this study borrowed from Thomas'(2006) general inductive approach. Student responses were examined evaluating first for correctness, and then strategies and explanations. Both correct and incorrect student responses were evaluated qualitatively. Student work was categorized based on the use of the same reasoning or the highlighting of the same misconception. Once categories were created, they were given names and definitions, and examples of student work highlighting each strategy were collected. Student work was examined a second time to ensure that it fit the definition and the work was comparable to the selected student work.

Each column in the matrix was then evaluated in order to determine whether or not certain task characteristics influenced the student strategies more than another. Student strategies within each construct of the matrix were also explored to determine how, if at all, task characteristics influenced student work or student strategies. Strategies were analyzed and compared to the foundational fraction understanding and fraction learning progressions discussed in Chapter 2 of this study in order to better understand the student thinking represented in both correct and incorrect responses. The student approaches are described in Chapter 4 of this study.

There were many similar strategies used for all of the tasks in this study. As such, the percentage of students using each strategy on each type of task was also calculated. Determining whether or not a certain strategy was more prevalent on one type of fraction task compared to another was necessary in order to draw conclusions about the tasks and their influence on student work.

Student Interviews

Ten students from one 4th grade classroom were interviewed after the analysis and categorization of student work. Mitchell and Clarke (2010) interviewed students after completing tasks to identify whether their solutions were representative of procedural or conceptual knowledge. Likewise, Brinker (1997) found that students could solve a problem using the same representations, but interpret those representations in different ways. As a result of these findings, interviews became a crucial part of this study in order to be sure that the student work being analyzed was being interpreted correctly. In order to avoid inaccurate generalizations, interviewing students helped clarify the student thinking used to complete the tasks, lending more confidence to the analysis of the student work on the tasks.

Students from the final classroom administration of tasks were observed as they completed the tasks. If the approaches were similar to the more-commonly used strategies of the students from the other classrooms, the students were selected for interviews. This allowed for greater insight to student thinking across the study. In the interviews, each student was asked to explain their process, why they chose that process, approach, or strategy, and any questions that were specific to that task.

Limitations

Teachers from all classrooms used in this study have completed additional professional development regarding teaching number concepts conceptually-including fractions. The professional development was part of a statewide initiative which was required of all teachers. The professional development each teacher completed may impact the way they teach number concepts and the language they use, which may impact the student strategies and student reasoning found in this study.

CHAPTER FOUR: FINDINGS

This chapter outlines the process of analyzing both the student strategies used on the tasks created. Both process and the outcomes of the quantitative and qualitative analysis are described in detail. The quantitative analysis is described first as it was a precursor to further exploration of the examination of the student strategies uncovered through the study.

Quantitative Analysis

The quantitative analysis of student work first looked at whether each problem was solved correctly or incorrectly. Each task was broken down by its characteristics and the percentage correct for each characteristic was calculated as shown in Table 4.1.

Percent Correct by Task Characteristic

While filled number line tasks were correct for more than 50% of students, tasks with empty number lines were solved correctly by less than 50% of the students. Partitioning tasks were solved correctly by more students than iterating tasks when working on empty number lines. The odd-numbered denominator tasks were solved correctly by more students than even-numbered tasks on iterating tasks, but the inverse is true for students solving partitioning tasks with odd-numbered denominators.

Table 4.1 Percent Correct by Task Characteristic

	Even- Numbered Denominators		Odd-Numbered Denominators	
Filled Number Lines	Iterating	.51	Iterating	.58
	Partitioning	.53	Partitioning	.44
Empty Number Lines	Iterating	.36	Iterating	.42
	Partitioning	.48	Partitioning	.40

Qualitative Analysis

A central aim of this study was to determine which task characteristics may help uncover student conceptual understanding of fractions, particularly looking at partitioning, iterating, and unitizing strategies. The analysis of student work found the ability to unitize often appeared to be intertwined with student partitioning or iterating strategies. The strategies for partitioning and iterating are discussed separately and the ability (or inability) to unitize is described within each partitioning or iterating strategy.

The student strategies used throughout these tasks often overlapped with some task characteristics. As such, each student strategy is discussed and explained in this chapter. The strategy is named, the task characteristics that influence the strategy are explained, and the strategy is described. An example of the student thinking is described through student reasoning used during interviews or their written explanations given on each task. An example of student work for each strategy is provided after the description of each strategy. The student strategies used in the tasks are described and illustrated with examples in Table 4.2. Figure 4.1 below outlines the structure of the qualitative analysis of each student strategy.

The strategy is named
The task characteristics that influence that strategy are named
The strategy is described
An example of student thinking/reasoning is given (from student interviews or written explanations)
A example of student work modeling the strategy is provided

Figure 4.1 Organization of Qualitative Analysis of Each Strategy

Student Partitioning Strategies

Student partitioning strategies were examined on all task types. Student partitioning strategies were found primarily in partitioning tasks and each strategy may be more influenced by additional task characteristics than others (more fully described below). In order to identify partitioning strategies, processes where students generated quantity as Lamon (1996) defined it. Most of these strategies demonstrated the "systematic splitting" of the number line that Confrey (1987) discussed. Partitioning strategies looked at how students partitioned the number line and what reasoning they provided for doing so on open number lined tasks. On filled number lines, student ability to partition looked at ways in which students viewed and/or treated each partition in a filled number line. Student strategies were analyzed and information about student reasoning from the task forms as well as student interviews was used to exemplify each strategy and better understand the student reasoning or thinking used when applying a strategy or working through each task.

Equipartitioning by Unit Fraction

Students who used or demonstrated the ability to equipartition correctly partitioned the line segment into equally sized pieces. Each unit fraction is partitioned into the same size and labeled with the value of the unit fraction. All partitioning tasks uncovered the ability to equipartition. Many of the partitioning tasks for this study used fractions greater than 1. In the student interviews, one student took $^{8}/_{6}$ and was able to partition that number line into 8 $^{1}/_{6}$ pieces. Students demonstrated unitizing through the explanation that the fraction was greater than 1 and therefore 6 $^{1}/_{6}$ths was equivalent to a unit of one. A sample of Equipartitioning by Unit Fraction is shown in Figure 4.2 on a task where students were given an empty number line with given distance of $^{8}/_{6}$ and were asked to find the location of "1."

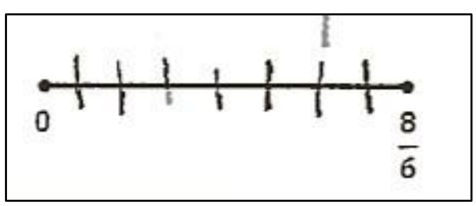


Figure 4.2 Equipartitioning by Unit Fraction

<u>Equipartitioning by Non-Unit Fraction</u>

Few students in this study demonstrated the ability to partition into non-unit amounts. These students partitioned $^{8}/_{6}$ into $^{2}/_{6}$ segments. The empty number line partitioning task with even-numbered denominators was the only task that uncovered this strategy. Students reasoned multiplicatively about this strategy stating they "just count by 2's." A sample of Equipartitioning by Non-Unit Fraction is shown in Figure 4.3 on a task where students were given an empty number line with given distance of $^{8}/_{6}$ and were asked to find the location of "1."

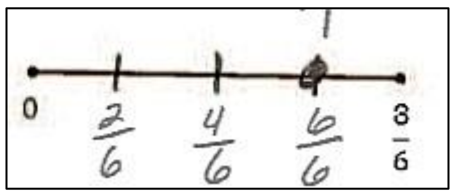


Figure 4.3 Equipartitioning by Non-Unit Fraction

<u>Unequal Partitioning</u>

Students demonstrate partitive reasoning by partitioning the correct number of times, but do not partition into equally sized pieces. On filled number line tasks, students using this strategy did not label the partitions on the number line correctly. Partitioning tasks on empty and filled number lines with even-numbered denominators uncovered this strategy. In tasks where the partitions are unequal, the student labels them as if they are. In tasks where the number line is empty, the student partitions unequally. Student thinking that exemplifies this strategy includes explanations such as: "Each cut is $^{1}/_{6}$." A student sample of this strategy is shown in Figure 4.4 on a task where students were given an empty number line with given distance of $^{8}/_{6}$ and were asked to find the location of "1."

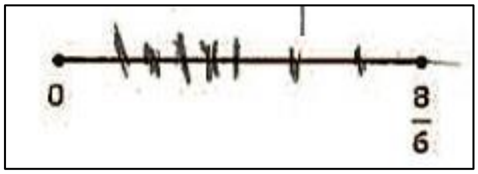


Figure 4.4 Unequal Partitioning

Inaccurate Partitioning

Students using this strategy do not make the correct *number of* partitions. Partitions are made and may or may not be equal in size, but the number of partitions made is not based on the number of unit fractions needed to make a unit of one. Students working on tasks with fractions greater than one, may make as many partitions as the numerator indicates rather than the denominator, but then used those partitions to rename the denominator. In filled number line tasks, these students label each partition with a denominator that matches the number of partitions. For example, in a number line partitioned into 8 $^{1}/_{6}$ths, the student labels a number line partitioned into eighths. This strategy was found in partitioning tasks with filled number lines and both even and odd-numbered denominators and in empty number line partitioning tasks with odd numbered denominators. One student interviewed explained this process as, "there are 8 pieces, so each piece is $^{1}/_{8}{}^{th}$." None of the students who used this strategy were able to correctly locate the unit of one. A sample of the Inaccurate Partitioning is found in Figure 4.5 on a task where students were given an filled number line, with eight partitions, given distance of $^{8}/_{6}$ and were asked to find the location of "1."

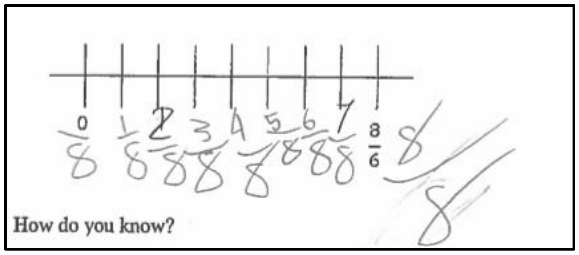


Figure 4.5 Inaccurate Partitioning

Comparative

Students demonstrating this strategy know that $8/6$ > 1 so they make a single partition or only label one partition made on the filled number line and place “1” somewhere before $8/6$ on the number line. These students do not demonstrate estimation of the number of unit fractions needed to count back or place the 1 before the fraction greater than 1.All partitioning tasks in this study uncovered student work or thinking using comparative reasoning, though it was more commonly used with open number line tasks. In student interviews, students who used comparative reasoning made statements such as, “This is bigger than 1 so 1 is closer to 0.” A sample of Comparative Reasoning is found in Figure 4.6 on a task where students were given an empty number line with given distance of $8/6$ and were asked to find the location of “1.”

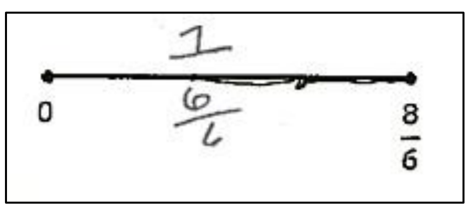


Figure 4.6 Comparative

Student Iterating Strategies

Student iterating strategies were examined on all task types. Iterating strategies were found primarily in iterating tasks and each strategy may be more influenced by additional task characteristics than others (more fully described below). With the research on student iterating strategies lacking in depth, the analysis of these strategies came from a combination of understanding the definition of iterating as making repeated copies of an amount and seeing iterating as copying rather than cutting. The examination of iterating strategies looked at how students iterated a line segment and what reasoning they provided for doing so on open number lined tasks. On filled number lines, student ability to iterate looked at ways in which students viewed and/or counted each iteration or partition on a number line. On filled number lines, student work treated each partition as a "copy" of a unit fraction rather than a piece that had been decomposed. Student strategies were analyzed and categorized. The information about student reasoning from the task forms as well as student interviews was used to exemplify each strategy and better understand the student reasoning or thinking used when applying a strategy or working through each task.

Equal Iteration

Students demonstrating equal iteration on open number lines made copies or iterations of the line segments accurately. Students made marks or used fingers to accurately copy the same size represented for each unit fraction or other fractional amount. Students demonstrating this strategy on filled number line tasks labeled each line segment as a copy of the unit fraction. This strategy was found on all iterating tasks. Students using this strategy reasoned that in order to create a unit of one they would need to "jump," "copy," or "iterate" enough unit fractions to build the unit of one. An example of Equal Iteration is shown in Figure 4.7 on a task where students were given a filled number line with five equipartitions and the first partition is given a value of $^1/_3$ and asked to find the location of "1."

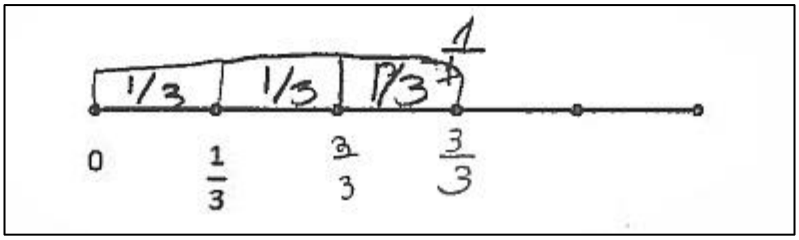


Figure 4.7 Equal Iteration

Unequal Iteration

Students demonstrating Unequal Iteration on open number lines made copies or iterations of the line segments that were not equal in size. Students demonstrating this strategy on filled number line tasks labeled each point on a number line incorrectly. This strategy was found on all iterating tasks. Students using this strategy demonstrated an understanding of how many unit fractions or non-unit fractions were required to make a unit of one, but did not show that the line segments representing each fractional amount should be the same size. Students using this strategy in the interviews, did not attempt to use their fingers or any other strategy to correctly make copies. An example of Equal Iteration is shown in Figure 4.8 on a task where students were given a line segment with a distance of $^2/_6$ and were asked to find the location of "1."

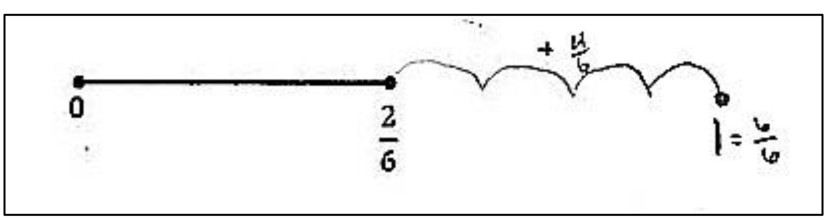


Figure 4.8 Unequal Iteration

Iteration by Non-Unit Fraction

Students demonstrating Iteration by Non-Unit Fraction copied a line segment with a value of a non-unit fraction, such as $^{2}/_{6}$ to build a unit of one. This strategy was found only on open number line iterating tasks with even-numbered denominators. Students reasoning that supported this strategy explained their process using additive or multiplicative reasoning with phrases like, “skip counting by $^{2}/_{6}$” or “ 2 x 3 = 6 so I need 3 copies of this (pointing to the line segment with a value of $^{2}/_{6}$.)” An example of Iteration by Non-Unit Fraction is shown in Figure 4.9 on a task where students were given a line segment with a distance of $^{2}/_{6}$ and were asked to find the location of “1.”

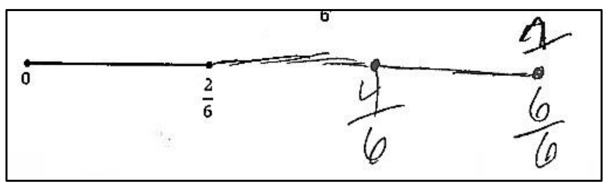


Figure 4.9 Iteration by Non-Unit Fraction

Unwritten Iteration

Students demonstrating Unwritten Iteration made imaginary or mental copies or iterations of the line segments until reaching a unit of one. Students only marked the unit of one and did not make any additional marks on their paper. Some students accurately placed the unit of one while others were not as accurate. This strategy was only found on open number line iterating tasks. Students using this strategy demonstrated an understanding of how many unit fractions or non-unit fractions were required to make a unit of one in interviews, but may not clearly show it on paper. During student interviews, some students would count over with their fingers to make copies, while other students using this strategy simply stated, “It would be about *here* because you would copy this amount 3 times." A sample of Unwritten Iteration is shown in Figure 4.10 on a task where students were given a line segment with a distance of $^2/_6$ and were asked to find the location of "1."

Figure 4.10 Unwritten Iteration

<u>Partitioning Instead of Iterating</u>

Students using this strategy partitioned on line segments that were meant to be iterated. For example, on a line segment labeled $^1/_3$, students would partition the segment and number it according to their understanding. Some students may place 0 at the end, $^1/_2$ in the middle, and 1 at the end. Other students placed "1" on the number line without explanation or reasoning. This strategy was found only on iterating tasks on open number lines. A sample of Partitioning Instead of Iterating is shown in Figure 4.11 on a task where students were given a line segment with a distance of $^1/_3$ and were asked to find the location of "1."

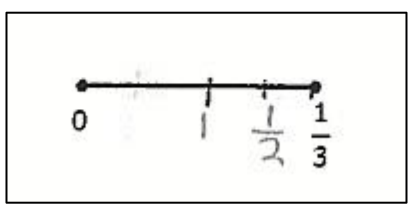


Figure 4.11 Partition Instead of Iterate

<u>Inaccurate Comparative</u>

Students demonstrating inaccurate comparative reasoning reasoned about the unit of one in relation to the given fraction, but did not accurately compare. Students with this strategy placed 1 *after* a fraction greater than one and added to (or made iterations on) the number line in order to demonstrate this. While some students equally iterated the unit fraction past the number line, others simply added to the number line without demonstrating strategic iterative reasoning. All partitioning tasks in this study uncovered student work or thinking using inaccurate comparative reasoning, though it was more commonly used with open number line tasks. One student reasoned "since this is a fraction, it is less so 1 should be out here." A sample of Inaccurate Comparative is found in Figure 4.12 on a task where students were given an empty number line with a distance of $^4/_3$ and were asked to find the location of "1."

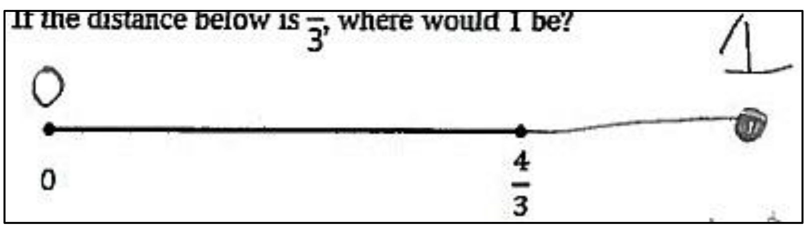


Figure 4.12 Inaccurate Comparative

Student Partitioning and Iterating Strategies

Norton et al. (2014) described "both partitioning and iterating" (p.354) as part of the progression of partitive reasoning and the development of fractional knowledge. Some student strategies in this study found that students used both partitive and iterative strategies in order to solve the tasks.

Partitioning then Iterating

Some fraction tasks uncovered the strategy of Partitioning then Iterating. With a given unit fraction and distance on filled number line, students partitioned to find the unit fraction between two points and then iterated the unit fraction to name the location of a new point or the unit of "1." On open number lines this strategy was shown as students partitioned a non-unit fraction into a unit fraction and then iterated the unit fraction to build the unit of "1." This strategy was found on all iterating tasks with even-numbered denominators. Students using this strategy understood that size was important in order to accurately iterate. One student in an interview described distance between point A and B "bigger than this (pointing to the space between 0 and Point A)." Students who recognized this, often correctly partitioned or correctly labeled the partition between A and B and then iterated that amount or "counted up" from that amount to name a new location. A sample of Partitioning then Iterating is shown in Figure 4.13 on a task where students were given a filled number line with non-equal partitions and given the distance of the first partition ($^1/_4$). Students were asked to find the location of the Point B as well as the location of "1."

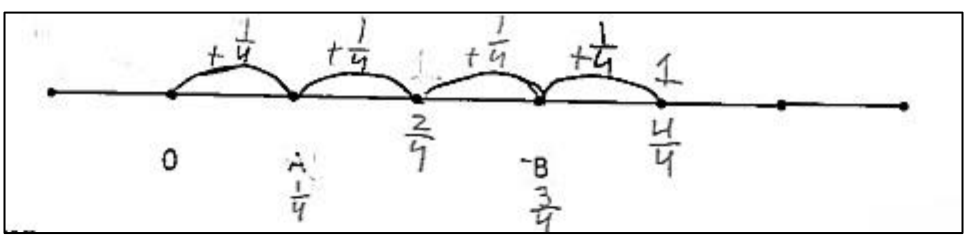


Figure 4.13 Partitioning then Iterating

Distance from Zero

Students demonstrated partitive and iterative reasoning with this strategy. First students equipartitioned or recognized and labeled each partition on a filled number line and then unitized, through counting (or iterating) by unit or non-unit fractions until reaching a unit of one. This strategy was influenced by partitioning tasks with both kinds of number lines and even and odd-numbered denominators. Students interviewed with this strategy used phrases like, "I cut into 4 $^1/_3$ rds and then counted by each third until I got to $^3/_3$.Which is the same as 1." A sample of this strategy is shown in Figure 4.14 on a task where students were given a filled number line with five equipartitions and the first partition is given a value of $^1/_3$ and asked to find the location of "1."

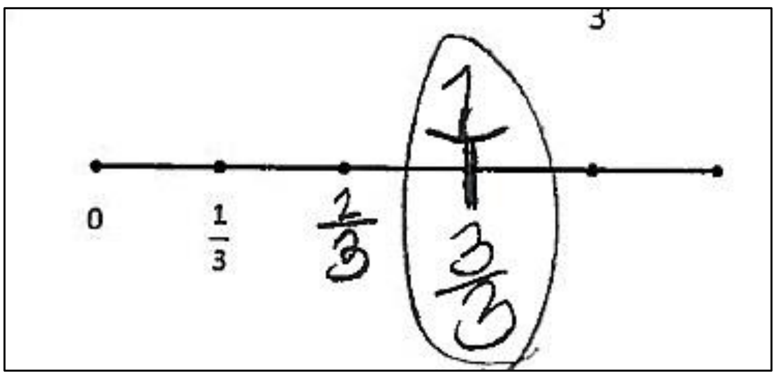


Figure 4.14. Distance from Zero Strategy

Distance from Fraction Greater than 1

Students who demonstrated partitive and iterative reasoning with this strategy did not equipartition the entire line segment or label and work with each partition of a filled number line. Instead, these students counted back from a fraction greater than one until reaching a unit of one. On empty number line tasks, students reasoned about the unit size and estimated or mentally partitioned in order to count back the spaces as accurately as possible. Unitization was demonstrated with reasoning about the amount greater than one that needed to be "counted back" in order to reach one. This strategy was influenced by partitioning tasks with both kinds of number lines and even and odd-numbered denominators. Students in interviews (and on the worksheets) used phrases like, "subtracting," "counting back," and "it's just $^1/_3$ *more than* 1." A sample of this strategy is shown in Figure 4.15 on a task where students were given an empty number line with a distance of $^4/_3$and were asked to find the location of "1."

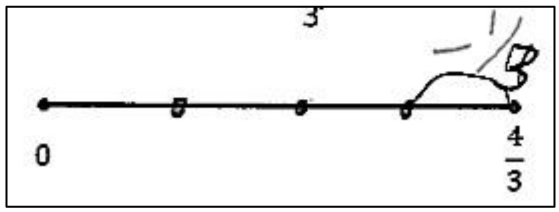


Figure 4.15 Distance from Fraction Greater than 1

Non Partitioning or Iterating Student Strategies

Some strategies did not demonstrate student ability to partition or iterate. These tasks highlighted student misconceptions, but those misconceptions were not in relation to partitioning or iterating strategies. These strategies are described below.

Whole Number Reasoning

Students using Whole Number Reasoning, ignored the fractions or treated the fractions as whole numbers. They either counted by whole numbers on the number line, or they saw the unit fraction (or numerator) as the same as "1." On a line segment with a length of $^2/_6$, for example, students using whole number reasoning would partition the segment to the unit fraction of $^1/_6$ and claim that $^1/_6 = 1$. If counting by whole numbers, students either treated the numerator, denominator, or both as whole numbers—increasing one or both when moving to the right of the number line. This strategy only showed up on open number line iterating tasks with odd-numbered denominators, filled number line partitioning tasks with even-numbered denominators, and filled number line partitioning tasks with odd-numbered denominators. Students using this strategy would point to the numerator of "1" and claim that was "1." A sample of Whole Number Reasoning is shown below in Figure 4.16 on a task where students were given an filled number line, with eight partitions, given distance of $^{8}/_{6}$ and were asked to find the location of "1."

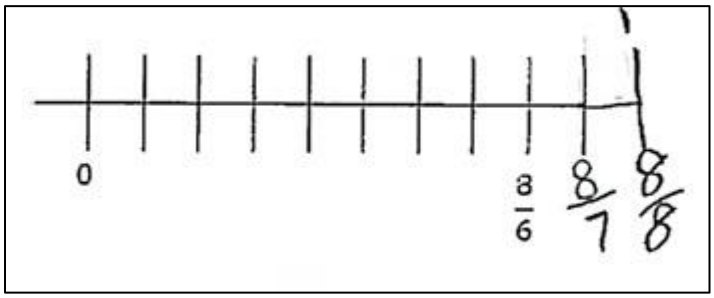


Figure 4.16 Whole Number Reasoning

<u>Inaccurate Number Line Construct</u>

Students demonstrating Inaccurate Number Line Construct strategy solve the problems with limited understanding of the constructs of a number line rather than reasoning about fractions. These students place "1" at the end of the number line because "One goes at the end." This strategy was found on all iterating tasks. A sample of the Inaccurate Number Line Construct strategy is shown below in Figure 4.17 on a task where students were given a filled number line with non-equal partitions and given the distance of Point A ($^{1}/_{5}$). Students were asked to find the location of the Point B as well as the location of "1."

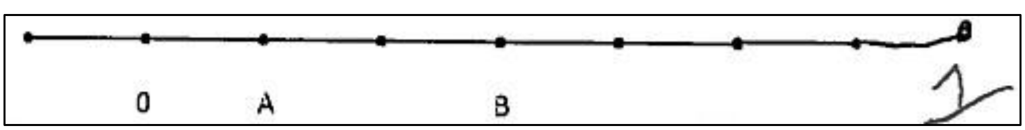


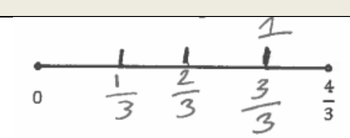
Figure 4.17 Inaccurate Number Line Construct

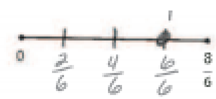
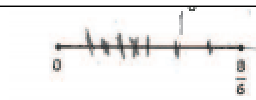
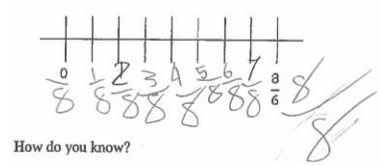
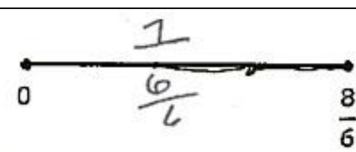
Student Strategy Summary

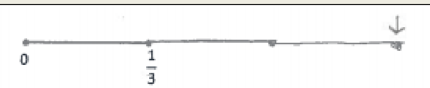
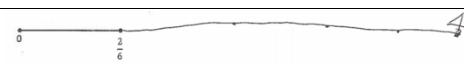
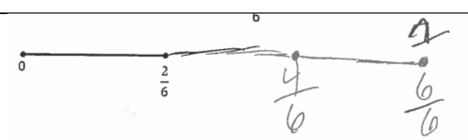
Sixteen different student strategies were used on the fraction tasks in this study. Some strategies were more common than others. The table below summarizes each strategy, the task characteristics that resulted in that strategy, and the frequency with which that strategy was used within each task characteristic. A brief description is given for each strategy, a student work sample, and whether or not that strategy indicates a potential to unitize.

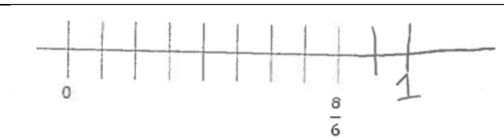
<u>Coding for Table</u>

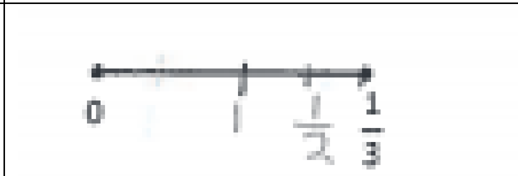
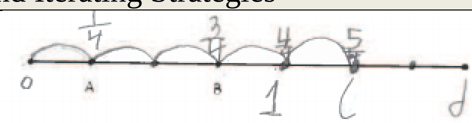
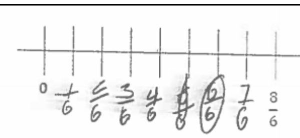
The task characteristics in the table are named with a coding system. The first letter of the name is E or F indicated Empty Number Line Tasks (E) or Filled Number Line Tasks (F). The next part of the name includes It or Pa indicating Iterating Tasks (It) or Partitioning Tasks (Pa). The last part of the name includes an E or O indicating Even-Numbered Denominators (E) or Odd-Numbered Denominators (O). For example a task code of : **E-It-E** indicates an empty number line iterating tasks with even-numbered denominators. The percentage following each task code indicates the percentage of each strategy found within each task characteristic.

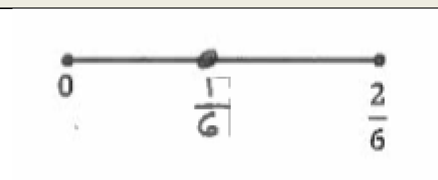
Strategy Name	**Found in —— Tasks**	**Description**	**Example**	**Indicates Ability to Unitize?**
		Partitioning Strategies		
Equipartitioning by Unit Fraction	E_Pa_E (25%) E_Pa_O (18%) F_Pa_E (27%)	Partitions are made into equally-sized segments to represent the unit fraction.		Yes

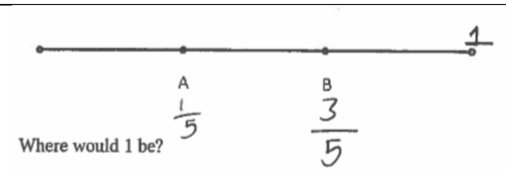
	F_Pa_O (10%)			
Equipartitioning by Non-Unit Fraction	E_Pa_E (2%)	Partitions are made into equally-sized non unit fractional line segments.		Yes
Unequal Partitioning	E_Pa_E (5%) E_Pa_O (8%) F_Pa_E (8%) F_Pa_O (3%)	The number of partitions made is correct, but the partitions are unequal in size.		Yes
Inaccurate Partitioning	E_Pa_O (8%) F_Pa_E (8%) F_Pa_O (3%)	An inaccurate number of partitions are made. Partitions may or may not be equal in size, but the number of partitions is not equivalent to the number needed to make a unit of one.		No
Comparative	E_Pa_E (16%) E_Pa_O (21%) F_Pa_E (10%) F_Pa_O (9%)	Reasoning about the value of the fraction in relation to the unit of one is used to determine whether or not the fraction is greater or less than 1 and then is placed either to the left or right of the fraction on the		No

		number line to indicate that comparative relationship.		
Iterating Strategies				
Equal Iteration	E_It_E (15%) E_It_O (16%) F_It_E (10%) F_It_O (8%)	Copies of the line segment representing a fractional value are made in order to compose a new fraction. The line is added to by same sized pieces copied as accurately as possible.		Yes
Unequal Iteration	E_It_E (10%) E_It_O (10%) F_It_E (4%) F_It_O (6%)	The correct number of iterations are made to create a unit of one, but each iteration is not the same size.		Yes
Iteration by Non-Unit Fraction	E_It_E (3%)	A correct number of iterations of non-unit fractions were		Yes

		made to create a unit of one. The non-unit fractions iterations were the same size for the same value.		
Unwritten Iteration	E_It_E (5%) E_It_O (7%)	Each iteration of the line segment is made with fingers or mentally iterating before placing and labeling a new location. Not every iteration is drawn or labeled.		Yes
Inaccurate Comparative	E_Pa_E (16%) E_Pa_O (17%) F_Pa_E (6%) F_Pa_O (9%)	The given fraction (fraction greater than 1) is seen as less than 1 and additions (or iterations) are made to the number line in order to		No

		place “1” to the right of the fraction.		
Partitioning Instead of Iterating	E_It_E (3%) E_It_O (6%)	The line segment is partitioned, numbers disregarded, and partitions are made to a line segment intended to be iterated.		No
Partitioning and Iterating Strategies				
Partitioning Then Iterating	E_It_E (17%) F_It_E (13%)	A line segment is first partitioned to find the unit fraction. The unit fraction is then iterated to build a specified amount or name a point.		Yes
Distance from Zero	E_Pa_E (25%) E_Pa_O (18%) F_Pa_E (27%) F_Pa_O (10%)	Partitions are made and students count iterate or label the partitions while counting up from		Yes

		zero. The number of iterations from zero indicate the value of the fraction.		
Distance from Fractions Greater than 1	E_Pa_E (25%) E_Pa_O (27%) F_Pa_E (39%) F_Pa_O (32%)	Partitions are only made to count the number of unit fraction iterations needed to subtract or move to the left on the number line from the fraction greater than 1.		Yes
Non Partitioning or Iterating Strategies				
Whole Number Reasoning	E_It_ O (9%) F_It_E (6%) F_Pa_ O (2%)	A unit fraction is treated as a unit of one. The numerator of 1 is seen as having a value of one. Some students locate the unit fraction and others partition	 **Student reasons that $^{1}/_{6}$ is equivalent to "1."**	No

		to find the unit fraction and name that location as "1."		
Inaccurate Number Line Construct	E_It_E (3%) E_It_O (6%) F_It_E (25%) F_It_O (31%)	The unit of "1" is treated as having the same location on all number lines—usually "at the end" and is placed at the end of a line segment or filled number line.		No

CHAPTER FIVE: DISCUSSION

Results

The exploration of the student strategies served to answer the research questions presented at the beginning of this study: What task characteristics influence student work or thinking around fractions and in what ways? What task characteristics highlight informal or intuitive understanding? While all tasks offered insight into student conceptual understanding of fractions, not all offered the same information. Additionally, some task characteristics better highlighted student understanding of partitioning, iterating, or unitizing-which are the key fractional understandings discussed during the review of literature for this study.

Quantitative Analysis

The quantitative analysis of this study was brief and provided a starting place for the analysis of the student work. The tasks were divided by characteristic and the percentage of correct responses on each task was calculated as shown in Table 4.1. The scores of percent correct were very similar—all hovering around 50%. These results may agree with the results of the study conducted by Tunk-Pekkan (2015) which found that students perform lower on fraction tasks on number lines compared to other types of fraction tasks. Additionally this supports the claims made by Cady, Hodges, and Collins (2015) and Niemi (1996) suggests that the lack of measurement contexts and number lines in fraction tasks impact student ability to operate on a number line.

One intention of calculating the percentage correct at the beginning of the analysis of student work was to highlight any differences in student scores that may indicate whether one task is easier to solve than another and to identify any differences in scores that may highlight some influential factors of particular task characteristics. The scores did not indicate large differences in the abilities for students to score correctly from one task type to another.

Qualitative Analysis

The qualitative analysis is the heart of this study. This section aims to answer the research questions that drove this study through describing themes that emerged in student work as influenced by different task characteristics. Some task characteristics more clearly impacted student work than others, those relationships are discussed in this section.

Partitioning Versus Iterating Tasks

Some interesting findings emerged regarding partitioning and iterating tasks such as

- Partitioning tasks highlight informal or intuitive understanding;
- Partitioning tasks result in reasoning with addition and subtraction;
- Iterating tasks highlight more sophisticated student thinking as well as more misconceptions;
- Both partitioning and iterating tasks highlight partitioning *and* iterating strategies;
- Many of the commonly used partitioning and iterating strategies students used were similar in nature to one another on both partitioning and iterating tasks

While all task characteristics seemed to highlight informal understanding on some level, strategies such as the Comparative and Inaccurate Comparative strategy, which were only found on partitioning tasks, highlight a very low level of formal understanding of fractions, a level of understanding that may not be assessed through other types of tasks. These informal and intuitive approaches are informative to teachers and researchers and were found only on partitioning tasks. In addition, some strategies encountered are more commonly found in school and in research texts such as unit and non-unit partitioning and iterating strategies, others such as the comparative or distance strategies are not as common if discussed at all. This suggests the comparative and distance strategies uncover informal or intuitive understanding. These strategies were only found on partitioning tasks. Perhaps partitioning is more intuitive than iterating and therefore tasks that are designed to get at partitioning are more likely to uncover informal or intuitive knowledge.

On partitioning tasks, students reasoned with addition and subtraction in ways they did not on iterating tasks. Strategies such as: Distance from Zero and Distance from Fractions Greater than 1 often used addition (Distance from Zero) and subtraction (Distance from Fractions Greater than 1). Students using these two strategies were very clearly adding or subtracting in both their paper/pencil work and in their language. Phrases like “more than,” “less than,” “counting up,” and “counting back” were most commonly found in these strategies which were only found on partitioning tasks.

It was only on iterating tasks that students did not iterate and used Partitioning Instead of Iterating as well as using the Inaccurate Number Line Construct. Iterating tasks also demonstrated student use of Unwritten Iteration and Iteration by Non-Unit Fraction--two strategies that may indicate greater sophistication of student understanding based on the work of Behr et al. (1985). Finding a potential hierarchy in the partitioning tasks was not as common. While Partitioning by Non-Unit Fraction may indicate greater sophistication of student understanding (Lamon, 1996), fewer students used the more sophisticated partitioning strategies (2%) than the more sophisticated iterating strategies (12%). This is surprising considering partitioning tends to be a more intuitive approach for students (Brizuela, 2006).

Some partitioning tasks and some iterating tasks highlighted strategies that demonstrated the ability to both partition and to iterate: Partitioning then Iterating, Distance from Zero, Distance from Fractions Greater than 1

Partitioning and iterating strategies seemed to mirror each other on the different task types. For example, while partitive strategies included approaches such as: Equipartitioning and Non Equipartitioning, student iterating strategies included approaches such as: Equal Iteration and Unequal Iteration. In addition, both partitioning and iterating tasks found students using Non-Unit iterating and partitioning

Student strategies were very similar and very telling of similar, but different conceptual understanding regarding unitizing. While students who iterated correctly were building or composing a unit of "1," partitioning tasks demonstrate student understanding of how many unit fractions are found within an already created unit of "1." Teachers selecting partitioning versus iterating tasks should be clear in their objectives and goals for what evidence they are looking for regarding unitizing as well as partitioning or iterating.

Empty Versus Filled Number Lines

Notable differences in student work on empty number line tasks compared to filled number were observed

- The work on empty number lines compared to filled number lines seemed more informative overall;
- Filled number line tasks do not clearly indicate whether a student reasoned iteratively or partitively;
- Empty number lines may better uncover intuitive thinking;
- Both empty and filled number lines influenced similar actions on the number line and unitizing language;

Tasks using empty number lines resulted in a better understanding of student *ability* to partition and iterate equally. Strategies that represented equipartitioning or equal iteration for students on empty number line tasks were clearer about student ability to equally partition and iterate as well as student demonstration that equal partitions or iterations were important when representing the same number. Only on empty number line tasks were strategies such as Unwritten Iteration and Iteration by Non-Unit found. On tasks with filled number lines, the ability to partition or iterate equally is not demonstrated, but the ability to see each segment as either equal or unequal in size is still evident. This evidence of equipartitioning and equal iteration on empty number lines is a very informative piece of student understanding. Regarding the work of Lamon (1996, 2007) and Pothier and Sawada (1983), these pieces are indicative of the level of sophistication in fraction understanding a student holds.

Another notable difference found on student strategies was in regards to the limitations for teachers when working with filled number lines. When operating with filled number lines, students could use either iterative or partitive reasoning because the number lines were similarly constructed. While the context did influence the ways in which students thought about and worked through each task, the thinking was much more difficult to understand through a simple analysis of the task. Student interviews regarding student thinking on filled number line tasks were needed in order to clearly understand whether students were using iterative or partitive reasoning on each task. Teachers using filled number lines should be aware of this limitation. Teachers using filled number line tasks should have clear goals of what type of student thinking they hope to uncover, carefully select tasks to get at that thinking, and allow for discussion of student approaches in order to understand the student strategy used.

The Inaccurate Number Line Construct approach where students placed 1 “at the end” of the number line because it “goes at the end” was found more often on filled number line tasks than empty number lines. This may also support the claim that empty number lines highlight a student’s intuitive thinking whether correct or not while a filled number line may limit the student conceptual knowledge that is uncovered by the task. The Inaccurate Number Line Construct approach did not uncover partitive or iterative reasoning whereas other informal or intuitive strategies found on empty number line tasks did uncover more partitive or iterative approaches.

In some ways student strategies and thinking were similar on both empty and filled number lines. On both empty and filled number lines, students used “jumps” or “copies” to iterate. They also used “slices,” or “pieces” when referring to partitioning. Students on both tasks used unitizing language such as, “It takes *n* number of these to make the whole,” or “we need *n* number to make 1.” Interestingly, this language used by students is iterative as it talks about building a whole or the number of copies needed, but was the same language used on both tasks types and with both partitioning and iterating strategies. Both empty and filled number lines provide insight into conceptual understanding of unitizing.

Even-Numbered Denominators Versus Odd-Numbered Denominators

The influence number choice had on the student work in this study can be summarized with:

- Number choice was the least influential characteristic

The research regarding number choice on fraction tasks indicates that students will find even-numbered denominator tasks easier to solve (Hunting, 1999). Because the even-numbered denominator task were non-unit fractions and the odd-numbered denominator tasks used unit fractions, the results found that each task was similar in difficulty. This may be due to the fact that the even-numbered denominator tasks in this study used non-unit fractions with even-numbered numerators as well. Non-unit fractions may be more difficult to operate with and may have increased the difficulty of working with even-numbered denominators.

Aside from the Partitioning by Non-Unit Fraction and Iteration by Non-Unit Fraction strategies that were used, no other strategies were influenced by number choice on these tasks. Strategies were similarly used at a similar rate regardless of number choice. In this study, the number choice of the denominators seemed to have the smallest impact on student work compared to other task characteristics.

Recommendations

Developing a hierarchy of student strategies was not an aim of this study. Through review of the student strategies used in this study as well as the review of literature completed for this study, there may be a hierarchical relationship between some of the strategies. These levels of sophistication were briefly touched in the results section of this chapter, but are not explored. The results and task framework for this study could provide insight into the learning trajectory students experience or the progression students go through in developing an understanding of fractions, particularly into the less-explored area of iterating.

While non-unit fractions were not an evaluated task characteristic in this framework, it is recommended that they be studied further. Students operating with non-unit fractions with even-numbered numerators *and* denominators in this study used strategies like Partitioning by Non-Unit Fraction and Iteration by Non-Unit Fraction. These fraction types also resulted in students using partitioning and iterating strategies to solve tasks—partitioning to find the unit fraction first and then iterating to unitize.

Another number choice fraction task characteristic to explore would be with fractions greater than 1. Some strategies that emerged in this study only found on tasks with fractions greater than 1. The Distance strategies on this study were used with fractions greater than 1. These strategies are telling—they are partitioning and iterating strategies that indicate an ability to unitize.

The main goal for this study was to contribute to informing teacher pedagogy. Teachers who better understand student thinking are able to address student misconceptions and press students to greater understanding and sophistication (Hunting, 1999). Through the creation of task framework for this study and the analysis of student work, teachers can make more informed decisions about the tasks they select for students to complete. Using the information in both the literature review for this study and from

Table 4.1 teachers and researchers can make more informed decisions about the tasks they select.

Conclusion

While this study opens up many more questions to be explored, it does answer the initial research questions of this study. This study discovered very little difference in approaches to solving fraction tasks with differences in number choice, but did discover differences in approaches when students worked on iterating tasks compared to partitioning tasks. There were also differences in student work and strategies on empty number line tasks compared to filled number line tasks. The ways in which these task characteristics influenced student work are discussed at the beginning of this final chapter.

There were some areas that were influenced in very clear ways by task characteristics, but not all areas were so influenced. This may provide further encouragement for the use of number line tasks in the classroom as all tasks in this study uncovered conceptual student understanding. The parts of conceptual understanding uncovered may vary from task characteristic to task characteristic, but overall, all the number line tasks gave more information about student understanding than simply whether the student was correct or incorrect.

The results of this study validate many of the claims found in the literature review of this study. The ideas about student misconceptions of fractions, particularly regarding whole number bias, shared by Behr et al. (1983) and Ni and Zhou (2005) were discovered in some student strategies. The partitioning strategies outlined in Pothier and Sawada's (1983) work were similar to those found in the student partitioning strategies in this study. The iterating strategies in this study are similar to the partitioning strategies and indicate a similar learning trajectory as outlined in partitioning studies conducted by Brizuela (2006) , Lamon (1996), and Pothier and Sawada (1983). The claims about number lines highlighting conceptual knowledge made by Hannula (2003) were supported in this study as well.

This study answers to the call from Mitchell and Clarke (2010) to refine fraction tasks through looking at how students view and respond to work on these tasks. The student work on this task provides insight that can be used to develop tasks with even clearer objectives aimed at uncovering student conceptual understanding.

Ultimately, the results of this study provide insight into the way task characteristics influence student work. This study also provides insight into potential learning trajectories of fractions. Student work regarding fractions is influenced in different ways by different task characteristics and there are certain task characteristics that do a better job of highlighting informal or intuitive knowledge, ability to unitize, and misconceptions. The work in this study is informative, but mathematics education research would benefit from further exploration of these findings.

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APPENDIX A:

CCSS Used in Task Framework

CCSS Used in Task Framework

CCSS Standards

The tasks will be designed to assess the third grade CCSS ("Common Core State Standards for Mathematics," 2010; 2013) for Number & Operations—Fractions. The standards that these tasks will address are:

3.NF.A.2.A

Represent a fraction 1/*b* on a number line diagram by defining the interval from 0 to 1 as the whole and partitioning it into *b* equal parts. Recognize that each part has size 1/*b* and that the endpoint of the part based at 0 locates the number 1/*b* on the number line.

3.NF.A.2.B

Represent a fraction *a*/*b* on a number line diagram by marking off a lengths 1/*b* from 0. Recognize that the resulting interval has size *a*/*b* and that its endpoint locates the number *a*/*b* on the number line.

The 4^{th} grade standard shown below is a progression from the 3^{rd} grade standards.

4.NF.B.3.B

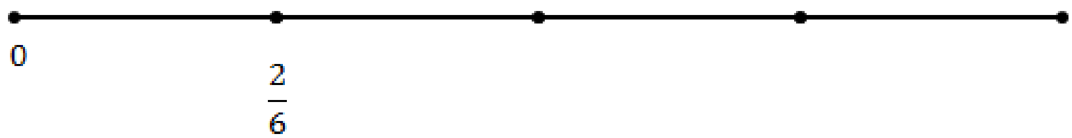
Decompose a fraction into a sum of fractions with the same denominator in more than one way, recording each decomposition by an equation. Justify decompositions, e.g., by using a visual fraction model. *Examples: 3/8 = 1/8 + 1/8 + 1/8 ; 3/8 = 1/8 + 2/8 ; 2 1/8 = 1 + 1 + 1/8 = 8/8 + 8/8 + 1/8*.

APPENDIX B

Fraction Task Form H

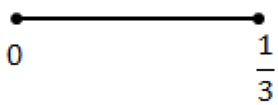
Fraction Task Form H

If the line segment below has a distance of $^2/_6$ where would 1 be?



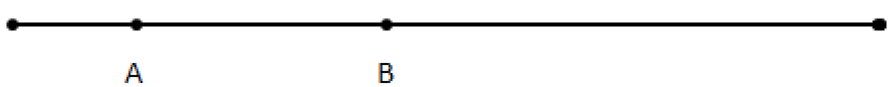
How do you know?

If the line segment below has a distance of $\frac{1}{3}$, where would 1 be?



How do you know?

If point A is equal to $\frac{1}{4}$, what is the value of Point B?



Where would 1 be?

Explain your strategy for finding each solution.

If Point A is equal to $\frac{1}{5}$, what is the value of Point B?

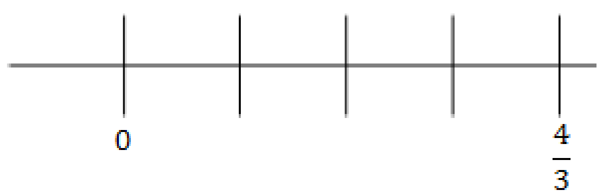
Where would 1 be?

Explain your strategy for finding each solution.

If the distance below is $\frac{8}{6}$, where would 1 be?

How do you know?

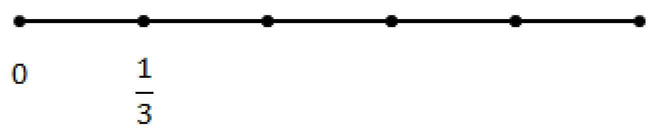
If the distance below is $\frac{4}{3}$, where would 1 be?



How do you know?

Fraction Task Form J

If the line segment below has a distance of $\frac{1}{3}$, where would 1 be?

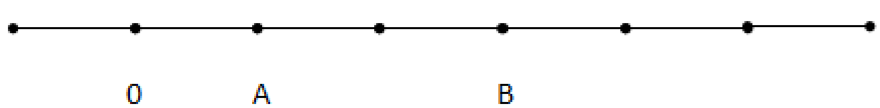


How do you know?

If the line segment below has a distance of $\frac{2}{6}$, where would 1 be?

How do you know?

If the value of Point A is $\frac{1}{4}$, what is the value of Point B?



Where would 1 be?

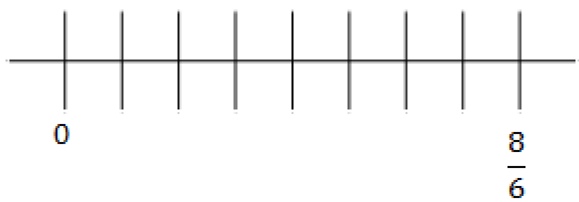
Explain your strategy for finding each solution.

If the value of Point A is $\frac{1}{5}$, what is the value of Point B?

Where would 1 be?

Explain your strategy for finding each solution.

If the distance below is $\frac{8}{6}$, where would 1 be?



How do you know?

If the distance below is $\frac{4}{3}$, where would 1 be?

How do you know?