THE CLASSICAL THEORY OF REARRANGEMENTS

by

Monica Josue Agana

A thesis

submitted in partial fulfillment

of the requirements for the degree of

Master of Science in Mathematics

Boise State University

December 2015

BOISE STATE UNIVERSITY GRADUATE COLLEGE

DEFENSE COMMITTEE AND FINAL READING APPROVALS

of the thesis submitted by

Monica Josue Agana

Thesis Title: The Classical Theory of Rearrangements

Date of Final Oral Examination: 9 October 2015

The following individuals read and discussed the thesis submitted by student Monica Josue Agana, and they evaluated her presentation and response to questions during the final oral examination. They found that the student passed the final oral examination.

Andrés Caicedo, Ph.D.	Co-Chair, Supervisory Committee
Zachariah Teitler, Ph.D.	Co-Chair, Supervisory Committee
Samuel Coskey, Ph.D.	Member, Supervisory Committee
Marion Scheepers, Ph.D.	Member, Supervisory Committee

The final reading approval of the thesis was granted by Andrés Caicedo, Ph.D., Co-Chair of the Supervisory Committee, and Zachariah Teitler, Ph.D., Co-Chair of the Supervisory Committee. The thesis was approved for the Graduate College by John R. Pelton, Ph.D., Dean of the Graduate College.

Dedicated to my parents, Edson and Maria.

ACKNOWLEDGMENTS

The author wishes to express gratitude to Dr. Andrés Caicedo, Dr. Zach Teitler, and Dr. Uwe Kaiser for advising her work. She would also like to thank the Math Department for the Summer Graduate Research Fellowship for the summer of 2014 and 2015.

ABSTRACT

One type of conditionally convergent series that has long been considered by mathematicians is the Alternating Harmonic Series and its sum under various types of rearrangements. The purpose of this thesis is to introduce results from the classical theory of rearrangements dating back to the 19th and early 20th century. We will look at results by mathematicians such as Ohm, Riemann, Schlömilch, Pringsheim, and Sierpiński. In addition, we show examples of each classical result by applying the Alternating Harmonic Series under the different types of rearrangements, and also introducing theorems by Lévy and Steinitz, and Wilczyński, which are modern extensions of the results of Sierpiński.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Preliminaries

Given a sequence (f_n) of real numbers, and a permutation $\pi : \mathbb{N} \to \mathbb{N}$, the rearrangement determined by π is the sequence $(f_{\pi(n)})$ whose n^{th} term is the $\pi(n)^{th}$ term of the original sequence. A rearrangement of a series $\sum f_n$ is $\sum f_{\pi(n)}$ for some such permutation π.

We demonstrate this by the following example.

Example 1.1.0.1. *Consider the Alternating Harmonic Series (see Section 1.2 below for an evaluation of the series),*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots = \ln(2).$$

Now consider another series,

$$\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \cdots = \frac{1}{2}\ln(2).$$

Note that adding 0 *between each term will not change the sum of the series. So we get,*

$$0 + \frac{1}{2} + 0 + \left(-\frac{1}{4}\right) + 0 + \frac{1}{6} + 0 + \left(-\frac{1}{8}\right) + 0 + \frac{1}{10} + 0 + \left(-\frac{1}{12}\right) + 0 + \cdots = \frac{1}{2}\ln(2).$$

Now let us add this series to the usual Alternating Harmonic Series, term by term, and we have

$$1+0+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+0+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+0+\frac{1}{11}-\cdots$$

$$=1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\frac{1}{13}+\frac{1}{15}-\cdots=\frac{3}{2}\ln(2).$$

Notice that the resulting series is a rearrangement of the Alternating Harmonic Series, but converges to another sum.

The goal of this thesis is to present some classical results illustrating the extent of this phenomenon.

1.2 Background

1.2.1 Evaluation of the Alternating Harmonic Series

Theorem 1.2.1. *If $\sum f_n$ converges, then $\lim_{n\to\infty} f_n = 0$. In fact, $\lim_{n\to\infty} \sum_{k=n+1}^{\infty} f_k = 0$ and, for any $\varepsilon > 0$, there is an N such that for all $n < m$ with $N < n$, we have*

$$\left| \sum_{k=n}^{m} f_k \right| < \varepsilon.$$

Proof. $\sum f_n$ converges means that the sequence of partial sums (S_k), where

$$S_k = \sum_{n=1}^{k} f_n$$

converges. In other words, the sequence is Cauchy. So for all $\varepsilon > 0$, there exists an N for all $n < m$ (if $n > N$, then $\mid S_n - S_m \mid < \varepsilon$). This is precisely the last statement. The first statement follows from this by taking $m = n+1$. The second statement follows by noting that if $\sum f_n = s$, then $S_n \to S$ if and only if $S_n - S \to 0$. $\square$

Abel's Theorem

Taken from [9, pg. 125], we have the following,

Theorem 1.2.2 (Abel (1826) [9])**.** *Suppose* $\sum\limits_{n\geq 0}^{\infty} f_n$ *converges. We then have that*

$$\lim_{x\to 1^-} \sum_{n\geq 0}^{\infty} f_n x^n = \sum_{n\geq 0}^{\infty} f_n.$$

Even if $f(x) = \sum f_n x^n$ has radius of convergence 1, it is still (left) continuous at the endpoint $x = 1$.

Proof. [18] Suppose $t_n = f_0 + f_1 + f_2 + \cdots + f_n$ and let $t_{-1} = 0$. Then, we have that

$$\sum_{n\geq 0}^{m} f_n x^n = \sum_{n\geq 0}^{m} (t_n - t - 1) x^n$$

$$= (1-x) \sum_{n\geq 0}^{m-1} t_n x^n + t_m x^m. \tag{1.1}$$

Now consider $\mid x \mid < 1$ and suppose $m \to \infty$. We want to show that

$$\sum_{n\geq 0}^{\infty} f_n x^n = (1-x) \sum_{n\geq 0}^{\infty} t_n x^n.$$

Since (t_n) converges, then it is bounded, meaning there is some value T such that $\mid t_n \mid < T$ for each n. Thus, $\mid t_m x^m \mid < T \mid x \mid^m$ for each m. Since $m \to \infty$, then $t_n x^m \to 0$. Also, by the fact that $\mid x \mid^m \to 0$, then $T \mid x \mid^m \to 0$. So from (1.1) we now have

$$= (1-x) \sum_{n \geq 0}^{m-1} t_n x^n + 0$$

which results from taking the limit of both sides as $m \to 0$. Now, let $t = \lim_{n\to\infty} t_n$, and let $\varepsilon > 0$. Pick an $N \in \mathbb{N}$ so that $n > N$ implies

$$\mid t - t_n \mid < \frac{\varepsilon}{2}. \tag{1.2}$$

Since

$$(1-x) \sum_{n \geq 0}^{\infty} x^n = 1 \tag{1.3}$$

for $\mid x \mid < 1$.

Then, from $\sum_{n \geq 0}^{\infty} f_n x^n = (1-x) \sum_{n \geq 0}^{\infty} t_n x^n$ we get

$$\begin{aligned} \mid \sum_{n \geq 0}^{\infty} f_n x^n - t \mid &= \mid (1-x) \textstyle\sum_{n \geq 0}^{\infty} (t_n - t) x^n \mid \\ &\leq (1-x) \sum_{n \geq 0}^{N} \mid t_n - t \mid \mid x \mid^n \\ &= (1-x) \sum_{n \geq 0}^{N} \mid t_n - t \mid \mid x \mid^n + (1-x) \sum_{n \geq N+1}^{\infty} \mid t_n - t \mid \mid x \mid^n . \end{aligned}$$

So by (1.2) and (1.3), we have

$$
\begin{aligned}
&< \quad (1-x)\sum_{n\geq 0}^{N} \mid t_n - t \mid \cdot 1^n + (1-x)\sum_{n\geq N+1}^{\infty} \frac{\varepsilon}{2} \mid x \mid^n \\
&= \quad (1-x)\sum_{n\geq 0}^{N} \mid t_n - t \mid + \frac{\varepsilon}{2}(1-x)\sum_{n\geq N+1}^{\infty} \mid x \mid^n \\
&\leq \quad (1-x)\sum_{n\geq 0}^{N} \mid t_n - t \mid + \frac{\varepsilon}{2}(1-x)\sum_{n\geq 0}^{\infty} \mid x \mid^n .
\end{aligned}
$$

Since $\mid x \mid < 1$, then

$$
= (1-x)\sum_{n\geq 0}^{N} \mid t_n - t \mid + \frac{\varepsilon}{2}. \tag{1.4}
$$

Let $B = \sum_{n\geq 0}^{N} \mid t_n - t \mid$ and let there be some $\delta = \frac{\varepsilon}{2B} > 0$. If

$$
1 - \delta < x < 1, \tag{1.5}
$$

then for equation (1.4) we get $\mid \sum_{n\geq 0}^{\infty} f_n x^n - t \mid < B(1-x) + \frac{\varepsilon}{2}$, and from $\delta = \frac{\varepsilon}{2B}$ and (1.5),

$$
\mid \sum_{n\geq 0}^{\infty} f_n x^n - t \mid < B \cdot \delta + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
$$

This proves that $\lim_{x\to 1^-} \sum_{\geq 0}^{\infty} f_n x^n = t = \sum_{n\geq 0}^{\infty} f_n$. $\square$

Geometric Series

We introduce the following theorem for a general geometric series:

Theorem 1.2.3 ([18])**.** *If* $|x| < 1$, *then* $\sum_{n\geq 0}^{\infty} x^n = \frac{1}{1-x}$. *If* $|x| \geq 1$, *then* $\sum_{n\geq 0}^{\infty} x^n = \infty$.

The proof to this theorem can be found in [18, page 61].

Now consider a variation of the geometric series

$$\sum_{n\geq 0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \cdots = \frac{1}{1+x},$$

which converges only when $|x| < 1$, and is divergent for $|x| \geq 1$.

Theorem 1.2.4. *If* $|x| < 1$, *then*

$$\sum_{n\geq 0}^{\infty} (-1)^n x^n = \frac{1}{1+x}.$$

If $|x| \geq 1$, *then the series diverges.*

Proof. Suppose we start with

$$\sum_{n\geq 0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \ldots.$$

We can rewrite the left side of the equation as

$$\sum_{n\geq 0}^{\infty} (-x)^n = 1 - x + x^2 - x^3 + x^4 - \ldots.$$

Notice that this is similar to the geometric series from Theorem 1.2.3 with $-x$ in place of x, and so we have

$$\sum_{n\geq 0}^{\infty} (-x)^n = \frac{1}{1-(-x)},$$

or

$$\sum_{n\geq 0}^{\infty}(-x)^n = \frac{1}{1-(-x)} = \frac{1}{1+x}$$

such that this series converges for $|x| < 1$, and diverges for $|x| \geq 1$. So,

$$\sum_{n\geq 0}^{\infty}(-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \cdots = \frac{1}{1+x}$$

when $\mid x \mid < 1$, and diverges for $\mid x \mid \geq 1$. $\square$

Considering only values of x within the interval of convergence, integrate both sides of

$$\frac{1}{1+x} = 1 - x + x^2 - x^3 + x^4 - \ldots$$

to get

$$\ln(x+1) + C_1 = \left(x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots\right) + C_2$$

or

$$\ln(x+1) + C_1 - C_2 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \tag{1.6}$$

for some constants C_1 and C_2. We show term by term that integration gives us $\sum_{n=0}^{\infty}(-1)^n x^n$:

For $|-x| < 1$, we have

$$\frac{1}{1+x} = \frac{1}{1-(-x)} = \sum_{n=0}^{\infty}(-x)^n = \sum_{n=0}^{\infty}(-1)^n x^n$$

for $|x| < 1$. Then,

$$\int \frac{1}{1+x}dx = \int \lim_{k\to\infty} \sum_{n=1}^{k} (-1)^n x^n dx = \lim_{k\to\infty} \int \sum_{n=1}^{k} (-1)^n x^n dx,$$

which is a consequence of [18, Theorem 7.16]. So

$$\int \frac{1}{1+x}dx = \sum_{n=0}^{\infty} (-1)^n \int x^n dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + C,$$

for $|x| < 1$.

In (1.6), let $C = C_1 - C_2$. Then, we have

$$\ln(x+1) + C = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots \tag{1.7}$$

To find C, evaluate both sides of (1.7) when $x = 0$. The result is

$$0 + C = 0 - 0 + 0 - \cdots,$$

or $C = 0$. Therefore,

$$\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \dots$$

for $-1 < x < 1$.

Now, when $x = 1$, we have the following power series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \cdots,$$

which we recognize as the Alternating Harmonic Series, and converges due to Leibniz's test.

Leibniz's Alternating Series Test

Theorem 1.2.5 (Leibniz's Alternating Series Test [9])**.** *If (a_n) is a sequence of positive numbers such that the a_n monotonically decrease and a_n converges to* 0*, then*

$$A = \sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots$$

converges.

Proof. Let A_n be the n^{th} partial sum of A. Since (a_n) are monotonically decreasing, then the following inequalities hold true:

$$a_1 \geq a_1 - a_2 + a_3 \geq a_1 - a_2 + a_3 - a_4 + a_5 \geq \cdots,$$

and

$$a_1 - a_2 \leq a_1 - a_2 + a_3 - a_4 \leq a_1 - a_2 + a_3 - a_4 + a_5 - a_6 \leq \cdots.$$

We find that (A_{2n}) monotonically increases and is bounded above by A_1, and that (A_{2n+1}) monotonically decreases and is bounded below by A_2. Since both sequences are bounded, they are also convergent being that (a_{2n}) and (a_{2n+1}) both converge to 0, or

$$\lim_{n\to\infty} (A_{2n+1} - A_{2n}) = \lim_{n\to\infty} a_{2n+1} = 0.$$

□

It now follows from Abel's Theorem 1.2.1 that

$$x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots,$$

converges to $\ln(1+x)$ for $\mid x \mid < 1$, and also converges to $\ln(2)$ when $x = 1$.

1.3 History

Definition 1.3.1. $\sum f_n$ *is a* conditionally convergent series *if* $\sum f_n$ *converges but* $\sum \mid f_n \mid$ *diverges.*

Lemma 1.3.2 (pg. 317, [10])**.** *Let* $\sum f_n$ *be a conditionally convergent series. Let* $\sum a_n$ *be the series of positive terms of* $\sum f_n$ *and let* $\sum b_n$ *be the series of negative terms of* $\sum f_n$*. Then, both* $\sum a_n$ *and* $\sum b_n$ *diverge.*

German mathematician Johann Peter Gustav Lejeune Dirichlet (1805 - 1859) [14] came up with a very important result involving the rearrangement of terms of certain series. Dirichlet was the first to notice that terms in certain series could be rearranged to a sum different from the original series (it was later founded by Bernhard Riemann that this was due to conditionally convergent series). In 1837, Dirichlet published a paper proving that the sum remains the same when rearranging terms in an absolutely convergent series [5].

We proceed to introduce Dirichlet's result, which we took from the chapter entitled, *Arbitrary Series and Infinite Products* in [7].

Theorem 1.3.3 ([7])**.** *If* $\sum f_n$ *is absolutely convergent and converges to* α*, then every rearrangement of* $\sum f_n$ *also converges to* α*.*

Proof. Assume $\sum \mid f_n \mid < \infty$, and that $\sum f_n = \alpha$. Now let $(f_{\pi(n)})$ be a rearrangement of (f_n). We need to show that $\sum_{n=1}^{\infty} f_{\pi(n)} = \alpha$.

Fix $\varepsilon > 0$. We must find N such that for all $n > N$,

$$\left| \sum_{k=1}^{n} f_{\pi(k)} - \alpha \right| < \varepsilon.$$

First find N_1 such that for every $n \geq N_1$,

$$\left| \sum_{k=1}^{n} f_k - \alpha \right| < \frac{\varepsilon}{2},$$

and find N_2 such that for every $n > N_2$,

$$\sum_{k=n}^{\infty} | f_k | < \frac{\varepsilon}{2}.$$

We may assume that $N_2 \geq N_1$.

Finally, let there be some N_3 large enough so that $\{1,2,3,\ldots,N_2\} \subseteq \{\pi(1),\pi(2),\pi(3),\ldots,\pi(N_3)\}$. We claim that $N = N_3$ works. To see this, note that if $n > N_3$, then $\sum_{k=1}^{n} f_{\pi(k)} = \sum_{k=1}^{N_2} f_k + \sum_{j\in A} f_j$, where $A = \{\pi(1),\pi(2),\pi(3),\ldots,\pi(n)\} \setminus \{1,2,3,\ldots,N_2\}$.

Therefore,

$$\begin{aligned}
\left| \sum_{k=1}^{n} f_{\pi(k)} - \alpha \right| &\leq \left| \sum_{k=1}^{N_2} f_k - \alpha \right| + \left| \sum_{j\in A} f_j \right| \\
&< \frac{\varepsilon}{2} + \sum_{j\in A} | f_j | \\
&\leq \frac{\varepsilon}{2} + \sum_{j=N_2+1}^{\infty} | f_j | < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\end{aligned}$$

$\square$

In 1839, a German mathematician by the name of Martin Ohm [1] came up with the following rearrangement theorem.

Theorem 1.3.4 ([11])**.** *For p and q positive integers, rearrange*

$$\sum_{n\geq 1}^{\infty}\frac{(-1)^{n-1}}{n}$$

by taking the first p positive terms, then the first q negative terms, then the next p positive terms, the next q negative terms, and so on. The rearranged series converges to

$$\ln(2)+\frac{1}{2}\ln\left(\frac{p}{q}\right). \tag{1.8}$$

First, we introduce some notation:

Denote by $A(p,q)$ the series resulting from this rearrangement. Let S_k be the partial sums of $A(p,q)$ where specifically $S'_k = S'_k(p,q)$ denotes the k^{th} partial sum of $A(p,q)$, and let $C_n = C_n(p,q)$ be the partial sum of $A(p,q)$, which is the result in S from precisely adding n blocks, where each block consists of p positive terms and q negative terms, consecutively.

Denote S_n to be the partial sums of the usual alternating series, where n denotes the terms. Call C_n as $C_n = S_{f(n)} + R_n$ for some $f(n)$ and some "remainder" R_n.

In Chapter 2, we analyze Ohm's rearrangement theorem by splitting up the proof into two parts. In part (1), we look at three possible cases that could happen in the rearrangement: when $p = q, p < q$, and $p > q$. We do this by first providing an example for each case, and then proving its generalization. In part (2), we show S_n and $C_n = S_{f(n)} + R_n$ converge to $\ln(2) + \frac{1}{2}\ln(\frac{p}{q})$ for each of the three possible cases.

In 1852, German mathematician Bernhard Riemann came up with the explanation for Peter Lejeune-Dirichlet's discovery [5] that one can change the sum of a conditionally convergent series by rearranging its terms. His explanation was the following theorem.

Theorem 1.3.5 (Riemann's Rearrangement Theorem [10])**.** *A series $\sum f_n$ is conditionally convergent if and only if for each real number α, there is a rearrangement of $\sum f_n$ that converges to α.*

In Chapter 3, we analyze Riemann's Rearrangement Theorem (also known as the Riemann Series Theorem) by first proving the result, then providing an example of the rearrangement of the Alternating Harmonic Series the sum of which $\frac{3}{2}\ln(2)$. We then compare it to $A(2,1)$ of Ohm's rearrangement (Chapter 2).

German mathematician Oscar Schlömilch came up with the following result.

Theorem 1.3.6 ([22])**.** *Let $f(n) = u_n$ be a decreasing and positive function, such that the asymptotic value $\lim_{n\to\infty} f(n) = 0$. If $K = \lim_{\omega\to\infty}(\omega u_\omega)$ or $K = \lim_{n\to\infty}(nu_n)$, the following holds: If in a series (see note below),*

$$s = u_0 - u_1 + u_2 - u_3 + \cdots$$

the terms are rearranged such that always p positive terms are followed by q negative terms, then the sum of the rearranged series is

$$S = s + \frac{\lim(nu_n)}{2} \cdot \log(\frac{p}{q}).$$

[Note: Schlömilch does not state it directly at this point, but the series that satisfies the theorem must be one of the form $s = \sum_{n=0}^{\infty}(-1)^n u_n$ such that $u_n > 0$ is strictly decreasing and $\lim_{n\to\infty} u_n = 0$, and by the Alternating Series Test it must converge.]

In Chapter 4, we present a translation of Schlömilch's original paper, and then proceed the rest of the chapter by providing a few examples. Note that the translation of

Schlömilch's result stems from what he addresses as the "well-known theorem," also known as the "Mean Value Theorem," which we also prove. German mathematician Alfred Pringsheim came up with a couple of important results in 1889.

First, we define the following:

Definition 1.3.7. *Let the number α be the* **asymptotic density** *of positive terms in $\sum_{n=1}^{\infty} f_n$. That is,*

$$\alpha = \lim_{n\to\infty} \frac{p_n}{n},$$

where p_n denotes the number of positive terms in the sequence $(f_n)_{n=1}^{k}$.

Theorem 1.3.8 ([2])**.** *The rearrangement $\sum_{k=1}^{\infty} f_n$ converges to an extended real number if and only if α, the asymptotic density of positive terms in $\sum_{k=1}^{\infty} f_n$ exists. Also, the sum of the rearrangement with asymptotic density α is*

$$\ln(2) + \frac{1}{2}\ln\left(\frac{\alpha}{1-\alpha}\right).$$

Theorem 1.3.9 ([2])**.** *Let $f_1, f_2, f_3, \ldots, f_k$ be a sequence such that each $f_i \in \mathbb{R}$ for $i \in \mathbb{N}$. If $\mid f_1 \mid \geq \mid f_2 \mid \geq \mid f_3 \mid \geq \ldots$, where*

$$\lim_{k\to\infty} f_k = 0,$$

and $f_{2j-1} > 0 > f_{2j}$ for $j \in \mathbb{N}$, then the following holds:

1. *If $\lim_{k\to\infty} k\cdot \mid f_k \mid = \infty$ and if $S \in \mathbb{R}$, then there exists a rearrangement of $\sum_{j=1}^{\infty} f_j$, call it $\sum_{j=1}^{\infty} f_j^*$, such that $\sum_{j=1}^{\infty} f_j^* = S$, and the asymptotic density of positive terms of the*

rearrangement is $\frac{1}{2}$,

whose $\alpha = \frac{1}{2}$, and whose sum is S.

2. *If $\lim_{k\to\infty} k \cdot f_k = 0$ and if $\sum_{n=1}^{\infty} g_n$ is a rearrangement of $\sum_{n=1}^{\infty} f_n$ for which the asymptotic density α exists where $0 < \alpha < 1$, then*

$$\sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} f_n.$$

In 1911, Wacław Sierpiński came up with the following theorem.

Theorem 1.3.10 ([21])**.** *Let (f_n) be conditionally convergent where $U = \sum f_n$ and $V = \sum f_{\pi(n)}$. If $V > U$, there exists an explicitly described rearrangement π with the property that each positive term of f_n is left in place (if $f_n > 0$ then $\pi(n) = n$). Similarly, if $V < U$, there exists an explicitly describe rearrangement π with the property that each negative term of f_n is left in place (if $f_n < 0$, then $\pi(n) = n$).*

[Note that Theorem 1.3.9 is our version of Sierpiński's original theorem.]

In Chapter 6, we provide a translation of Sierpiński's paper [21] along with his original theorem. Later, we analyze his result by providing a couple examples.

In Chapter 7, we briefly introduce several modern rearrangements by various mathematicians, which extend Sierpiński's theorem.

CHAPTER 2

OHM'S THEOREM

2.1 Martin Ohm

Martin Ohm (May 6, 1792 - April 1, 1872) was a German mathematician who earned his doctorate in 1811 at Friedrich-Alexander-University, Erlangen-Nuremberg, under his advisor, Karl Christian von Langsdorf [1]. By the early 1800s he became a professor in the gymnasium at Thorn for mathematics and physics. In 1839, he was was chosen to be a professor at the University of Berlin, and delivered lectures at the academy of architecture, and at the school of artillery and engineering [12].

2.2 Ohm's Rearrangement Theorem

2.2.1 Part I

Theorem 2.2.1 ([11])**.** *For p and q positive integers, rearrange* $\sum_{n\geq 1}^{\infty} \frac{(-1)^{n-1}}{n}$ *by taking the first p positive terms, then the first q negative terms, then the next p positive terms, the next q negative terms, and so on. The rearranged series converges to*

$$\ln(2) + \frac{1}{2}\ln\left(\frac{p}{q}\right). \tag{2.1}$$

Denote by $A(p,q)$ the series resulting from this rearrangement. Let S_k be the partial sums of $A(p,q)$ where specifically $S'_k = S'_k(p,q)$ denotes the k^{th} partial sum of $A(p,q)$, and let $C_n = C_n(p,q)$ be the partial sum of $A(p,q)$, which is the result in S from precisely adding n blocks, where each block consists of p positive terms and q negative terms, consecutively.

Denote S_n to be the partial sums of the usual alternating series, where n denotes the terms. Call C_n as $C_n = S_{f(n)} + R_n$ for some $f(n)$ and some "remainder" R_n.

It suffices to break the proof of Theorem 2.2.1 into the following:

I Verifying the explicit formula of $C_n(p,q)$ for each of the three types of rearrangements:

(a) $p = q$,

(b) $p > q$, and

(c) $p < q$.

II Arguing that S_n and C_n converge to $\ln(2) + \frac{1}{2}\ln\left(\frac{p}{q}\right)$.

Example 2.2.1.1. *Consider $A(2,2)$. We want to show that $C_n(2,2) = S_{4n}$. Note that*

$$
\begin{aligned}
C_1 &= 1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{4} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = S_4 = S_{4\cdot 1} \\
C_2 &= 1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{4} + \left[\frac{1}{5} + \frac{1}{7} - \frac{1}{6} - \frac{1}{8}\right] \\
&= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} = S_8 = S_{4\cdot 2}.
\end{aligned}
$$

Proof. We proceed by induction with the base case shown in Example 2.2.1.1. So suppose that

$$
C_n(2,2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{4n-3} - \frac{1}{4n-4} + \frac{1}{4n-1} - \frac{1}{4n} = S_{4n}.
$$

Then by the inductive hypothesis,

$$\begin{aligned} C_{n+1}(2,2) &= C_n(2,2) + \left(\frac{1}{4n+1} + \frac{1}{4n+3}\right) - \left(\frac{1}{4n+2} + \frac{1}{4n+4}\right) \\ &= C_n(2,2) + \frac{1}{4n+1} - \frac{1}{4n+2} + \frac{1}{4n+3} - \frac{1}{4n+4} = S_{4n+4}, \end{aligned}$$

as wanted, so $f(n) = 4n$ and $R_n = 0$. □

General Case: $A(p,q), p = q$**.**

We now proceed with the more general case:

Theorem 2.2.2. *For positive integer p,* $C_n(p,p) = S_{2pn}$ *for all n.*

Proof. Consider $A(p,p)$. We argue that

$$C_n := C_n(p,p) = S_{2pn}$$

by induction. Note that

$$\begin{aligned} C_1 &= \left(1 + \frac{1}{3} + \cdots + \frac{1}{2p-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2p}\right) \\ &= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2p-1} - \frac{1}{2p} = S_{2p}, \\ C_2 &= \left(1 + \frac{1}{3} + \cdots + \frac{1}{2p-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2p}\right) \\ &\quad + \left[\left(\frac{1}{2p+1} + \frac{1}{2p+3} + \cdots + \frac{1}{4p-1}\right) - \left(\frac{1}{2p+2} + \frac{1}{2p+4} + \cdots + \frac{1}{4p}\right)\right] \end{aligned}$$

$$= \quad C_1 + \frac{1}{2p+1} - \frac{1}{2p+2} + \frac{1}{2p+3} - \frac{1}{2p+4} + \cdots + \frac{1}{4p-1} - \frac{1}{4p}$$

$$= \quad S_{2p} + \frac{1}{2p+1} - \frac{1}{2p+2} + \frac{1}{2p+3} - \frac{1}{2p+4} + \cdots + \frac{1}{4p-1} - \frac{1}{4p} = S_{4p}.$$

Now suppose that

$$C_n(p,p) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2pn-1} + \frac{1}{2pn} = S_{2pn}.$$

By the inductive hypothesis,

$$\begin{aligned} C_{n+1}(p,p) = C_n(p,p) \quad &+ \quad \left(\frac{1}{2pn+1} + \frac{1}{2pn+3} + \cdots + \frac{1}{2pn+2p-1}\right) \\ &- \quad \left(\frac{1}{2pn+2} + \frac{1}{2pn+4} + \cdots + \frac{1}{2pn+2p}\right) \end{aligned}$$

$$\begin{aligned} = C_n(p,p) + \frac{1}{2pn+1} - \frac{1}{2pn+2} + \frac{1}{2pn+3} - \frac{1}{2pn+4} + \\ \cdots + \frac{1}{2pn+2p-1} - \frac{1}{2pn+2p} \end{aligned}$$

$$= S_{2np} + \frac{1}{2pn+1} - \frac{1}{2pn+2} + \frac{1}{2pn+3} - \frac{1}{2pn+4} + \cdots + \frac{1}{2pn+2p-1} - \frac{1}{2pn+2p}$$

$$= S_{2pn+2p} = S_{2p(n+1)}$$

Therefore, $f(n) = 2p(n+1)$ and $R_n = 0$. $\square$

Example 2.2.2.1. *Consider $A(3,1)$. We have*

$$
\begin{aligned}
C_1 &= 1+\frac{1}{3}+\frac{1}{5}+\left(-\frac{1}{2}\right) \\
&= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\left(\frac{1}{4}+\frac{1}{6}\right) \\
&= S_6+\frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right)=S_{6\cdot 1}+\frac{1}{2}\left(\frac{1}{2}+\frac{1}{3}\right) \\
C_2 &= 1+\frac{1}{3}+\frac{1}{5}+\left(-\frac{1}{2}\right)+\left[\frac{1}{7}+\frac{1}{9}+\frac{1}{11}+\left(-\frac{1}{4}\right)\right] \\
&= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}+\left(\frac{1}{6}+\frac{1}{8}+\frac{1}{10}+\frac{1}{12}\right) \\
&= S_{12}+\frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)=S_{6\cdot 2}+\frac{1}{2}\left(\frac{1}{3}+\frac{1}{4}+\frac{1}{5}+\frac{1}{6}\right)
\end{aligned}
$$

We claim from our analysis of C_1 and C_2 that

$$C_n(3,1)=S_{6n}+\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{3n}\right).$$

So, $f(n)=6n$ and

$$R_n=\frac{1}{2}\left(\frac{1}{n+1}+\frac{1}{n+2}+\frac{1}{n+3}+\cdots+\frac{1}{3n}\right).$$

We also find that the last block of $C_n(3,1)$ is

$$\frac{1}{6n-5}+\frac{1}{6n-3}+\frac{1}{6n-1}-\frac{1}{2n}.$$

We proceed to verify this.

Theorem 2.2.3. *(1) The last block of* $C_n(3,1)$ *is* $\dfrac{1}{6n-5}+\dfrac{1}{6n-3}+\dfrac{1}{6n-1}-\dfrac{1}{2n}$, *and*
(2) $C_n(3,1) = S_{6n} + \dfrac{1}{2}\left(\dfrac{1}{n+1}+\dfrac{1}{n+2}+\dfrac{1}{n+3}+\cdots+\dfrac{1}{3n}\right)$.

Proof. Let us proceed by induction on n. We already have seen that (1) and (2) hold when $n = 1,2$. Now suppose that (1) and (2) hold for n. We need to show that (1) and (2) hold for $n+1$.

Proof of (1): By the inductive hypothesis, we know that the positive terms added in the last block of $C_n(3,1)$ are

$$\frac{1}{6n-5}+\frac{1}{6n-3}+\frac{1}{6n-1}.$$

Hence, the three positive terms in the last block of $C_{n+1}(3,1)$ that are added together are

$$\frac{1}{6n+1}+\frac{1}{6n+3}+\frac{1}{6n+5}.$$

Similarly, by the inductive hypothesis, we know that the term subtracted in the last block of $C_n(3,1)$ is $\frac{1}{2n}$. Hence, then the term subtracted in $C_{n+1}(3,1)$ must be $\frac{1}{2n+2}$. We continue to show that these terms added and subtracted in the last block of $C_{n+1}(3,1)$ by noting that $6(n+1)-5 = 6n+1, 6(n+1)-3 = 6n+3, 6(n+1)-1 = 6n+5, 2(n+1) = 2n+2$. This proves (1) by induction, and we have

$$C_{n+1}(3,1) = C_n(3,1) + \frac{1}{6n+1}+\frac{1}{6n+3}+\frac{1}{6n+5}-\frac{1}{2n+2}.$$

Proof of (2): By the inductive hypothesis,

$$C_n(3,1) = S_{6n} + \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{3n}\right).$$

We want to show that

$$C_{n+1}(3,1) = S_{6(n+1)} + \frac{1}{2}\left(\frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{3(n+1)}\right).$$

From (1), we have shown that

$$C_{n+1}(3,1) = C_n(3,1) + \frac{1}{6n+1} + \frac{1}{6n+3} + \frac{1}{6n+5} - \frac{1}{2n+2}.$$

Or in other words,

$$C_{n+1}(3,1) - C_n(3,1) = \frac{1}{6n+1} + \frac{1}{6n+3} + \frac{1}{6n+5} - \frac{1}{2n+2}.$$

So, to prove

$$C_{n+1}(3,1) = S_{6(n+1)} + \frac{1}{2}\left(\frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{3(n+1)}\right),$$

it is enough to show that

$$\begin{aligned}\Bigg(S_{6(n+1)} \quad &+ \quad \frac{1}{2}\left(\frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{3(n+1)}\right)\Bigg)\\ &- \quad \left(S_{6n} + \frac{1}{2}\left(\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{3n}\right)\right)\\ &= \quad \frac{1}{6n+1} + \frac{1}{6n+3} + \frac{1}{6n+5} - \frac{1}{2n+2}.\end{aligned}$$

By definition of S_k, we know that

$$S_{6(n+1)} - S_{6n} = \frac{1}{6n+1} - \frac{1}{6n+2} + \frac{1}{6n+3} - \frac{1}{6n+4} + \frac{1}{6n+5} - \frac{1}{6n+6},$$

and since the expression we are considering equals

$$S_{6(n+1)} - S_{6n} + \frac{1}{2}\left(\frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{3(n+1)} - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{3n}\right),$$

we need to prove the identity obtained from substituting the expression $S_{6(n+1)} - S_{6n}$:

$$\frac{1}{6n+1} - \frac{1}{6n+2} + \frac{1}{6n+3} - \frac{1}{6n+4} + \frac{1}{6n+5} - \frac{1}{6n+6}$$

$$+\frac{1}{2}\left(\frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{3(n+1)} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3} - \cdots - \frac{1}{3n}\right)$$

$$= \frac{1}{6n+1} + \frac{1}{6n+3} + \frac{1}{6n+5} - \frac{1}{2n+2}.$$

Notice that the terms $\frac{1}{6n+1}, \frac{1}{6n+3}, \frac{1}{6n+5}$ get canceled out and we are left with

$$-\frac{1}{6n+2} - \frac{1}{6n+4} - \frac{1}{6n+6} + \frac{1}{2}\left(\frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \right.$$

$$\left. \frac{1}{3(n+1)} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3} - \cdots - \frac{1}{3n}\right) = -\frac{1}{2n+2}.$$

Expanding and simplifying the left side of the equation, we need to show,

$$-\frac{1}{6n+2} \quad - \quad \frac{1}{6n+4} - \frac{1}{6n+6} + \frac{1}{2}\left(\frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} + \cdots + \frac{1}{2n} + \frac{1}{2n+1} + \ldots\right.$$

$$+ \quad \frac{1}{3n} + \frac{1}{3n+1} + \frac{1}{3n+2} + \frac{1}{3n+3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3} - \frac{1}{n+4} - \ldots$$

$$-\quad \frac{1}{2n}-\frac{1}{2n+1}-\frac{1}{2n+2}-\frac{1}{2n+3}-\frac{1}{2n+4}-\cdots-\frac{1}{3n}\Bigg)=-\frac{1}{2n+2},$$

or

$$-\frac{1}{6n+2}-\frac{1}{6n+4}-\frac{1}{6n+6}+\frac{1}{2}\left(\frac{1}{3n+1}+\frac{1}{3n+2}+\frac{1}{3n+3}-\frac{1}{n+1}\right)=-\frac{1}{2n+2}$$

$$=-\frac{1}{6n+2}-\frac{1}{6n+4}-\frac{1}{6n+6}+\frac{1}{6n+2}+\frac{1}{6n+4}+\frac{1}{6n+6}-\frac{1}{2n+2}=-\frac{1}{2n+2},$$

which simplifies to $-\frac{1}{2n+2}=-\frac{1}{2n+2}$, and so (2) follows. $\square$

General Case: $A(p,q), p > q$.

Theorem 2.2.4. *Suppose that $p > q$. (1) The last block of $C_n(p,q)$ is*
$\left(\frac{1}{2np-2p+1}+\frac{1}{2np-2p+3}+\cdots+\frac{1}{2np-1}\right)-\left(\frac{1}{2nq-2q+2}+\frac{1}{2nq-2q+4}+\dots\right.$
$\left.+\frac{1}{2nq}\right)$*, and*
(2) $C_n(p,q) = S_{2np}+\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\cdots+\frac{1}{2np}\right)$.

Proof. We proceed by induction. Note first that

$$\begin{aligned}
C_1 &= \left(1+\frac{1}{3}+\cdots+\frac{1}{2p-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2q}\right) \\
&= \left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2p-1}-\frac{1}{2p}\right)+\left(\frac{1}{2q+2}+\frac{1}{2q+4}+\cdots+\frac{1}{2p}\right) \\
&= S_{2p}+\left(\frac{1}{2q+2}+\frac{1}{2q+4}+\cdots+\frac{1}{2p}\right) = S_{2\cdot 1\cdot p}+\left(\frac{1}{2q+2}+\frac{1}{2q+4}+\cdots+\frac{1}{2p}\right) \\
C_2 &= C_1+\left[\left(\frac{1}{2p+1}+\frac{1}{2p+3}+\cdots+\frac{1}{4p-1}\right)-\left(\frac{1}{2q+2}+\frac{1}{2q+4}+\cdots+\frac{1}{4q}\right)\right] \\
&= \left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{4p-1}-\frac{1}{4p}\right)+\left(\frac{1}{4q+2}+\frac{1}{4q+4}+\cdots+\frac{1}{4p}\right) \\
&= S_{4p}+\left(\frac{1}{4q+2}+\frac{1}{4q+4}+\cdots+\frac{1}{4p}\right) = S_{2\cdot 2\cdot p}+\left(\frac{1}{4q+2}+\frac{1}{4q+4}+\cdots+\frac{1}{4p}\right).
\end{aligned}$$

This shows that (1) and (2) hold when $n = 1,2$.

Now suppose that (1) and (2) hold for n. We need to show that (1) and (2) hold for $n+1$.

Proof of (1): By the inductive hypothesis, we know that the positive terms added in the last block of $C_n(p,q)$ are

$$\left(\frac{1}{2np-2p+1}+\frac{1}{2np-2p+3}+\cdots+\frac{1}{2np-1}\right).$$

Hence, the p positive terms in the last block of $C_{n+1}(p,q)$ that are added together are

$$\frac{1}{2np+1}+\frac{1}{2np+3}+\cdots+\frac{1}{2np+2p+1}.$$

Similarly, by the inductive hypothesis, we know that the q terms subtracted in the last block of $C_n(p,q)$ are $\frac{1}{2nq-2q+2}+\frac{1}{2nq-2q+4}+\cdots+\frac{1}{2nq}$.

Hence, the terms subtracted in $C_{n+1}(p,q)$ must be $\frac{1}{2nq+2}+\frac{1}{2nq+4}+\cdots+\frac{1}{2nq+2q}$. This proves (1) by noting that

$$\begin{aligned}
2(n+1)p-2p+1 &= 2np+1,\\
2(n+1)p-2p+3 &= 2np+3,\ldots,2(n+1)p-1=2np+2p-1\\
2(n+1)q-2q+2 &= 2nq+2,\\
2(n+1)q-2q+4 &= 2nq+4,\ldots,2(n+1)q=2nq+2q.
\end{aligned}$$

Note also that

$$\begin{aligned}
C_{n+1}(p,q)=C_n(p,q) &+ \left(\frac{1}{2np+1}+\frac{1}{2np+3}+\cdots+\frac{1}{2np+2p-1}\right)\\
&- \left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\cdots+\frac{1}{2nq+2q}\right).
\end{aligned}$$

Proof of (2): By the inductive hypothesis,

$$C_n(p,q) = S_{2np} + \left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np}\right).$$

We want to show that

$$C_{n+1}(p,q) = S_{2(n+1)p} + \left(\frac{1}{2(n+1)q+2} + \frac{1}{2(n+1)q+4} + \cdots + \frac{1}{2(n+1)p}\right).$$

From (1), we have already shown that

$$\begin{aligned} C_{n+1}(p,q) = C_n(p,q) \quad &+ \quad \left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2np+2p-1}\right) \\ &- \quad \left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2nq+2q}\right). \end{aligned}$$

Since

$$\begin{aligned} C_{n+1}(p,q) = C_n(p,q) \quad &+ \quad \left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2np+2p-1}\right) \\ &- \quad \left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \frac{1}{2nq+2q}\right), \end{aligned}$$

it is enough to show that

$$C_{n+1}(p,q) = \left(S_{2np} + \left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np}\right)\right)$$

$$+\left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2np+2p-1}\right) - \left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2nq+2q}\right).$$

By definition of S_k, we have that

$$\begin{aligned} S_{2(n+1)p} - S_{2np} &= \left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2np+2p-1}\right) \\ &- \left(\frac{1}{2np+2} + \frac{1}{2np+4} + \cdots + \frac{1}{2np+2p}\right) \end{aligned}$$

$$= \frac{1}{2np+1} - \frac{1}{2np+2} + \frac{1}{2np+3} - \frac{1}{2np+4} + \cdots + \frac{1}{2np+2p-1} - \frac{1}{2np+2p}$$

$$= \frac{1}{2np+1} - \frac{1}{2np+2} + \frac{1}{2np+3} - \cdots - \frac{1}{2np+2p}.$$

Notice that S_{2np} can be written as

$$S_{2np} = \left(S_{2(n+1)p} - \left(\frac{1}{2np+1} - \frac{1}{2np+2} + \frac{1}{2np+3} - \cdots - \frac{1}{2np+2p}\right)\right).$$

So substituting S_{2np} into our expression for $C_{n+1}(p,q)$, we get

$$C_{n+1}(p,q) = \left(S_{2(n+1)p} - \left(\frac{1}{2np+1} - \frac{1}{2np+2} + \frac{1}{2np+3} - \cdots - \frac{1}{2np+2p}\right)\right)$$

$$+ \left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np}\right) + \left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2np+2p-1}\right)$$

$$- \left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2nq+2q}\right).$$

Rewriting this so that the positive and negative terms in

$$\left(\frac{1}{2np+1} - \frac{1}{2np+2} + \frac{1}{2np+3} - \cdots - \frac{1}{2np+2p}\right)$$

are grouped together, we have

$$C_{n+1}(p,q) = \left(S_{2(n+1)p} - \left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2np+2p-1}\right)\right.$$

$$+ \quad \left.\left(\frac{1}{2np+2} + \frac{1}{2np+4} + \cdots + \frac{1}{2np+2p}\right)\right) + \left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np}\right)$$

$$+ \quad \left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2np+2p-1}\right)$$
$$- \quad \left(\tfrac{1}{2nq+2} + \tfrac{1}{2nq+4} + \cdots + \tfrac{1}{2nq+2q}\right).$$

Simplifying we have,

$$C_{n+1}(p,q) = \left(S_{2(n+1)p} + \left(\frac{1}{2np+2} + \frac{1}{2np+4} + \cdots + \frac{1}{2np+2p}\right)\right)$$

$$+\left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np}\right) - \left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2nq+2q}\right).$$

Notice by expansion and simplification we get

$$\begin{aligned} C_{n+1}(p,q) &= S_{2(n+1)p} + \left(\frac{1}{2nq+2q+2} + \frac{1}{2nq+2q+4} + \cdots + \frac{1}{2np+2p}\right) \\ &= S_{2(n+1)p} + \left(\frac{1}{2(n+1)q+2} + \frac{1}{2(n+1)q+4} + \cdots + \frac{1}{2(n+1)p}\right), \end{aligned}$$

which is what we wanted. $\square$

Example 2.2.4.1. *Consider $A(2,3)$. We have*

$$\begin{aligned}
C_1 &= 1+\frac{1}{3}-\frac{1}{2}-\frac{1}{4}-\frac{1}{6} \\
&= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}-\left(\frac{1}{5}\right) \\
&= S_6-\left(\frac{1}{5}\right)=S_{6\cdot 1}-\left(\frac{1}{5}\right) \\
C_2 &= 1+\frac{1}{3}-\frac{1}{2}-\frac{1}{4}-\frac{1}{6}+\left[\frac{1}{5}+\frac{1}{7}-\frac{1}{8}-\frac{1}{10}-\frac{1}{12}\right] \\
&= 1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\frac{1}{5}-\frac{1}{6}+\frac{1}{7}-\frac{1}{8}+\frac{1}{9}-\frac{1}{10}+\frac{1}{11}-\frac{1}{12}-\left(\frac{1}{9}+\frac{1}{11}\right) \\
&= S_{12}-\left(\frac{1}{9}+\frac{1}{11}\right)=S_{6\cdot 2}-\left(\frac{1}{9}+\frac{1}{11}\right).
\end{aligned}$$

This leads us to claim general formulas for $C_n(2,3)$, which we proceed to show:

Theorem 2.2.5. *$C_n(2,3)$ is (1) the last block of $C_n(2,3)$ is $\dfrac{1}{4n-3}+\dfrac{1}{4n-1}-\dfrac{1}{6n-4}-\dfrac{1}{6n-2}-\dfrac{1}{6n}$, and (2) $C_n(2,3)=S_{6n}+-\left(\dfrac{1}{4n+1}+\dfrac{1}{4n+3}+\cdots+\dfrac{1}{6n-1}\right)$.*

Proof. Let us proceed by induction on n. We already have seen that (1) and (2) hold when $n=1,2$. Now suppose that (1) and (2) hold for n. We need to show that (1) and (2) hold for $n+1$.

Proof of (1): By the inductive hypothesis, we know that the positive terms added in the last block of $C_n(2,3)$ are $\frac{1}{4n-3}$ and $\frac{1}{4n-1}$. Since we know we need to add two positive terms, then the positive terms added in the last block of $C_{n+1}(2,3)$ are $\frac{1}{4n+1}$ and $\frac{1}{4n+3}$. Similarly, by the inductive hypothesis, the three terms subtracted in the last block of $C_n(2,3)$, which are, respectively, $\frac{1}{6n-4}, \frac{1}{6n-2}$, and $\frac{1}{6n}$. So, the three negative terms subtracted in $C_{n+1}(2,3)$ are $\frac{1}{6n+2}, \frac{1}{6n+4}$, and $\frac{1}{6n+6}$. (1) follows from noting that $4(n+1)-3 = 4n+1, 4(n+1)-1 = 4n+3, 6(n+1)-4 = 6n+2, 6(n+1)-2 = 6n+4, 6(n+1) = 6n+6$.

Also,

$$C_{n+1}(2,3) = C_n(2,3) + \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{6n+2} - \frac{1}{6n+4} - \frac{1}{6n+6}.$$

Proof of (2): By the inductive hypothesis,

$$C_n(2,3) = S_{6n} + -\left(\frac{1}{4n+1} + \frac{1}{4n+3} + \cdots + \frac{1}{6n-1}\right).$$

We want to show that

$$C_{n+1}(2,3) = S_{6(n+1)} + -\left(\frac{1}{4n+5} + \frac{1}{4n+7} + \cdots + \frac{1}{6n+5}\right).$$

From (1), we have shown that

$$C_{n+1}(2,3) = C_n(2,3) + \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{6n+2} - \frac{1}{6n+4} - \frac{1}{6n+6}$$

or

$$C_{n+1}(1,2) - C_n(2,3) = \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{6n+2} - \frac{1}{6n+4} - \frac{1}{6n+6}.$$

To prove $C_{n+1}(2,3) = S_{6(n+1)} + -\left(\frac{1}{4n+5} + \frac{1}{4n+7} + \cdots + \frac{1}{6n+5}\right)$, it is enough to show

$$\begin{aligned}
\left(S_{6(n+1)}\right. &+ \left.-\left(\frac{1}{4n+5}+\frac{1}{4n+7}+\cdots+\frac{1}{6n+5}\right)\right) \\
&- \left(S_{6n}+-\left(\frac{1}{4n+1}+\frac{1}{4n+3}+\cdots+\frac{1}{6n-1}\right)\right) \\
&= \frac{1}{2n+1}-\frac{1}{4n+2}-\frac{1}{4n+4}.
\end{aligned}$$

By definition of S_k, we know that

$$S_{6(n+1)}-S_{6n}=\frac{1}{6n+1}-\frac{1}{6n+2}+\frac{1}{6n+3}-\frac{1}{6n+4}+\frac{1}{6n+5}-\frac{1}{6n+6},$$

and since the expression we are considering equals

$$S_{6(n+1)}-S_{6n}+-\left(\frac{1}{4n+5}+\frac{1}{4n+7}+\cdots+\frac{1}{6n+5}-\frac{1}{4n+1}-\frac{1}{4n+3}-\cdots-\frac{1}{6n-1}\right),$$

we need to prove the identity obtained from substituting the expression $S_{6(n+1)}-S_{6n}$:

$$\begin{aligned}
\frac{1}{6n+1} &- \frac{1}{6n+2}+\frac{1}{6n+3}-\frac{1}{6n+4}+\frac{1}{6n+5}-\frac{1}{6n+6}+ \\
&- \left(\frac{1}{4n+5}+\frac{1}{4n+7}+\cdots+\frac{1}{6n+5}-\frac{1}{4n+1}-\frac{1}{4n+3}-\cdots-\frac{1}{6n-1}\right) \\
&= \frac{1}{4n+1}+\frac{1}{4n+3}-\frac{1}{6n+2}-\frac{1}{6n+4}-\frac{1}{6n+6},
\end{aligned}$$

or,

$$\frac{1}{6n+1}-\frac{1}{6n+2} + \frac{1}{6n+3}-\frac{1}{6n+4}+\frac{1}{6n+5}-\frac{1}{6n+6}-\frac{1}{4n+5}-\frac{1}{4n+7}-$$

$$\cdots \quad -\frac{1}{6n+5}+\frac{1}{4n+1}+\frac{1}{4n+3}+\cdots+\frac{1}{6n-1}$$

$$= \quad \frac{1}{4n+1}+\frac{1}{4n+3}-\frac{1}{6n+2}-\frac{1}{6n+4}-\frac{1}{6n+6}.$$

Notice that the terms $-\frac{1}{6n+2}, -\frac{1}{6n+4}$, and $-\frac{1}{6n+6}$ get canceled out and we are left with

$$\frac{1}{6n+1} \quad + \quad \frac{1}{6n+3}+\frac{1}{6n+5}-\frac{1}{4n+5}-\frac{1}{4n+7}-\cdots$$

$$- \quad \frac{1}{6n+5}+\frac{1}{4n+1}+\frac{1}{4n+3}+\cdots+\frac{1}{6n-1}$$

$$= \quad \frac{1}{4n+1}+\frac{1}{4n+3}.$$

Expanding and simplifying the left side, we need to show

$$\frac{1}{6n+1}+\frac{1}{6n+3} \quad + \quad \frac{1}{6n+5}-\frac{1}{4n+5}-\frac{1}{4n+7}-\cdots-\frac{1}{6n-1}-\frac{1}{6n+1}-\frac{1}{6n+3}$$

$$- \quad \frac{1}{6n+5}+\frac{1}{4n+1}+\frac{1}{4n+3}+\frac{1}{4n+5}+\frac{1}{4n+7}+\cdots+\frac{1}{6n-1}$$

$$= \quad \frac{1}{4n+1}+\frac{1}{4n+3}.$$

This reduces to $\frac{1}{4n+1}+\frac{1}{4n+3}=\frac{1}{4n+1}+\frac{1}{4n+3}$. $\square$

General Case: $A(p,q), p<q$.

Thus, we come up with the following theorem.

Theorem 2.2.6. *Suppose that $p<q$. (1) The last block of $C_n(p,q)$ is* $\left(\frac{1}{2np-2p+1}+\frac{1}{2np-2p+3}+\cdots+\frac{1}{2np-1}\right)-\left(\frac{1}{2nq-2q+2}+\frac{1}{2nq-2q+4}+\cdots+\frac{1}{2nq}\right)$*, and* $C_n(p,q)=S_{2nq}+-\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\cdots+\frac{1}{2nq-1}\right)$.

Proof. We proceed by induction. Note first that

$$\begin{aligned}
C_1 &= \left(1+\frac{1}{3}+\cdots+\frac{1}{2p-1}\right)-\left(\frac{1}{2}+\frac{1}{4}+\cdots+\frac{1}{2q}\right) \\
&= \left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{2q-1}-\frac{1}{2q}\right)+-\left(\frac{1}{2p+1}+\frac{1}{2p+3}+\cdots+\frac{1}{2q-1}\right) \\
&= S_{2q}+-\left(\frac{1}{2p+1}+\frac{1}{2p+3}+\cdots+\frac{1}{2q-1}\right) \\
&= S_{2\cdot 1\cdot q}+-\left(\tfrac{1}{2p+1}+\tfrac{1}{2p+3}+\cdots+\tfrac{1}{2q-1}\right)
\end{aligned}$$

$$\begin{aligned}
C_2 &= C_1+\left[\left(\frac{1}{2p+1}+\frac{1}{2p+3}+\cdots+\frac{1}{4p-1}\right)-\left(\frac{1}{2q+2}+\frac{1}{2q+4}+\cdots+\frac{1}{4q}\right)\right] \\
&= \left(1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots+\frac{1}{4q-1}-\frac{1}{4q}\right)+-\left(\frac{1}{4p+1}+\frac{1}{4p+3}+\cdots+\frac{1}{4q-1}\right) \\
&= S_{4q}+-\left(\frac{1}{4p+1}+\frac{1}{4p+3}+\cdots+\frac{1}{4q-1}\right) \\
&= S_{2\cdot 2\cdot q}+-\left(\frac{1}{4p+1}+\frac{1}{4p+3}+\cdots+\frac{1}{4q-1}\right).
\end{aligned}$$

This shows that (1) and (2) hold when $n=1,2$. Suppose that (1) and (2) hold for n. We need to show that (1) and (2) hold for $n+1$.

Proof of (1): By the inductive hypothesis, we know that the positive terms added in the last block of $C_n(p,q)$ are

$$\frac{1}{2np-2p+1}+\frac{1}{2np-2p+3}+\cdots+\frac{1}{2np-1}.$$

Since we need to add p positive terms, then the p positive terms in the last block of $C_{n+1}(p,q)$ that are added together are

$$\frac{1}{2np+1}+\frac{1}{2np+3}+\cdots+\frac{1}{2np+2p-1}.$$

Similarly, by the inductive hypothesis, we know that the q terms subtracted in the last block of $C_n(p,q)$ are

$$\frac{1}{2nq-2q+2}+\frac{1}{2nq-2q+4}+\cdots+\frac{1}{2nq},$$

and therefore the new terms subtracted in $C_{n+1}(p,q)$ must be

$$\frac{1}{2nq+2}+\frac{1}{2nq+4}+\cdots+\frac{1}{2nq+2q}.$$

(1) follows from noting that $2(n+1)p-2p+1=2np+1$, $2(n+1)p-2p+3=2np+3,\ldots,2(n+1)p-1=2np+2p-1$, $2(n+1)q-2q+2=2nq+2$, $2(n+1)q-2q+4=2nq+4,\ldots,2(n+1)q=2nq+2q$. Also,

$$C_{n+1}(p,q)=C_n(p,q)+\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\cdots+\frac{1}{2np+2p-1}\right)$$
$$-\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\cdots+\frac{1}{2nq+2q}\right).$$

Proof of (2): By the inductive hypothesis,

$$C_n(p,q) \quad = \quad S_{2nq} + -\left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq-1}\right).$$

We want to show that

$$C_{n+1}(p,q) = \quad S_{2(n+1)q} + -\left(\frac{1}{2(n+1)p+1} + \frac{1}{2(n+1)p+3} + \cdots + \frac{1}{2(n+1)q-1}\right).$$

From (1), we have already shown

$$C_{n+1}(p,q) = C_n(p,q) + \left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2np+2p-1}\right)$$

$$-\left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2nq+2q}\right).$$

Since

$$C_{n+1}(p,q) = C_n(p,q) + \left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2np+2p-1}\right)$$

$$-\left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2nq+2q}\right),$$

It suffices to show that

$$C_{n+1}(p,q) = \left(S_{2nq} + -\left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq-1}\right)\right.$$

$$\left. + \left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2np-1}\right) - \left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2nq+2q}\right)\right).$$

By definition of S_k, we have that

$$S_{2(n+1)q} - S_{2nq} = \left(\frac{1}{2nq+1} + \frac{1}{2nq+3} + \cdots + \frac{1}{2nq+2q-1}\right)$$

$$-\left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2nq+2q}\right).$$

Notice that S_{2nq} can be written as

$$S_{2nq} = S_{2(n+1)q} - \left(\frac{1}{2nq+1} - \frac{1}{2nq+2} + \frac{1}{2nq+3} - \cdots - \frac{1}{2nq+2q}\right).$$

So, substituting S_{2nq} into our expression for $C_{n+1}(p,q)$, we have that

$$C_{n+1}(p,q) = S_{2(n+1)q} - \left(\frac{1}{2nq+1} - \frac{1}{2nq+2} + \frac{1}{2nq+3} - \cdots - \frac{1}{2nq+2q}\right)$$

$$-\left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq-1}\right) + \left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2np-1}\right)$$

$$-\left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2nq+2q}\right).$$

Rewriting this so that the positive and negative terms in $\left(\frac{1}{2nq+1} - \frac{1}{2nq+2} + \frac{1}{2nq+3} - \cdots - \frac{1}{2nq+2q}\right)$ are grouped together,

$$C_{n+1}(p,q) = S_{2(n+1)q} - \left(\frac{1}{2nq+1} \quad \frac{1}{2n+3} + \cdots + \frac{1}{2nq+2q-1}\right)$$

$$+\left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2nq+2q}\right) - \left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq-1}\right)$$

$$+\left(\tfrac{1}{2np+1} + \tfrac{1}{2np+3} + \cdots + \tfrac{1}{2np-1}\right) - \left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2nq+2q}\right).$$

Simplifying,

$$
\begin{aligned}
C_{n+1}(p,q) &= S_{2(n+1)q} + -\left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq+2q-1}\right) \\
&+ \left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2np-1}\right).
\end{aligned}
$$

By expansion and again simplification,

$$
\begin{aligned}
C_{n+1}(p,q) &= S_{2(n+1)q} + -\left(\frac{1}{2np+2p+1} + \frac{1}{2np+2p+3} + \cdots + \frac{1}{2nq+2q-1}\right) \\
&= S_{2(n+1)q} + -\left(\frac{1}{2(n+1)p+1} + \frac{1}{2(n+1)p+3} + \cdots + \frac{1}{2(n+1)q-1}\right).
\end{aligned}
$$

□

2.2.2 Part II

Recall $C_n = C_n(p,q)$ to be the partial sum of $A(p,q)$, which is the result in S from precisely adding n blocks, where each block consists of p positive terms and q negative terms, consecutively. Also, recall S_n to be the partial sums of the usual alternating series, where n denotes the terms. Call C_n as $C_n = S_{f(n)} + R_n$ for some $f(n)$ and some "remainder" R_n.

We aim to show that S_n and $C_n = S_{f(n)} + R_n$ for some $f(n)$ converging to $\ln(2) + \frac{1}{2}\ln(\frac{p}{q})$. For our argument, we begin by splitting up the proof for each of the three cases. In our argument, we label each case as $1, 2$, and 3, respectively.

Proof. 1. Consider $C_n(p,p) = S_{2np}$, where $f(n) = 2np$ and $R_n = 0$.

We show that $S_{f(n)}$ converges to $\ln(2)$ as $n \to \infty$, and that for each S'_k, if n is defined by

$$n(p+q) \leq k < (n+1)(p+q),$$

then letting $r_k = S'_k - C_n$, we have $r_k \to 0$ as $k \to \infty$. Note that $n \to \infty$ if $k \to \infty$, and so $S'_k \to \ln(2)$.

First, r_k consists of a sum of $x_1 + x_2 + \cdots$, which is at most $p + p - 1 = 2p - 1$ terms where each term is

$$| x_i | \leq \frac{1}{2np+1}.$$

So, letting n be such that $2np \leq k < 2(n+1)p$. Then,

$$| x_1 + x_2 + \cdots | \leq \frac{2p-1}{2np+1}.$$

But taking the limit of $\dfrac{2p-1}{2np+1}$, we have

$$\lim_{n\to\infty} \frac{2p-1}{2np+1} = 0,$$

which means that these extra terms are negligible. Now, since S_k is defined as the partial sums of the usual Alternating Harmonic Series, we know that

$$\lim_{n\to\infty} S_n = \ln(2).$$

By the fact that S_{2np} are the partial sums of the usual alternating series up to $2np$ terms, then

$$\lim_{n\to\infty} S_{2np} = \ln(2).$$

2. Consider

$$C_n(p,q) = S_{2np} + \left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np}\right),$$

where $f(n) = 2np$ and

$$R_n = \left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np}\right).$$

We have already shown from 1. that S_{2np} converges to $\ln(2)$ as $n \to \infty$. Now we want to show both that

$$\lim_{n\to\infty} R_n = \lim_{n\to\infty}\left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np}\right) = \frac{1}{2}\ln\left(\frac{p}{q}\right)$$

and that for each S'_k, if n is defined by $n(p+q) \leq k < (n+1)(p+q)$, then, letting $r_k = S'_k - C_n$, we have that $r_k \to 0$ as $k \to \infty$. But $n \to \infty$ if $k \to \infty$, and so $S'_k \to \ln(2)$.

First, notice that r_k consists of a sum of $x_1 + x_2 + \cdots$, which is at most $p+q-1$ terms where each term is

$$\mid x_i \mid \leq \frac{1}{2nq+2}$$

terms. So, for all k, there exists an n such that $n(p+q) \leq k < (n+1)(p+q)$, which means

$$\mid x_1 + x_2 + \cdots \mid \leq \frac{p+q-1}{2nq+2}.$$

But taking the limit of $\frac{p+q-1}{2nq+2}$, we have

$$\lim_{n\to\infty} \frac{p+q-1}{2nq+2} = 0.$$

Therefore, the size of r_k is 0. In other words, r_k is so small we can consider these terms to be negligible. Now,

$$\frac{1}{2}\ln\left(\frac{p}{q}\right) = \frac{1}{2}\ln\left(\frac{np}{nq}\right) = \frac{1}{2}\Big(\ln(np) - \ln(nq)\Big) = \frac{1}{2}\int_{nq}^{np} \frac{1}{x}\,dx$$

where

$$\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np} < \frac{1}{2}\int_{nq}^{np} \frac{1}{x}\,dx.$$

Also,

$$\begin{aligned} \frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np} > \frac{1}{2}\int_{nq+1}^{np+1} \frac{1}{x}\,dx &= \frac{1}{2}\Big(\ln(np+1) - \ln(nq+1)\Big) \\ &= \frac{1}{2}\ln\left(\frac{np+1}{nq+1}\right). \end{aligned}$$

Thus,

$$\frac{1}{2}\int_{nq+1}^{np+1} \frac{1}{x}\,dx < \frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np} < \frac{1}{2}\int_{nq}^{np} \frac{1}{x}\,dx,$$

or

$$\frac{1}{2}\ln\left(\frac{np+1}{nq+1}\right) < \frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np} < \frac{1}{2}\ln\left(\frac{np}{nq}\right).$$

Now taking the limit of the left and right side, applying L'Hopitals rule, we get

$$\frac{1}{2}\lim_{n\to\infty}\ln\left(\frac{np+1}{nq+1}\right) = \frac{1}{2}\ln\left(\frac{p}{q}\right) \qquad \text{and} \qquad \frac{1}{2}\lim_{n\to\infty}\ln\left(\frac{np}{nq}\right) = \frac{1}{2}\ln\left(\frac{p}{q}\right).$$

Therefore,

$$\begin{aligned}
\frac{1}{2}\lim_{n\to\infty}\ln\left(\frac{np+1}{nq+1}\right) &= \frac{1}{2}\ln\left(\frac{p}{q}\right) \\
&\leq \lim_{n\to\infty}\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\cdots+\frac{1}{2np}\right) \\
&\leq \frac{1}{2}\lim_{n\to\infty}\ln\left(\frac{np}{nq}\right) \\
&= \frac{1}{2}\ln\left(\frac{p}{q}\right).
\end{aligned}$$

Since both limits converge to $\frac{1}{2}\ln\left(\frac{p}{q}\right)$ as $n\to\infty$, then it is true that

$$\begin{aligned}
\lim_{n\to\infty} R_n &= \lim_{n\to\infty}\left[\frac{1}{2nq+2}+\frac{1}{2nq+4}+\cdots+\frac{1}{2np}\right] \\
&= \frac{1}{2}\ln\left(\frac{p}{q}\right).
\end{aligned}$$

We conclude that

$$\begin{aligned}
\lim_{n\to\infty} C_n(p,q) &= \lim_{n\to\infty}\left[S_{2np}+\left(\frac{1}{2nq+2}+\frac{1}{2nq+4}+\cdots+\frac{1}{2np}\right)\right] \\
&= \ln(2)+\frac{1}{2}\ln\left(\frac{p}{q}\right)
\end{aligned}$$

(by properties of convergent sequences).

3. Consider

$$C_n(p,q) = S_{2nq} + -\left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq-1}\right),$$

where $f(n) = 2nq$ and

$$R_n = -\left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq-1}\right).$$

We have already shown from 1. that since S_n is defined as the partial sums of the usual Alternating Harmonic Series, then S_{2nq} are the partial sums of the usual series up to $2nq$ terms, converging to $\ln(2)$ as $n \to \infty$. Now we want to show both that

$$\begin{aligned} \lim_{n\to\infty} R_n &= \lim_{n\to\infty} -\left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq-1}\right) \\ &= \frac{1}{2}\ln\left(\frac{p}{q}\right), \end{aligned}$$

and that for each S'_k, if n is defined by $n(p+q) \leq k < (n+1)(p+q)$, then, letting $r_k = S'_k - C_n$, we have that $r_k \to 0$ as $k \to \infty$. (Note that $n \to \infty$ if $k \to \infty$, and so $S'_k \to \ln(2)$ as well.) First, notice that r_k consists of a sum of $x_1 + x_2 + \cdots$, which is at most $p+q-1$ terms where each term is $\mid x_i \mid \leq \dfrac{1}{2np+1}$ terms. So, for all k, there exists an n such that $n(p+q) \leq k < (n+1)(p+q)$, which means that $\mid x_1 + x_2 + \cdots \mid \leq \dfrac{p+q-1}{2np+1}$. But taking the limit of $\dfrac{p+q-1}{2np+1}$, we have

$$\lim_{n\to\infty} \frac{p+q-1}{2np+1} = 0.$$

Therefore, the size of r_k is 0, which means that these extra terms are negligible, and so do not converge to any value. Note that

$$\frac{1}{2}\ln\left(\frac{p}{q}\right) = \frac{1}{2}\ln\left(\frac{np}{nq}\right) = \frac{1}{2}\Big(\ln(np) - \ln(nq)\Big) = \frac{1}{2}\int_{nq}^{np}\frac{1}{x}\,dx,$$

where

$$-\left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq-1}\right) > \frac{1}{2}\int_{nq}^{np}\frac{1}{x}\,dx.$$

Also

$$\begin{aligned}
-\left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq-1}\right) &< \frac{1}{2}\int_{nq+1}^{np+1}\frac{1}{x}\,dx \\
&= \frac{1}{2}\Big(\ln(np+1) - \ln(nq+1)\Big) \\
&= \frac{1}{2}\ln\left(\frac{np+1}{nq+1}\right).
\end{aligned}$$

So,

$$\frac{1}{2}\int_{nq+1}^{np+1}\frac{1}{x}\,dx > -\left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq-1}\right) > \frac{1}{2}\int_{nq}^{np}\frac{1}{x}\,dx,$$

or

$$\frac{1}{2}\ln\left(\frac{np+1}{nq+1}\right) > -\left(\frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq-1}\right) > \frac{1}{2}\ln\left(\frac{np}{nq}\right).$$

Now taking the limit of the left and right side, applying L'Hopitals rule, we get

$$\frac{1}{2}\lim_{n\to\infty}\ln\left(\frac{np+1}{nq+1}\right)=\frac{1}{2}\ln\left(\frac{p}{q}\right)$$

and

$$\frac{1}{2}\lim_{n\to\infty}\ln\left(\frac{np}{nq}\right)=\frac{1}{2}\ln\frac{p}{q}.$$

Then, it is true that

$$\begin{aligned}\frac{1}{2}\lim_{n\to\infty}\ln\left(\frac{np+1}{nq+1}\right)=\frac{1}{2}\ln\left(\frac{p}{q}\right) &\geq \lim_{n\to\infty}\left[-\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\cdots+\frac{1}{2nq-1}\right)\right]\\ &\geq \frac{1}{2}\lim_{n\to\infty}\ln\left(\frac{np}{nq}\right)=\frac{1}{2}\ln\left(\frac{p}{q}\right).\end{aligned}$$

So, since both limits converge to $\frac{1}{2}\ln\left(\frac{p}{q}\right)$ as n tends to ∞, then it is true that

$$R_n=-\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\cdots+\frac{1}{2nq-1}\right)$$

converges to $\frac{1}{2}\ln\left(\frac{p}{q}\right)$ as $n\to\infty$. So we can conclude that

$$C_n(p,q)=S_{2nq}+-\left(\frac{1}{2np+1}+\frac{1}{2np+3}+\cdots+\frac{1}{2nq-1}\right)$$

converges to $\ln(2)+\frac{1}{2}\ln\left(\frac{p}{q}\right)$ as $n\to\infty$ (by properties of convergent sequences). □

CHAPTER 3

RIEMANN'S THEOREM

3.1 Bernhard Riemann

Bernhard Riemann (17 September 1826 - 20 July 1866) was a German mathematician. In 1846, he attended the University of Göttingen to study theology as his father had encouraged him to do. Eventually, his father gave him permission to study mathematics under Moritz Stern and Carl Friedrich Gauss. In 1852, Riemann began working on the results of Dirichlet involving the Fourier series. Dirichlet found that certain types of series could be rearranged to a sum different from the sum of the original series. Later, Riemann discovered that this works for any conditionally convergent series. This result became known as Riemann's Rearrangement Theorem in his Fourier series paper, "On the Representation of a Function by a Trigonometric Series," which he completed in 1853. However, his paper was not published until after his death [13].

3.2 Riemann's Rearrangement Theorem

Theorem 3.2.1 ([10])**.** *A series $\sum f_n$ is conditionally convergent if and only if for each real number α, there is a rearrangement of $\sum f_n$ that converges to α.*

Proof. ($\Leftarrow$) Note that this direction follows from Dirichlet's Theorem 1.3.3, which states that an absolutely convergent series converges to the same value no matter how it is rearranged.

($\Rightarrow$) Suppose $\sum f_n$ is conditionally convergent. We want to show that there is a rearrangement, $(f_{\pi(n)})$, of (f_n) whose series converges to the real number α.

First, consider the subsequence of positive terms of (f_n), call it (a_n), and the subsequence of negative terms of (f_n), call it (b_n). Then, by Lemma 1.3.2, $\sum a_n = +\infty$ and $\sum b_n = -\infty$.

We know that $\sum a_n = \infty$. This implies that there exists some natural number N such that

$$\sum_{k=1}^{N} a_k > \alpha.$$

Now let $N_1 = N$ be the least such number, and consider the partial sum $S_1 = \sum_{k=1}^{N_1} a_k$, so

$$S_1 = \sum_{k=1}^{N_1} a_k > \alpha,$$

but

$$\sum_{k=1}^{N_1-1} a_k \leq \alpha,$$

so that

$$0 < S_1 - \alpha \leq a_{N_1}.$$

Now to S_1, add just enough terms from (b_n) (in order) so that the resulting partial sum

$$\sum_{k=1}^{N_1} a_k + \sum_{i=1}^{M} b_i$$

is now less than or equal to α. Note that this is possible since $\sum b_i = -\infty$.

Letting M_1 be the least such number M, and setting

$$S_2 = \sum_{k=1}^{N_1} a_k + \sum_{i=1}^{M_1} b_i,$$

we get that

$$0 \le \alpha - S_2 < -b_{M_1}.$$

Continuing this process, we get partial sums that alternate between being larger and smaller than α, and each time choosing the next smallest N_k or M_k, we get the following rearrangement for (f_n),

$$a_1, a_2, \ldots, a_{N_1}, b_1, b_2, \ldots, b_{M_1}, a_{N_1+1}, \ldots, a_{N_2}, b_{M_1+1}, \ldots, b_{M_2}, a_{N_2+1}, \ldots$$

Note that for all odd i, we have that $\mid S_i - \alpha \mid \le a_{N_i}$, and for all even j, we have that $\mid S_j - \alpha \mid \le -b_{M_j}$. Now, for any n

$$S_{2n+1} > S_{2n+1} + b_{M_n+1} > S_{2n+1} + b_{M_n+1} + b_{M_n+2} > \cdots > S_{2n+2} - b_{M_n+1} > \alpha$$

and

$$S_{2n+2} < S_{2n+2} + a_{N_{n+1}+1} < S_{2n+2} + a_{N_{n+1}+2} < \cdots < S_{2n+3} - a_{N_{n+2}} \le \alpha.$$

So all the partial sums of the rearrangement $(f_{\pi(n)})$ between S_{2n+1} and $S_{2n+2} - b_{M_{n+1}}$ are bounded between α and $\alpha + a_{N_{n+1}}$, and all partial sums between S_{2n+2} and $S_{2n+3} - a_{N_{n+2}}$ are bounded between $\alpha + b_{M_{n+1}}$ and α.

Since $\sum f_n$ converges, notice that (f_n) converges to 0. Therefore, (a_{N_i}) and (b_{M_j}) also converge to 0. Hence, the partial sums of $(f_{\pi(n)})$ converge to α; that is, $\sum f_{\pi(n)} = \alpha$, as wanted. $\square$

3.3 Examples

Example 3.3.0.1. *Consider the usual Alternating Harmonic Series*

$$\sum \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.$$

We already know that this sum converges to $\ln(2)$. *However, let us show that this is true by applying Riemann's Rearrangement Theorem. So first consider the series of positive terms of* $\sum \frac{(-1)^{n-1}}{n}$, *call it* $\sum a_n$, *and the series of negative terms of* $\sum \frac{(-1)^{n-1}}{n}$, *call it* $\sum b_n$.

Now using Riemann's Rearrangement Theorem, we get the partial sum S_1 *by adding just enough terms from* $\sum a_n$ *so that* $S_1 > \ln(2) \approx .6931$*:*

$$S_1 = 1 > \ln(2).$$

To get S_2, *we add just enough terms from* $\sum b_n$ *to* S_1 *so that* $S_2 < \ln(2)$*:*

$$S_2 = 1 - \frac{1}{2} = \frac{1}{2} < \ln(2).$$

To get S_3, we again add just enough terms from $\sum a_n$ to S_2 so that $S_3 > \ln(2)$:

$$S_3 = 1 - \frac{1}{2} + \frac{1}{3} \approx .8333 > \ln(2),$$

and likewise, we add just enough terms from $\sum b_n$ to S_3 to get S_4 so that $S_4 < \ln(2)$:

$$S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \approx .5833 < \ln(2).$$

Continuing this process, we get the following partial sums:

$$S_{10} \approx .6456 < \ln(2)$$

$$S_{21} \approx .7164 > \ln(2)$$

$$S_{30} \approx .6768 < \ln(2)$$

$$S_{41} \approx .7052 > \ln(2)$$

$$S_{50} \approx .6832 < \ln(2)$$

$$S_{71} \approx .7018 > \ln(2)$$

$$S_{80} \approx .6886 < \ln(2)$$

$$S_{91} \approx .7002 > \ln(2)$$

$$S_{100} \approx .6898 < \ln(2)$$

$$S_{111} \approx .6993 > \ln(2)$$

$$S_{120} \approx .6906 < \ln(2)$$

$$S_{131} \approx .6986 > \ln(2).$$

Each time we add just enough negative terms to get the new corresponding partial sum so that it is less than $\ln(2)$*, the sum increases and approaches the sum of the Alternating Harmonic Series. Likewise, each time we add just enough positive terms to get the new partial sum, the partial sums decrease and approach* $\ln(2)$*. In other words, as k increases, for all* $k \in \mathbb{N}$, $S_{2k} \to \ln(2)^-$ *(from the left), and as k increases,* $S_{2k+1} \to \ln(2)^+$ *(from the right).*

Example 3.3.0.2. *Consider again the Alternating Harmonic Series. Similar to Example 3.3.0.1., we want to show that there exists a rearrangement that converges to* $\alpha = \frac{3}{2}\ln(2) \approx 1.0397$*. Applying the formula of Riemann's Theorem we get the following partial sums:*

$$S_1 = 1 + \frac{1}{3} \approx 1.3333 > \frac{3}{2}\ln(2)$$

$$S_2 = 1 + \frac{1}{3} - \frac{1}{2} \approx 0.8333 < \frac{3}{2}\ln(2)$$

$$S_3 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} \approx 1.1762 > \frac{3}{2}\ln(2)$$

$$S_4 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} \approx 0.926190476 < \frac{3}{2}\ln(2)$$

$$S_5 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} \approx 1.2191 > \frac{3}{2}\ln(2).$$

Recall that for every odd partial sum, we are adding just enough positive terms not already used to make the partial sum larger than $\frac{3}{2}\ln(2)$*, and for every even partial sum, we are adding just enough negative terms not already used so that the partial sum is less than* $\frac{3}{2}\ln(2)$*. Continuing, we get the following partial sums:*

$$S_6 = 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6} \approx 0.9615 < \frac{3}{2}\ln(2)$$

$$S_7 = 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\frac{1}{13}+\frac{1}{15} \approx 1.1051 > \frac{3}{2}\ln(2)$$

$$S_8 = 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\frac{1}{13}+\frac{1}{15}-\frac{1}{8} \approx 0.9801 < \frac{3}{2}\ln(2)$$

$$S_9 = 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\frac{1}{13}+\frac{1}{15}-\frac{1}{8}+\frac{1}{17}+\frac{1}{19} \approx 1.0916 > \frac{3}{2}\ln(2)$$

$$S_{10} = 1+\frac{1}{3}-\frac{1}{2}+\frac{1}{5}+\frac{1}{7}-\frac{1}{4}+\frac{1}{9}+\frac{1}{11}-\frac{1}{6}+\cdots+\frac{1}{17}+\frac{1}{19}-\frac{1}{10} \approx 0.9916 < \frac{3}{2}\ln(2)$$

$$S_{35} = 1+\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{7}+\cdots-\tfrac{1}{32}+\tfrac{1}{65}+\tfrac{1}{67}-\tfrac{1}{34}+\tfrac{1}{69}+\tfrac{1}{71} \approx 1.0538 > \tfrac{3}{2}\ln(2).$$

An obvious pattern is found: for every odd partial sum, we need to add the first two positive terms not already used from $\sum a_n$ *so that the partial sum is greater than* $\frac{3}{2}\ln(2)$*, and for each even partial sum, the first negative term not already used from* $\sum b_n$ *so that the partial sum is less than* $\frac{3}{2}\ln(2)$*.*

For clarity, we will show sums S_{35} *to* S_{40}*:*

$$S_{36} = 1+\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{7}+\cdots-\tfrac{1}{32}+\tfrac{1}{65}+\tfrac{1}{67}-\tfrac{1}{34}+\tfrac{1}{69}+\tfrac{1}{71}-\tfrac{1}{36} \approx 1.02598 < \tfrac{3}{2}\ln(2)$$

$$S_{37} = 1+\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{7}+\cdots-\tfrac{1}{34}+\tfrac{1}{69}+\tfrac{1}{71}-\tfrac{1}{36}+\tfrac{1}{73}+\tfrac{1}{75} \approx 1.05301 > \tfrac{3}{2}\ln(2)$$

$$S_{38} = 1+\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{7}+\cdots-\tfrac{1}{34}+\tfrac{1}{69}+\tfrac{1}{71}-\tfrac{1}{36}+\tfrac{1}{73}+\tfrac{1}{75}-\tfrac{1}{38} \approx 1.02669 < \tfrac{3}{2}\ln(2)$$

$$S_{39} = 1+\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{7}+\cdots-\tfrac{1}{36}+\tfrac{1}{73}+\tfrac{1}{75}-\tfrac{1}{38}+\tfrac{1}{77}+\tfrac{1}{79}\approx 1.05234 > \tfrac{3}{2}\ln(2)$$

$$S_{40} = 1+\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{7}+\cdots-\tfrac{1}{36}+\tfrac{1}{73}+\tfrac{1}{75}-\tfrac{1}{38}+\tfrac{1}{77}+\tfrac{1}{79}-\tfrac{1}{40}\approx 1.02734 < \tfrac{3}{2}\ln(2).$$

Notice that if we continue this process, the partial sums eventually converge to $\dfrac{3}{2}\ln(2)$. *Thus,* $S_2 \le S_4 \le S_6 \le S_8 \le \cdots \le \dfrac{3}{2}\ln(2) \le \cdots \le S_7 \le S_5 \le S_3 \le S_1$.

So the rearrangement $1+\tfrac{1}{3}-\tfrac{1}{2}+\tfrac{1}{5}+\tfrac{1}{7}+\cdots-\tfrac{1}{36}+\tfrac{1}{73}+\tfrac{1}{75}-\tfrac{1}{38}+\tfrac{1}{77}+\tfrac{1}{79}-\tfrac{1}{40}+\cdots = \tfrac{3}{2}\ln(2)$.

Recall Ohm' Theorem in Chapter 2 and consider $A(2,1)$. *Using Ohm's Theorem to look at the partial sums* $A(2,1)$, *we get the following first five partial sums:*

$$C_1 = 1+\tfrac{1}{3}+\left(-\tfrac{1}{2}\right)\approx 0.83 = S_2 < \tfrac{3}{2}\ln(2)$$

$$C_2 = 1+\tfrac{1}{3}+\left(-\tfrac{1}{2}\right)+\left[\tfrac{1}{5}+\tfrac{1}{7}+\left(-\tfrac{1}{4}\right)\right]\approx 0.93 = S_4 < \tfrac{3}{2}\ln(2)$$

$$C_3 = 1+\tfrac{1}{3}+\left(-\tfrac{1}{2}\right)+\tfrac{1}{5}+\tfrac{1}{7}+\left(-\tfrac{1}{4}\right)+\left[\tfrac{1}{9}+\tfrac{1}{11}+\left(-\tfrac{1}{6}\right)\right]\approx 0.96 = S_6 < \tfrac{3}{2}\ln(2)$$

$$C_4 = 1+\tfrac{1}{3}+\left(-\tfrac{1}{2}\right)+\tfrac{1}{5}+\tfrac{1}{7}+\left(-\tfrac{1}{4}\right)+\cdots+\left[\tfrac{1}{13}+\tfrac{1}{15}+\left(-\tfrac{1}{8}\right)\right]\approx 0.98 = S_8 < \tfrac{3}{2}\ln(2)$$

$$C_5 = 1+\tfrac{1}{3}+\left(-\tfrac{1}{2}\right)+\tfrac{1}{5}+\tfrac{1}{7}+\left(-\tfrac{1}{4}\right)+\cdots+\left[\tfrac{1}{17}+\tfrac{1}{19}+\left(-\tfrac{1}{10}\right)\right]\approx 0.99 = S_{10} < \tfrac{3}{2}\ln(2).$$

Observe that each partial sum C_n *from Ohm's Theorem is equal to each even partial sum* S_{2n} *from Riemann's Rearrangement Theorem, and we know that by Ohm's Theorem,* C_n *converges to* $\dfrac{3}{2}\ln(2)$.

CHAPTER 4

SCHLÖMILCH'S THEOREM

In Section 4.3, we provide a modern English-language translation of Schlömilch's paper [22]. Since the basis of his argument comes from the Mean Value Theorem for Integrals, which he refers to as the "well-known" theorem, we will provide a few preliminaries in Section 4.2 by introducing and arguing the following three results: (1) Rolle's Theorem, which will be used to prove (2) the Mean Value Theorem, then extend that to (3) the Mean Value Theorem for Integrals.

4.1 Oscar Schlömilch

Oscar Xavier Schlömilch (13 April 1823 - 7 February 1901) was a German mathematician. He studied mathematics and physics primarily in Jena, Berlin and Vienna. Most of his work was strongly influenced by Johann Peter Gustav Lejeune Dirichlet (1805-1859). In 1844, Schlömilch received his doctorate from Friedrich-Schiller-Universität. From 1851 to 1874, he was a professor at Dresden teaching Higher Mathematics and Analytic Mechanics [13].

4.2 Mean Value Theorem

Theorem 4.2.1 (Rolle's Theorem [18]). *Let $f : [a,b] \to \mathbb{R}$ be a continuous function such that f is differentiable on (a,b) where $a < b$. If $f(a) = f(b)$, then there exists some point $x \in (a,b)$ where $f'(x) = 0$.*

Proof. Let $f(a) = f(b)$. Since f is continuous on $[a,b]$, then f attains a maximum at some point $t \in [a,b]$ and a minimum at some point $s \in [a,b]$.

Suppose first that s,t are both endpoints of $[a,b]$. Since $f(a) = f(b)$, then the maximum and the minimum are equivalent, which means f is a constant function on $[a,b]$. In other words, $f(x) = 0$, then for each $x \in (a,b)$, $f'(x) = 0$, in which case, we are done.

But now consider when (I) s is not an endpoint of $[a,b]$, or when (II) t is not an endpoint of $[a,b]$.

(I) If s is not an endpoint of $[a,b]$, then $s \in (a,b)$, where f has a local maximum at s, and therefore $f'(s) = 0$.

(II) If t is not an endpoint in $[a,b]$, then $t \in (a,b)$ and f has a local minimum at t, and therefore $f'(t) = 0$.

We have proved that in all cases for some point $x \in (a,b)$, $f'(x) = 0$. $\square$

Theorem 4.2.2 (Mean Value Theorem [18]). *Suppose $f : [a,b] \to \mathbb{R}$ be a continuous function such that f is differentiable on (a,b). Then, there is some point $t \in (a,b)$ such that*

$$f(b) - f(a) = f'(t)(b-a).$$

Proof. Let

$$y(x) = \frac{f(b) - f(a)}{b - a}(x - a) + f(a).$$

Notice that this is the slope of the secant of the graph of f on $[a,b]$. Now let

$$h(x) = f(x) - \frac{f(b) - f(a)}{b - a}(x - a) - f(a),$$

and note that

$$h(a) = h(b) = 0$$

and h is continuous on $[a,b]$ and differentiable on (a,b). Applying Rolle's Theorem, there is some $t \in (a,b)$ such that $h'(t) = 0$. But since

$$h'(t) = f'(t) - \frac{f(b) - f(a)}{b - a} = 0,$$

then

$$f'(t) = \frac{f(b) - f(a)}{b - a}.$$

$\square$

Theorem 4.2.3 (Mean Value Theorem for Integrals [8])**.** *If f is a continuous function on the closed interval $[a,b]$, then there exists some number c on (a,b) so that*

$$\int_a^b f(x)dx = f(c)(b - a).$$

Proof. Define $f(x) = f(a)$ for values of $x < a$ and $f(x) = f(b)$ for values of $x > b$. Then, f has the integral of F by the Fundamental Theorem of Calculus (see (ii) from [8, pg. 193]). So there is some point c such that $a < c < b$ where

$$F'(c) = \frac{F(b) - F(a)}{b - a}.$$

By the definition of F, we know that $F'(c) = f(c)$ and

$$F(a) = \int_a^a f(x)dx = 0,$$

or

$$F(b) - F(a) = \int_a^b f(x)dx.$$

Thus,

$$f(c) = \frac{F(b)}{b - a} = \frac{1}{b - a}\int_a^b f(x)dx$$

or

$$f(c)(b - a) = \int_a^b f(x)dx.$$

$\square$

4.3 On Conditionally Converging Series

[Note: This section is a modern English-language translation of Schlömilch's original paper [22].]

Dr. Scheibner proved a proposition that a necessary and sufficient condition for absolute convergence of a series (i.e., the series converges to a number no matter how the terms are rearranged) is that the series of absolute values of its terms will also converge to that number.

[In the original paper, Scheibner uses the term "Moduli" to refer to the absolute values of complex numbers. He is stating that this property also holds for series of complex numbers. In particular, when a series of complex numbers is rearranged, it will converge to some limit, and likewise, its absolute value series will also converge to that limit.

Schlömilch points out from the proof of Scheibner's proposition it is clear that the condition is not a simplification as long as the notion of absolute convergence is defined by arbitrarily reordering of its terms. But consider a special condition where you rearrange the terms only so that p positive terms are always followed by q negative terms. We will see that this requirement can, without changing the values, be met even if the absolute value series is not convergent.]

Let $f(x)$ be a decreasing and positive function, such that the asymptotic value $\lim_{x\to\infty} f(x) = 0$ [Schlömilch uses $f(\infty) = 0$ to denote $\lim_{x\to\infty} f(x) = 0$]. Then, we have that the following corresponding series converges:

$$f(0) - f(1) + f(2) - f(3) + \cdots ;$$

[The series converges because of the Alternating Series Test (Theorem 1.2.5).]

Let $f(n) = u_n$ [he uses $f(x) = u_x$], and let s be the sum of the original series; for $n \to \infty$

[he uses $f(\infty) = 0$],

$$s_{2m} = u_0 - u_1 + u_2 - u_3 + \cdots + u_{2m-2} - u_{2m-1}.$$

Let S denote the sum of the rearranged series defined by $n = \infty$ [as n goes to ∞] for:

$$\begin{aligned}
&S_{(p+q)n} = u_0 + u_2 + u_4 + \cdots + u_{2p-2} \\
&-u_1 - u_3 - u_5 - \cdots - u_{2q-1} \\
&+u_{2p} + u_{2p+2} + u_{2p+4} + \cdots + u_{4p-2} \\
&-u_{2q+1} - u_{2p+3} - u_{2q+5} - \cdots - u_{4q-1} \\
&+u_{4p} + u_{4p+2} - u_{4q+4} + \cdots + u_{6p-2} \\
&-u_{4q+1} - u_{4q+3} - u_{4q+5} - \cdots - u_{6q-1} \\
&\qquad\qquad \cdots\cdots\cdots\cdots \\
&+u_{(2n-2)p} + u_{(2n-2)p+2} + \cdots + u_{2np-2} \\
&-u_{(2n-2)q+1} - u_{(2n-2)q+3} - \cdots - u_{2nq-1}.
\end{aligned}$$

[$S_{(p+q)n}$ is the sum of the rearranged series up to $(p+q)n$ terms.]

For $m = nq$, $s_{2m} = s_{2nq}$ contains all negative terms occurring in $S_{(p+q)n}$. So we have

$$S_{(p+q)n} - s_{2qn} = u_{2nq} + u_{2nq+2} + u_{2nq+4} + \cdots + u_{2np-2}, \tag{4.1}$$

[Note that Schlömilch is taking the difference between $S_{(p+q)n}$ and s_{2qn} to compare the two so as to say something about the convergence of the rearranged series. Also note that Schlömilch knows s_{2qn} converges since it is an alternating series and we assume that it passes the Alternating Series Test.] where in the series $n(p-q)$ is the number of terms of equation (4.1), assuming $p > q$. Since $f(x)$ is decreasing, we have the following inequality

$$\frac{1}{h}\int_a^{a+rh} f(x)dx \quad < \quad f(a)+f(a+h)+f(a+2h)+\cdots+f(a+[r-1]h)$$

$$< \quad \frac{1}{h}\int_a^{a+rh} f(x)dx+f(a)-f(a+rh).$$

[Above, he uses the Integral Test.]

If we set $f(n)=u_n, a=2nq, h=2$, and $r=n(p-q)$, then the $f(a)+f(a+h)+f(a+2h)+\cdots+f(a+[r-1]h)$, from above, is equal to the right side of equation (4.1). Thus, we obtain the following inequality [note that $a+rh=2nq+n(p-q)2=2np$],

$$\frac{1}{2}\int_{2nq}^{2np} f(x)dx < S_{(p+q)n}-s_{2qn} < \frac{1}{2}\int_{2nq}^{2np} f(x)dx+u_{2nq}-u_{2np},$$

using the substitution $x=n\xi$ and $dx=nd\xi$,

$$\frac{1}{2}\int_{2q}^{2p}\frac{n\xi f(n\xi)}{\xi}d\xi < S_{(p+q)n}-s_{2qn} < \frac{1}{2}\int_{2q}^{2p}\frac{n\xi f(n\xi)}{\xi}d\xi+u_{2qn}-u_{2pn}.$$

Recall the Mean Value Theorem for Integrals [he refers to this theorem as the "known" theorem], when $\varphi(\xi)$ is finite and continuous, $\psi(\xi)$ finally becomes constant and positive for $\xi=\alpha$ and $\xi=\beta$ [$\alpha<\mu<\beta$],

$$\int_\alpha^\beta \varphi(\xi)\psi(\xi)d\xi = \varphi(\mu)\int_\alpha^\beta \varphi(\xi)\psi(\xi)d\xi, \qquad \alpha<\mu<\beta;$$

and one makes use of this equation [substituting the following values into the equation above] by applying this theorem with $\varphi(\xi)=n\xi f(n\xi)$, $\psi(\xi)=\frac{1}{\xi}, \alpha=2q, \beta=2p$, so that we can arrive at the following inequality:

$$\frac{1}{2}n\mu\cdot f(n\mu)\cdot\log(\frac{p}{q}) < S_{(p+q)n}-s_{2qn} < \frac{1}{2}n\mu\cdot f(n\mu)\cdot\log(\frac{p}{q})+u_{2qn}-u_{2pn}.$$

[Schlömilch uses the notation l for log, and . for $\cdot$]

Now for $n \to \infty$, we have $u_{2qn} \to 0$ and $u_{2pn} \to 0$. Also, $n\mu \to \infty$ since $n\mu > n2q$. Then

$$\lim_{\omega \to \infty} f(\omega) = K.$$

[He uses $\lim[\omega f(\omega)] = K$.] So one obtains from the previous inequality

$$S - s = \frac{1}{2} K \cdot \log(\frac{p}{q}).$$

The same result follows when $p < q$, and we consider the difference is $s_{2qn} - S_{(p+q)n}$, by proceeding analogously.

Theorem 4.3.1 ([22]). *Let $f(n) = u_n$ be a decreasing and positive function, such that the asymptotic value $\lim_{n\to\infty} f(n) = 0$. If $K = \lim_{\omega\to\infty} (\omega u_\omega)$ or $K = \lim_{n\to\infty} (nu_n)$, the following holds: If in a series (see note below)*

$$s = u_0 - u_1 + u_2 - u_3 + \cdots$$

the terms are rearranged such that always p positive terms are followed by q negative terms, then the sum of the rearranged series is

$$S = s + \frac{\lim(nu_n)}{2} \cdot \log(\frac{p}{q}).$$

[Note: He does not state it directly at this point, but the series that satisfies the theorem must be one of the form $s = \sum_{n=0}^{\infty} (-1)^n u_n$ such that $u_n > 0$ is strictly decreasing and $\lim_{n\to\infty} u_n = 0$, and by the Alternating Series Test it must converge.]

Following amongst other things is the Mean Value Theorem for Integrals for the Alternating Harmonic Series.

If $\lim(nu_n) = \infty$, as for example for

$$s = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots,$$

notice that the new series is divergent.

If $\lim(nu_n) = 0$, then reordering as above does not change the sum. For an example of this, consider the series

$$s = \frac{1}{2\log 2} - \frac{1}{3\log 3} + \frac{1}{4\log 4} - \cdots,$$

for which Scheibner's Criterion does not give a result because the absolute values is divergent.

CHAPTER 5

PRINGSHEIM'S THEOREM

5.1 Alfred Pringsheim

Alfred Pringsheim (2 September 1850 - 25 June 1941) was a German mathematician who studied mathematics in Berlin, receiving his doctorate in 1872. In 1877, he began teaching at the University of Munich as a privatdozent. Two years after he married Hedwig Dohm, an actress from Berlin. The two of them had four sons and one daughter. In 1886, Pringsheim was promoted as an "extraordinary" professor at the Ludwig-Maximilians University of Munich. He continued to work there for the rest of his career starting at the Barvarian Academy of Sciences in 1898. By 1901, he became a full professor, then retired in 1922 [13].

5.2 Pringsheim's Rearrangement

Let $\sum\limits_{n=1}^{\infty} f_n$ be a rearrangement of the Alternating Harmonic Series. Let p_k denote the number of positive terms in the sequence $(f_n)_{n=1}^{k}$, and let q_k denote the number of negative terms in $(f_n)_{n=1}^{k}$.

Recall the following definition.

Definition 5.2.1. *Let the number α be the* **asymptotic density** *of the positive terms in $\sum\limits_{n=1}^{\infty} f_n$. That is,*

$$\alpha = \lim_{n\to\infty} \frac{p_n}{n},$$

where p_n denotes the number of positive terms in the sequence $(f_n)_{n=1}^{k}$.

Note that $\alpha = \frac{1}{2}$ for the usual Alternating Harmonic Series.

The following is a generalization of Ohm's Theorem.

Theorem 5.2.2 ([2]). *Let $\sum f_n$ be a rearrangement of the AHS. The rearrangement $\sum_{k=1}^{\infty} f_n$ converges to an extended real number if and only if α, the asymptotic density of positive terms in $\sum_{k=1}^{\infty} f_n$ exists, and in that case*

$$\sum_{n=1}^{\infty} f_n = \ln(2) + \frac{1}{2}\ln\left(\frac{\alpha}{1-\alpha}\right).$$

Proof. Consider $\sum_{k=1}^{\infty} f_k$, and let $q_k = k - p_k$ with

$$\sum_{k=1}^{k} f_k = \sum_{m=1}^{p_k} \frac{1}{2m-1} - \sum_{m=1}^{q_k} \frac{1}{2m}.$$

Then, for each $k \in \mathbb{N}$, let

$$a_k = \left(\sum_{n=1}^{k} \frac{1}{n}\right) - \ln(k).$$

So $(a_k)_{k=1}^{\infty}$ is a decreasing sequence such that

$$\lim_{k\to\infty} (a_k)_{k=1}^{\infty} = \gamma$$

(known as Euler's constant). Then,

$$\begin{aligned}
\sum_{m=1}^{q_k} \frac{1}{2n} &= \frac{1}{2}\sum_{m=1}^{q_k} \frac{1}{m} \\
&= \frac{1}{2}\ln(q_k) + \frac{1}{2}a_{q_k},
\end{aligned}$$

and

$$\begin{aligned}
\sum_{m=1}^{p_k} \frac{1}{2m-1} &= \sum_{m=1}^{2p_k} \frac{1}{m} - \sum_{m=1}^{p_k} \frac{1}{2m} \\
&= \ln(2p_k) + a_{2p_k} - \frac{1}{2}\ln(p_k) - \frac{1}{2}a_{p_k}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\lim_{k\to\infty}\sum_{n=1}^{k} f_n &= \lim_{k\to\infty}\left[\ln(2_{p_n}) - \frac{1}{2}\ln(p_k) - \frac{1}{2}\ln(q_k) + a_{2p_k} - \frac{1}{2}a_{p_k} - \frac{1}{2}a_{q_k}\right] \\
&= \ln(2) + \lim_{k\to\infty}\frac{1}{2}\ln\left(\frac{p_k}{q_k}\right) + \gamma - \frac{1}{2}\gamma - \frac{1}{2}\gamma \\
&= \ln(2) + \frac{1}{2}\ln\left(\lim_{k\to\infty}\frac{p_k}{q_k}\right),
\end{aligned}$$

which is what we wanted. □

We introduce another important generalization of Pringsheim's results:

Theorem 5.2.3 ([2]). *Let $f_1, f_2, f_3, \ldots, f_k$ be a sequence such that each $f_i \in \mathbb{R}$ for $i \in \mathbb{N}$. If $|f_1| \geq |f_2| \geq |f_3| \geq \ldots$, where*

$$\lim_{k\to\infty} f_k = 0,$$

and $f_{2j-1} > 0 > f_{2j}$ *for* $j \in \mathbb{N}$, *then the following holds:*

1. *If* $\lim_{k\to\infty} k\cdot \mid f_k \mid = \infty$ *and if* $S \in \mathbb{R}$, *then there exists a rearrangement of* $\sum_{j=1}^{\infty} f_j$, *call it* $\sum_{j=1}^{\infty} f_j^*$, *such that* $\sum_{j=1}^{\infty} f_j^* = S$, *and the asymptotic density of positive terms of the rearrangement is* $\frac{1}{2}$,

   whose $\alpha = \frac{1}{2}$, *and whose sum is S.*

2. *If* $\lim_{k\to\infty} k \cdot f_k = 0$ *and if* $\sum_{n=1}^{\infty} g_n$ *is a rearrangement of* $\sum_{n=1}^{\infty} f_n$ *for which the asymptotic density* α *exists where* $0 < \alpha < 1$, *then*

$$\sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} f_n.$$

For the proof, refer to [15].

CHAPTER 6

SIERPIŃSKI'S THEOREM

In Section 6.2 we will provide a modern English-language translation of Sierpiński's result. In Section 6.3 we show some examples that demonstrate his result.

For clarity, we provide our version of Sierpiński's original theorem (Theorem 6.2.1):

Theorem 6.0.4 ([21])**.** *Let (f_n) be conditionally convergent where $U = \sum f_n$, and let $V \neq U$ be a real number. If $V > U$, there exists an explicitly described rearrangement π with the property that each positive term of f_n is left in place (if $f_n > 0$, then $\pi(n) = n$) (as π is defined in Section 1.1) and $\sum f_{\pi(n)} = V$. Similarly, if $V < U$, there exists an explicitly describe rearrangement π with the property that each negative term of f_n is left in place (if $f_n < 0$, then $\pi(n) = n$) and $\sum f_{\pi(n)} = V$.*

6.1 Wacław Sierpiński

Wacław Sierpiński (14 March 1882 - 21 October 1969) was a Polish mathematician who began his college career in 1899 at the University of Warsaw in the Department of Mathematics and Physics. In 1904, Sierpiński received the gold medal in a prize essay contest regarding Georgy Fedoseevich Voronoy's contributions to number theory. In 1904, Sierpiński graduated from the University of Warsaw, and immediately began working for some time at an all-girls school in Warsaw teaching mathematics and physics. By 1908, he received his doctorate and started working at the University of Lvov. Sierpiński became very interested in the study of set theory, and so in 1909 he taught the first lecture on pure set theory [13].

6.2 On a Property of a Non-Absolutely Convergent Series

[Note: This section is a modern English-language translation of Sierpiński's paper [21]. His paper's original references come from [16], [20], [19].]

Presented by S. Zaremba in the meeting on March 6, 1911.

After a well-known theorem of Riemann [16], the order of terms under any convergent series that is not absolutely convergent can always be modified so that the sum of the series is a value arbitrarily given *in principle*. It is possible to demonstrate his theorem by changing appropriately the *relative frequency* of positive and negative terms in the series given.

I have demonstrated, in a recent note [20], that we can even produce an arbitrary variation of the sum of a series that is not absolutely convergent with the aid of a change in the order of its terms which do not change the disposition of their signs.

We now show that it is enough to modify the order of the terms of a *determined sign* to experience a change, in advance, which gives the sum of a series that is not absolutely convergent.

Theorem 6.2.1 ([21])**.** *Let U be the sum of a non-absolutely convergent series. For the series to have an arbitrary sum $V < U$, given in advance, it suffices to modify the order of the positive terms of the series leaving each negative term in its place. On the other hand, it suffices to modify the order of the negative terms, leaving each positive term in its place, for its sum to have a value $V > U$.*

[Sierpiński's theorem may be a bit unclear for the reader. He is defining to two properties that must hold for the rearrangement to occur, but later in his paper he will be defining the explicit rearrangement that is involved.

At the start of his paper, he simply wants to make two claims before he starts defining the process: The series is some conditionally convergent series, which we will denote as $\sum u_i$, given that this series converges to U (for $u_i \in \mathbb{R}$). Then, Claim (i) states that when the positive terms are rearranged while the negative terms are left in place, the rearranged series of $\sum +u_i$ converges to a sum $V \leq U$, and Claim (ii) likewise, when the negative terms are rearranged while the positive terms are left in place, the rearranged series $\sum -u_i$ converges to some sum $V \geq U$.]

It is sufficient to establish only the first part of the proposal; in fact,

$$u_1 + u_2 + u_3 + \cdots \tag{6.1}$$

is a series given which is not absolutely convergent, U its sum, $T > U$, applying the second part of the theorem to the series (6.1) and the number $V = T$ is obviously equivalent to the application of the first part of this theorem to the series

$$-u_2 - u_2 - u_3 - \ldots \tag{6.2}$$

and the number $V = -T$.

Let

$$u_1 + u_2 + u_3 + \cdots$$

be a given non-absolutely convergent series, U its sum, V a number $< U$, given in advance, and let

$$a_1 + a_2 + a_3 + \cdots \tag{6.3}$$

be the series of positive consecutive terms of the series (6.1).

Set

$$U - V = l \qquad a_1 + a_2 + a_3 + \cdots + a_n = A_n \qquad A_0 = 0;$$

we have:

$$l > 0, \qquad \lim_{n\to\infty} a_n = \infty, \qquad \lim_{n\to\infty} A_n = 0.$$

Modifying the order of the terms in the series (6.3), we have a different series

$$c_1 + c_2 + c_3 + \cdots \tag{6.4}$$

To determine the law of formation of an explicit expression for the terms of (6.4), suppose we have already determined its n (≥ 0) first terms and set

$$c_1 + c_2 + c_3 + \cdots + c_n = C_n \qquad C_0 = 0.$$

Let n be a positive integer or zero: only one of three following cases may occur:

$$\text{I. } A_n - C_n \leq l.$$

II^a. $A_n - C_n > l$ and the term a_{n+1} is not in the sum C_n.

II^b. $A_n - C_n > l$ and the term a_{n+1} is in the sum C_n.

In case I, we choose the index r to be the smallest, for which a_r is not part of the sum C_n, and such that

$$a_r < \frac{1}{2^n} \qquad \text{and} \qquad a_r < \frac{a_{n+1}}{2}.$$

Such an index r exists always since

$$\lim_{n\to\infty} a_n = 0.$$

We will let

$$c_{n+1} = a_r.$$

In case II^a, we set

$$c_{n+1} = a_{n+1}.$$

Now consider case II^b. By virtue of

$$A_n - C_n > l > 0$$

we have

$$A_n > C_n,$$

which shows that the sum C_n cannot contain *all* the terms of the series A_n. Let a_r be the first term of the sum A_n, which does not appear in the sum C_n. We set:

$$c_{n+1} = a_r.$$

The conditions presented define perfectly the series (6.4), and clearly each term in the series (6.3) appears once more in the series (6.4), since we picked c_{n+1} to be the term of (6.3) not in C_n. I say that *every* term in the series (6.3) is in the series (6.4).

Denote by q_m the number that expresses how many indices $n \leq m$ for which we have case II^b. The sum C_{m+1} will obviously contain all the terms of the sum A_{q_m}. If therefore we show that

$$\lim_{m\to\infty} q_m = \infty,$$

it will follow that (6.4) contains all the terms of the series (6.3).

In the case that the equation

$$\lim_{m\to\infty} q_m = \infty$$

is not satisfied, the number of indices n for which we have the case II^b is finite; v being a fixed number, we have then for $n \geq v$ the case I or II^a. Suppose there exists an index $i \geq v$ for which we have the case II^a; we have $A_i > C_i$. The number of terms of the sums A_i and C_i being the same, the first may not contain all the terms of the second except in the case $A_i = C_i$; hence, the sum C_i contains terms that do not fall within the sum A_i. These terms belonging to the series (6.3) (as all terms of the series (6.4)) but not included in the sum

A_i have in the series (6.3) an index $> i$. The sum C_i contains therefore some terms a_n for which $n > i$. Let j be the smallest index $> i$ for which a_j is in the sum C_i. We can easily demonstrate that for the indices $i, i+1, i+2, \cdots, j-1$ we have case II^a, while the case II^b will hold for the index j. However, this is inconsistent with the hypothesis that for $n \geq v$ we always have case I or case II^a. Now this hypothesis requires that we always have case I for $n \geq v$. This being admitted, we have for $n \geq v$ constantly

$$A_n - C_n \leq l \qquad \text{and} \qquad c_{n+1} < \frac{a_{n+1}}{2},$$

from which for all natural x:

$$l \geq A_{v+x} - C_{v+x} > A_v - C_v + \frac{a_{v+1} + a_{v+2} + \cdots + a_{v+x}}{2}.$$

Or:

$$a_{v+1} + a_{v+2} + \cdots + a_{v+x} < 2(l - A_v + C_v),$$

which is inconsistent with the divergence of the series (6.3).

We have therefore shown that

$$\lim_{n\to\infty} q_m = \infty.$$

Moreover we can consider also as demonstrated that the series (6.4) differs from the series (6.3) only by the order of its terms.

Now denote p_m to be the number expressing how many indices $n \leq m$ there are for which case I is realized. I say that

$$\lim_{m\to\infty} p_m = \infty.$$

Suppose this proposal to be inaccurate; v being a fixed number, we then have case II for all $n \geq v$. Denote by τ_n, the number of terms of the sum A_n not included in the sum C_n. We obviously have

$$\tau_{n+1} = \tau_n \qquad \text{or} \qquad \tau_{n+1} = \tau_n - 1,$$

as case II^a or II^b will take place for the index n. As we have shown above, case II^b is realized for infinitely many indices; so τ_n would be negative for sufficiently large values of n, which is clearly absurd. The equality

$$\lim_{m\to\infty} p_m = \infty$$

is thus established.

The sequence a_n tends to 0 and the sequence c_n differs from the sequence a_n only by the order of its terms; we therefore also have

$$\lim_{n\to\infty} c_n = 0.$$

Imagine a number h sufficiently large so that we have

$$\frac{1}{2^{h-1}} < \varepsilon, \tag{6.5}$$

ε a positive number given in advance. As we have

$$\lim_{n\to\infty} a_n = 0 \qquad \text{and} \qquad \lim_{n\to\infty} c_n = 0$$

and as all the terms of the series (6.4) are at the same time the terms of the series (6.3), we set the number ε to match a number v such that the inequality $n > v$ implies the inequalities:

$$a_n < \varepsilon, \qquad c_i < \varepsilon \qquad \text{(Sierpiński (2))}$$

and that the sum A_v contains all the terms of C_h.

On the other hand, as we have

$$\lim_{m\to\infty} p_m = \infty, \qquad \lim_{m\to\infty} q_m = \infty$$

we can match to the number v a number μ such as the inequality $m > \mu$ gives:

$$p_m > v \qquad \text{and} \qquad q_m > v. \qquad \text{(Sierpiński (3))}$$

Now let m be an index $> \mu$. We examine separately the case

$$A_m - C_m \leq l$$

and the case

$$A_m - C_m > l.$$

Suppose, in the first place,

$$A_m - C_m \leq l. \tag{6.6}$$

Denote by k the largest index $k < m$ for which we have

$$A_k - C_k > l. \tag{6.7}$$

For $n = k+1, k+2, \cdots, m$, we evidently have the inequality

$$A_n - C_n \leq l,$$

which corresponds to case I; by the meaning of the symbols q_k and q_m, we arrive at the immediate conclusion: $q_k = q_m$. On the other hand, obviously $q_k \leq k$. We have:

$$A_{k+1} - C_{k+1} = A_k - C_k - a_{k+1} - c_{k+1} > A_k - C_k - c_{k+1} > l - c_{k+1}$$

by (6.7); and by (3):

$$k \geq q_k = q_m > v,$$

finally, by (2):

$$c_{k+1} < \varepsilon$$

Consequently:

$$A_{k+1} - C_{k+1} > l - \varepsilon. \tag{6.8}$$

For $n = k+1, k+2, \cdots, m$, we have case I; consequently,

$$A_m - C_m - (A_{k+1} - C_{k+1}) > \frac{a_{k+2} + a_{k+3} + \cdots + a_m}{2} > 0,$$

whence:

$$A_m - C_m > A_{k+1} - C_{k+1} > l - \varepsilon,$$

by (6.8). Following (6.6), we can write:

$$-\varepsilon < A_m - C_m - l \leq 0. \tag{6.9}$$

Now let

$$A_m - C_m > l. \tag{6.10}$$

Denote by k the largest index $< m$ for which we have

$$A_k - C_k \leq l.$$

For $n = k+1, k+2, \cdots, m$, we have do not case I; therefore, $p_k = p_m$; on the other hand, evidently $p_k \leq k$. We have:

$$A_{k+1} - C_{k+1} = A_k - C_k + a_{k+1} - c_{k+1} < A_k - C_k + a_{k+1} \leq l + a_{k+1};$$

but by (2): $a_{k+1} < \varepsilon$ since

$$k \geq p_k = p_m > v.$$

We therefore have

$$A_{k+1} - C_{k+1} \leq l + \varepsilon. \tag{6.11}$$

Denote by f_i $(i = 1, 2, \cdots, s)$ the indices included in between k and m for which case II^b holds; for the other indices n between k and m, we therefore have case II^a. Now:

$$a_{n+1} - c_{n+1} = 0.$$

It follows that

$$A_m - C_m - (A_{k+1} - C_{k+1}) = \sum_{i=1}^{s} (a_{f_i+1} - c_{f_i+1}) < \sum_{i=1}^{s} a_{f_i+1}. \tag{6.12}$$

If for an index f we have case II^b, the term a_{f+1} is included in the sum C_f, then we have

$$a_{f+1} = c_{g+1} \qquad \text{or} \qquad g < f.$$

I say that case I will occur for the index g. Indeed, if we had case II for the index g, it would set

$$c_{g+1} = a_{j+1} \qquad \text{or} \qquad j \leq g;$$

however, we have

$$c_{g+1} = a_{f+1} \qquad \text{with} \qquad f > g.$$

We therefore have for the index g case I; or

$$c_{g+1} < \frac{1}{2^g}.$$

For $f = f_i$ $(i = 1, 2, \cdots, s)$, we obviously have $f_i > k > v$ and as in the sum A_v are included all the terms of the sum C_h, the term a_{f_i+1}, having an index greater than v, cannot be in the sum A_v nor C_h. The index $g_i + 1$ of the term

$$c_{g_i+1} = a_{f_i+1}$$

is therefore greater than h and it follows that: $g_i \geq h$. We have case II^b for f_i; consequently, as we have demonstrated above, we have

$$c_{g_i+1} < \frac{1}{2^{g_i}}.$$

Thus, we have:

$$\sum_{i=1}^{s} a_{j_i+1} = \sum_{i=1}^{s} c_{g_i+1} < \sum_{i=1}^{s} \frac{1}{2^{g_i}}; \tag{6.13}$$

g_i $(i = 1, 2, \cdots, s)$ represents s different numbers, all $\geq h$. From that we conclude,

$$\sum_{i=1}^{s} \frac{1}{2^{g_i}} < \sum_{j=h}^{\infty} \frac{1}{2^j} = \frac{1}{2^{h-1}} < \varepsilon, \tag{6.14}$$

by (6.5).

We therefore have, by (6.12), (6.13), and (6.14): Hence, by (6.12), (6.13), and (6.14), we get that,

$$A_m - C_m - (A_{k+1} - C_{k+1}) < \varepsilon;$$

Thus, by (6.10) and (6.11):

$$0 < A_m - C_m - l < 2\varepsilon. \tag{6.15}$$

For all $m > \mu$, we therefore have one or the other of the inequalities (6.9) or (6.15); so, for all $m > \mu$, we have:

$$| A_m - C_m - l | < 2\varepsilon,$$

from which it follows immediately that

$$\lim_{m\to\infty} (A_m - C_m) = l. \tag{6.16}$$

Now that

$$-b_1 - b_2 - b_3 - \cdots$$

be the series of consecutive negative terms of the series (6.1), $-B_n$ the sum of the first n terms.

We have for all natural numbers n:

$$U_n = A_{r_n} - B_{s_n},$$

r_n and s_n being two non-decreasing sequences and such that

$$\lim_{n\to\infty} r_n = \infty \qquad \textit{and} \qquad \lim_{n\to\infty} s_n = \infty. \tag{6.17}$$

Now form a new series

$$v_1 + v_2 + v_3 + \cdots \tag{6.18}$$

by replacing each positive term $u_n = a_{r_n}$ of the series (6.1) by the term $c_n = c_{r_n}$, and maintaining without any modification the negative terms of this series.

The series (6.18) differs from (6.1) only by the order of its positive terms. Upon designating by V_n to be the sum of the first n terms of the series (6.18), we evidently have:

$$V_n = C_{r_n} - B_{s_n} = U_n - (A_{r_n} - C_{r_n}).$$

By (6.16) and (6.17), we have:

$$\lim_{n\to\infty}(A_{r_n} - C_{r_n}) = l,$$

Therefore,

$$\lim_{n\to\infty} V_n = \lim_{n\to\infty} U_n - \lim_{n\to\infty}(A_{r_n} - C_{r_n}) = U - l = V.$$

Our theorem is thus demonstrated.

Note again that by suitably modifying the order of the terms of a *single sign* one could obtain a *divergent series* starting with a non-absolutely convergent series; this results almost immediately from the following theorem I demonstrated in a recent publication [19]:

"By changing the order of the terms of a divergent series whose terms are positive and tend to 0, we can always get a series which diverges more slowly than a divergent series, given in advance, whose terms are also positive."

6.3 Examples

Example 6.3.0.1. Consider again the Alternating Harmonic Series

$$\sum \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots$$

whose sum is $U = \ln(2)$. Suppose $V = \dfrac{\ln(2)}{2}$. Then, $l = U - V = \ln(2) - \dfrac{\ln(2)}{2} = \dfrac{\ln(2)}{2}$.

Now we have that $a_1 = 1, a_2 = \frac{1}{3}, a_3 = \frac{1}{5}, a_4 = \frac{1}{7}, a_5 = \frac{1}{9}, a_6 = \frac{1}{11}, \ldots$.

Given that $C_0 = 0$, and $A_0 = C_0$, then $A_0 - C_0 = 0 < \dfrac{\ln(2)}{2}$. So we are in case I and we choose index r to be the smallest, for which a_r is not part of the sum C_0, and such that $a_r < \frac{1}{2^0} = 1$ and $a_r < \frac{a_1}{2} = \frac{1}{2}$. Set $c_1 = a_r$. Since $a_r = \frac{1}{3}$, then $c_1 = \frac{1}{3}$.

Now consider $A_1 = 1$ and $C_1 = \frac{1}{3}$. Then, $A_1 - C_1 = \frac{2}{3} > \frac{\ln(2)}{2}$, and since $a_2 = \frac{1}{3}$ is in the sum C_1, then we are in case II^b. So let a_r be the first term of the sum A_n, which is not in the sum C_1. We set $a_r = c_2$. Since $a_r = 1$, then $c_2 = 1$.

Now

$$A_2 = 1 + \frac{1}{3} = \frac{4}{3}$$

and

$$C_2 = \frac{1}{3} + 1 = \frac{4}{3}.$$

So

$$A_2 - C_2 = \frac{4}{3} - \frac{4}{3} = 0 < \frac{\ln(2)}{2}.$$

So we are in case I. Thus, we choose r to the smallest for which a_r is not part of the sum C_2, and $a_r < \frac{1}{2^2} = \frac{1}{4}$ and $a_r < \frac{a_3}{2} = \frac{1}{10}$. Since 1 and $\frac{1}{3}$ are in the sum C_2, then $a_r = \frac{1}{11}$, which means $c_3 = \frac{1}{11}$.

Then

$$A_3 = 1 + \frac{1}{3} + \frac{1}{5} = \frac{4}{3} + \frac{1}{5} = \frac{23}{15}$$

and

$$C_3 = \frac{1}{3} + 1 + \frac{1}{11} = \frac{47}{33}.$$

So

$$A_3 - C_3 = \frac{23}{15} - \frac{47}{33} = \frac{6}{55},$$

so we are in case I, which means we will choose r to be the smallest for which a_r is not part of the sum C_n, and such that $a_r < \frac{1}{2^3} = \frac{1}{8}$, and $a_r < \frac{a_4}{2} = \frac{1}{14}$. So let $c_3 = a_r$. Since $a_r = \frac{1}{15}$, then $c_4 = \frac{1}{15}$.

Next, we have

$$A_4 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}$$

and

$$C_4 = \frac{1}{3} + 1 + \frac{1}{11} + \frac{1}{15}.$$

Hence, we are in case I again. So choose r to be the smallest for which a_r is not part of the sum C_n, and such that $a_r < \frac{1}{16}$ and $a_r < \frac{a_5}{2} = \frac{1}{18}$. Let $c_5 = a_r$, which means $c_5 = \frac{1}{19}$.

Note we are in case I until $n = 7$ (or C_7). For the sake of time, we will skip the evaluation of c_6 and c_7, but do note that $c_6 = \frac{1}{23}$ and $c_7 = \frac{1}{65}$.

Thus,

$$A_7 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13},$$

and

$$C_7 = \frac{1}{3} + 1 + \frac{1}{11} + \frac{1}{15} + \frac{1}{19} + \frac{1}{23} + \frac{1}{65}.$$

Notice we have

$$A_7 - C_7 > \frac{\ln(2)}{2},$$

and that $a_8 = \frac{1}{15}$ is in the sum C_7. So we are in case II^b. Now let a_r be the first term of A_7 not in the sum C_7. Set $c_8 = a_r$. Since $a_r = \frac{1}{5}$, then $c_8 = \frac{1}{5}$.

So

$$A_8 - C_8 > \frac{\ln(2)}{2}$$

and $a_9 = \frac{1}{17}$ is not in the sum C_8. So we are in case II^a, thus since we let $c_9 = a_9$, then $c_9 = \frac{1}{17}$.

Now

$$A_9 - C_9 < \frac{\ln(2)}{2},$$

which means we are in case I. So we choose a_r to be the smallest, for which it is not in C_9 and $a_r < \frac{1}{2^9} = \frac{1}{512}$ and $a_r < \dfrac{a_{10}}{2} = \dfrac{1}{38}$. Since $a_r = \frac{1}{513}$, then $c_{10} = \frac{1}{513}$.

We note that case I continues again for a while before it switches to another case. Nonetheless, the process for the formation of the rearrangement is the same.

Taking the $c_1, c_2, c_3, c_4, \dots$ terms we found and using them to replace the positive terms of the original series, respectively, we get the following new series:

$$\frac{1}{3} - \frac{1}{2} + 1 - \frac{1}{4} + \frac{1}{11} - \frac{1}{6} + \frac{1}{15} - \frac{1}{8} + \frac{1}{19} - \frac{1}{10} + \cdots + \frac{1}{17} - \frac{1}{16} + \frac{1}{513} - \frac{1}{18} + \frac{1}{1025} - \frac{1}{20} + \cdots$$

.

Example 6.3.0.2. Consider the series,

$$\sum (-1)^n \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{7}} + \cdots,$$

where $a_1 = \dfrac{1}{\sqrt{2}}, a_2 = \dfrac{1}{\sqrt{4}}, a_3 = \dfrac{1}{\sqrt{6}}, a_4 = \dfrac{1}{\sqrt{8}}, \dots$.

Consider again that $V = \frac{\ln(2)}{2}$. Given that $C_0 = 0$ and $A_0 = 0$, then

$$A_0 - C_0 < \frac{\ln(2)}{2}.$$

So we are in case I. Let us choose r to be the smallest index, for which a_r is not part of the sum C_0, and $a_r < \frac{1}{2^0} = 1$ and $a_r < \frac{a_1}{2} = \frac{1}{2\sqrt{2}}$. Since $a_r = \frac{1}{\sqrt{10}}$, then $c_1 = \frac{1}{\sqrt{10}}$.

Since $A_1 = \frac{1}{\sqrt{2}}$ and $C_1 = \frac{1}{\sqrt{10}}$, then $A_1 - C_1 > \frac{\ln(2)}{2}$, and because $a_2 = \frac{1}{\sqrt{4}}$ is not in C_1, then we are in case II^a. Let $c_2 = a_2$. Since $a_2 = \frac{1}{\sqrt{4}}$, then $c_2 = \frac{1}{\sqrt{4}}$.

Next, we have that

$$A_2 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}}$$

and

$$C_2 = \frac{1}{\sqrt{10}} + \frac{1}{\sqrt{4}}.$$

Then, we get that

$$A_2 - C_2 > \frac{\ln(2)}{2},$$

and $a_3 = \frac{1}{\sqrt{6}}$ is not in C_2. Hence, we are in case II^a again. Since $a_3 = \frac{1}{\sqrt{6}}$, then $c_3 = \frac{1}{\sqrt{6}}$.

Then

$$A_3 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{6}}$$

and

$$C_3 = \frac{1}{\sqrt{10}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{6}}.$$

Since

$$A_3 - C_3 > \frac{\ln(2)}{2},$$

and $a_4 = \frac{1}{\sqrt{8}}$ is not in C_3, then we are in case II^a. So we have that $a_4 = c_4 = \frac{1}{\sqrt{8}}$.

Now notice that

$$A_4 - C_4 > \frac{\ln(2)}{2}$$

and $a_5 = \frac{1}{\sqrt{10}}$, which is in C_4. Thus, we are in case II^b. So we choose a_r to be the first term in A_n that is not in C_4. Let $a_r = c_5$. Since $a_r = \frac{1}{\sqrt{2}}$, then $c_5 = \frac{1}{\sqrt{2}}$.

Since

$$A_5 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{10}}$$

and

$$C_5 = \frac{1}{\sqrt{10}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{2}},$$

then it is easy to see that

$$A_5 - C_5 = 0 < \frac{\ln(2)}{2}.$$

So we are in case I. We pick the index r to be the smallest for which a_r is not in C_5, and $a_r < \frac{1}{2^5} = \frac{1}{32}$ and $a_r < \frac{a_6}{2} = \frac{1}{2\sqrt{12}}$. Let $c_6 = a_r$. Since $a_r = \frac{1}{\sqrt{1020}}$, then $c_6 = \frac{1}{\sqrt{1020}}$.

Note that we stay in case I for a while. Continuing, the process for the formation of the new rearranged series is the same. Then, taking the $c_1, c_2, c_3, c_4, \ldots$ terms and replacing them with all the positive terms of the original series, we get the following rearrangement:

$$\frac{1}{\sqrt{10}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{8}} - \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{11}} + \frac{1}{\sqrt{1020}} - \cdots = \frac{\ln 2}{2}.$$

CHAPTER 7

ADDITIONAL RESULTS

In this chapter, we introduce some modern results complementing the classical theory we have presented throughout this thesis.

7.1 Lévy and Steinitz

We know that by Riemann's Theorem a conditionally convergent series of real numbers can be rearranged to sum to any $\alpha \in \mathbb{R}$.

In [17], Rosenthal formulated Riemann's result another way:

The set of all sums of rearrangements of a given series of real numbers is either empty, a single point (if the series is absolutely convergent), or spans the entirety of $\mathbb{R}$.

We can extend this idea to complex numbers:

Theorem 7.1.1 ([17])**.** *The set of all sums of rearrangements of a given series of complex numbers is the empty set, a single point, a line in the complex plane, or the whole complex plane.*

More generally, the Levy-Steinitz Theorem gives a similar result in *n* dimensions.

Theorem 7.1.2 (Lévy-Steinitz Theorem [17])**.** *Let $\sum T$ be a given series in $\mathbb{R}^n$. Let R be the set of all sums of rearrangements of $\sum T$ in $\mathbb{R}^n$. Then, R is either $\emptyset$ or a translate of a subspace (that is, $R = v + M$ for some vector v and some linear subspace M).*

For a proof, refer to [6, pg. 54-61].

We proceed with a few examples.

Example 7.1.2.1. *Consider the following in $\mathbb{R}^2$: $\sum(\frac{(-1)^{n-1}}{n}, 0)$, in which we are considering the sums of all the rearrangements of this particular series of vectors. Notice that the every x-coordinate is a sum from a rearrangement of the alternating harmonic series, which we know to be conditionally convergent.*

By Riemann's Theorem, since we can get rearrangements that converge to any real number, then the set of all the x-coordinates in this series of vectors is $\mathbb{R}$. Since the y-coordinate is always 0, then it is easy to see that the set of $\sum(\frac{(-1)^{n-1}}{n}, 0)$ spans all of $\mathbb{R}$ along the x-axis for every $y = 0$. In other words, $\sum(\frac{(-1)^{n-1}}{n}, 0)$ in $\mathbb{R}^2$ is the line $y = 0$, which is an affine space[1].

Example 7.1.2.2. *Now consider the following series in $\mathbb{R}^2$: $\sum(\frac{(-1)^{n-1}}{n}, \frac{1}{2^n})$.*

Again, we see that every x-coordinate is a sum from a rearrangement of the AHS. From the previous example, we have already verified that the set of all the x-coordinates in this series of vectors is the set $\mathbb{R}$. Notice that the y-coordinates come from $\sum \frac{1}{2^n}$, where

$$\sum \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1.$$

[1] An *Affine Space* is a translation of a subspace. For example, in $\mathbb{R}^2$, a line in a one-dimensional subspace is the line $y = mx$ or the vertical line $x = 0$, while a one-dimensional affine space is the line $y = mx + b$ or the line $x = b$.

Also note that

$$\begin{aligned}\sum \left| \frac{1}{2^n} \right| &= \left|\frac{1}{2}\right| + \left|\frac{1}{4}\right| + \left|\frac{1}{8}\right| + \left|\frac{1}{16}\right| + \cdots \\ &= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1\end{aligned}$$

By definition, $\sum \frac{1}{2^n}$ *is absolutely convergent. It follows from Dirichlet's theorem that every rearrangement of* $\sum \frac{1}{2^n}$ *converges to* 1. *Thus, in* $\mathbb{R}^2$, $\sum(\frac{(-1)^{n-1}}{n}, \frac{1}{2^n})$ *is the line* $y = 1$, *which is an affine space.*

Example 7.1.2.3. *Last, take the example* $\sum(\frac{(-1)^{n-1}}{n}, \frac{1}{2^n} - \frac{(-1)^{n-1}}{n})$. *We already know that the set of all the x-coordinates in this series of vectors spans* $\mathbb{R}$. *So now consider the y-coordinates,* $\sum(\frac{1}{2^n} - \frac{(-1)^{n-1}}{n})$. *Since we know that the sum of every rearrangement of* $\frac{1}{2^n}$ *is always* 1, *then the difference* $\frac{1}{2^n} - \frac{(-1)^{n-1}}{n}$ *in the set of sums of all rearrangements produces the set* $\mathbb{R}$. *In other words, the y-coordinates spans* $\mathbb{R}$, *just as the x-coordinates do.*

Important to note is that for a chosen rearrangement yielding some particular sum in the x-coordinate, that same sum is used in the difference of $\frac{1}{2^n} - \frac{(-1)^{n-1}}{n}$ *of the y-coordinate. In other words, say we let* $V_n = (\frac{(-1)^{n-1}}{n}, \frac{1}{2^n} - \frac{(-1)^{n-1}}{n})$, $A_n = \frac{(-1)^{n-1}}{n}$, *and* $B_n = \frac{1}{2^n}$.

Then, for a permutation $\pi : \mathbb{N} \to \mathbb{N}$,

$$\begin{aligned}\sum V_{\pi(n)} &= \sum(\frac{(-1)^{n-1}}{n}, \frac{1}{2^n} - \frac{(-1)^{n-1}}{n}) \\ &= (\sum A_{\pi(n)}, \sum B_{\pi(n)} - \sum A_{\pi(n)}).\end{aligned}$$

Now let $\sum A_{\pi(n)} = S$.

Then, we have $(S, 1-S)$. *So* $x + y = 1$, *which means that* $\sum(\frac{(-1)^{n-1}}{n}, \frac{1}{2^n} - \frac{(-1)^{n-1}}{n})$

is the affine space $x + y = 1$.

Lévy and Steinitz also came up with the following rearrangement theorem in terms of vectors, which serves as an important part in proving the Lévy-Steinitz Theorem.

Theorem 7.1.3 (The Rearrangement Theorem [17])**.** *In* $\mathbb{R}^n$*, if a subsequence of the sequence of partial sums of a series of vectors converges to S, and if the sequence of terms of the series converges to* 0*, then there is a rearrangement of the series that sums to S.*

For a proof of this, refer to [17, pg. 346].

7.2 Extensions of Pringsheim's Results

In this section, we will introduce some additional generalization, due to Scheepers [23], of Pringsheim's rearrangement results [15].

First recall the definition of asymptotic density from Chapter 5.

Definition 7.2.1. *A number* α *is the* **asymptotic density** *of the positive terms in* $\sum_{n=1}^{\infty} f_n$*, provided that*

$$\alpha = \lim_{n\to\infty} \frac{P_n}{n}$$

where P_n *denotes the number of positive terms in the sequence* $(f_n)_{n=1}^{k}$.

Let us provide an example to make the concept of asymptotic density more clear to the reader.

Example 7.2.1.1. *Consider the usual Alternating Harmonic Series. By [2],*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n},$$

has $\alpha = \frac{1}{2}$. *Again, consider the rearrangement* $A(2,1)$. *We will illustrate that* $A(2,1)$ *has* $\alpha = \frac{2}{3}$ *by showing that quotients* $\dfrac{P_{3n}}{3n}$ *for* $n \leq 5$ *actually equals* $\frac{2}{3}$*:*

$$\begin{aligned}
C_1 &= 1+\tfrac{1}{3}+\left(-\tfrac{1}{2}\right) \Rightarrow \tfrac{P_3}{3} = \tfrac{2}{3}, \\
C_2 &= 1+\tfrac{1}{3}+\left(-\tfrac{1}{2}\right)+\tfrac{1}{5}+\tfrac{1}{7}+\left(-\tfrac{1}{4}\right) \Rightarrow \tfrac{P_6}{6} = \tfrac{4}{6} = \tfrac{2}{3}, \\
C_3 &= 1+\tfrac{1}{3}+\left(-\tfrac{1}{2}\right)+\tfrac{1}{5}+\tfrac{1}{7}+\left(-\tfrac{1}{4}\right)+\tfrac{1}{9}+\tfrac{1}{11}+\left(-\tfrac{1}{6}\right) \Rightarrow \tfrac{P_9}{9} = \tfrac{6}{9} = \tfrac{2}{3}, \\
C_4 &= 1+\tfrac{1}{3}+\left(-\tfrac{1}{2}\right)+\tfrac{1}{5}+\tfrac{1}{7}+\left(-\tfrac{1}{4}\right)+\cdots+\tfrac{1}{13}+\tfrac{1}{15}+\left(-\tfrac{1}{8}\right) \Rightarrow \tfrac{P_{12}}{12} = \tfrac{8}{12} = \tfrac{2}{3}, \\
C_5 &= 1+\tfrac{1}{3}+\left(-\tfrac{1}{2}\right)+\tfrac{1}{5}+\tfrac{1}{7}+\left(-\tfrac{1}{4}\right)+\cdots+\tfrac{1}{17}+\tfrac{1}{19}+\left(-\tfrac{1}{10}\right) \Rightarrow \tfrac{P_{15}}{15} = \tfrac{10}{15} = \tfrac{2}{3}.
\end{aligned}$$

Next let us bring forth some notation from [23] [Note that some of our notation may be slightly different]:

Let $\sum f$ be a conditionally convergent series if it is convergent. Then, both $\{n : f(n) > 0\}$ and $\{n : f(n) < 0\}$ are infinite sets.

Let $a_1, a_2, a_3, \cdots$ be positive terms of f in the order that they occur, and let $-b_1, -b_2, -b_3, \cdots$ be the negative terms of f in the order that they occur.

Now suppose f to be signwise monotonic. Suppose S to be an infinite subset of $\mathbb{N}$ where $T = \mathbb{N} \setminus S$ is an infinite set. Let f_S be such that

$$f_S(n) = \begin{cases} j^{th} \text{ positive term of } f \text{ if } n \text{ is the } j^{th} \text{ element of } S, \\ j^{th} \text{negative term of } f \text{ if } n \text{ is the } j^{th} \text{ element of } T. \end{cases}$$

So f_S is a Riemann (Chapter 3) rearrangement of f. For any set X, let $\mid X \mid$ denote the size of X. Then, for some subset S of $\mathbb{N}$, let $\pi_S(n) = \mid S \cap \{1,2,3,\ldots,n\} \mid$ be the pre-density of S, let $d_+(S) = \limsup \dfrac{\pi_S(n)}{n}$ be the upper density of S, and let $d_-(S) = \liminf \dfrac{\pi_S(n)}{n}$ be the lower density of S.

Note that $0 \leq d_-(S) \leq d_+(S) \leq 1$ [23], and if the equality of $d_+(S)$ and $d_-(S)$ exists, then we can let $d(S) = d_+(S) = d_-(S)$ be the asymptotic density of S. Now suppose that S is a subset of the natural numbers and for any $\sum f_S$ is a conditionally convergent series. Let R be a subset of the natural numbers such that for any finitely many n, we have $\pi_S(n) < \pi_R(n)$. So for each n, pick some k_n small enough so that

$$n - \pi_R(n) = k_n - \pi_S(k_n).$$

Then, for each n we have

$$\sum_{m \leq n} f_R(m) - \sum_{i \leq k_n} f_S(i) = a_{\pi_S(n)+1} + \cdots + a_{\pi_R(n)}.$$

Note that $\sum f_S$ and $(a_{\pi_S(k_n)+1} + \cdots + a_{\pi_R(n)})$, $n \in \mathbb{N}$, determines whether $\sum f_R$ converges, and if the convergence exists, $\sum f_S$ and $(a_{\pi_S(k_n)+1} + \cdots + a_{\pi_R(n)})$ also determines the limit of $\sum f_R$. Also note that for each n, the n^{th} term of $(a_{\pi_S(k_n)+1} + \cdots + a_{\pi_R(n)})$ is bounded from below by

$$(\pi_R(n) - \pi_S(k_n) - 1) \cdot a_{\pi_R(n)},$$

and bounded from above by

$$(\pi_R(n) - \pi_S(k_n) - 1) \cdot a_{\pi_S(k_n)}.$$

Next, the following are results (taken from [23]) of Pringsheim, which are extensions of Schlömilch's results (Chapter 4) and Ohm's theorem (Chapter 2).

Let $S \subset \mathbb{N}$. Define

$$\omega_f = \{x \in (0,1) : (\exists S \subset \mathbb{N})(d(S) = x \text{ and } \sum f_S(n) \text{ converges })\}.$$

Lemma 7.2.2 ([23])**.** *Let f be signwise monotonic. If $|\ \omega_f\ | > 1$, then for all $A, B \subset \mathbb{N}$ where $d(A) = d(B)$ and $\sum f_A$ converges, $\sum f_B$ converges and*

$$\sum f_A = \sum f_B.$$

Theorem 7.2.3 ([23])**.** *Let f be signwise monotonic, converging to 0. Let $0 < x < 1$ be given. Then, the following are equivalent: 1. $x \in \omega_f$ (asymptotic density) and $\lim n \cdot a_n = \infty$. 2. For each set B, such that $\sum f_B$ converges, $d(B) = x$.*

Statement 7.2.3.1 (Pringsheim I [23])**.** *Let $(n \cdot a_n)$, $n \in \mathbb{N}$, diverge to ∞. Then $\sum f_R$ converges if and only if $(a_{\pi_S(k_n)} \cdot (\pi_R(n) - \pi_S(k_n))$ converges. Also, if*

$$\lim(a_{\pi_A(k_n}) \cdot (\pi_R(n) - \pi_S(k_n)) = a,$$

then

$$\sum_{n<\infty} f_R(n) = \sum_{n<\infty} f_S(N) + a.$$

Note: This statement is false. A counterexample is described in [23, pg. 422].

Theorem 7.2.4 (Pringsheim II [23])**.** *Suppose* $\lim_{n\to\infty} n \cdot a_n = 0$. *If* $(a_{\pi_S(k_n)} \cdot (\pi_R(n) - \pi_S(k_n))$ *is bounded, then*

$$\sum_{n<\infty} f_R = \sum_{n<\infty} f_S.$$

Theorem 7.2.5 (Pringsheim-Schlömilch [23])**.** *Suppose* $(n \cdot a_n) \to t$ *such that* $t \in \mathbb{R} \setminus \{0\}$. *Then,* $\sum f_R$ *converges if and only if* $(a_{\pi_S(k_n)} \cdot (\pi_R(n) - \pi_S(k_n))$ *converges. Also, if*

$$\lim_{n\to\infty} a_{\pi_S(k_n)} \cdot (\pi_R(n) - \pi_S(k_n)) = a,$$

then

$$\sum_{n<\infty} f_R = \sum_{n<\infty} f_S + t \cdot \ln(1 + \frac{a}{t}).$$

In summary, mathematicians found: (1) the convergence criteria of $\sum f_B$ when $\lim n \cdot a_n = \infty$, (2) the convergence criteria of $\sum f_B$ when $\lim n \cdot a_n = 0$, and (3) the change in value of $\sum f_B$ for all B with $0 < d(B) < 1$ when $\lim n \cdot a_n = t \neq 0$ [23].

7.3 Władyslaw Wilczyński

Recall that Riemann's Rearrangement Theorem states that for each conditionally convergent series $\sum f_n$ (of real numbers) and every real number α, there is a permutation $\pi : \mathbb{N} \to \mathbb{N}$, such that $\sum f_{\pi(n)} = \alpha$. A problem addressed in [4, pg. 64] is whether or not one can extend this result of Riemann by always taking a permutation π so that it only changes a small set of terms in $\sum f_n$.

Let us begin by introducing some notation from [4].

Let $\mid X \mid$ be the cardinality of a set X. Let $(a_n)_{n\in\omega}$ be a given sequence and define $a_n^+ = \max\{a_n, 0\}$ and $a_n^- = \min\{a_n, 0\}$. Consider the series $\sum_{n\in\omega} a_n$ and $A \subset \omega$, by $\sum_{n\in A} a_n$ we denote the series $\sum_{n\in\omega} \chi_A(n) \cdot a_n$.

An ideal on ω is a family $\mathcal{I} \subset \mathcal{P}(\omega)$ ($\mathcal{P}(\omega)$ being the power set of ω) such that it is closed under taking subsets of finite unions. Assume that all considered ideals are proper (unless stated otherwise) and contain all finite sets. We are able to talk about ideals on any countable set by identifying this set with ω in terms of a fixed $1-1$ and *onto* occurrence.

The ideal of all finite sets of natural numbers is denoted Fin. An ideal $\mathcal{I}$ is *dense* if each $A \notin \mathcal{I}$ has an infinite subset that belongs to the ideal.

The notion of smallness of the set of terms in a series can be considered in terms of ideals of subsets of $\mathbb{N}$, denoted $\mathcal{I}$. Then, for $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$, where $\mathcal{P}(\mathbb{N})$ denotes the powerset of $\mathbb{N}$, there is a permutation $\pi : \mathbb{N} \to \mathbb{N}$ that changes a small set of terms of $\mathbb{N}$ when $\{n \in \mathbb{N} : \pi(n) \neq n\} \in \mathcal{I}$.

Definition 7.3.1 ([3]). *An ideal of the form*

$$\mathcal{I}_f = \{A : \sum_{n\in A} f(n) < \infty\}$$

is a **summable ideal***, where f is some positive function, such that $\sum_n f(n) = \infty$.*

Now we proceed to introduce two properties introduced in [4].

(R) Property: $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ has the (R) property if for each conditionally convergent series $\sum_n f_n$ and $r \in \mathbb{R}$, there is a permutation $\pi_r : \mathbb{N} \to \mathbb{N}$ where $\sum_n f_{\pi_r(n)} = r$ and $\{n \in \mathbb{N} : \pi_r(n) \neq n\} \in \mathcal{I}$.

(W) Property: $\mathcal{I} \subset \mathcal{P}(\mathbb{N})$ has the (W) property if for each conditionally convergent series $\sum_n f_n$ there exists an $A \in \mathcal{I}$ where $\sum_{n \in A} f_n$ is conditionally convergent.

Now we introduce a couple of applicable theorems by Wilczyński.

Theorem 7.3.2 ([4])**.** *Let $\mathcal{I}$ be an ideal in $\mathbb{N}$. Then, $\mathcal{I}$ has the (R) property if and only if $\mathcal{I}$ cannot be extended to a summable ideal if and only if $\mathcal{I}$ has the (W) property.*

Theorem 7.3.3 ([24])**.** *If $\sum_{n=1}^{\infty} f_n$ in a conditionally convergent series, then there is a set $A \subset \mathbb{N}$, where the asymptotic density of A is equal to zero and $\sum_{n=1}^{\infty} \chi_A(n) \cdot f_n$ is conditionally convergent.*

A detailed proof of Theorem 7.3.2 and Theorem 7.3.3 can be found in [24, pg. 80].

Thus, there are many results regarding rearrangements of series.

REFERENCES

[1] Cajori, Florian. *A History of Mathematics.* 5th ed. Chelsea Pub Co., 1991.

[2] Cowen, C. C., K. R. Davidson, and R. P. Kaufman. *Rearranging the Alternating Harmonic Series.* The American Mathematical Monthly. 87.10 (1980): 817-819.

[3] Farah, I. *Analytic Ideals and Their Quotients.*, phD thesis, University of Toronto, 1997.

[4] Filipów, Rafał, and Piotr Szuca. *Rearrangement of Conditionally Convergent Series on a Small Set.* Journal of Mathematical Analysis and Applications 362.1 (2010): 64-71.

[5] Galanor, Stewart. *Riemann's Rearrangement Theorem.* The Mathematics Teacher. 80.8 (1987): 675-81.

[6] Gödel, Kurt. *Collected Works.* Vol. 3. Unpublished Essays and Lectures. New York: Oxford University Press, 1995.

[7] Gupta, S. L., and Nisha Rani. *Fundamental Real Analysis.* Delhi: Vikas, 1975.

[8] Keisler, H. Jerome. *Elementary Calculus: An Infinitesimal Approach*. Boston, MA: Prindle, Weber & Schmidt, 1986.

[9] Knopp, Konrad. *Elementary Calculus: An Infinitesimal Approach.* Dover Books on Mathematics. New York: Dover Publications, 1956.

[10] Knopp, Konrad. *Theory and Application of Infinite Series.* Dover Books on Mathematics. New York: Dover Publications, 1990.

[11] Ohm, Martin, *De nonnullis seriebus infinitis summandis*. typis Trowitzschii et filii, 1839.

[12] Dana, Charles A., and Ripley, George. *The American Cyclopaedia.* Appleton, 1879.

[13] *Oscar Xavier Schlömilch.* Schlomilch Biography.
<http://www-groups.dcs.st-and.ac.uk/ history/Biographies/Schlomilch.html>.

[14] *Peter Gustav Lejeune Dirichlet.* Australian Mathematics Trust. <http://www-groups.dcs.st-and.ac.uk/ history/Biographies/Dirichlet.html'>.

[15] Pringsheim, Alfred, *Üeber die Werthveränderungen bedingt convergenter Reihen und Producte*. Mathematische Annalen, 22 (1883): 445-503.

[16] Riemann, Bernhard, *Gesammelte Mathematische Werke*. Leipzig, 1892.

[17] Rosenthal, Peter. *The Remarkable Theorem of Lévy and Steinitz.* The American Mathematical Monthly 94.4 (1987): 342-51.

[18] Rudin, Walter. *Principles of Mathematical Analysis.* 2nd ed. New York: McGraw-Hill, 1964. Web.

[19] Sierpiński, Wacław, *Contribution à la théorie des séries divergentes*, Comp. Rend. Soc. Sci. Varsovie 3 (1910): 89-93.

[20] Sierpiński, Wacław, *Remarque sur la théorème de Riemann relatif aux sries semi-convergentes*, Prac. Mat. Fiz., XXI (1910): 17-20.

[21] Sierpiński, Wacław, *Sur une propriété des séries qui ne sont pas absolument convergentes*. Bull. Intern. Acad. Sci.: Cracovie A (1911): 149-58.

[22] Schlömilch, Oscar. *For Conditionally Converging Series.* Zeitschrift Für Angewandte Mathematik Und Physik ZAMP 18 (1873): 520-22.

[23] Scheepers, Marion. *On the Pringsheim Rearrangement Theorems.* Journal of Mathematical Analysis and Applications 267 (2002): 418-33.

[24] Wilczyński, Wacław, *On Riemann Derangement Theorem*, Słup. Pr. Mat.-Fitz. 4 (2007): 79-82.