SOLUTION TECHNIQUES AND ERROR ANALYSIS OF GENERAL CLASSES OF PARTIAL DIFFERENTIAL EQUATIONS

by

Wijayasinghe Arachchige Waruni Nisansala Wijayasinghe

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Wijayasinghe Arachchige Waruni Nisansala Wijayasinghe

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The following individuals read and discussed the thesis submitted by student Wijayasinghe Arachchige Waruni Nisansala Wijayasinghe, and they evaluated her presentation and response to questions during the final oral examination. They found that the student passed the final oral examination.

Barbara Zubik-Kowal, Ph.D.	Chair, Supervisory Committee
M. Randall Holmes, Ph.D.	Member, Supervisory Committee
Uwe Kaiser, Ph.D.	Member, Supervisory Committee

The final reading approval of the thesis was granted by Barbara Zubik-Kowal, Ph.D., Chair of the Supervisory Committee. The thesis was approved for the Graduate College by John R. Pelton, Ph.D., Dean of the Graduate College.

DEDICATION

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ඔබටයි මෙහි පිදුම!

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ABSTRACT

While constructive insight for a multitude of phenomena appearing in the physical and biological sciences, medicine, engineering and economics can be gained through the analysis of mathematical models posed in terms of systems of ordinary and partial differential equations, it has been observed that a better description of the behavior of the investigated phenomena can be achieved through the use of functional differential equations (FDEs) or partial functional differential equations (PFDEs). PFDEs or functional equations with ordinary derivatives are subclasses of FDEs. FDEs form a general class of differential equations applied in a variety of disciplines and are characterized by rates of change that depend on the state of the system. As opposed to traditional partial differential equations (PDEs), the formulation of PFDEs, and hence, their methods of solution, are generally significantly complicated by the functional dependence of the system. Consequently, mathematical analysis has become essential to address important questions on PFDEs, their properties and solutions. This thesis is devoted to a general class of parabolic PFDEs and works out the details of the proof techniques of a related paper that help to address these questions. In particular, we examine error bounds of approximate solutions with the aim to address whether or not they converge to the exact solutions as a result of refining the associated discretizations.

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CHAPTER 1

INTRODUCTION

Many real-life problems can be adequately modeled in terms of systems of differential equations, creating a general class into which all ordinary and partial differential equations fall. A variety of models written in terms of differential equations feature the independent time variable t, which plays a significant role in predicting the future behavior of certain phenomena in question with the aid of a variety of different available data, such as clinical, experimental, or field data. Although it is collected in a limited period of time, this data can be used in tandem with systems of differential equations to provide information about the future.

However, in the context of mathematical models formulated in terms of differential equations, the derivative of the solution with respect to the time variable t depends only on the solution at the present time. To obtain more realistic models that better approximate the behavior of the investigated phenomena, models written in terms of differential equations are very often improved by incorporating information about the past history of the solution, giving rise to functional differential equations. As opposed to traditional differential equations, in models written in terms of functional differential equations, the derivative of the solution depends on its values at earlier times in addition to its value at the present time. This flexible property provides an important tool in mathematical modeling with functional differential equations broadly applied in many scientific disciplines such as biology, medicine, physics, engineering, economics, etc. Throughout its long history, functional differential equations have been investigated by many authors with respect to a multitude of aspects, about which we refer the reader to [1], [2], [5]-[10], [12], [14]-[21], [23]-[28], [30], [31], [38] for ordinary functional differential equations and [3], [4], [11], [13], [22], [29]-[38] for partial functional differential equations. Aspects connected with modeling with functional differential equations are presented in [3], [4], [13], [14], [24]. One of the main problems is that many of the equations have no analytical formulae for the exact solutions and it has become essential to study their approximations in order to gain insight about the solutions to the model equations and to conduct numerical simulations. In order to get reliable approximate solutions, careful mathematical analysis of their errors has to be conducted. In this thesis, we expand on the development of [37] by filling in the details of the proofs to investigate errors of approximate solutions to a class of parabolic partial functional differential equations. For similar developments and proof techniques for partial functional differential equations as well as numerical experiments for this class of equations, we refer the reader to [4, 22, 34, 35, 36, 38].

Originating from a multitude of areas of application to the real world around us, partial functional differential equations form a general class of problems that includes partial differential equations as one if its subclasses.

This thesis is devoted to partial functional differential equations written in the form

$$\frac{\partial u}{\partial t}(x,t) = f\Big(x, t, u_{(x,t)}, \frac{\partial u}{\partial x}(x,t), \frac{\partial^2 u}{\partial x^2}(x,t)\Big), \tag{1.1}$$

where $u \in C(B, \mathbb{R})$ is an unknown function (see below for B), $x \in [-L, L]$, $t \in [0, T]$ and $f : [-L, L] \times [0, T] \times C(D, \mathbb{R}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$, represents any given continuous function. Another generalization in (1.1) is introduced by the argument $u_{(x,t)}$; for any fixed $x \in [-L, L]$ and $t \in [0, T]$, $u_{(x,t)}$ is a function. Such a functional argument allows to generate differential equations with e.g. a time delay and shift in space. Unlike for classical partial differential equations, the third argument $u_{(x,t)}$ in (1.1) is not a real value but a real function defined on D (see Figure 1 for D) and called a functional argument. The functional argument $u_{(x,t)} \in C(D, \mathbb{R})$ for $x \in [-L, L], t \in [0, T]$, and $u \in C(B, \mathbb{R})$, with $L, T > 0$, $B = [-\hat{L}, \hat{L}] \times [-\tau_0, T]$, $D = [-\hat{\tau}, \hat{\tau}] \times [-\tau_0, 0]$, $\hat{\tau}, \tau_0 \geq 0$, $\hat{L} = L + \hat{\tau}$, is defined as

$$u_{(x,t)}(s, \tau) = u(x + s, t + \tau), \quad (s, \tau) \in D.$$

Equation (1.1) is supplemented with the following initial conditon

$$u(x, t) = u_0(x, t), \quad t \in [-\tau_0, 0],\ x \in [-\hat{L}, \hat{L}], \tag{1.2}$$

and boundary condition

$$u(x, t) = g(x, t), \quad t \in [0, T],\ x \in [-\hat{L}, -L] \cup [L, \hat{L}]. \tag{1.3}$$

Here, $f : [-L, L] \times [0, T] \times C(D, \mathbb{R}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ is a continuous function, and u_0, g are given initial and boundary functions, respectively.

The partial differential equation (1.1) describes a general class of problems. For example, if f is defined by

$$f(x, t, \gamma, p, q) = \epsilon q + \gamma(0, 0)(1 - \gamma(0, -\tau_0)), \tag{1.4}$$

where ϵ is a positive constant, then (1.1) can be written in the following form

$$\frac{\partial u}{\partial t}(x,t) = \epsilon \frac{\partial^2 u}{\partial x^2}(x,t) + u(x,t)(1 - u(x, t - \tau_0)).$$

Here, the functional argument is given by $t - \tau_0$. Another class of examples can be generated by defining f by

$$f(x,t,\omega,p,q) = a(t)q + a(t) \int_{-\tau_0}^{0} \int_{-\hat{\tau}}^{\hat{\tau}} \omega(s,\tau) ds d\tau, \tag{1.5}$$

where $a \in C([0,T], \mathbb{R}_+)$. Then, (1.1) can be written in the general form

$$\frac{\partial u}{\partial t}(x,t) = a(t)\frac{\partial^2 u}{\partial x^2}(x,t) + a(t) \int_{-\tau_0}^{0} \int_{-\hat{\tau}}^{\hat{\tau}} u(x+s, t+\tau) ds d\tau.$$

An important subclass of the class of partial functional differential equations captured by (1.1) that may be most familiar to most readers is the entire class of partial differential equations written in the form

$$\frac{\partial u}{\partial t}(x,t) = \tilde{f}\Big(x, t, u(x,t), \frac{\partial u}{\partial x}(x,t), \frac{\partial^2 u}{\partial x^2}(x,t)\Big),$$

where $\tilde{f} : [-L, L] \times [0,T] \times \mathbb{R}^3 \to \mathbb{R}$ is any given function. This entire subclass is another example that can be generated from the general class of equations (1.1) by suitably defining f appearing on the right-hand side.

We have seen that not only do partial functional differential equations offer a modeling approach that more realistically portrays a wide class of real-world phenomena, but also that their generality encompasses wide classes of subproblems, some of which many readers have been acquainted with already in various real-world contexts.

The following figure illustrates the domain of the functional argument.

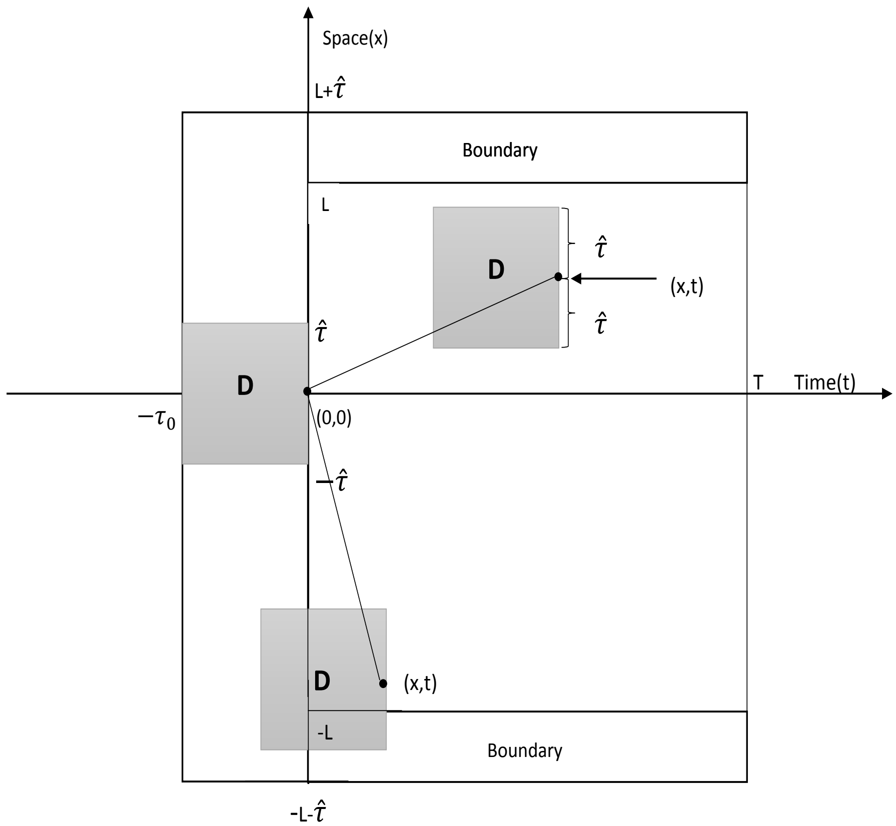

Figure 1.1: Graphical illustration of initial and boundary sets.

CHAPTER 2

NUMERICAL SOLUTIONS FOR A GENERAL CLASS OF SYSTEMS OF ORDINARY FUNCTIONAL DIFFERENTIAL EQUATIONS: APPROXIMATIONS AND DEFINITIONS

The general class of equations given by (1.1) is written in terms of arbitrary functions f and, in many cases, analytic solutions to these equations defined in the continuum sense are unknown and approximated by numerical solutions computed on discrete subsets. For any element of any of the discrete subsets (such elements are referred to as *grid-points*), there exists an open neighborhood that is disjoint from the other grid-points. The discrete subsets are finite and determined by parameters, affecting the coarseness of the corresponding discretizations. In the literature, the process of semi-discretization has been also referred to as the *Method of Lines*. Letting the values of these parameters approach zero causes the corresponding discretizations to become finer. The goal of the thesis is to study error bounds of the approximate solutions defined on the discrete subsets and to address the question of whether or not they get closer to the exact solutions as the discretization becomes finer – a property desired of discretization.

In this chapter, we construct approximate solutions to the general problem (1.1)–(1.3). With this aim, we introduce spatial grid-points x_j that we will use to replace the spatial derivatives in (1.1) by discrete operators. Let the spatial step-size $h > 0$ and M, $\hat{M}$ be such that $Mh = L$, $\hat{M}h = \hat{\tau}$ and $M, \hat{M} \in \mathbb{N}$. Then, we define $x_j = jh$, for $j = 0, \pm 1, \pm 2, .., \pm \tilde{M}$, where $\tilde{M} = M + \hat{M}$. Henceforth, we also use the notation $M' = M - 1$ and $n = 2M - 1$.

For each discretization parameter h, we define the vector function $F = (F_{-M'}, \ldots, F_{M'}) : [0,T] \times \mathbb{R}^n \times C([-\tau_0, 0], \mathbb{R}^n) \to \mathbb{R}^n$, the initial function $\tilde{u}_0 : [-\tau_0, 0] \to \mathbb{R}^n$, and the initial value problem

$$\begin{cases} \dot{\upsilon}(t) &= F(t, \upsilon(t), \upsilon_t), \quad t \in [0,T], \\ \upsilon(t) &= \tilde{u}_0(t), \quad t \in [-\tau_0, 0], \end{cases} \tag{2.1}$$

whose solution $\upsilon(t) \in \mathbb{R}^n$ depends on h and (as it will be shown in the next chapters) converges to $(u(x_{-M'}, t), ..., u(x_{M'}, t))$, as $h \to 0$. Note that $n \to \infty$, as $h \to 0$, and that the dimension of the system (2.1) increases as the discretization becomes finer.

The third input $\upsilon_t \in C([-\tau_0, 0], \mathbb{R}^n)$ in (2.1) is defined by

$$\upsilon_t(\tau) = \upsilon(t + \tau),$$

for $\tau \in [-\tau_0, 0]$, where $t \in [0,T]$ and $\upsilon \in C([-\tau_0, T], \mathbb{R}^n)$.

The vector function F in (2.1) can be defined as follows

$$F_i(t, z, \omega) = f(x_i, t, \mathcal{L}_{i,t}\omega, \delta_{i,t}z, \delta^2_{i,t}z), \tag{2.2}$$

where $i = 0, \pm 1, ..., \pm M'$, $t \in [0,T]$, $z \in \mathbb{R}^n$, and $\omega = (\omega_{-M'}, ..., \omega_{M'}) \in C([-\tau_0, 0], \mathbb{R}^n)$. The operators $\mathcal{L}_{i,t} : C([-\tau_0, 0], \mathbb{R}^n) \to C(D, \mathbb{R})$ are defined in the following way

$$[\mathcal{L}_{i,t}\omega](s,\tau) = \frac{x_{k+1}-s}{h}\omega^t_{k+i}(\tau) + \frac{s-x_k}{h}\omega^t_{k+1+i}(\tau),$$

where $s \in [-\hat{\tau}, \hat{\tau}]$, $\tau \in [-\tau_0, 0]$, and $k \in \mathbb{N}$ is such that $x_k \leqslant s \leqslant x_{k+1}$ and

$$\omega^t_j(\tau) = \begin{cases} \omega_j(\tau), & \text{for} \quad j = 0, \pm 1, ..., \pm M', \\ g(x_j, t+\tau), & \text{for} \quad j = \pm M, \ldots, \pm \tilde{M}, \end{cases}$$

where g is defined in (1.3). The discrete operators $\delta_{i,t}$ and $\delta^2_{i,t}$ in (2.2) are defined for $t \in [0,T]$, $i = 0, \pm 1, ..., \pm M'$, and $z = (z_{-M'}, ..., z_{M'}) \in \mathbb{R}^n$, by

$$\delta_{i,t} z = \frac{z^t_{i+1} - z^t_{i-1}}{2h}, \tag{2.3}$$

$$\delta^2_{i,t} z = \frac{z^t_{i+1} - 2z^t_i + z^t_{i-1}}{h^2}, \tag{2.4}$$

where the vector $z^t = (z^t_{-M}, ... z^t_M) \in \mathbb{R}^{n+2}$ is defined by

$$z^t_i = \begin{cases} g(x_i, t), & \text{for } i = \pm M, \\ z_i, & \text{for } i = 0, \pm 1, \ldots, \pm (M-1). \end{cases}$$

As it will be shown in Chapter 4, the operators (2.3) and (2.4) approximate the first and second order derivatives (respectively) at the point x_i.

The initial function $\tilde{u}_0 : [-\tau_0, 0] \to \mathbb{R}^n$ in (2.1) is defined by

$$\tilde{u}_0(t) = (u_0(x_{-M'}, t), \ldots, u_0(x_{M'}, t)),$$

for $t \in [-\tau_0, 0]$, where $u_0 : [-\hat{L}, \hat{L}] \times [-\tau_0, 0] \to \mathbb{R}$ is the initial function given in the original problem. The goal of this thesis is to work through the proof techniques of [37] to help address the question of whether or not the components of the solution

$v(t) \in \mathbb{R}^n$ to (2.1) converge, as $h \to 0$, to the values $u(x_i, t)$ of the exact solution to the problem (1.1) with (1.2), (1.3).

CHAPTER 3

ITERATIVE PROCESSES WITH GENERAL SPLITTING FUNCTIONS FOR SEMI-DISCRETE DIFFERENTIAL FUNCTIONAL SYSTEMS

In this chapter, we construct iterative procedures for solving the general problem (2.1) and thus (1.1)–(1.3). The process is summarised in the form of the following algorithm. Let $\upsilon^{(0)} : [-\tau_0, T] \to \mathbb{R}^n$ be an arbitrary function. We define the sequence of vector functions $\upsilon^{(k)} : [-\tau_0, T] \to \mathbb{R}^n$, where $k = 0, 1, 2, \ldots$, recursively, by

$$\begin{aligned} \dot{\upsilon}^{(k+1)}(t) &= G(t, \upsilon^{(k+1)}(t), \upsilon^{(k)}(t), \upsilon_t^{(k)}), \quad t \in [0, T], \\ \upsilon^{(k+1)}(t) &= \tilde{u}_0(t), \quad t \in [-\tau_0, 0]. \end{aligned} \tag{3.1}$$

The functions G are chosen according to the given F and are referred to as *splitting functions*. The function $\upsilon^{(0)}$ is referred to as a starting function, and the functions $\upsilon^{(k)}$ are referred to as the successive iterates.

For example, if G is defined by

$$G_i(t, \zeta, z, w) = F_i(t, z_1, \ldots, z_{i-1}, \zeta_i, z_{i+1}, \ldots, z_n, w),$$

for $i = 1, 2, \ldots, n$, where F_i is defined, for example, by (2.2), then (3.1) generates the following iterative process of the Picard type in the functional sense

$$\dot{\upsilon}_i^{(k+1)}(t) = F_i\Big(t, \upsilon_1^{(k)}(t), ..., \upsilon_{i-1}^{(k)}, \upsilon_i^{(k+1)}(t), \upsilon_{i+1}^{(k)}(t), ..., \upsilon_n^{(k)}(t), \upsilon_t^{(k)}\Big) \qquad (3.2)$$

or if G is defined by

$$G_i(t, \zeta, z, w) = F_i(t, \zeta_1, \ldots, \zeta_i, z_{i+1}, \ldots, z_n, w),$$

then (3.1) generates another iterative process of the Picard type in the functional sense

$$\dot{\upsilon}_i^{(k+1)}(t) = F_i\Big(t, \upsilon_1^{(k+1)}(t), ..., \upsilon_{i-1}^{(k+1)}(t), \upsilon_i^{(k+1)}(t), \upsilon_{i+1}^{(k)}(t), ...\upsilon_n^{(k)}(t), \upsilon_t^{(k)}\Big). \qquad (3.3)$$

A common feature of these two processes is that the functional argument in F is given by the previous iterate denoted by the superscript k, as is the case for standard Picard iterations. The first of these processes has been referred to in the literature also by terms such as *Jacobi-Picard scheme, Jacobi-Picard waveform relaxation scheme, Jacobi-Picard waveform method, Jacobi-Picard iteration scheme, or simply Jacobi waveform relaxation.* The second of these processes has been referred to in the literature also by terms such as *Gauss-Seidel-Picard scheme, Gauss-Seidel-Picard waveform relaxation scheme, Gauss-Seidel-Picard waveform method, Gauss-Seidel-Picard iteration scheme, or simply Gauss-Seidel waveform relaxation .* Both of these processes have also been collectively referred to by terms such as *waveform relaxation*, *dynamic iteration* or simply, *iterative processes.* See e.g. [26], [35] for different naming conventions. If the equaton in question is a classical equation without the functional argument, then the name 'Picard' is not used in the above terms. The two iterative processes of the Picard type in the functional sense differ in the dependence of the indices of the successive iterates, and one may be preferable to the other depending on the problem.

In our error analysis, we will use the following definitions:

$$\begin{aligned} e(t) &= U(t) - \upsilon(t), \\ e^{(k)}(t) &= \upsilon(t) - \upsilon^{(k)}(t), \\ E^{(k)}(t) &= U(t) - \upsilon^{(k)}(t), \end{aligned}$$

where $t \in [-\tau_0, T]$, $k = 0, 1, \ldots$, $U(t) = \big(U_1(t), \ldots, U_n(t)\big)$, $U_i(t) = u(x_i, t)$, and the functions $u(x,t)$, $\upsilon(t)$, $\upsilon^{(k)}(t)$ are solutions to the three problems (1.1)–(1.3), (2.1), (3.1), respectively. Notice that $e(t)$ is the error of semi-discretization (2.1) and $e^{(k)}(t)$ is the error of iterative process (3.1), while $E^{(k)}(t)$ is the error of both the semi-discretization (2.1) and iterative process (3.1). The prefix *semi* indicates that the discretization corresponds to the spatial variable only. As mentioned earlier, another name for the process of semi-discretization is the *Method of Lines*.

Henceforth, we will use the following assumptions. Suppose that, for

$$F : [0, T] \times \mathbb{R}^n \times C([0, T], \mathbb{R}^n) \to \mathbb{R}^n,$$

there exist positive continuous functions $\gamma_n \in C([0, T], \mathbb{R}_+)$ such that

$$\lim_{n \to \infty} \gamma_n(t) = 0 \tag{3.4}$$

and

$$\big|\dot{U}_i(t) - F_i(t, U(t), U_t)\big| \le \gamma_n(t), \tag{3.5}$$

for all $t \in [0, T]$ and $i = -M', \ldots, M'$. Moreover, suppose that $u \in C^{(4)}(B, \mathbb{R})$ (class of 4-times continuously differentiable funtions from B to $\mathbb{R}$) and that for the given function

$$f : [-L, L] \times [0, T] \times C(D, \mathbb{R}) \times \mathbb{R} \times \mathbb{R} \to \mathbb{R},$$

there exist positive continuous functions $\kappa_1, \kappa_2, \kappa_3 \in C([0, T], \mathbb{R}_+)$ such that

$$\begin{aligned} |f(x, t, \omega, p, q) - f(x, t, \bar{\omega}, \bar{p}, \bar{q})| &\leq \kappa_1(t)|p - \bar{p}| + \kappa_2(t)|q - \bar{q}| \\ &+ \kappa_3(t) \max_{(s,\tau)\in D} |\omega(s, \tau) - \bar{\omega}(s, \tau)|, \end{aligned} \tag{3.6}$$

for all $x \in [-L, L]$, $t \in [0, T]$, $\omega, \bar{\omega} \in C(D, \mathbb{R})$, $p, q, \bar{p}, \bar{q} \in \mathbb{R}$.

For the iterative processes applied to (2.1), we assume that the functions

$$G : [0, T] \times \mathbb{R}^n \times \mathbb{R}^n \times C([-\tau_0, 0], \mathbb{R}^n) \to \mathbb{R}^n$$

satisfy

$$G(t, r(t), r(t), r_t) = F(t, r(t), r_t), \tag{3.7}$$

for all $t \in [0, T]$, $r \in C([-\tau_0, T], \mathbb{R}^n)$ (note that for any $t \in [0, T]$, $r_t \in C([-\tau_0, 0], \mathbb{R}^n)$) and there exist continuous functions $\mu_1 \in C([0, T], \mathbb{R})$, $\mu_2, \mu_3 \in C([0, T], \mathbb{R}_+)$ such that

$$\|\varsigma - \bar{\varsigma} - \varepsilon[G(t, \varsigma, z, \omega) - G(t, \bar{\varsigma}, z, \omega)]\|_n \geq (1 - \varepsilon\mu_1(t))\|\varsigma - \bar{\varsigma}\|_n, \tag{3.8}$$

$$\|G(t, \varsigma, z, \omega) - G(t, \varsigma, \bar{z}, \omega)\|_n \leq \mu_2(t)\|z - \bar{z}\|_n, \tag{3.9}$$

$$\|G(t, \varsigma, z, \omega) - G(t, \varsigma, z, \bar{\omega})\|_n \leq \mu_3(t)\|\omega - \bar{\omega}\|_n^0, \tag{3.10}$$

for $\varepsilon \geq 0$, $t \in [0, T]$, $\varsigma, \bar{\varsigma}, z, \bar{z} \in \mathbb{R}^n$, $\omega, \bar{\omega} \in C([-\tau_0, 0], \mathbb{R}^n)$, where $\|\cdot\|_n$ is an arbitrary norm in $\mathbb{R}^n$ and

$$\|\omega\|_n^0 = \max_{\tau\in[-\tau_0,0]} \|\omega(\tau)\|_n,$$

for $\omega \in C([-\tau_0, 0], \mathbb{R}^n)$.

In the following chapters, we will show that the conditions (3.7)–(3.10) are satisfied for the iterative processes (3.2) and (3.3) of the Picard type in the functional sense and we will use them to derive error bounds for the general numerical schemes (2.1) and (3.1).

CHAPTER 4

CONSISTENCY PROPERTIES OF NUMERICAL SCHEMES FOR PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

In this chapter, we present results that we will apply to derive error bounds for numerical solutions to partial functional differential equations. In the theorem below, the notation $C^{(4)}(B,\mathbb{R})$ refers to the class of 4-times continuously differentiable functions from B to $\mathbb{R}$.

Theorem 4.1 ([37], Lemma 3.1)**.** *If $u \in C^{(4)}(B,\mathbb{R})$ and f satisfies condition (3.6), then F defined by (2.2) satisfies condition (3.5) with*

$$\gamma_n(t) = \frac{c}{n^2}\big(\kappa_1(t) + \kappa_2(t) + \kappa_3(t)\big), \tag{4.1}$$

for $t \in [0,T]$, where c is a positive constant that is independent on n.

Proof. Let $i \in \{0, \pm 1, \ldots, \pm M'\}$ and $t \in [0,T]$ be arbitrary. From the definition of $U(t)$, equation (1.1), and definition (2.2), we have

$$
\begin{aligned}
\left|\dot{U}_i(t) - F_i(t, U(t), U_t)\right| &= \left|\frac{\partial u}{\partial t}(x_i, t) - F_i(t, U(t), U_t)\right| \\
&= \left|f\Big(x_i, t, u_{(x_i,t)}, \frac{\partial u}{\partial x}(x_i, t), \frac{\partial^2 u}{\partial x^2}(x_i, t)\Big) - F_i\big(t, U(t), U_t\big)\right| \\
&= \Big|f\Big(x_i, t, u_{(x_i,t)}, \frac{\partial u}{\partial x}(x_i, t), \frac{\partial^2 u}{\partial x^2}(x_i, t)\Big) \\
&\qquad -f\big(x_i, t, \mathcal{L}_{i,t} U_t, \delta_{i,t} U(t), \delta_{i,t}^2 U(t)\big)\Big|.
\end{aligned}
$$

Hence

$$
\begin{aligned}
\left|\dot{U}_i(t) - F_i(t, U(t), U_t)\right| &\leq \kappa_1(t)\left|\frac{\partial u}{\partial x}(x_i, t) - \delta_{i,t} U(t)\right| \\
&+ \kappa_2(t)\left|\frac{\partial^2 u}{\partial x^2}(x_i, t) - \delta_{i,t}^2 U(t)\right| \qquad (4.2) \\
&+ \kappa_3(t) \max_{(s,\tau)\in D}\left|u_{(x_i,t)}(s, \tau) - \mathcal{L}_{i,t} U_t(s, \tau)\right|.
\end{aligned}
$$

By Taylor's Theorem applied to $u(x_{i+1}, t)$ and $u(x_{i-1}, t)$, since $u \in C^{(4)}(B, \mathbb{R})$, we have,

$$
u(x_{i+1}, t) = u(x_i, t) + \frac{h}{1!} \cdot \frac{\partial u}{\partial x}(x_i, t) + \frac{h^2}{2!} \cdot \frac{\partial^2 u}{\partial x^2}(x_i, t) + \frac{h^3}{3!} \cdot \frac{\partial^3 u}{\partial x^3}(\theta_i, t), \qquad (4.3)
$$

with $\theta_i \in (x_i, x_{i+1})$, and

$$
u(x_{i-1}, t) = u(x_i, t) - \frac{h}{1!} \cdot \frac{\partial u}{\partial x}(x_i, t) + \frac{h^2}{2!} \cdot \frac{\partial^2 u}{\partial x^2}(x_i, t) - \frac{h^3}{3!} \cdot \frac{\partial^3 u}{\partial x^3}(\xi_i, t), \qquad (4.4)
$$

with $\xi_i \in (x_{i-1}, x_i)$. From (4.3) and (4.4), we have

$$\left|\frac{\partial u}{\partial x}(x_i,t)-\delta_{i,t}U(t)\right| = \left|\frac{\partial u}{\partial x}(x_i,t)-\frac{U_{i+1}(t)-U_{i-1}(t)}{2h}\right|$$
$$= \left|\frac{\partial u}{\partial x}(x_i,t)-\frac{u(x_{i+1},t)-u(x_{i-1},t)}{2h}\right|$$

$$= \left|\frac{\partial u}{\partial x}(x_i,t)-\frac{1}{2h}\left(2h\cdot\frac{\partial u}{\partial x}(x_i,t)+\frac{h^3}{6}\cdot\frac{\partial^3 u}{\partial x^3}(\theta_i,t)+\frac{h^3}{6}\cdot\frac{\partial^3 u}{\partial x^3}(\xi_i,t)\right)\right|$$
$$= \frac{h^2}{12}\left|\frac{\partial^3 u}{\partial x^3}(\theta_i,t)+\frac{\partial^3 u}{\partial x^3}(\xi_i,t)\right|.$$

Since $u\in C^{(4)}(B,\mathbb{R})$, there exists a constant $C>0$ such that

$$\left|\frac{\partial^3 u}{\partial x^3}(x,t)\right|\le C,$$

for all $(x,t)\in B$, and

$$\left|\frac{\partial u}{\partial x}(x_i,t)-\delta_{i,t}U(t)\right| \le \frac{Ch^2}{6}.$$

From this and $n+1=\dfrac{2L}{h}$, we get

$$\left|\frac{\partial u}{\partial x}(x_i,t)-\delta_{i,t}U(t)\right|\le\frac{C}{6}\cdot\frac{4L^2}{(n+1)^2}<\frac{2CL^2}{3n^2}. \tag{4.5}$$

For the second term of (4.2), we obtain

$$\begin{aligned}\left|\frac{\partial^2 u}{\partial x^2}(x_i,t)-\delta^2_{i,t}U(t)\right| &= \left|\frac{\partial^2 u}{\partial x^2}(x_i,t)-\frac{U_{i+1}(t)-2U_i(t)+U_{i-1}(t)}{h^2}\right|\\ &= \left|\frac{\partial^2 u}{\partial x^2}(x_i,t)-\frac{u(x_{i+1},t)-2u(x_i,t)+u(x_{i-1},t)}{h^2}\right|\end{aligned} \tag{4.6}$$

and apply

$$
\begin{aligned}
u(x_{i+1},t) &= u(x_i,t) + \frac{h}{1!}\cdot\frac{\partial u}{\partial x}(x_i,t) + \frac{h^2}{2!}\cdot\frac{\partial^2 u}{\partial x^2}(x_i,t) + \frac{h^3}{3!}\cdot\frac{\partial^3 u}{\partial x^3}(x_i,t) \\
&+ \frac{h^4}{4!}\cdot\frac{\partial^4 u}{\partial x^4}(\tilde{\theta}_i,t), \\
u(x_{i-1},t) &= u(x_i,t) - \frac{h}{1!}\cdot\frac{\partial u}{\partial x}(x_i,t) + \frac{h^2}{2!}\cdot\frac{\partial^2 u}{\partial x^2}(x_i,t) - \frac{h^3}{3!}\cdot\frac{\partial^3 u}{\partial x^3}(x_i,t) \\
&+ \frac{h^4}{4!}\cdot\frac{\partial^4 u}{\partial x^4}(\tilde{\xi}_i,t),
\end{aligned}
\tag{4.7}
$$

with $\tilde{\theta}_i \in (x_i, x_{i+1})$ and $\tilde{\xi}_i \in (x_{i-1}, x_i)$, respectively. From (4.6) and (4.7), we have

$$
\begin{aligned}
&\left|\frac{\partial^2 u}{\partial x^2}(x_i,t) - \delta^2_{i,t}U(t)\right| \\
&\qquad = \left|\frac{\partial^2 u}{\partial x^2}(x_i,t) - \frac{1}{h^2}\left(h^2\frac{\partial^2 u}{\partial x^2}(x_i,t) + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4}(\tilde{\theta}_i,t) + \frac{h^4}{24}\frac{\partial^4 u}{\partial x^4}(\tilde{\xi}_i,t)\right)\right| \\
&\qquad = \frac{h^2}{24}\left|\frac{\partial^4 u}{\partial x^4}(\tilde{\theta}_i,t) + \frac{\partial^4 u}{\partial x^4}(\tilde{\xi}_i,t)\right|.
\end{aligned}
$$

Since $u \in C^{(4)}(B,\mathbb{R})$, there exists a constant $C > 0$ such that

$$
\left|\frac{\partial^4 u}{\partial x^4}(x,t)\right| \leq C,
$$

for all $(x,t) \in B$. From this and $h = \dfrac{2L}{n+1}$, we get

$$
\left|\frac{\partial^2 u}{\partial x^2}(x_i,t) - \delta^2_{i,t}U(t)\right| \leq \frac{Ch^2}{12} = \frac{4L^2C}{12(n+1)^2} < \frac{L^2C}{3n^2}. \tag{4.8}
$$

Finally, for the third term in (4.2), we get

$$\left|u_{(x_i,t)}(s,\tau) - [\mathcal{L}_{i,t}U_t](s,\tau)\right| = \left|u(x_i+s,t+\tau) - \frac{(x_{k+1}-s)}{h}\cdot(U_t)^t_{k+i}(\tau) - \frac{(s-x_k)}{h}\cdot(U_t)^t_{k+1+i}(\tau)\right|, \tag{4.9}$$

for $(s,\tau) \in D$ and $k \in \mathbb{N}$ such that $x_k \le s \le x_{k+1}$. Since $(U_t)^t_j(\tau) = (U_t)_j(\tau) = U_j(t+\tau) = u(x_j, t+\tau)$, for $j = k+i$ and $j = k+1+i$, from (4.9), we get

$$\begin{aligned}
&\left|u_{(x_i,t)}(s,\tau) - [\mathcal{L}_{i,t}U_t](s,\tau)\right| \\
&= \left|u(x_i+s,t+\tau) - \frac{(x_{k+1}-s)}{h}u(x_k+x_i,t+\tau) - \frac{(s-x_k)}{h}u(x_{k+1}+x_i,t+\tau)\right|
\end{aligned}$$

and, by Taylor's Theorem, we obtain

$$\begin{aligned}
&\left|u_{(x_i,t)}(s,\tau) - [\mathcal{L}_{i,t}U_t](s,\tau)\right| \\
&= \Bigg|u(x_i+s,t+\tau) - \frac{(x_{k+1}-s)}{h}\Bigg[u(x_i+s,t+\tau) + \frac{(x_i+x_k-x_i-s)}{1!}\frac{\partial u}{\partial x}(x_i+s,t+\tau) \\
&\quad + \frac{(x_i+x_k-x_i-s)^2}{2!}\frac{\partial^2 u}{\partial x^2}(\hat{\theta}_i,t+\tau)\Bigg] - \frac{(s-x_k)}{h}\Bigg[u(x_i+s,t+\tau) \\
&\quad + \frac{(x_i+x_{k+1}-x_i-s)}{1!}\frac{\partial u}{\partial x}(x_i+s,t+\tau) + \frac{(x_i+x_{k+1}-x_i-s)^2}{2!}\frac{\partial^2 u}{\partial x^2}(\hat{\xi}_i,t+\tau)\Bigg]\Bigg| \\
&= \Bigg|u(x_i+s,t+\tau)\left[1 - \frac{x_{k+1}-s}{h} - \frac{s-x_k}{h}\right] \\
&\quad - \frac{\partial u}{\partial x}(x_i+s,t+\tau)\left[\frac{x_{k+1}-s}{h}(x_k-s) + \frac{s-x_k}{h}(x_{k+1}-s)\right] \\
&\quad - \frac{\partial^2 u}{\partial x^2}(\hat{\theta}_i,t+\tau)\frac{(x_k-s)^2}{2}\frac{(x_{k+1}-s)}{h} - \frac{\partial^2 u}{\partial x^2}(\hat{\xi}_i,t+\tau)\frac{(s-x_k)}{h}\frac{(x_{k+1}-s)^2}{2}\Bigg|.
\end{aligned}$$

Upon rearranging this and using the inequality

$$\left|\frac{\partial^2 u}{\partial x^2}(x,t)\right| \leq C,$$

with $C > 0$, we get

$$\begin{aligned}
\left|u_{(x_i,t)}(s,\tau) - [\mathcal{L}_{i,t}U_t](s,\tau)\right| &\leq \frac{C}{2h}\left[(x_k - s)^2(x_{k+1} - s) + (s - x_k)(x_{k+1} - s)^2\right] \\
&\leq \frac{C}{2h}\left[h^2(x_{k+1} - s) + h^2(s - x_k)\right] \\
&= \frac{Ch}{2}\left[x_{k+1} - s + s - x_k\right] = \frac{Ch^2}{2}
\end{aligned}$$

Therefore, since $h = \dfrac{2L}{n+1}$, we get

$$\left|u_{(x_i,t)}(s,\tau) - [\mathcal{L}_{i,t}U_t](s,\tau)\right| \leq \frac{2CL^2}{(n+1)^2} < \frac{2CL^2}{n^2}. \tag{4.10}$$

From (4.2), (4.5), (4.8), and (4.10), we get

$$|\dot{U}_i(t) - F_i(t, U(t), U_t)| \leq \kappa_1(t)\frac{2CL^2}{3n^2} + \kappa_2(t)\frac{CL^2}{3n^2} + \kappa_3(t)\frac{2CL^2}{n^2},$$

which shows that (3.5) holds with, for example, $c = 2CL^2$ in (4.1), and finishes the proof. □

Corollary 4.1 ([37], Corollary 3.1)**.** *Suppose that there exists a constant $d > 0$ such that*

$$\max\left\{\left|u(x,t)\right| : \quad x \in [-L, L], \quad t \in [0,T]\right\} \leq d$$

where u is a solution of equation (1.1) with f defined by (1.4) for a class of functions $w \in C(D, \mathbb{R})$ such that $max\{|w(s,\tau)| : (s,\tau) \in D\} \le d$. Then the function f satisfies condition (3.6). Moreover, if u is of class $C^4(B, \mathbb{R})$, then F defined by (2.2) satisfies condition (3.5).

Proof. In order for Theorem 4.1 to be applied, it suffices to show (3.6). Let $x \in [-L, L]$, $t \in [0, T]$, $w, \bar{w} \in C(D, \mathbb{R})$, $p, q, \bar{p}, \bar{q} \in \mathbb{R}$ be arbitrary. From the definition of f, we get

$$\begin{aligned}
& f(x,t,w,p,q) - f(x,t,\bar{w},\bar{p},\bar{q}) \\
& = \varepsilon q + w(0,0)\big(1 - w(0,-\tau_0)\big) - \varepsilon\bar{q} - \bar{w}(0,0)\big(1 - \bar{w}(0,-\tau_0)\big) \\
& = \varepsilon(q-\bar{q}) + w(0,0) - \bar{w}(0,0) - w(0,0)w(0,-\tau_0) + w(0,0)\bar{w}(0,-\tau_0) \\
& -w(0,0)\bar{w}(0,-\tau_0) + \bar{w}(0,0)\bar{w}(0,-\tau_0) \\
& = \varepsilon(q-\bar{q}) + \big(w(0,0) - \bar{w}(0,0)\big) - w(0,0)\big(w(0,-\tau_0) - \bar{w}(0,-\tau_0)\big) \\
& -\bar{w}(0,-\tau_0)\big(w(0,0) - \bar{w}(0,0)\big).
\end{aligned}$$

and

$$\begin{aligned}
& |f(x,t,w,p,q) - f(x,t,\bar{w},\bar{p},\bar{q})| \\
& \le \varepsilon|q-\bar{q}| + \big|w(0,0) - \bar{w}(0,0)\big| + \big|w(0,0)\big| \cdot \big|w(0,-\tau_0) - \bar{w}(0,-\tau_0)\big| \\
& +\big|\bar{w}(0,-\tau_0)\big| \cdot \big|w(0,0) - \bar{w}(0,0)\big| \\
\le \; & \varepsilon|q-\bar{q}| + \Big(1 + \big|\bar{w}(0,-\tau_0)\big| + \big|w(0,0)\big|\Big) \max_{(s,\tau)\in D} \big|w(s,\tau) - \bar{w}(s,\tau)\big| \\
\le \; & \varepsilon|q-\bar{q}| + (1+2d) \max_{(s,\tau)\in D} \big|w(s,\tau) - \bar{w}(s,\tau)\big|.
\end{aligned}$$

Therefore, f satisfies (3.6) with

$$\kappa_1(t) = 0, \quad \kappa_2(t) = \varepsilon, \quad \kappa_3(t) = 1 + 2d.$$

We now apply Theorem 4.1 and conclude that F defined by (2.2) satisfies (3.5) with

$$\gamma_n(t) = \frac{c}{n^2}(\varepsilon + 1 + 2d),$$

which finishes the proof. □

Corollary 4.2 ([37], Corollary 3.2)**.** *Let $d > 0$ be as in Corollary 4.1 with f defined by (1.5). Then f satisfies condition (3.6). Moreover, if u is of class $C^4(B, \mathbb{R})$, then F defined by (2.2) and (1.5) satisfies condition (3.5).*

Proof. We apply Theorem 4.1 with f defined by (1.5). Let $x \in [-L, L]$, $t \in [0, T]$, $\omega, \hat{\omega} \in C(D, \mathbb{R})$, $p, q, \hat{p}, \hat{q} \in \mathbb{R}$ be arbitrary. Since a is a positive function, from (1.5), we get

$$\begin{aligned}
|f(x,t,\omega,p,q) - f(x,t,\hat{\omega},\hat{p},\hat{q})| &= \Bigg| a(t)q + a(t)\int_{-\tau_0}^{0}\int_{-\hat{\tau}}^{\hat{\tau}} \omega(s,\tau)dsd\tau \\
&\qquad - a(t)\hat{q} - a(t)\int_{-\tau_0}^{0}\int_{-\hat{\tau}}^{\hat{\tau}} \hat{\omega}(s,\tau)dsd\tau \Bigg| \\
&\le a(t)|q-\hat{q}| + a(t)\left|\int_{-\tau_0}^{0}\int_{-\hat{\tau}}^{\hat{\tau}} (\omega(s,\tau) - \hat{\omega}(s,\tau))dsd\tau\right| \\
&\le a(t)|q-\hat{q}| + a(t)\int_{-\tau_0}^{0}\int_{-\hat{\tau}}^{\hat{\tau}} |\omega(s,\tau) - \hat{\omega}(s,\tau)|\, dsd\tau \\
&\le a(t)|q-\hat{q}| + a(t)\int_{-\tau_0}^{0}\int_{-\hat{\tau}}^{\hat{\tau}} \max_{(s,\tau)\in D} |\omega(s,\tau) - \hat{\omega}(s,\tau)|\, dsd\tau \\
&= a(t)|q-\hat{q}| + a(t)\max_{(s,\tau)\in D} |\omega(s,\tau) - \hat{\omega}(s,\tau)| \int_{-\tau_0}^{0}\int_{-\hat{\tau}}^{\hat{\tau}} dsd\tau \\
&= a(t)|q-\hat{q}| + 2\hat{\tau}\tau_0 a(t)\max_{(s,\tau)\in D} |\omega(s,\tau) - \hat{\omega}(s,\tau)|\,,
\end{aligned}$$

which shows that f satisfies (3.6) with

$$\kappa_1(t) = 0, \quad \kappa_2(t) = a(t), \quad \kappa_3(t) = 2\hat{\tau}\tau_0 a(t).$$

By Theorem 4.1, the function F defined by (2.2) satisfies (3.5) with

$$\gamma_n(t) = \frac{c\, a(t)}{n^2}(1 + 2\tau_0\hat{\tau})$$

and the proof is finished. □

We have proved some preliminary results that will provide useful stepping stones in deriving error bounds for numerical solutions to general partial functional differential equations in the next chapters.

CHAPTER 5

GENERALIZED LIPSCHITZ CONDITIONS FOR NUMERICAL SCHEMES APPLIED TO PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

In this chapter, we will show that the iterations (3.2) of the Picard type in the functional sense satisfy conditions (3.7)–(3.10), which will be useful in deriving error bounds for the general numerical schemes (2.1) and (3.1). The proof for iterations of the type (3.3) is similar.

The main result can be summarised by the following theorem.

Theorem 5.1 ([37], Lemma 3.2)**.** *Suppose that f satisfies condition (3.6) and*

$$\frac{\partial f}{\partial q}(x, t, \omega, p, q) \geq 0, \tag{5.1}$$

where $x \in [-L, L]$, $[0, T]$, $\omega \in C(D, \mathbb{R})$, $p, q \in \mathbb{R}$. Moreover, suppose that $\nu \in C([0, T], \mathbb{R})$ is a function such that

$$\nu(t) \leq \frac{\partial f}{\partial q}(x, t, \omega, p, q) \tag{5.2}$$

for all inputs as in (5.1). Then, G defined by

$$G_i(t,\varsigma,z,\omega) = f\left(x_i, t, \mathcal{L}_{i,t}\omega, \frac{z^t_{i+1} - z^t_{i-1}}{2h}, \frac{z^t_{i+1} - 2\varsigma_i + z^t_{i-1}}{h^2}\right)$$

satisfies condition (3.7). Moreover, if additionally $\|\cdot\|_n$ is the infinity norm, then G satisfies conditions (3.8)–(3.10).

Proof. First, we apply the definition of G_i and (2.2), obtaining

$$G_i(t,r(t),r(t),r_t) = f\Big(x_i, t, \mathcal{L}_{i,t}r_t, \frac{r_{i+1}(t) - r_{i-1}(t)}{2h}, \frac{r_{i+1}(t) - 2r_i(t) + r_{i-1}(t)}{h^2}\Big)$$
$$= f\Big(x_i, t, \mathcal{L}_{i,t}r_t, \delta_{i,t}r(t), \delta^2_{i,t}r(t)\Big) = F_i(t, r(t), r_t),$$

where $t \in [0,T]$ and $r \in C([-\tau_0, T], \mathbb{R})$ are arbitrary. This shows that (3.7) holds.

In order to prove (3.8), we begin by using the definition of G_i and then we apply the mean value theorem

$$\begin{aligned} G_i(t,\varsigma,z,\omega) - G_i(t,\bar{\varsigma},z,\omega) &= f\Big(x_i, t, \mathcal{L}_{i,t}r_t, \frac{z^t_{i+1} - z^t_{i-1}}{2h}, \frac{z^t_{i+1} - 2\varsigma^t_i + z^t_{i-1}}{h^2}\Big) \\ &- f\Big(x_i, t, \mathcal{L}_{i,t}r_t, \frac{z^t_{i+1} - z^t_{i-1}}{2h}, \frac{z^t_{i+1} - 2\bar{\varsigma}^t_i + z^t_{i-1}}{h^2}\Big) \end{aligned}$$

$$\begin{aligned} &= \frac{\partial f}{\partial q}(Q)\Big(\frac{z^t_{i+1} - 2\varsigma^t_i + z^t_{i-1}}{h^2} - \frac{z^t_{i+1} - 2\bar{\varsigma}^t_i + z^t_{i-1}}{h^2}\Big) \\ &= \frac{-2(\varsigma^t_i - \bar{\varsigma}^t_i)}{h^2}\frac{\partial f}{\partial q}(Q) = \frac{-2(\varsigma_i - \bar{\varsigma}_i)}{h^2}\frac{\partial f}{\partial q}(Q), \end{aligned}$$

where $t \in [0,T]$, $\varsigma, \bar{\varsigma}, z \in \mathbb{R}^n$, $\omega \in C([-\tau_0, 0], \mathbb{R}^n)$ and $Q \in [-L, L] \times [0,T] \times C(D,\mathbb{R}) \times \mathbb{R}^2$. From (5.1), we get

$$\frac{\partial f}{\partial q}(Q) \geq 0$$

and conclude further that

$$\begin{aligned}\Big|\varsigma_i - \bar{\varsigma}_i - \varepsilon\big(G_i(t,\varsigma,z,\omega) - G_i(t,\bar{\varsigma},z,\omega)\big)\Big| &= \Big|\big(1 + \frac{2\varepsilon}{h^2}\frac{\partial f}{\partial q}(Q)\big)\big(\varsigma_i - \bar{\varsigma}_i\big)\Big| \\ &= \big(1 + \frac{2\varepsilon}{h^2}\frac{\partial f}{\partial q}(Q)\big)\big|\varsigma_i - \bar{\varsigma}_i\big|,\end{aligned}$$

where $\varepsilon \geq 0$. Upon taking the infinity norm on both sides of the above equation, we deduce that

$$\Big\|\varsigma - \bar{\varsigma} - \varepsilon\big(G(t,\varsigma,z,\omega) - G(t,\bar{\varsigma},z,\omega)\big)\Big\|_n = \big(1 + \frac{2\varepsilon}{h^2}\frac{\partial f}{\partial q}(Q)\big)\big\|\varsigma - \bar{\varsigma}\big\|_n.$$

From this and (5.2), we obtain

$$\begin{aligned}\Big\|\varsigma - \bar{\varsigma} - \varepsilon\big(G(t,\varsigma,z,\omega) - G(t,\bar{\varsigma},z,\omega)\big)\Big\|_n &\geq \big(1 + \frac{2\varepsilon\nu(t)}{h^2}\big)\big\|\varsigma - \bar{\varsigma}\big\|_n \\ &= \big(1 - \varepsilon\mu_1(t)\big)\big\|\varsigma - \bar{\varsigma}\big\|_n,\end{aligned}$$

with

$$\mu_1(t) = -\frac{2\nu(t)}{h^2},$$

which shows that (3.8) holds.

In order to prove (3.9), we apply (3.6) and obtain

$$\begin{aligned}\big|G_i(t,\varsigma,z,\omega) - G_i(t,\varsigma,\bar{z},\omega)\big| &= \Bigg|f\Big(x_i, t, \mathcal{L}_{i,t}\omega, \frac{z^t_{i+1} - z^t_{i-1}}{2h}, \frac{z^t_{i+1} - 2\varsigma_i + z^t_{i-1}}{h^2}\Big) \\ &- f\Big(x_i, t, \mathcal{L}_{i,t}\omega, \frac{\bar{z}^t_{i+1} - \bar{z}^t_{i-1}}{2h}, \frac{\bar{z}^t_{i+1} - 2\varsigma_i + \bar{z}^t_{i-1}}{h^2}\Big)\Bigg|\end{aligned}$$

$$\leq \kappa_1(t)\left|\frac{z^t_{i+1}-z^t_{i-1}}{2h}-\frac{\bar{z}^t_{i+1}-\bar{z}^t_{i-1}}{2h}\right|+\kappa_2(t)\left|\frac{z^t_{i+1}-2\varsigma_i+z^t_{i-1}}{h^2}-\frac{\bar{z}^t_{i+1}-2\varsigma_i+\bar{z}^t_{i-1}}{h^2}\right|$$
$$\leq \frac{\kappa_1(t)}{2h}\Big(|z^t_{i+1}-\bar{z}^t_{i+1}|+|z^t_{i-1}-\bar{z}^t_{i-1}|\Big)+\frac{\kappa_2(t)}{h^2}\Big(|z^t_{i+1}-\bar{z}^t_{i+1}|+|z^t_{i-1}-\bar{z}^t_{i-1}|\Big)$$
$$= \Big(\frac{\kappa_1(t)}{2h}+\frac{\kappa_2(t)}{h^2}\Big)\Big(|z^t_{i+1}-\bar{z}^t_{i+1}|+|z^t_{i-1}-\bar{z}^t_{i-1}|\Big)$$

where $t\in[0,T]$, $\varsigma, z, \bar{z}\in\mathbb{R}^n$, $\omega\in C([-\tau_0,0],\mathbb{R}^n)$. Since

$$\frac{\kappa_1(t)}{2h}+\frac{\kappa_2(t)}{h^2}\geq 0,$$

upon taking the infinity norm (with $i=1,\ldots,n$) on both sides of the above inequality, we get

$$\|G(t,\varsigma,z,\omega)-G(t,\varsigma,\bar{z},\omega)\|_n \leq 2\left(\frac{\kappa_1(t)}{2h}+\frac{\kappa_2(t)}{h^2}\right)\|z-\bar{z}\|_n,$$

which shows that (3.9) holds with

$$\mu_2(t)=2\left(\frac{\kappa_1(t)}{2h}+\frac{\kappa_2(t)}{h^2}\right).$$

In order to prove (3.10), we apply (3.6) with $\kappa_1(t)=\kappa_2(t)=0$ and obtain

$$\begin{aligned}\big|G_i(t,\varsigma,z,\omega)-G_i(t,\varsigma,z,\bar{\omega})\big| &= \bigg|f\Big(x_i,t,\mathcal{L}_{i,t}\omega,\frac{z^t_{i+1}-z^t_{i-1}}{2h}-\frac{z^t_{i+1}-2\varsigma_i+z^t_{i-1}}{h^2}\Big)\\ &\quad - f\Big(x_i,t,\mathcal{L}_{i,t}\bar{\omega},\frac{z^t_{i+1}-z^t_{i-1}}{2h}-\frac{z^t_{i+1}-2\varsigma_i+z^t_{i-1}}{h^2}\Big)\bigg|\\ &\leq \kappa_3(t)\max_{(s,\tau)\in D}\Big|\mathcal{L}_{i,t}\omega(s,\tau)-\mathcal{L}_{i,t}\bar{\omega}(s,\tau)\Big|,\end{aligned}$$

for $t\in[0,T]$, $\varsigma, z\in\mathbb{R}^n$, $\omega,\bar{\omega}\in C([-\tau_0,0],\mathbb{R}^n)$. Then, from the definition of the operator $\mathcal{L}_{i,t}$ we get

$$\begin{aligned}
&\mathcal{L}_{i,t}\omega(s,\tau) - \mathcal{L}_{i,t}\bar{\omega}(s,\tau) = \mathcal{L}_{i,t}(\omega - \bar{\omega})(s,\tau) \\
&= \frac{x_{k+1}-s}{h}\Big(\omega^t_{k+i}(\tau) - \bar{\omega}^t_{k+i}(\tau)\Big) + \frac{s-x_k}{h}\Big(\omega^t_{k+i+1}(\tau) - \bar{\omega}^t_{k+i+1}(\tau)\Big),
\end{aligned}$$

where $x_k \le s \le x_{k+1}$, and

$$\begin{aligned}
\Big|\mathcal{L}_{i,t}\omega(s,\tau) - \mathcal{L}_{i,t}\bar{\omega}(s,\tau)\Big| &\le \frac{x_{k+1}-s}{h}\Big|\omega^t_{k+i}(\tau) - \bar{\omega}^t_{k+i}(\tau)\Big| \\
&\quad + \frac{s-x_k}{h}\Big|\omega^t_{k+i+1}(\tau) - \bar{\omega}^t_{k+i+1}(\tau)\Big| \\
&\le \frac{x_{k+1}-s}{h}\max_{\tau\in[-\tau_0,0]}\|(\omega-\bar{\omega})(\tau)\| \\
&\quad + \frac{s-x_k}{h}\max_{\tau\in[-\tau_0,0]}\|(\omega-\bar{\omega})(\tau)\| \\
&= \frac{x_{k+1}-s+s-x_k}{h}\|\omega-\bar{\omega}\|^0_n = \|\omega-\bar{\omega}\|^0_n
\end{aligned}$$

Upon taking the maximum over D on both sides of the above relations, we get

$$\max_{(s,\tau)\in D}\Big|\mathcal{L}_{i,t}\omega(s,\tau) - \mathcal{L}_{i,t}\bar{\omega}(s,\tau)\Big| \le \|\omega-\bar{\omega}\|^0_n.$$

Therefore,

$$\Big\|G(t,\varsigma,z,\omega) - G(t,\varsigma,z,\bar{\omega})\Big\|_n \le \kappa_3(t)\|\omega-\bar{\omega}\|^0_n,$$

which shows that (3.10) holds with

$$\mu_3(t) = \kappa_3(t),$$

and finishes the proof of the theorem. □

Having proven that G satisfies the conditions (3.7)–(3.10) under the assumptions of Theorem 5.1, we are now in a position to investigate the numerical schemes (2.1) and (3.1) designed to find numerical solutions to general partial functional differential equations and to derive appropriate bounds on their errors, from which we can deduce important convergence properties of the numerical algorithm.

CHAPTER 6

THEOREMS ON ERROR BOUNDS FOR NUMERICAL SOLUTIONS TO GENERAL PARTIAL FUNCTIONAL DIFFERENTIAL EQUATIONS

In this chapter, we prove a sequence of results that can be used to deduce appropriate bounds on the errors that we can expect to get numerically by applying the numerical schemes (2.1) and (3.1) to solve a class of general partial functional differential equations.

The first of these results is a main theorem specifying a bound on the norm of the error in terms of an integral that holds as long as the right-hand-side function f satisfies an appropriate condition. In particular, we require a sharper condition than (5.1).

Theorem 6.1 ([37], Theorem 4.1)**.** *Suppose the given function f satisfies conditions (3.6) and (5.1), the step-size $h > 0$ is chosen in such a way that*

$$\frac{\partial f}{\partial q}(x,t,\omega,p,q) - \frac{h}{2}\left|\frac{\partial f}{\partial p}(x,t,\omega,p,q)\right| \geq 0, \tag{6.1}$$

for all $x \in [-L, L]$, $t \in [0, T]$, $\omega \in C(D, \mathbb{R})$, $p, q \in \mathbb{R}$, and $\frac{\partial f}{\partial p}(x,t,\omega,\cdot)$, $\frac{\partial f}{\partial q}(x,t,\omega,\cdot)$ are continuous functions for each $x \in [-L, L]$, $t \in [0, T]$, and $\omega \in C([0, T], \mathbb{R})$. Moreover, suppose that F satisfies (2.2) and $u \in C^{(4)}(D, \mathbb{R})$. Then the errors $e(t)$

satisfy

$$\|e(t)\|_n \leq \int_0^t \gamma_n(s) exp\Bigg(\int_s^t \kappa_3(\tau) d\tau \Bigg) ds,$$

for $t \in [0,T]$*, where* $\|\cdot\|_n$ *is the infinity norm in* $\mathbb{R}^n$*.*

Proof. We apply Theorem 6.7 from the Appendix with $\rho(x_i,t) = e_i(t) = U_i(t) - \upsilon_i(t)$, $t \in [-\tau_0, T]$ and $i = 0, \pm 1, ..., \pm \tilde{M}$, where $\upsilon_i(t) = g(x_i,t)$, for $i = \pm M, \ldots, \pm \tilde{M}$. Then, $\dfrac{\partial \rho}{\partial t}(x_i,t) = \dot{e}_i(t) = \dot{U}_i(t) - \dot{\upsilon}_i(t)$. We want to show that

$$\max_{i=0,\pm 1,\ldots,\pm M'} |e_i(t)| = \max_{i=0,\pm 1,\ldots,\pm M'} |\rho(x_i,t)| \leq \int_0^t \gamma_n(s) exp\Bigg(\int_s^t \kappa_3(\tau) d\tau \Bigg) ds.$$

Since

$$\dot{\upsilon}_i(t) = F_i(t, \upsilon(t), \upsilon_t) = f\big(x_i, t, \mathcal{L}_{i,t}\upsilon_t, \delta_{i,t}\upsilon(t), \delta^2_{i,t}\upsilon(t)\big),$$

we get

$$\begin{aligned}
\frac{\partial \rho}{\partial t}(x_i,t) &= \dot{U}_i(t) - F_i(t, U(t), U_t) \\
&+ f\big(x_i, t, \mathcal{L}_{i,t}U_t, \delta_{i,t}U(t), \delta^2_{i,t}U(t)\big) - f\big(x_i, t, \mathcal{L}_{i,t}\upsilon_t, \delta_{i,t}U(t), \delta^2_{i,t}U(t)\big) \\
&+ f\big(x_i, t, \mathcal{L}_{i,t}\upsilon_t, \delta_{i,t}U(t), \delta^2_{i,t}U(t)\big) - f\big(x_i, t, \mathcal{L}_{i,t}\upsilon_t, \delta_{i,t}\upsilon(t), \delta^2_{i,t}\upsilon(t)\big).
\end{aligned}$$

From this and from the Mean Value Theorem, we get

$$\begin{aligned}
&\frac{\partial \rho}{\partial t}(x_i,t) - \tilde{\delta}\rho(x_i,t) \int_0^1 \frac{\partial f}{\partial p}(Q_s) ds - \tilde{\delta}^{(2)}\rho(x_i,t) \int_0^1 \frac{\partial f}{\partial q}(Q_s) ds \\
&= \dot{U}_i(t) - F_i(t, U(t), U_t) \\
&+ f\big(x_i, t, \mathcal{L}_{i,t}U_t, \delta_{i,t}U(t), \delta^2_{i,t}U(t)\big) - f\big(x_i, t, \mathcal{L}_{i,t}\upsilon_t, \delta_{i,t}U(t), \delta^2_{i,t}U(t)\big)
\end{aligned}$$

and

$$\Big|\frac{\partial\rho}{\partial t}(x_i,t)-\tilde{\delta}\rho(x_i,t)\int_0^1\frac{\partial f}{\partial p}(Q_s)ds-\tilde{\delta}^{(2)}\rho(x_i,t)\int_0^1\frac{\partial f}{\partial q}(Q_s)ds\Big|$$
$$\le\Big|\dot{U}_i(t)-F_i(t,U(t),U_t)\Big|$$
$$+\Big|f\big(x_i,t,\mathcal{L}_{i,t}U_t,\delta_{i,t}U(t),\delta^2_{i,t}U(t)\big)-f\big(x_i,t,\mathcal{L}_{i,t}\upsilon_t,\delta_{i,t}U(t),\delta^2_{i,t}U(t)\big)\Big|,$$

where $\tilde{\delta}\rho(x_i,t)$, $\tilde{\delta}^{(2)}\rho(x_i,t)$ are defined by (6.12) in the Appendix and

$$Q_s=(x_i,t,\mathcal{L}_{i,t}\upsilon_t,\delta_{i,t}\upsilon(t)+s\tilde{\delta}\rho(x_i,t),\delta^2_{i,t}\upsilon(t)+s\tilde{\delta}^{(2)}\rho(x_i,t))$$

is a point from the domain of the function f. We now apply conditions (3.5) and (3.6) to obtain

$$\Big|\frac{\partial\rho}{\partial t}(x_i,t)-\tilde{\delta}\rho(x_i,t)\int_0^1\frac{\partial f}{\partial p}(Q_s)ds-\tilde{\delta}^{(2)}\rho(x_i,t)\int_0^1\frac{\partial f}{\partial q}(Q_s)ds\Big|$$
$$\le\gamma_n(t)+\kappa_3(t)\max_{(s,\tau)\in D}\Big|\mathcal{L}_{i,t}U_t(s,\tau)-\mathcal{L}_{i,t}\upsilon_t(s,\tau)\Big|.$$

Since

$$\max_{(s,\tau)\in D}|\mathcal{L}_{i,t}U_t(s,\tau)-\mathcal{L}_{i,t}\upsilon_t(s,\tau)|$$
$$=\max_{(s,\tau)\in D}\left|\frac{x_{k+1}-s}{h}\Big((U_t)_{k+i}(\tau)-(\upsilon_t)_{k+i}(\tau)\Big)+\frac{s-x_k}{h}\Big((U_t)_{k+1+i}(\tau)-(\upsilon_t)_{k+1+i}(\tau)\Big)\right|$$
$$=\max_{(s,\tau)\in D}\left|\frac{x_{k+1}-s}{h}(e_t)_{k+i}(\tau)+\frac{s-x_k}{h}(e_t)_{k+1+i}(\tau)\right|$$
$$=\max_{(s,\tau)\in D}\left|\frac{x_{k+1}-s}{h}e_{k+i}(t+\tau)+\frac{s-x_k}{h}e_{k+1+i}(t+\tau)\right|$$
$$=\max_{(s,\tau)\in D}\left|\frac{x_{k+1}-s}{h}\rho(x_{k+i},t+\tau)+\frac{s-x_k}{h}\rho(x_{k+1+i},t+\tau)\right|$$

$$= \max_{(s,\tau)\in D} \left| \frac{x_{k+1}-s}{h}\rho(x_i+x_k, t+\tau) + \frac{s-x_k}{h}\rho(x_i+x_{k+1}, t+\tau) \right|$$

$$\leq \max_{(s,\tau)\in D} \left(\frac{x_{k+1}-s}{h} + \frac{s-x_k}{h} \right) \cdot \max_{j=k,k+1} |\rho(x_i+x_j, t+\tau)|$$

$$= \max_{(s,\tau)\in D} \cdot \max_{j=k,k+1} |\rho(x_i+x_j, t+\tau)|$$

$$= \max\{|\rho(x_i+x_j, t+\tau)| : j = 0, \pm 1, \ldots, \pm \hat{M}, \tau \in [-\tau_0, 0]\}$$

$$= \|\rho_{(x_i,t)}\|_n^h,$$

where k is such that $x_k \leq s \leq x_{k+1}$, we have

$$\left| \frac{\partial \rho}{\partial t}(x_i,t) - \tilde{\delta}\rho(x_i,t) \int_0^1 \frac{\partial f}{\partial p}(Q_s)ds - \tilde{\delta}^{(2)}\rho(x_i,t) \int_0^1 \frac{\partial f}{\partial q}(Q_s)ds \right|$$
$$\leq \gamma_n(t) + \kappa_3(t)\|\rho_{(x_i,t)}\|_n^h.$$

We now apply Theorem 6.7 from the Appendix with

$$\mathcal{G}(x_i, t, \rho_{(x_i,t)}) = \int_0^1 \frac{\partial f}{\partial p}(Q_s)ds, \quad \mathcal{H}(x_i, t, \rho_{(x_i,t)}) = \int_0^1 \frac{\partial f}{\partial q}(Q_s)ds,$$

and from the above inequality combined with (6.1), we get

$$\|e(t)\|_n = \max_{i=0,\pm 1,\ldots,\pm M'} |e_i(t)| = \max_{i=0,\pm 1,\ldots,\pm M'} |\rho(x_i,t)| \leq \eta(t), \tag{6.2}$$

for $t \in [0,T]$, where η is the solution to the initial value problem

$$\begin{cases} \dot{\eta}(t) &= \kappa_3(t)\eta(t) + \gamma_n(t), \\ \eta(0) &= 0. \end{cases} \tag{6.3}$$

From (6.2) and (6.3), we have

$$\|e(t)\|_n \leq \int_0^t \exp\left(\int_s^t \kappa_3(r)dr\right) \cdot \gamma_n(s)ds$$

and the proof is finished. □

The next theorem will be applied to derive an error bound for iterative processes applied to problem (1.1)–(1.3).

Theorem 6.2 ([37], Lemma 5.1)**.** *Suppose F and G satisfy conditions (3.5) and (3.7)–(3.10). Then,*

$$\|E^{(k+1)}(t)\|_n \leq \int_0^t exp\left(\int_\tau^t \mu_1(s)ds\right)\Big(\big(\mu_2(\tau)+\mu_3(\tau)\big)\|E_\tau^{(k)}\|_n^0 + \gamma_n(\tau)\Big)d\tau, \quad (6.4)$$

for all $t \in [0,T]$ and $k = 0,1,2,\dots,$ where $\|\cdot\|_n$ is the infinity norm in $\mathbb{R}^n$.

Proof. From the definition of the error $E^{(k+1)}(t)$, we get

$$\begin{aligned}
\dot{E}^{(k+1)}(t) &= \dot{U}(t) - \dot{\upsilon}^{(k+1)}(t) \\
&= \dot{U}(t) - G(t, \upsilon^{(k+1)}(t), \upsilon^{(k)}(t), \upsilon_t^{(k)}) \\
&= \dot{U}(t) - F(t, U(t), U_t) + F(t, U(t), U_t) - G(t, \upsilon^{(k+1)}(t), U(t), U_t) \\
&+ G(t, \upsilon^{(k+1)}(t), U(t), U_t) - G(t, \upsilon^{(k+1)}(t), \upsilon^{(k)}(t), U_t) \\
&+ G(t, \upsilon^{(k+1)}(t), \upsilon^{(k)}(t), U_t) - G(t, \upsilon^{(k+1)}(t), \upsilon^{(k)}(t), \upsilon_t^{(k)})
\end{aligned}$$

We now multiply both sides of the above relation by an arbitrary $\varepsilon < 0$ and evaluate the following infinity norm

$$\begin{aligned}
&\big\|E^{(k+1)}(t)+\varepsilon\dot{E}^{k+1}(t)\big\|_n \\
&= \Big\|U(t)-\upsilon^{(k+1)}(t)-(-\varepsilon)\Big(G(t,U(t),U(t),U_t)-G(t,\upsilon^{(k+1)}(t),U(t),U_t)\Big) \\
&-(-\varepsilon)\Big(\dot{U}(t)-F(t,U(t),U_t)+G(t,\upsilon^{(k+1)}(t),U(t),U_t)-G(t,\upsilon^{(k+1)}(t),\upsilon^{(k)}(t),U_t) \\
&+G(t,\upsilon^{(k+1)}(t),\upsilon^{(k)}(t),U_t)-G(t,\upsilon^{(k+1)}(t),\upsilon^{(k)}(t),\upsilon_t^{(k)})\Big)\Big\|_n \\
&\geq \Big\|U(t)-\upsilon^{(k+1)}(t)-(-\varepsilon)[G(t,U(t),U(t),U_t)-G(t,\upsilon^{(k+1)}(t),U(t),U_t)]\Big\|_n \\
&-\Big\|(-\varepsilon)\Big(\dot{U}(t)-F(t,U(t),U_t)+G(t,\upsilon^{(k+1)}(t),U(t),U_t)-G(t,\upsilon^{(k+1)}(t),\upsilon^{(k)}(t),U_t) \\
&+G(t,\upsilon^{(k+1)}(t),\upsilon^{(k)}(t),U_t)-G(t,\upsilon^{(k+1)}(t),\upsilon^{(k)}(t),\upsilon_t^{(k)})\Big)\Big\|_n
\end{aligned}$$

From this and (3.8), we get

$$\begin{aligned}
&\big\|E^{(k+1)}(t)+\varepsilon\dot{E}^{(k+1)}(t)\big\|_n \geq \big(1-(-\varepsilon)\mu_1(t)\big)\big\|U(t)-\upsilon^{(k+1)}(t)\big\|_n \\
&-\Big\|(-\varepsilon)\Big(\dot{U}(t)-F(t,U(t),U_t)+G(t,\upsilon^{(k+1)}(t),U(t),U_t)-G(t,\upsilon^{(k+1)}(t),\upsilon^{(k)}(t),U_t) \\
&+G(t,\upsilon^{(k+1)}(t),\upsilon^{(k)}(t),U_t)-G(t,\upsilon^{(k+1)}(t),\upsilon^{(k)}(t),\upsilon_t^{(k)})\Big)\Big\|_n.
\end{aligned}$$

Since $-\varepsilon > 0$, we use the scaling property of the norm and further conclude that the following inequality holds:

$$\begin{aligned}
&\big\|E^{(k+1)}(t)+\varepsilon\dot{E}^{(k+1)}(t)\big\|_n-\big\|E^{(k+1)}(t)\big\|_n \geq \varepsilon\mu_1(t)\big\|E^{(k+1)}(t)\big\|_n \\
&+\varepsilon\Big\|\dot{U}(t)-F(t,U(t),U_t)+G(t,\upsilon^{(k+1)}(t),U(t),U_t)-G(t,\upsilon^{(k+1)}(t),\upsilon^{(k)}(t),U_t) \\
&+G(t,\upsilon^{(k+1)}(t),\upsilon^{(k)}(t),U_t)-G(t,\upsilon^{(k+1)}(t),\upsilon^{(k)}(t),\upsilon_t^{(k)})\Big\|_n.
\end{aligned}$$

Upon dividing the above inequality by $\varepsilon < 0$, we obtain

$$\begin{aligned}
&\frac{1}{\varepsilon}\Big(\big\|E^{(k+1)}(t)+\varepsilon\dot{E}^{(k+1)}(t)\big\|_n-\big\|E^{(k+1)}(t)\big\|_n\Big) \leq \mu_1(t)\big\|E^{(k+1)}(t)\big\|_n \\
&+\Big\|\dot{U}(t)-F(t,U(t),U_t)+G(t,\upsilon^{(k+1)}(t),U(t),U_t)-G(t,\upsilon^{(k+1)}(t),\upsilon^{(k)}(t),U_t) \\
&+G(t,\upsilon^{(k+1)}(t),\upsilon^{(k)}(t),U_t)-G(t,\upsilon^{(k+1)}(t),\upsilon^{(k)}(t),\upsilon_t^{(k)})\Big\|_n.
\end{aligned}$$

Notice that the right-hand side of the above inequality does not depend on ε and by taking $\varepsilon \to 0^-$ on both sides, we deduce that

$$\begin{aligned} & D^-\|E^{(k+1)}(t)\|_n = \lim_{\varepsilon\to 0^-} \frac{1}{\varepsilon}\Big(\big\|E^{(k+1)}(t) + \varepsilon \dot{E}^{(k+1)}(t)\big\|_n - \big\|E^{(k+1)}(t)\big\|_n\Big) \\ & \le \mu_1(t)\big\|E^{(k+1)}(t)\big\|_n \\ & + \big\|\dot{U}(t) - F(t, U(t), U_t)\big\|_n + \big\|G(t, \upsilon^{(k+1)}(t), U(t), U_t) - G(t, \upsilon^{(k+1)}(t), \upsilon^{(k)}(t), U_t)\big\|_n \\ & + \big\|G(t, \upsilon^{(k+1)}(t), \upsilon^{(k)}(t), U_t) - G(t, \upsilon^{(k+1)}(t), \upsilon^{(k)}(t), \upsilon_t^{(k)})\big\|_n, \end{aligned}$$

where D^- is the left-hand side derivative with respect to t. We now apply conditions (3.5), (3.9), (3.10) to get that

$$\begin{aligned} & D^-\|E^{(k+1)}(t)\|_n \\ & \le \mu_1(t)\big\|E^{(k+1)}(t)\big\|_n + \gamma_n(t) + \mu_2(t)\big\|U(t) - \upsilon^{(k)}(t)\big\|_n + \mu_3(t)\big\|U_t - \upsilon_t^{(k)}\big\|_n^0 \\ & = \mu_1(t)\big\|E^{(k+1)}(t)\big\|_n + \gamma_n(t) + \mu_2(t)\big\|E^{(k)}(t)\big\|_n + \mu_3(t)\big\|E_t^{(k)}\big\|_n^0 \\ & \le \mu_1(t)\big\|E^{(k+1)}(t)\big\|_n + \big(\mu_2(t) + \mu_3(t)\big)\|E_t^{(k)}\|_n^0 + \gamma_n(t). \end{aligned}$$

Also, note that $\|E^{(k+1)}(0)\big\|_n = 0$ as the successive iterates satisfy the same initial condition as $U(0)$. Therefore, we conclude that

$$\|E^{(k+1)}(t)\big\|_n \le \lambda(t),$$

where $\lambda(t)$ solves the following problem

$$\begin{cases} \lambda'(t) &= \mu_1(t)\lambda(t) + \xi(t), \\ \lambda(0) &= 0, \end{cases}$$

and

$$\xi(t) = \big(\mu_2(t) + \mu_3(t)\big)\|E_t^{(k)}\|_n^0 + \gamma_n(t).$$

Notice that

$$\lambda(t) = \int_0^t \xi(s) \exp\Big(\int_s^t \mu_1(\tau)d\tau\Big)ds,$$

for $t \in [0, T]$. Therefore,

$$\big\|E^{(k+1)}(t)\big\|_n \leq \int_0^t \xi(s) \exp\Big(\int_s^t \mu_1(\tau)d\tau\Big)ds,$$

which implies (6.4) and finishes the proof. □

In what follows, we will prove a sequence of preliminary results that will be useful in proving Theorem 6.5.

Henceforth, we will use the following notation. Let $t \in [0, T]$. Then, the maximum starting error will be denoted by

$$E(t) = \max_{\tau \in [0,t]} \|E^{(0)}(\tau)\|_n.$$

We assume that the function μ_1 has no roots in $[0, T]$, that is, either $\operatorname{sign}(\mu_1) = 1$ or $\operatorname{sign}(\mu_1) = -1$, and we define

$$\begin{aligned} r(t) &= \operatorname{sign}(\mu_1) \max_{\tau \in [0,t]} \frac{\mu_2(\tau) + \mu_3(\tau)}{|\mu_1(\tau)|}, \\ \varGamma_n(t) &= \operatorname{sign}(\mu_1) \max_{\tau \in [0,t]} \frac{\gamma_n(\tau)}{|\mu_1(\tau)|}. \end{aligned}$$

For the sake of simplicity of the subsequent proofs, we also introduce the following definitions of four functions:

$$\begin{aligned} A(t) &= \int_0^t \mu_1(\tau)d\tau, \\ \alpha_k(t) &= 1 - e^{A(t)} \sum_{j=0}^{k-1} \frac{\big(-A(t)\big)^j}{j!}, \\ S_k(t) &= -\big(-r(t)\big)^{k-1} \alpha_k(t), \\ \lambda_k(t) &= \Gamma_n(t) \sum_{i=1}^{k} S_i(t). \end{aligned}$$

Theorem 6.3 ([37], Lemma 5.2)**.** *If μ_1 has no roots in $[0,T]$, then all functions*

$$\big(-\operatorname{sign}(\mu_1)\big)^k \alpha_k(t),$$

where $k = 1, 2, \dots$ and $t \in [0,T]$, are nondecreasing and nonnegative.

Proof. Throughout the proof, we use the notation

$$\tilde{\alpha}_k(t) = (-\operatorname{sign}\big(\mu_1)\big)^k \alpha_k(t)$$

and firstly show that $\tilde{\alpha}_k'(t) \geq 0$, for all $t \in [0,T]$. From the definition of the function A, we get

$$\begin{aligned} \alpha_k'(t) &= -\exp\big(A(t)\big) \sum_{j=1}^{k-1} \frac{(-1)^j \big(A(t)\big)^{j-1} A'(t)}{(j-1)!} - \exp\big(A(t)\big) A'(t) \sum_{j=0}^{k-1} \frac{\big(-A(t)\big)^j}{j!} \\ &= -\exp(A(t))A'(t) \left(\sum_{j=0}^{k-2} \frac{(-1)^{j+1}\big(A(t)\big)^j}{j!} + \sum_{j=0}^{k-1} \frac{(-1)^j \big(A(t)\big)^j}{j!} \right) \\ &= \exp\big(A(t)\big) A'(t) \frac{(-1)^k \big(A(t)\big)^{k-1}}{(k-1)!}. \end{aligned}$$

We now consider two opposite cases, $\operatorname{sign}(\mu_1) = 1$ and $\operatorname{sign}(\mu_1) = -1$. Suppose firstly that $\operatorname{sign}(\mu_1) = 1$. Then, from the definition, $A(t) \geq 0$ and $A'(t) = \mu_1(t) > 0$.

Since

$$\begin{aligned}\tilde{\alpha}_k'(t) &= \big(-\operatorname{sign}(\mu_1)\big)^k \alpha_k'(t) = (-1)^k \exp\big(A(t)\big) A'(t) \frac{(-1)^k \big(A(t)\big)^{k-1}}{(k-1)!} \\ &= \exp\big(A(t)\big)\mu_1(t)\frac{\big(A(t)\big)^{k-1}}{(k-1)!} \geq 0,\end{aligned}$$

$\tilde{\alpha}_k(t)$ is shown to be nondecreasing in the first case.

We now suppose that $\operatorname{sign}(\mu_1) = -1$. Then, $A(t) \leq 0$ and we get

$$\begin{aligned}\tilde{\alpha}_k'(t) &= \big(-\operatorname{sign}(\mu_1)\big)^k \alpha_k'(t) = \alpha_k'(t) = \exp\big(A(t)\big)\mu_1(t)\frac{(-1)^k \big(A(t)\big)^{k-1}}{(k-1)!} \\ &= \exp\big(A(t)\big)\big(-\mu_1(t)\big)\frac{\big(-A(t)\big)^{k-1}}{(k-1)!} \geq 0\end{aligned}$$

showing that $\tilde{\alpha}_k(t)$ is nondecreasing also in the second case.

Since $A(0) = 0$, $\tilde{\alpha}_k(0) = \alpha_k(0) = 0$ and since $\tilde{\alpha}_k(t)$ is nondecreasing on $[0, T]$, we conclude that $\tilde{\alpha}_k(t) \geq 0$, for $t \in [0, T]$, which finishes the proof. □

We now apply Theorem 6.3 to prove the next theorem on the nonnegativity and monotonicity of $r(t)S_k(t)$ and $\Gamma_n(t)S_k(t)$ for $k = 1, 2, \ldots$.

Theorem 6.4 ([37], Corollary 5.1)**.** *If μ_1 has no roots in $[0, T]$, then the functions $r(t)S_k(t)$ and $\Gamma_n(t)S_k(t)$, with $k = 1, 2, \ldots,$ are nondecreasing and nonnegative for all $t \in [0, T]$.*

Proof. Let $\tilde{r}(t) = r(t)S_k(t)$. Then, from the definitions of $r(t)$ and $S_k(t)$, we get

$$\begin{aligned}
\tilde{r}(t) &= r(t)\cdot\Big(-\big(-r(t)\big)^{k-1}\alpha_k(t)\Big)\\
&= \big(-r(t)\big)^k\alpha_k(t)\\
&= \left(-\operatorname{sign}(\mu_1)\max_{\tau\in[0,t]}\frac{\mu_2(\tau)+\mu_3(\tau)}{|\mu_1(\tau)|}\right)^k\cdot\alpha_k(t)\\
&= \big(-\operatorname{sign}(\mu_1)\big)^k\alpha_k(t)\cdot\left(\max_{\tau\in[0,t]}\frac{\mu_2(\tau)+\mu_3(\tau)}{|\mu_1(\tau)|}\right)^k.
\end{aligned}$$

Notice that the function

$$\left(\max_{\tau\in[0,t]}\frac{\mu_2(\tau)+\mu_3(\tau)}{|\mu_1(\tau)|}\right)^k$$

is nondecreasing and nonnegative for $t\in[0,T]$. Moreover, by Theorem 6.3, the function $(-\operatorname{sign}(\mu_1))^k\alpha_k(t)$ is also nondecreasing and nonnegative for $t\in[0,T]$. Therefore, we conclude the same about $\tilde{r}(t)$.

We now define $\tilde{\Gamma}(t)=\Gamma_n(t)S_k(t)$. From the definitions of the functions $\Gamma_n(t)$ and $S_k(t)$ we get

$$\begin{aligned}
\tilde{\Gamma}(t) &= \left(\operatorname{sign}(\mu_1)\max_{\tau\in[0,t]}\frac{\gamma_n(\tau)}{|\mu_1(\tau)|}\right)\cdot\Big(-\big(-r(t)\big)^{k-1}\alpha_k(t)\Big)\\
&= \big(-\operatorname{sign}(\mu_1)\big)\big(-r(t)\big)^{k-1}\alpha_k(t)\left(\max_{\tau\in[0,t]}\frac{\gamma_n(\tau)}{|\mu_1(\tau)|}\right)\\
&= \big(-\operatorname{sign}(\mu_1)\big)\left(-\operatorname{sign}(\mu_1)\max_{[0,t]}\frac{\mu_2(\tau)+\mu_3(\tau)}{|\mu_1(\tau)|}\right)^{k-1}\alpha_k(t)\left(\max_{\tau\in[0,t]}\frac{\gamma_n(\tau)}{|\mu_1(\tau)|}\right)\\
&= \big(-\operatorname{sign}(\mu_1)\big)^k\alpha_k(t)\cdot\left(\max_{[0,t]}\frac{\mu_2(\tau)+\mu_3(\tau)}{|\mu_1(\tau)|}\right)^{k-1}\cdot\left(\max_{\tau\in[0,t]}\frac{\gamma_n(\tau)}{|\mu_1(\tau)|}\right).
\end{aligned}\tag{6.5}$$

It can be observed that

$$\left(\max_{[0,t]}\frac{\mu_2(\tau)+\mu_3(\tau)}{|\mu_1(\tau)|}\right)^{k-1}$$

and

$$\max_{\tau\in[0,t]}\frac{\gamma_n(\tau)}{|\mu_1(\tau)|}$$

are nondecreasing and nonnegative as functions of $t \in [0,T]$. Moreover, by Theorem 6.3, we conclude that $\big(-\operatorname{sign}(\mu_1)\big)^k\alpha_k(t)$ is nondecreasing and nonnegative on $[0,T]$. Therefore, we further conclude from (6.5), that $\tilde{\Gamma}(t)$ has the same properties, which finishes the proof. □

The last result that is necessary in order to prove Theorem 6.5 can be summarised by the following lemma.

Lemma 6.1 ([37], Lemma 5.3)**.** *The relation*

$$\int_0^t \exp\big(-A(\tau)\big)\alpha_k(\tau)\mu_1(\tau)d\tau = -\exp\big(-A(t)\big)\alpha_{k+1}(t),$$

is satisfied for all $t \in [0,T]$ *and* $k = 1,2,\dots$.

Proof. Since $A'(t) = \mu_1(t)$, we get

$$\begin{aligned}\int_0^t \exp\big(-A(\tau)\big)\mu_1(\tau)\alpha_k(\tau)d\tau &= \int_0^t \exp\big(-A(\tau)\big)A'(\tau)\alpha_k(\tau)d\tau \\ &= -\int_0^t \frac{d}{d\tau}\Big(\exp\big(-A(\tau)\big)\Big)\cdot\alpha_k(\tau)d\tau\end{aligned}$$

From $\alpha_k(0) = 0$, we further conclude that

$$\int_0^t \exp\big(-A(\tau)\big)\mu_1(\tau)\alpha_k(\tau)d\tau$$

$$
\begin{aligned}
&= -\Big[\exp\big(-A(\tau)\big)\alpha_k(\tau)\Big]_{\tau=0}^{\tau=t} + \int_0^t \exp\big(-A(\tau)\big)\alpha_k'(\tau)d\tau \\
&= -\exp\big(-A(t)\big)\alpha_k(t) + \exp\big(-A(0)\big)\alpha_k(0) + \int_0^t \exp\big(-A(\tau)\big)\alpha_k'(\tau)d\tau \\
&= -\exp\big(-A(t)\big)\alpha_k(t) + \int_0^t \exp\big(-A(\tau)\big)\alpha_k'(\tau)d\tau.
\end{aligned}
$$

Since

$$
\alpha_k'(\tau) = \exp\big(A(\tau)\big)A'(\tau)(-1)^k \frac{\big(A(\tau)\big)^{k-1}}{(k-1)!},
$$

we get

$$
\int_0^t \exp\big(-A(\tau)\big)\mu_1(\tau)\alpha_k(\tau)d\tau
$$

$$
\begin{aligned}
&= -\exp\big(-A(t)\big)\alpha_k(t) + \int_0^t A'(\tau)(-1)^k \frac{\big(A(\tau)\big)^{k-1}}{(k-1)!}d\tau \\
&= -\exp\big(-A(t)\big)\alpha_k(t) + \frac{(-1)^k}{(k-1)!}\int_0^t A'(\tau)\big(A(\tau)\big)^{k-1}d\tau \\
&= -\exp\big(-A(t)\big)\alpha_k(t) + \frac{(-1)^k}{(k-1)!}\left[\frac{\big(A(\tau)\big)^k}{k}\right]_{\tau=0}^{\tau=t}
\end{aligned}
$$

From $A(0) = 0$ and the definition of $\alpha_k(t)$, we further conclude that

$$
\begin{aligned}
\int_0^t \exp\big(-A(\tau)\big)\mu_1(\tau)\alpha_k(\tau)d\tau &= -\exp(-A(t))\alpha_k(t) + \frac{\big(-A(t)\big)^k}{k!} \\
&= -\exp\big(-A(t)\big)\left(\alpha_k(t) - \frac{\exp\big(A(t)\big)\big(-A(t)\big)^k}{k!}\right) \\
&= -\exp\big(-A(t)\big)\alpha_{k+1}(t),
\end{aligned}
$$

which finishes the proof. □

We now apply Lemma 6.1 and the previous two Theorems 6.3 and 6.4 to prove the following theorem that supplies an explicit error bound for the successive iterates.

Theorem 6.5 ([37], Theorem 5.1)**.** *Suppose that the function F satisfies condition (3.5) and G satisfies conditions (3.7)–(3.10). Moreover, suppose that the function μ_1 has no roots in $[0, T]$. Then,*

$$\left\|E^{(k)}(t)\right\|_n \leq r(t)E(t)S_k(t) + \lambda_k(t), \tag{6.6}$$

for $t \in [0, T]$ and $k = 1, 2, \ldots$.

Proof. We first show (6.6) for $k = 1$. By Theorem 6.2, we get

$$\begin{aligned}\|E^{(1)}(t)\|_n &\leq \int_0^t exp\Bigg(\int_\tau^t \mu_1(s)ds\Bigg)\Big(\big(\mu_2(\tau) + \mu_3(\tau)\big)\|E_\tau^{(0)}\|_n^0 + \gamma_n(\tau)\Big)d\tau \\ &\leq \int_0^t \exp\left(\int_0^t \mu_1(s)ds - \int_0^\tau \mu_1(s)ds\right)\Big(\big(\mu_2(\tau) + \mu_3(\tau)\big)E(\tau) + \gamma_n(\tau)\Big)d\tau.\end{aligned}$$

Then, using the definition of the function $A(t)$, we further get

$$\begin{aligned}\|E^{(1)}(t)\|_n &\leq \int_0^t \exp\big(A(t) - A(\tau)\big)\Big(\big(\mu_2(\tau) + \mu_3(\tau)\big)E(\tau) + \gamma_n(\tau)\Big)d\tau \\ &= \exp\big(A(t)\big)\int_0^t \exp\big(-A(\tau)\big)\big|\mu_1(\tau)\big|\left(\frac{\mu_2(\tau) + \mu_3(\tau)}{\big|\mu_1(\tau)\big|}E(\tau) + \frac{\gamma_n(\tau)}{\big|\mu_1(\tau)\big|}\right)d\tau.\end{aligned}$$

We now reduce the above integrand by considering its maximum and deduce that

$$\begin{aligned}\|E^{(1)}(t)\|_n \;\leq\; & \exp\big(A(t)\big)\int_0^t \exp\big(-A(\tau)\big)\big|\mu_1(\tau)\big| \max_{s\in[0,t]}\left(\frac{\mu_2(s) + \mu_3(s)}{\big|\mu_1(s)\big|}\right)E(\tau)d\tau \\ &+ \exp\big(A(t)\big)\int_0^t \exp\big(-A(\tau)\big)\big|\mu_1(\tau)\big| \max_{s\in[0,t]}\left(\frac{\gamma_n(s)}{\big|\mu_1(s)\big|}\right)d\tau.\end{aligned}$$

Therefore, since both maxima do not depend on τ and the function $E(t)$ is nondecreasing, we deduce that

$$\begin{aligned}\|E^{(1)}(t)\|_n &\leq \exp\big(A(t)\big)\max_{s\in[0,t]}\left(\frac{\mu_2(s)+\mu_3(s)}{\big|\mu_1(s)\big|}\right)E(t)\int_0^t \exp\big(-A(\tau)\big)\big|\mu_1(\tau)\big|d\tau \\ &+ \exp\big(A(t)\big)\max_{s\in[0,t]}\left(\frac{\gamma_n(s)}{\big|\mu_1(s)\big|}\right)\int_0^t \exp\big(-A(\tau)\big)\big|\mu_1(\tau)\big|d\tau,\end{aligned}$$

which further implies that

$$\|E^{(1)}(t)\|_n \leq \exp\big(A(t)\big)\,\mathrm{sign}(\mu_1)\left(\max_{s\in[0,t]}\left(\frac{\mu_2(s)+\mu_3(s)}{\big|\mu_1(s)\big|}\right)E(t)+\max_{s\in[0,t]}\left(\frac{\gamma_n(s)}{\big|\mu_1(s)\big|}\right)\right)\cdot$$
$$\cdot\int_0^t \exp\big(-A(\tau)\big)\mu_1(\tau)d\tau.$$

We now use the definitions of the functions $r(t)$, Γ_n to deduce that

$$\|E^{(1)}(t)\|_n \leq \exp\big(A(t)\big)\Big(r(t)E(t)+\Gamma_n(t)\Big)\int_0^t \exp\big(-A(\tau)\big)\mu_1(\tau)d\tau.$$

Therefore, since

$$\int_0^t \exp\big(-A(\tau)\big)\mu_1(\tau)d\tau = \int_0^t \exp\big(-A(\tau)\big)A'(\tau)d\tau = \Big[-\exp\big(-A(\tau)\big)\Big]_{\tau=0}^{\tau=t}$$
$$= 1-\exp\big(-A(t)\big)$$

we find that

$$\begin{aligned}\|E^{(1)}(t)\|_n &\leq \exp\big(A(t)\big)\Big(r(t)E(t)+\Gamma_n(t)\Big)\Big(1-\exp\big(-A(t)\big)\Big) \\ &= \Big(r(t)E(t)+\Gamma_n(t)\Big)\Big(\exp\big(A(t)\big)-1\Big).\end{aligned} \tag{6.7}$$

On the other hand, notice that, for $k = 1$, the right-hand side of inequality (6.6) is written in the form

$$r(t)E(t)S_1(t) + \lambda_1(t) = -r(t)E(t)\alpha_1(t) + \Gamma_n(t)S_1(t) = -\alpha_1(t)\Big(r(t)E(t) + \Gamma_n(t)\Big).$$

Moreover, from the definition of the function $\alpha_1(t)$, we get

$$\alpha_1(t) = 1 - \exp\big(A(t)\big).$$

Therefore, from (6.7), we have

$$\|E^{(1)}(t)\|_n \le r(t)E(t)S_1(t) + \lambda_1(t),$$

which shows that (6.6) is satisfied for $k = 1$.

We now suppose that (6.6) is satisfied for a certain $k > 1$. From the definition of the maximum norm $\|\cdot\|_n^0$, we have

$$\|E_\tau^{(k)}\|_n^0 \;=\; \max_{s\in[-\tau_0,0]} \|E_\tau^{(k)}(s)\|_n = \max_{s\in[-\tau_0,0]} \|E^{(k)}(\tau+s)\|_n,$$

for $\tau \in [0, T]$. Therefore, from (6.6), we find that

$$\|E_\tau^{(k)}\|_n^0 \;\le\; \max_{s\in[-\tau_0,0]} \Big(r(\tau+s)E(\tau+s)S_k(\tau+s) + \lambda_k(\tau+s)\Big).$$

Since, by Theorem 6.4, all functions $\Gamma_n(t)S_i(t)$, where $i = 1, 2, \ldots$, are nondecreasing, from the definition of the function λ_k, we conclude that λ_k is also nondecreasing. Moreover, the function $E(t)$ is nondecreasing and, by Theorem 6.4, the functions $r(t)S_k(t)$ have the same feature. Therefore, the function that is being maximized on the right-hand side of the above inequality is also nondecreasing, which implies that

$$\|E_\tau^{(k)}\|_n^0 \;\le\; r(\tau)E(\tau)S_k(\tau) + \lambda_k(\tau).$$

Therefore, by Theorem 6.2, we find that

$$\|E^{(k+1)}(t)\|_n \le \int_0^t \exp\left(\int_\tau^t \mu_1(s)ds\right)\biggl(\bigl(\mu_2(\tau)+\mu_3(\tau)\bigr)\Bigl(r(\tau)E(\tau)S_k(\tau) + \lambda_k(\tau)\Bigr) + \gamma_n(\tau)\biggr)d\tau.$$

From this and the definition of the function $A(t)$, we further deduce that

$$\begin{aligned}\|E^{(k+1)}(t)\|_n \;&\le\; e^{A(t)}\int_0^t e^{-A(\tau)}\left(\frac{\mu_2(\tau)+\mu_3(\tau)}{\bigl|\mu_1(\tau)\bigr|}\Bigl(r(\tau)E(\tau)S_k(\tau)+\lambda_k(\tau)\Bigr)\right.\\ &\quad+\left.\frac{\gamma_n(\tau)}{\bigl|\mu_1(\tau)\bigr|}\right)\bigl|\mu_1(\tau)\bigr|d\tau.\end{aligned}$$

We now consider maxima of the two quotients in the above integrand and obtain that

$$\|E^{(k+1)}(t)\|_n \le e^{A(t)}\max_{s\in[0,t]}\left(\frac{\mu_2(s)+\mu_3(s)}{\bigl|\mu_1(s)\bigr|}\right)\int_0^t e^{-A(\tau)}\Bigl(r(\tau)E(\tau)S_k(\tau)+\lambda_k(\tau)\Bigr)\bigl|\mu_1(\tau)\bigr|d\tau + e^{A(t)}\max_{s\in[0,t]}\left(\frac{\gamma_n(s)}{\bigl|\mu_1(s)\bigr|}\right)\int_0^t e^{-A(\tau)}\bigl|\mu_1(\tau)\bigr|d\tau.$$

From this and the definitions of $r(t)$ and $\Gamma_n(t)$, we deduce that

$$\begin{aligned}\|E^{(k+1)}(t)\|_n \;&\le\; e^{A(t)}r(t)\int_0^t e^{-A(\tau)}\Bigl(r(\tau)E(\tau)S_k(\tau)+\lambda_k(\tau)\Bigr)\mu_1(\tau)d\tau\\ &\quad+\; e^{A(t)}\Gamma_n(t)\int_0^t e^{-A(\tau)}\mu_1(\tau)d\tau.\end{aligned} \tag{6.8}$$

We now consider the first term, $\mathcal{T}_1$, on the right-hand side of (6.8) and, from the definitions of $S_i(t)$ and $\lambda_k(t)$, we obtain the following expression for $\mathcal{T}_1$

$$\mathcal{T}_1 = e^{A(t)}(\operatorname{sign}\mu_1)r(t)\Bigg(\int_0^t e^{-A(\tau)}(-\operatorname{sign}\mu_1)^k\alpha_k(\tau)\left(\max_{s\in[0,\tau]}\frac{\mu_2(s)+\mu_3(s)}{|\mu_1(s)|}\right)^k$$

$$E(\tau)|\mu_1(\tau)|d\tau + \int_0^t e^{-A(\tau)}\sum_{i=1}^k \max_{s\in[0,\tau]}\left(\frac{\gamma_n(s)}{|\mu_1(s)|}\right)\left(\max_{s\in[0,\tau]}\frac{\mu_2(s)+\mu_3(s)}{|\mu_1(s)|}\right)^{i-1}$$

$$(-\operatorname{sign}\mu_1)^i\alpha_i(\tau)|\mu_1(\tau)|d\tau\Bigg).$$

By extending the above maxima from $[0,\tau]$ to $[0,t]$ and interchanging the order of summation and integration, we deduce that

$$\mathcal{T}_1 \le e^{A(t)}(\operatorname{sign}\mu_1)r(t)\Bigg(\left(\max_{s\in[0,t]}\frac{\mu_2(s)+\mu_3(s)}{|\mu_1(s)|}\right)^k E(t)\int_0^t e^{-A(\tau)}(-\operatorname{sign}\mu_1)^k\alpha_k(\tau)$$

$$|\mu_1(\tau)|d\tau + \max_{s\in[0,t]}\left(\frac{\gamma_n(s)}{|\mu_1(s)|}\right)\sum_{i=1}^k\left(\max_{s\in[0,t]}\frac{\mu_2(s)+\mu_3(s)}{|\mu_1(s)|}\right)^{i-1}\int_0^t e^{-A(\tau)}(-\operatorname{sign}\mu_1)^i$$

$$\alpha_i(\tau)|\mu_1(\tau)|d\tau\Bigg).$$

We now apply Lemma 6.1 and obtain

$$\mathcal{T}_1 \le e^{A(t)}r(t)\Bigg(\left(\max_{s\in[0,t]}\frac{\mu_2(s)+\mu_3(s)}{|\mu_1(s)|}\right)^k E(t)(-\operatorname{sign}\mu_1)^k(-1)e^{-A(t)}\alpha_{k+1}(t)$$

$$+\max_{s\in[0,t]}\left(\frac{\gamma_n(s)}{|\mu_1(s)|}\right)\sum_{i=1}^k\left(\max_{s\in[0,t]}\frac{\mu_2(s)+\mu_3(s)}{|\mu_1(s)|}\right)^{i-1}(-\operatorname{sign}\mu_1)^i(-1)e^{-A(t)}\alpha_{i+1}(t)\Bigg).$$

From the definitions of $r(t)$, $\Gamma_n(t)$, and $S_i(t)$, we obtain

$$\begin{aligned}\mathcal{T}_1 &\leq \big(-r(t)\big)^{k+1}E(t)\alpha_{k+1}(t)+\Gamma_n(t)\sum_{i=1}^{k}(-1)\big(-r(t)\big)^{i}\alpha_{i+1}(t)\\ &= r(t)S_{k+1}(t)E(t)+\Gamma_n(t)\sum_{i=1}^{k}S_{i+1}(t).\end{aligned}\tag{6.9}$$

We now consider the second term, $\mathcal{T}_2$, on the right-hand side of (6.8) and obtain the following expression for it:

$$\begin{aligned}\mathcal{T}_2 &= e^{A(t)}\Gamma_n(t)\int_0^t e^{-A(\tau)}A'(\tau)d\tau = e^{A(t)}\Gamma_n(t)\Big[-e^{-A(\tau)}\Big]_{\tau=0}^{\tau=t} = \Gamma_n(t)\big(e^{A(t)}-1\big)\\ &= -\Gamma_n(t)\alpha_1(t) = \Gamma_n(t)S_1(t).\end{aligned}\tag{6.10}$$

From (6.8), (6.9), and (6.10), we deduce that

$$\begin{aligned}\|E^{(k+1)}(t)\|_n &\leq \mathcal{T}_1+\mathcal{T}_2 \leq r(t)S_{k+1}(t)E(t)+\Gamma_n(t)\sum_{i=2}^{k+1}S_i(t)+\Gamma_n(t)S_1(t)\\ &= r(t)S_{k+1}(t)E(t)+\Gamma_n(t)\sum_{i=1}^{k+1}S_i(t) = r(t)S_{k+1}(t)E(t)+\lambda_{k+1}(t),\end{aligned}$$

which shows that (6.6) holds and finishes the proof. □

We now compare error bounds by means of the following theorem.

Theorem 6.6 ([37], Lemma 6.1)**.** *Suppose*

$$\kappa_1(t)=0,\quad \kappa_3(t)=\varrho\kappa_2(t),\quad \gamma_n(t)=c_0h^2(1+\varrho)\kappa_2(t),$$
$$\mu_1(t)=-\sigma h^{-2}\kappa_2(t),\quad \mu_2(t)=\sigma h^{-2}\kappa_2(t),\quad \mu_3(t)=\varrho\kappa_2(t)$$

where $\varrho,\sigma,c_0>0$. *Then,*

$$\lambda_k(t) < \int_0^t \gamma_n(s)\exp\Big(\int_s^t \kappa_3(\tau)d\tau\Big)ds,$$

for all $t \in [0,T],\ k = 1,2,\ldots$.

Proof. Let

$$\eta(t) = \int_0^t \gamma_n(s)\exp\left(\int_s^t \kappa_3(\tau)d\tau\right)ds,$$

for $t \in [0,T]$. Then, from the definition of $\gamma_n(t)$ and $\kappa_3(t)$, we deduce that

$$\begin{aligned}\eta(t) &= c_0h^2(1+\varrho)\int_0^t \kappa_2(s)\exp\left(\int_0^t \varrho\kappa_2(\tau)d\tau - \int_0^s \varrho\kappa_2(\tau)d\tau\right)ds\\ &= c_0h^2(1+\varrho)\exp\left(\int_0^t \varrho\kappa_2(\tau)d\tau\right)\int_0^t \kappa_2(s)\exp\left(-\int_0^s \varrho\kappa_2(\tau)d\tau\right)ds.\end{aligned}$$

Since

$$\begin{aligned}\int_0^t \kappa_2(s)\exp\left(-\int_0^s \varrho\kappa_2(\tau)d\tau\right)ds &= \frac{-1}{\varrho}\left[\exp\left(-\int_0^s \varrho\kappa_2(\tau)d\tau\right)\right]_{s=0}^{s=t}\\ &= \frac{1}{\varrho}\left(1-\exp\left(-\int_0^t \varrho\kappa_2(\tau)d\tau\right)\right),\end{aligned}$$

we further obtain

$$\begin{aligned}\eta(t) &= c_0h^2(1+\varrho)\exp\left(\int_0^t \varrho\kappa_2(\tau)d\tau\right)\frac{1}{\varrho}\left(1-\exp\left(-\int_0^t \varrho\kappa_2(\tau)d\tau\right)\right)\\ &= c_0h^2\frac{1+\varrho}{\varrho}\left(\exp\left(\int_0^t \varrho\kappa_2(\tau)d\tau\right)-1\right)\end{aligned}$$

and

$$\frac{d\eta}{dt}(t) = c_0h^2(1+\varrho)\kappa_2(t)\exp\left(\int_0^t \varrho\kappa_2(\tau)d\tau\right).$$

Let

$$\mu(t) = \max_{\tau\in[0,t]}\frac{\mu_2(\tau)+\mu_3(\tau)}{|\mu_1(\tau)|}, \qquad \gamma(t) = \max_{\tau\in[0,t]}\frac{\gamma_n(\tau)}{|\mu_1(\tau)|}.$$

Then, from the definition of $\lambda_k(t)$, $\Gamma_n(t)$, $S_i(t)$, and $r(t)$, we find that

$$\begin{aligned}\lambda_k(t) &= \Gamma_n(t)\sum_{i=1}^{k} S_i(t) = -\operatorname{sign}(\mu_1)\gamma(t)\sum_{i=1}^{k}\big(-r(t)\big)^{i-1}\alpha_i(t)\\ &= -\operatorname{sign}(\mu_1)\gamma(t)\sum_{i=1}^{k}\big(-\operatorname{sign}(\mu_1)\mu(t)\big)^{i-1}\alpha_i(t)\\ &= \sum_{i=1}^{k}\gamma(t)\big(\mu(t)\big)^{i-1}\big(-\operatorname{sign}(\mu_1)\big)^{i}\alpha_i(t).\end{aligned}$$

Since

$$\max_{\tau\in[0,t]}\frac{\gamma_n(\tau)}{|\mu_1(\tau)|} = c_0 h^4(1+\varrho)\sigma^{-1}, \qquad \max_{\tau\in[0,t]}\frac{\mu_2(\tau)+\mu_3(\tau)}{|\mu_1(\tau)|} = 1+\varrho h^2\sigma^{-1},$$

the function $\lambda_k(t)$ can be written in the form

$$\lambda_k(t) = \sum_{i=1}^{k} c_0 h^4(1+\varrho)\sigma^{-1}\big(1+\varrho h^2\sigma^{-1}\big)^{i-1}\big(-\operatorname{sign}(\mu_1)\big)^{i}\alpha_i(t).$$

Therefore, from the relation

$$\big(-\operatorname{sign}(\mu_1)\big)^{i}\frac{d\alpha_i}{dt}(t) = \frac{|\mu_1(t)|}{(i-1)!}e^{A(t)}\Big(\int_0^t |\mu_1(\tau)|d\tau\Big)^{i-1}$$

$$= \frac{\sigma h^{-2}\kappa_2(t)}{(i-1)!}\Big(\sigma h^{-2}\int_0^t \kappa_2(\tau)d\tau\Big)^{i-1}\exp\Big(-\sigma h^{-2}\int_0^t \kappa_2(\tau)d\tau\Big)$$

we obtain

$$\begin{aligned}\frac{d\lambda_k}{dt}(t) &= c_0 h^2(1+\varrho)\kappa_2(t)\exp\Big(-\sigma h^{-2}\int_0^t \kappa_2(\tau)d\tau\Big)\\ &\cdot \sum_{i=1}^{k}\frac{1}{(i-1)!}\Big((\sigma h^{-2}+\varrho)\int_0^t \kappa_2(\tau)d\tau\Big)^{i-1}\end{aligned}$$

and

$$\begin{aligned}\frac{d\lambda_k}{dt}(t) &< c_0 h^2 (1+\varrho)\kappa_2(t)\exp\Big(-\sigma h^{-2}\int_0^t \kappa_2(\tau)d\tau\Big)\exp\Big((\sigma h^{-2}+\varrho)\int_0^t \kappa_2(\tau)d\tau\Big)\\ &= c_0 h^2 (1+\varrho)\kappa_2(t)\exp\Big(\varrho\int_0^t \kappa_2(\tau)d\tau\Big) = \frac{d\eta}{dt}(t).\end{aligned} \tag{6.11}$$

Notice that $\lambda_k(0) = 0$ and $\eta(0) = 0$. Therefore, from (6.11), we conclude that $\lambda_k(t) < \eta(t)$, for all $t \in [0, T]$ and $k = 1, 2, \ldots$, which finishes the proof. □

In this last theorem, we have demonstrated a relation between error bounds, specifically, that the latter bound is sharper. Particularly, we conclude from the form of the error bounds that the numerical schemes indeed converge towards the exact solutions as the step-size h tends to zero – a necessity for any numerical scheme – from which we deduce that the schemes produce robust results.

In this thesis, we investigated general parabolic partial functional differential equations and their approximate solutions constructed by means of spatial discretization and iterative processes of the Picard type in the functional sense. We used differential inequalities and proved a variety of theorems on the approximate solutions and their errors. Particularly, on the basis of the results that we have proved, we have arrived at important convergence properties of the considered numerical schemes.

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APPENDIX

In this thesis, we apply the following one-dimensional version of [34, Theorem 1]. For this application, we use the symbol $\mathcal{F}_c(\{x_i : i = 0, \pm 1, \ldots, \pm\tilde{M}\} \times [-\tau_0, 0], \mathbb{R})$ to denote a class of functions continuously differentiable with respect to the second argument. We also use similar notation for functions on similar domains.

Theorem 6.7. *We assume that the following conditions are satisfied.*

(i) $\sigma : [0,T] \times \mathbb{R}_+ \to \mathbb{R}_+$ *is continuous, nondecreasing with respect to the second input and such that there exists the right-hand maximum solution* ω *on* $[0,T]$ *of the initial-value problem*

$$\begin{cases} \dot{\omega}(t) &= \sigma(t, \omega(t)), \\ \omega(0) &= 0, \end{cases}$$

(ii) $\mathcal{G}, \mathcal{H} : [0,T] \times \{x_i : i = 0, \pm 1, \ldots, \pm M'\} \times \mathcal{F}_c(\{x_i : i = 0, \pm 1, \ldots, \pm\hat{M}\} \times [-\tau_0, 0], \mathbb{R}) \to \mathbb{R}$ *satisfy the inequality*

$$\frac{h}{2}\Big|\mathcal{G}(P)\Big| \leq \mathcal{H}(P),$$

where P *is any point in the domain of* $\mathcal{G}$ *and* $\mathcal{H}$*,*

(iii) $\rho \in \mathcal{F}_c(\{x_i : i = 0, \pm 1, \ldots, \pm\tilde{M}\} \times [-\tau_0, T], \mathbb{R})$ *is such that* $\rho(x_i, t) = 0$ *if* $t \in [-\tau_0, 0]$ *and* $i = 0, \pm 1, \ldots, \pm\tilde{M}$ *or if* $t \in [-\tau_0, T]$ *and* $i = \pm M, \ldots, \pm\tilde{M}$*, is*

differentiable with respect to the second input and satisfies the inequality

$$\Big|\frac{\partial \rho}{\partial t}(x_i,t) - \tilde{\delta}\rho(x_i,t)\mathcal{G}(x_i,t,\rho_{(x_i,t)}) - \tilde{\delta}^{(2)}\rho(x_i,t)\mathcal{H}(x_i,t,\rho_{(x_i,t)})\Big|$$
$$\leq \sigma(t, \|\rho_{(x_i,t)}\|_n^h),$$

for $i = 0, \pm 1, \dots, \pm M'$, *where*

$$\begin{aligned}
\tilde{\delta}\rho(x_i,t) &= \frac{1}{2h}\Big(\rho(x_{i+1},t) - \rho(x_{i-1},t)\Big),\\
\tilde{\delta}^{(2)}\rho(x_i,t) &= \frac{1}{h^2}\Big(\rho(x_{i+1},t) - 2\rho(x_i,t) + \rho(x_{i-1},t)\Big),\\
\|\rho_{(x_i,t)}\|_n^h &= \max\{|\rho(x_i + x_j, t+\tau)| : j = 0, \pm 1, \dots, \pm \hat{M}, \tau \in [-\tau_0, 0]\}.
\end{aligned} \tag{6.12}$$

Then,

$$|\rho(x_i,t)| \leq \omega(t),$$

for all $t \in [0,T], i = 0, \pm 1, \dots, \pm M'$.

For clarity, we omit the proof, which is too technical for the purpose of this thesis. We refer the reader to [34], where the proof is presented for the multi-dimensional case.