THE IMPACT OF A QUANTITATIVE REASONING INSTRUCTIONAL APPROACH TO LINEAR EQUATIONS IN TWO VARIABLES ON STUDENT ACHIEVEMENT AND STUDENT THINKING ABOUT LINEARITY

by

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The following individuals read and discussed the thesis submitted by student Paul Thomas Belue, and they evaluated his presentation and response to questions during the final oral examination. They found that the student passed the final oral examination.

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DEDICATION

I dedicate this thesis to my mother, Dian Rouse Belue, a math instructor herself, and the greatest woman I have ever known.

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AUTOBIOGRAPHICAL SKETCH OF AUTHOR

Paul Belue was born on April 15, 1969, in Missoula, Montana. He graduated from Brigham Young University with a bachelor of science in mathematics in 1993 and from the University of Utah with a bachelor of science and masters in architecture in 1995 and 1997 respectively. He worked in architecture for approximately 6 years and earned his teacher's certificate at Montana State University – Billings in 2006. He taught developmental mathematics at Montana State University – Billings for two years and has been teaching mathematics at the College of Western Idaho in Boise since January of 2010.

ABSTRACT

A control group and an experimental group of college students at a community college in the Pacific Northwest were taught a unit on linear equations in two variables. The control group was taught using a traditional instructional approach that focused on learning procedures and the experimental group was taught using a quantitative reasoning instructional approach that focused on learning proportional and functional reasoning. Both groups were then given the same unit assessment that had 10 procedural understanding items and 10 conceptual understanding items related to linear equations in two variables. The assessment was given to determine the impact of the quantitative reasoning instructional approach on performance of linear equations in two variables tasks. A quantitative and qualitative analysis of items on the unit assessment found that in relation to the control group, the experimental students demonstrated better performance in proportional reasoning ability, and had no difference in either functional reasoning ability or performance on procedural tasks. The experimental students received more instruction focused on conceptual understanding compared to the control group without sacrificing their procedural understanding. Analysis also revealed that community college students enrolled in beginning algebra can have misconceptions about what constitutes a linear relationship and the meaning of a variable quantity.

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CHAPTER ONE: INTRODUCTION

In my work as a community college instructor, I have observed my students doing well with solving linear equations with one variable but struggling to understand the concepts associated with linear equations in two variables. In fact, after teaching a unit on linear equations in two variables to 94 Beginning Algebra students during Spring semester of 2014, only 18% of the students correctly answered all six problems involving linear equations in two variables, and only 44% correctly answered four or more of the six problems (internal consistency reliability for the six items was 0.8583).

Solving linear equations in one variable only requires that students follow a step-by-step procedure for solving an equation where the variable usually represents one particular unknown value. With linear equations in two variables, there are an infinite number of solutions, each solution is composed of two values that are related, and the solutions can be represented in tables and graphs. This requires students to attend to and coordinate two values, understand how the two values are related, and work between tables, graphs, and equations. I observed that my students struggled to accomplish these tasks, making it difficult for them to transition from linear equations in one variable to linear equations in two variables.

I realized my students’ difficulty with making this transition was preventing them from being successful in my course and would make it difficult for them to be successful working with functions and the concepts of calculus in the future. As a result, I reflected on how I had been teaching linear equations in two variables and searched for a more effective way to help students make the transition and understand the concepts.

As I reflected upon my instructional approach and researched other instructional approaches, I first discovered my instructional approach was mainly what Baroody (2003) calls a skills approach to teaching mathematics. This approach is based on the behavioristic learning theory that students are empty vessels needing to be filled with the knowledge and procedures of mathematics. In a skills approach, instruction focuses on students memorizing rote procedures for doing particular problems and practicing these procedures until they are mastered.

Second, I realized my instruction was focused on the manipulation of symbols with very little connection to the real world. Of the 21 objectives I taught in my unit on linear equations in two variables only four of the objectives were focused on doing problems that related to the real world (See Appendix A & B, and Trigsted, Bodden, & Gallaher, 2014). I found that researchers believe instruction should move beyond a focus on symbol manipulation and include a focus on reasoning with patterns, quantities, and real world situations (Ellis, 2007a).

Third, I learned that Baroody (2003) identifies three other categories of instructional approaches in addition to the skills approach: a conceptual approach, a problem-solving approach, and an investigative approach. In a conceptual approach, students learn procedures and practice these procedures but they also learn why the procedures work and explore different ways to do problems. In a problem-solving approach, students are immersed in trying to solve real problems through their own inquiry efforts. The instructor is a participant in that inquiry and guides the inquiry process. The problem-solving approach is based on the constructivist learning theory which recognizes students' prior knowledge of mathematical concepts and builds on that knowledge and supports students in constructing their own understandings. The investigative approach is also based on constructivism but more moderately. It involves real world problems but the teacher provides more scaffolding as students construct understandings. Research shows problem-solving and investigative constructivist teaching approaches have positive effects on student achievement (Ross & Wilson 2012; Hickey, Moore, & Pellegrino, 2001).

Fourth, I was intrigued by a particular constructivist instructional approach, advocated by Thompson (1994b), where students develop mathematical concepts through quantitative reasoning. In a quantitative reasoning approach, instruction always starts with a real world situation followed by the identification of relevant quantities within the situation. The instruction then proceeds by exploring relationships between these quantities to understand concepts and answer questions about the situation. The approach was developed in reaction to mathematics instruction focused only on number, shape, and relationship with no connection to the real world (Thompson, 1994b). One study on a quantitative reasoning approach to teaching linear equations in two variables showed it supported students making generalizations about relationships, connections between situations, and dynamic phenomena (Ellis, 2007a).

Fifth, I learned about two specific types of quantitative reasoning the development of which can help students understand linear equations in two variables: covariational reasoning and proportional reasoning. Covariational reasoning is "the cognitive activities involved in coordinating two varying quantities while attending to the ways in which they change in relation to each other" (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). It is important for understanding linear equations in two variables because linear equations in two variables involve two quantities that vary in tandem and the way these two quantities change in relation to each other (their rate of change is constant) is the key to understanding linear relationships. Proportional reasoning involves forming ratios, producing ratios that are equivalent, and solving problems that involve ratios (Lobato, Ellis, & Charles, 2010). It is important for understanding linear equations in two variables because linear equations in two variables involve an invariant ratio called the slope and solutions that are related to this slope.

My reflection and research caused me to wonder: Could a quantitative reasoning instructional approach with a focus on developing covariational and proportional reasoning help my students better understand the concepts associated with linear equations in two variables?

To answer this question, I conducted a teaching experiment where I taught linear equations in two variables using a quantitative reasoning instructional approach with a focus on developing covariational and proportional reasoning. I then examined the effects of this approach on student performance on tasks related to linear equations in two variables. I also examined the effects of this approach on students' thinking about linearity.

The next chapter explains quantitative reasoning and summarizes research on quantitative reasoning, covariational reasoning, and proportional reasoning.

CHAPTER TWO: LITERATURE REVIEW

In this chapter, I first explain quantitative reasoning and why quantitative reasoning is important. I then summarize some of the important results of research on quantitative reasoning. The last two sections of the chapter are a review of the relevant research on covariational and proportional reasoning as it relates to understanding linear equations in two variables.

Quantitative Reasoning

Quantitative reasoning is reasoning with quantities and their relationships (Ellis, 2009). To fully understand quantitative reasoning we must first clearly define a quantity. A quantity is a conception in one's mind of a measurable attribute of an object or event (Thompson, 1994a). It is composed of four parts: an object or event, an attribute of the object or event, an appropriate unit or dimension, and a process for assigning a numerical value to the attribute. For example, a quantity associated with a rectangular floor is its width. The object is the floor, the attribute is the linear extent of the width, an appropriate unit is feet, and the process for assigning a numerical value is using a tape measure. Other quantities associated with a rectangular floor that can be constructed in one's mind are its length, perimeter, and area. A quantity has no numerical value, only the conception in one's mind of a process for assigning a numerical value.

Quantities are either directly measured (e.g., the width and length associated with a rectangular floor) or indirectly measured as the result of operating on quantities to produce new quantities (e.g., the perimeter and area associated with a rectangular floor).

There are four quantitative operations involved in constituting new quantities: combining quantities additively, comparing quantities additively, combining quantities multiplicatively, and comparing quantities multiplicatively (Thompson, 1994a; Thompson, 2011).

A rectangular floor provides examples of the four quantitative operations:

1. The perimeter of a rectangular floor is an example of a quantity that is the result of combining quantities additively. It involves thinking about combining the two widths and two lengths of the floor.
2. How much greater the length of a rectangular floor is than the width of a rectangular floor is an example of a quantity that is the result of comparing quantities additively. It is thinking about the excess of the length of the floor compared to the width of the floor.
3. The area of a rectangular floor is an example of a quantity that can be the result of combining quantities multiplicatively. It is thinking about how to combine the length and width to produce a region.
4. How many times greater the length of a rectangular floor is than the width of a rectangular floor is an example of a quantity that is the result of comparing quantities multiplicatively. It is thinking about the ratio of the length to the width.

Quantitative operations and their results are non-numerical and only have to do with the comprehension of the situation: i.e., comprehending how the new quantity relates to the quantities used to create it (Thompson, 1994a). However, each quantitative operation has an associated numerical operation used to evaluate or assign a value to the new quantity.

For example, the quantitative operation of comparing additively the length and the width of a rectangular floor is only the comprehension in one's mind of the quantity (which has no numerical value) that is the excess of the length of the floor compared to the width of the floor. The numerical operation used to evaluate this quantity is subtraction, taking the value (number) of the width and subtracting it from the value (number) of the length. The numerical operations used to evaluate the four quantitative operations are: addition for an additive combination, subtraction for an additive comparison, multiplication for a multiplicative combination, and division for a multiplicative comparison (Thompson, 2011).

Hence quantitative reasoning involves using quantitative operations to measure indirectly attributes of objects or phenomenon not easily measured directly. Quantitative reasoning activities return mathematics to the reason why it was invented: to measure that which cannot be easily measured directly in the real world. Auguste Comte wrote,

> In the light of previous experience, we must acknowledge the impossibility of determining, by direct measurement, most of the heights and distances we should like to know. It is this general fact which makes the science of mathematics necessary. For in renouncing the hope, in almost every case, of measuring great heights or distances directly, the human mind has had to attempt to determine them indirectly, and it is thus that philosophers were led to invent mathematics. (Serres, 1982, p. 85)

When students see the need for mathematics to indirectly measure quantities in real world situations, they are more likely to learn and understand the mathematics (Harel, 2001).

Research on Quantitative Reasoning

Research on a quantitative reasoning instructional approach has revealed many benefits and effective teaching strategies of the approach. Benefits include motivation to search for relationships, motivation to reason covarationally, and the ability to generalize correctly. Teaching strategies include exploring with students how changing quantities affect an emergent quantity, giving students time to reflect on a problem's context, and helping students identify a quantity by the name of the measurable attribute. This section describes in more detail the benefits and teaching strategies that research on quantitative reasoning has unveiled.

An important research study involving quantitative reasoning was conducted by Ellis (2007a) examining the generalizations of middle school students taught linear functions in two different learning and teaching environments. In the first environment, students were taught linear functions using an approach that emphasized number pattern finding from tables of linear data, procedures, skills, and direct computational questions and responses. In the second environment, students were taught linear functions using an approach that emphasized quantitative reasoning: i.e. exploring quantities and their relationships in two real world situations about linear growth.

Seven students from each environment were observed in the classroom and through interviews and their generalizing actions and reflection generalizations were counted and placed into the categories of Ellis' generalization taxonomy (Ellis, 2007b). The most salient results are summarized in Table 1 that follows.

Table 1 Counts of Students' Generalizations in Two Instructional Approaches

Students' Generalizations		Counts of Seven Number Pattern Approach Students	Counts of Seven Quantitative Reasoning Approach Students
Generalizing Actions	Searching for relationships	7	48
	Searching for patterns	21	15
Reflection Generalizations	Statements of continuing phenomena	3	50
	Statements of global rules	41	26

Searching for relationships meant the students performed a repeated action in order to detect a stable relationship between two or more objects (Ellis, 2007b). Searching for patterns meant the students checked to see if a detected pattern remained stable across all cases. Statements of continuing phenomena were statements made by the students identifying a dynamic property extending beyond a specific instance like "every time x goes up 1, y goes up 5." A statement of global rules is a statement of the meaning of an object or idea such as "If the rate of change stays the same, the data are linear" (Ellis, 2007b).

Although the *number* of global rules statements made by the number-pattern students was more than the number made by the quantitative reasoning students, analysis of the interviews of the students showed that the *correctness* of the global rules statements favored the quantitative reasoning students (Ellis, 2009). The number pattern students incorrectly generalized that a table of data representing a situation is linear only if the numbers down a column in the table increased or decreased by the same amount between rows (See Uniform Table in Figure 1 below). They made this incorrect statement mainly because all the tables they examined had this uniform quality. When they encountered a non-uniform $y = mx$ or $y = mx + b$ table that represented a linear situation, they struggled and declared it nonlinear (See Non-Uniform Table in Figure 1 below).

x	**y**
0	5
1	12
2	19
3	26

Uniform Table

S	L
7 ½	5
27	18
4 ½	3
16	10 2/3

Non-Uniform Table

Figure 1 Uniform & non-uniform table

The equation that describes the relationship between *x* and *y* for the real world situation represented by the Uniform Table above is $y = 7x + 5$. The table's linearity and equation is readily apparent from the table by first observing that the values in the y column increase by 7 down the column in tandem with the values in the x column increasing by 1. This means the slope is constant and its value is 7/1 = 7. Second, we observe that the *y*-intercept is 5 since the first row tells us *y* is 5 when *x* is 0. We can easily see that replacing *m* with the slope 7 and *b* with the *y*-intercept 5 in the equation $y = mx + b$ gives us the equation $y = 7x + 5$. The equation that describes the relationship between *S* and *L* for the real world situation represented by the Non-Uniform Table above is $L = (2/3)S$. This table's linearity and equation is not so readily apparent from the table because it does not fit the form of the Uniform Table and we therefore cannot use the method we used for the Uniform Table to determine its linearity, slope, and y-intercept. It does not fit because the values down a column of the Non-Uniform Table are not increasing or decreasing and even if the rows are rearranged so that values down a column are increasing or decreasing, the values down a column do not increase or decrease by the same amount; e.g. 4 ½, 7 ½, 16, 27, is an increase of 3, 8 ½, and 11. But does this mean the relationship represented by the Non-Uniform Table is not linear? No. The test for linearity of the Non-Uniform Table (and any table for that matter) is: are all quotients of differences of L values and corresponding differences of S values constant or can the relationship between the L value and S value for every row be expressed by the equation $L = mS + b$ for some constant m and b? The answer to this question is easy for the Uniform Table because its form makes it simple to see that all of the quotients have a constant value. With the Non-Uniform table it is not so easy to see each quotient must be calculated individually to determine if all quotients are constant. Furthermore, once it is determined that the table is linear, finding the y-intercept requires more work than just looking at a row in the table like we did for the Uniform table. In the study by Ellis (2009), the quantitative reasoning students seemed to better understand this test of linearity and were able to extend this understanding to both uniform and non-uniform $y = mx$ or $y = mx + b$ tables that represented real world situations.

The understanding of linearity the quantitative reasoning students were able to obtain makes sense considering the greater number of statements of continuing phenomena made by the quantitative reasoning students. A continuing phenomena statement involves the co-varying nature of the quantities (Ellis, 2007b). Covariational reasoning is critical to understanding rate of change and a constant rate of change is what makes a relationship linear (Jacobson, 2014; Carlson et al., 2002).

Furthermore, when students focused on quantities in real world contexts instead of numbers in tables, they searched for a new relationship between the direct measure quantities, experienced the necessity to create a ratio as a measure of the new relationship, and understood the ratio remains constant even when the direct measure quantities changed. Ellis (2007a) named reasoning leading to the construction of a ratio as emergent-ratio quantitative reasoning.

In summary, the research by Ellis (2007a) revealed three important things about a quantitative reasoning approach to teaching linear equations in two variables:

1. A quantitative reasoning approach motivates students to search for relationships among quantities.
2. A quantitative reasoning approach motivates students to think covariationally about quantities involved in a situation.
3. A quantitative reasoning approach contributes to students making correct global rules about what constitutes linearity.

Ellis (2007a) also points out that simply placing students in a quantitatively rich situation does not necessarily guarantee students will develop correct understandings of the concepts. Instead, the type of quantitative reasoning students engage in is what results in correct understandings. To insure the correct type of quantitative reasoning, Ellis (2007a) suggested students engage in activities such as:

1. Exploring how changing one or both initial quantities will affect the emergent quantity.
2. Determining how to adjust the initial quantities while keeping the emergent quantity constant.

3. Determining how to double, halve, or otherwise manipulate the emergent quantity in relationship to the initial quantity.

Other studies have also shed light on a quantitative reasoning instructional approach. Lobato and Thanheiser (2002) found that in identifying a quantity in a situation, students struggle to isolate an attribute from other attributes found in a situation. For example, in reasoning quantitatively about a ramp, students confused the attribute of steepness with the attribute of work required to climb the ramp. Furthermore, students struggle to determine which quantities affect an attribute and how. Again in the ramp situation, students thought changing the length of the top horizontal platform, extending the end of the horizontal platform not connected to the top of the ramp, would affect the steepness of the ramp (see Figure 2 below).

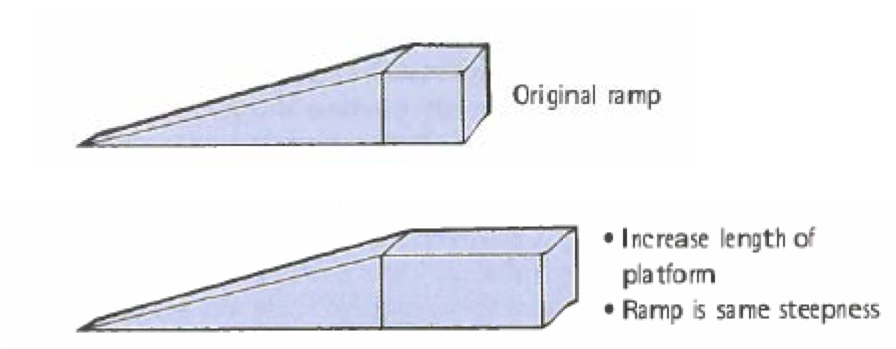

Figure 2 The effect on the steepness of a ramp when changing length of platform

Students also need to be clear and specific on what is being measured in order to eventually write meaningful expressions and formulas (Carlson, Oehrtman, & Moore, 2013).

Research by Moore and Carlson (2012) focused on students' mental images of a problem's context when engaged in quantitative reasoning. They discovered the following:

1. A student's mental image of a problem's context will either help or hinder the student's ability to come up with a correct graph or equation.
2. Different and underdeveloped images of the relative quantitative relationships of a problem possibly lead to the inability to reason covariationally and construct the correct graph or equation.
3. Helpful mental actions in solving problems are imagining the situation with diagrams, mentally playing out variations in quantities, and using specific values to explore relationships between quantities.
4. Students should be given time to develop the correct image or quantitative structure of a problem's context before attempting to provide a correct formula or graph.
5. Teachers should attend to the emergent nature of a student's image by prompting them to first reason about the quantities and how they change together and are related.
6. Reflecting and focusing on a problem's context can help check or correct a solution to the problem.

Research on quantitative reasoning by Thompson (1988) indirectly confirmed that students deficient in quantitative reasoning are poor at algebraic problem solving and identified three main obstacles in reasoning quantitatively:

1. Students name quantities by their numerical value only instead of by the name of the measurable quality of the object, e.g. "100 dollars" instead of "Paul's savings." They do not distinguish between the quantity and its measure. This

leads to difficulty identifying quantities that have an unknown value and deciding which operation to perform when computing a third quantity.

2. Students struggle to identify multiplicative quantities like products, ratios, and rates.
3. Students struggle to know when quantities can or cannot be combined or compared.

Another important part of research related to quantitative reasoning and linear equations in two variables was research done on what are called ratio-as-measure tasks. A ratio-as-measure task is a task requiring students to identify a ratio as an appropriate measure of a given attribute and to create that ratio (Simon & Blume, 1994). Such tasks are relevant to linear equations in two variables because the slope of linear equations in two variables is a ratio and a ratio-as-measure task can be beneficial in helping understand slope.

In a ratio-as-measure task, the ratio is not given at the outset and students must recognize the need for a ratio and create the ratio themselves. The result is a deeper understanding of ratio and for purposes of this study a deeper understanding of slope. Simon and Blume (1994) wrote,

> When a ratio is not given at the outset, as is typically done, but rather is developed by the classroom mathematics community as a useful mathematization of the situation, greater understanding of mathematization and of ratio are likely to result. The perspective of developer and evaluator of mathematical models is certainly going to be broader than the perspective of someone who is only the consumer of models of others.

When students are given a ratio as a measure of a given attribute (consumer of models of others) they do not see a need for a ratio to measure a real world attribute and why the ratio is appropriate. As Harel's (2001) Necessity Principle asserts, students are most likely to learn if they see an intellectual need for what we intend to teach them.

Several studies have been conducted involving ratio-as-measure tasks (Simon & Blume, 1994; Lobato & Thanheiser, 2002; Smith et al., 2013). These studies have been qualitative in nature and have shown students to have difficulty coming up with a ratio to measure a given attribute, isolating the attribute being measured, determining which quantities affect the attribute and how, and understanding the characteristics of a measure.

In teaching using a quantitative reasoning approach, understanding slope requires careful attention to how slope is defined. Many textbooks define slope as the measure of the steepness of a line (Lial, Hornsby, & McGinnis, 2012). Tuescher and Reys (2010) point out that this definition can be misleading because steepness is usually the attribute of a physical object that has no sign while slope can have a sign because it is usually referencing a graph in the coordinate plane. For example, a mountainside that is rising from left to right and a mountainside that is falling from left to right with the same ratio of height to length have the same steepness. We usually do not think of the steepness of the rising mountainside being positive and the falling mountainside being negative. But a line in the coordinate plane that is rising from left to right and a line in the coordinate plane falling from left to right with the same ratio of height to length, have different slopes. The first slope is positive and the second slope is negative. Therefore, we have to be cautious in applying the term steepness to the graph of a line. The term "slope" is more appropriate in discussing the graph of a line and a better definition of slope in this context is the change of one quantity relative to the change of another quantity, where the two quantities covary (Lobato & Thanheiser, 2002).

Covariational Reasoning

Jacobson (2014) asserts there are two types of quantitative reasoning relevant for beginning algebra students learning the concepts associated with linear equations in two variables:

1. *Covariational reasoning,* which involves analyzing how two quantities vary in tandem or simultaneously: How does one quantity change as another quantity changes?
2. *Correspondence reasoning,* which involves identifying a correlation between two quantities: How is one quantity related to another? Correspondence reasoning is part of proportional reasoning and is further discussed in the section on proportional reasoning.

Jacobson (2014) points out that covariational reasoning is essential to understanding the rate of change of a situation and correspondence reasoning is essential to understanding function rules or equations that describe how two quantities are related. He also points out that covariational reasoning is examining relationships *between* rows of data in a table and correspondence reasoning is examining relationships *within* a row of data in a table. He then goes on to give the following important strategies for supporting students' covariational reasoning (Jacobson, 2014):

1. Use a sequence of specific pairs of values to support students' reasoning about the problem.

2. Ask students to describe and explain their thinking about a single pair of values and to compare different pairs of values. Listen carefully for covariational or correspondence reasoning.
3. If students make incorrect claims, ask for students' ideas. Model quantitative reasoning by providing a mathematical reason or counterexample based in the problem situation that explains why the claim is incorrect.
4. Ask students the following kinds of questions about problem situations (adapted from Carlson & Oehrtman, 2005):
    a. What quantities are changing together, and how are they changing?
    b. As one quantity increases, does the other quantity increase or decrease?
    c. As one quantity increases in constant increments, by what amount does the other quantity change?
    d. As one quantity increases in constant increments, what is the rate of change in the other quantity?

These suggestions based on research on covariational reasoning are important to use in a quantitative reasoning instructional approach in order to help students develop an understanding of the rate of change of a linear equations in two variables.

Proportional Reasoning

Comparing quantities multiplicatively to form a ratio and thinking about the ratio's equivalent forms is a special type of quantitative reasoning called proportional reasoning. Proportional reasoning is an important aspect of understanding linear equations in two variables and critical to success in higher mathematics. In this section, I describe proportional reasoning, some of the challenges students face in reasoning proportionally, and how proportional reasoning relates to linear equations in two variables.

A proportion is a statement of equality between two ratios; two quantities are proportional when the ratio of the quantities remains constant as the corresponding values of the two quantities change (Lobato et al., 2010). For example, if a person walks 4 feet in 2 seconds, we can form the ratio 4 feet in 2 seconds and make the statement (a proportion) that 4 feet in 2 seconds is in the same ratio as 2 feet in 1 second (4/2 = 2/1). In this situation, we say the distance (in feet) is proportional to the time (in seconds).

Knowing how to reason proportionally requires understanding ratios. A ratio is a multiplicative comparison of two or more quantities or the joining of the quantities in a composed unit (Lobato et al., 2010). Ratios are formed to tell us how many times larger or how many times smaller a quantity is than another (multiplicative comparison), or the intensity of a quantity compared to another (composed unit). For example, with a multiplicative comparison, if we have a rectangle with a width of 4 feet and a length of 6 feet, forming the ratio 4 feet to 6 feet tells us the width of the rectangle is 2/3 the length (since 4/6 = 2/3) and the length of the rectangle is 1 1/2 times the width (since 6/4 = 3/2). With a composed unit, if a swimming pool is being filled at a rate of 3 gallons every 6 seconds, forming the ratio 3 gallons to 6 seconds tells us the swimming pool is being filled at an intensity of 0.5 gallons every second (since 3/6 = 0.5) or 1 gallon every 2 seconds (since 6/3 = 2).

Knowing how to reason proportionally also requires recognizing that there are an infinite number of ratios that tell us the same thing as a given ratio and knowing how to produce these equivalent ratios. In the rectangle example above, other ratios that tell us the same thing as the ratio 4 to 6 are the ratios 8 to 12 and 2 to 3. They are produced by iterating or repeating 4 to 6 to get, for example, 8 to 12 or 12 to 18, or by partitioning or splitting 4 to 6 to get, for example, 2 to 3 or 1 to 1.5. They can also be produced by combining iterating and partitioning. For example iterate (triple) 4 to 6 to get the ratio 12 to 18 and then partition (half) this ratio to get the ratio 6 to 9. A fourth way to produce equivalent ratios is to find the unit ratios of a given ratio, which tells us how many times greater one quantity is than another, and use these ratios to produce equivalent ratios. For example the unit ratios for 4 to 6 in the rectangle example are 2/3 to 1 and 1 to 1.5. These tell us that the width is 2/3 of the length and the length is 1.5 times the width. We can then pick any length and multiply it by 2/3 or pick any width and multiply it by 1.5 to form an equivalent ratio. For example, let the width be 14, multiply 14 by 1.5 to get 21, and the equivalent ratio is 14 to 21 (Lobato et al., 2010, p. 10-11). Finally, an equivalent ratio can also be produced by multiplying or dividing both quantities of a ratio by the same factor (Lobato et al., 2010).

Becoming proficient in proportional reasoning requires understanding ratio and being able to produce equivalent ratios to solve problems found in different situations. The best way to acquire this skill is to experience many different types of problems that require proportional thinking. Lobato et al. (2010) recommend five types of problems: comparison problems, transformation problems, mean value problems, part-part-whole and containment problems, and geometric similarity and scaling problems.

Challenges Related to Understanding Ratio

The first challenge students may face in understanding ratio is inappropriately applying an additive reasoning strategy to a proportional situation. This may be a result of textbooks and teachers defining a ratio as a comparison of two quantities without specifying that the comparison is multiplicative opposed to additive (Lobato et al., 2010, p. 17, 21). For example, suppose a student is asked how many gallons of red paint should be mixed with 6 gallons of yellow paint to produce a color of orange if the ratio of red to yellow is 3 to 2. If the student is thinking that the ratio 3 to 2 tells us how the red and yellow compare additively (how much more red there is than yellow) they will think that the red is one unit more than the yellow unit (3-2), probably add one to 6, and give an incorrect answer of 7 gallons. But if they are thinking that the ratio 3 to 2 tells us how the red and yellow compare multiplicatively (how many times more the red is than the yellow) they will think that the red is 1.5 times more than the yellow, hopefully multiply 6 by 1.5, and give a correct answer of 9 gallons. Therefore, it is important that textbooks and teachers clearly explain that a ratio and tasks that involve applying the same ratio to different situations involves a multiplicative comparison of the two quantities not an additive comparison. This could be done by giving students examples of a multiplicative comparison and examples that are not a multiplicative comparison.

The second challenge students may face in understanding ratio is thinking that forming a ratio is just writing a ratio. But forming a ratio is a cognitive task not a writing task (Lobato et al., 2010, p. 22). It is thinking that the two quantities compare multiplicatively, i.e. 4 compares multiplicatively to 6, or the first quantity is 2/3 of the second quantity.

Furthermore, students may also struggle with using two numbers to measure an attribute instead of one, using ratios to compare attributes instead of using whole numbers, and knowing whether changing a quantity in a situation affects a ratio or not.

For example, research shows that students struggled to know whether changing the length of the platform at the end of a ramp affected the slope of the ramp or not (Lobato et al., 2010).

Challenges Related to Understanding Algebraic Statements of Proportion

The first challenge students may face in understanding proportion is thinking a proportion is setting two ratios equal to each other where one term is missing and cross multiplying to find the missing term (the algorithm). This is how you can find the missing term of a proportion but it does not demonstrate an understanding of what a proportion is and what proportional means.

In addition, students may have difficulty conceiving that something may remain the same while the values of the two quantities change. In the proportional statement 1/2 = 4/8, although the corresponding quantities of the ratios have changed, the equal sign means 1/2 is in the same ratio as 4/8: i.e., 1 and 2 compares multiplicatively the same as 4 and 8 (in both cases the second number is two times the first). The statement also means that although the corresponding quantities have changed, the equal sign means 1/2 and 4/8 as quotients are the same, namely 0.5.

Challenges Related to Recognizing Ratio Relationships

The first challenge students may face in recognizing ratio relationships is difficulty isolating attributes in a situation that a ratio measures from other attributes (Lobato et al., 2010, p. 23). For example, Lobato et al., explain how students confused the steepness of a ramp with the work required to climb the ramp. These are two different attributes, and the work required to climb the ramp depends on the length of the ramp while the steepness does not.

The second challenge students may face in recognizing ratio relationships is being misled by superficial cues in the context of a problem that seem to indicate there is a proportional relationship between quantities when there is not (Lobato et al., 2010). Examples of these cues are:

1. The problem has three numbers and one is missing.
2. The problem uses words like per, rate, or speed.
3. The problem appears in a chapter on ratio and proportions.

<u>Linear Equations in Two Variables and Proportional Reasoning</u>

One of the simplest linear equations in two variables is an equation of the form $y = mx$ where m is an invariant ratio that gives the slope of the line when graphed in the coordinate plane. If you solve the equation for m you get the equation $\frac{y}{x} = m$. The solutions to this equation are all ordered pairs (x,y) such that their ratios $\frac{y}{x}$ are equivalent to m. Therefore, to find all the ordered pair solutions of this equation, you must find all the equivalent ratios of the ratio m. Proportional reasoning provides methods for finding ratios equivalent to m (e.g., by iterating and partitioning m) and provides an explanation for how the ratios relate to each other and the slope m. In summary, proportional reasoning contributes the following to working with the linear equation $y = mx$:

1. Understanding a way to find the ordered pair solutions of the linear equation through reasoning and not just substituting a value for x and finding the corresponding y.
2. Understanding how the ordered pair solutions of the linear equation relate to each other and the slope m (each ordered pair forms a ratio that is equivalent to m).

3. Understanding why the linear equation has solutions that form equivalent ratios. The equation $y = mx$ is a statement that the slope is the ratio of y to x.

For the linear equation in two variables of the form $y = mx + b$, solving for m gives $\frac{y-b}{x} = m$. The ordered pair solutions of this equation with b subtracted from the y-coordinates form ratios that are equivalent to m. Proportional reasoning still provides a method for finding ordered pair solutions (iterate or partition m and add b to the numerator) and understanding how the ratios relate to each other and the slope m.

If we start with a real world task that involves a ratio and finding equivalent ratios, we can construct a linear equation in two variables for the situation using the multiplicative comparison interpretation of the ratio. For example, suppose we are given the height of a ramp as 2 feet, the length of its base as 5 feet, and the problem involves constructing ramps with the same slope. The slope of the ramp is the ratio of the height to the length of the base or 2 to 5. Thinking about this ratio as a multiplicative comparison, 2 to 5 means that the height is 2/5 of the length of the base or the length of the base is 2.5 times the height. These two expressions can be translated to equations (by correspondence reasoning) with y representing height and x representing the length of the base, as $y = (2/5)x$ or $x = 2.5(y)$. Their ordered pair solutions are the terms of all ratios equivalent to 2 to 5.

Using proportional reasoning in a real world situation can also lead to seeing the straight-line pattern of the graph of the two quantities in a proportional relationship. Using the same example of a ramp with height 2 feet and length of base 5 feet, if we iterate the ratio 2 to 5 we get ratios such as 4 to 10, 6 to 15, and 8 to 20. Partitioning gives us ratios like 1 to 2.5 and 0.5 to 1.25. If we put these ratios in a table and then plot the ordered pairs in the coordinate plane with axes of height and length of base, we see that the ordered pairs form a straight line in the plane. The graph helps us see that there are an infinite number of ordered pairs that can be produced by iterating and partitioning that represent height and length of base with the same ratio as 2 to 5.

In conclusion, the research on quantitative reasoning has revealed the benefits of a quantitative reasoning approach and many tasks and strategies for effectively teaching with a quantitative reasoning approach. What is lacking is an experimental research study examining the effects of such tasks and strategies on student performance on linear equations in two variables tasks. This thesis attempts to address this need.

CHAPTER THREE: METHODOLOGY

In this chapter, I first describe the research design, participants, and setting of the study. This is followed by an explanation of the instructional approaches and unit assessment used in the study. I then discuss how the quantitative and qualitative data for the study were collected and analyzed. The research question for the study was: What is the impact of a quantitative reasoning instructional approach to teaching linear equations in two variables on student performance on linear equations in two variables tasks and students' thinking about linearity?

Research Design

The design of this study is a mixed method of both quantitative and qualitative research. A control group was taught a unit on linear equations in two variables using a traditional, skills-based instructional approach (See Appendix A for outline of objectives and Appendix B for day 1 lesson plan) and an experimental group was taught a unit on linear equations in two variables using mainly a quantitative reasoning approach (See Appendix C for outline of objectives, Appendix D for day 1 lesson plan, and Appendix E for day 2 & 3 outline). The lesson plans for the control group were taken from Trigsted et al. (2014) and the lesson plans for the experimental group were adapted from the Connected Mathematics 2 curriculum and Pathways Precalculus curriculum (discussed below).

The instructional approach to teaching linear equations in two variables was the independent variable and the dependent variable was performance on linear equations in two variables tasks. Performance on linear equations in two variables tasks was measured on a unit assessment given immediately after instruction and taken by the students of both groups. The unit assessment consisted of 20 items. The first ten items were designed to assess students' procedural understanding of linear equations in two variables and the other ten items were designed to assess students' conceptual understanding of linear equations in two variables (See Appendix F for the 20 assessment items).

The results of the test for the two groups were analyzed and compared to determine the effects of the intervention (quantitative component) and also identify any themes in student thinking about linearity (qualitative component). See Data Collection and Analysis below for more detail.

Participants and Setting

The participants in the study for both the control and experimental group were enrolled in Beginning Algebra at a community college in the Pacific Northwest. Beginning Algebra is a 3-credit developmental math course that has five main units: linear equations in one variable, linear equations in two variables, exponents and polynomials, factoring polynomials, and rational expressions. The research for this study focused on the unit for linear equations in two variables. The linear equations in two variables unit for both the control group and the experimental group consisted of five 75 minute long class periods taught by the same instructor using the two different instructional approaches.

Participants in the control group were enrolled in Beginning Algebra during the Fall of 2014. Participants in the experimental group were enrolled in Beginning Algebra during the Spring of 2015. The participants for both groups were placed in Beginning

Algebra based on their score on a placement test or by earning a C or better in a Prealgebra course that was a pre-requisite to Beginning Algebra. The control group had 34 participants. The experimental group had 37 participants.

Instructional Approaches

A traditional instructional approach was used with the control group. During each class session, the teacher began the lesson by explaining a concept, doing an example problem on the board, and then allowing time for the students to practice a problem in class. The focus of instruction was on students learning the correct procedures so that they could correctly execute them on similar problems. Students were also given homework to do for added practice in course software (See Appendix A for outline of objectives and Appendix B for day 1 lesson plan).

The control group received instruction and did homework problems that were based on the book Developmental Mathematics by Trigsted et al. (2014). This curriculum focuses on learning the procedures for doing particular types of problems and provides examples in the book on how to do the problems. This curriculum and instructional approach was used on all five days of instruction for the control group and on one day for the experimental group.

A quantitative reasoning instructional approach was used with the experimental group on four out of the five days of instruction of the unit on linear equations in two variables. The quantitative reasoning approach involved students participating in whole class discussions about investigations of real world problems and completing worksheets about the investigations in small groups on four of the five class sessions. The worksheets included exercises intended to help students develop their proportional, covariational, and correspondence reasoning ability. The worksheets were adapted from the Connected Mathematics 2 unit *Moving Straight Ahead* and the Pathways Precalculus curriculum (discussed below). Students in the experimental group also did some procedural based homework in the course software (See Appendix C for outline of learning objectives of quantitative reasoning instructional approach, Appendix D for day 1 lesson plan and worksheet, and Appendix E for day 2 & 3 worksheet outline).

Connected Mathematics 2 is a curriculum that uses a problem-based instructional approach in real world contexts by engaging students in investigations that are designed to build a conceptual understanding of the mathematics (Lappan, Fey, Fitzgerald, Friel, & Phillips, 2009). The *Moving Straight Ahead* unit contains investigations that are designed to help students build a deep understanding of linear equations in two variables (linear relationships). The unit starts by engaging students in finding patterns among tables of real world data that have a linear relationship and finding the rate of change from these data. Students then graph the data and see how the tables of data, graph, and an equation can be formed to represent the linear relationship. The unit then engages students in a ratio-as-measure task that involves creating a ratio to measure the slope of stairs as a means to build a conceptual understanding of slope. Parts of this curriculum were used on the first four days of instruction for the experimental group.

Pathways Precalculus is a research-based curriculum using a quantitative reasoning approach to solving word problems. The lesson plans and worksheets for day 1 and day 4 of the experimental instruction were adapted from part of Module 2, Investigation 1, on quantities and covariation of quantities (Carlson et al., 2013). This investigation is designed to help students identify and understand fixed and variable quantities in a situation. It also addresses understanding variables and how to use variables to create equations to model a situation.

Unit Assessment

The same unit assessment was given to the experimental and control groups and consisted of 10 procedural items and 10 conceptual items (See Appendix F). The 10 procedural items align with the skills based approach unit objectives as shown in the table found in Appendix A. The 10 conceptual items align with the quantitative reasoning approach unit objectives as shown in the table found in Appendix C.

The procedural items were designed to assess students' ability to perform a procedure in order to answer the question. For example, item 4 asked students to find the slope of the line containing the given points. This tested students' ability to either use the slope formula to find the slope or to graph the points and find the slope graphically. Both of these are step-by-step procedures the student must perform to correctly find the slope.

The conceptual items were designed to assess students' understanding of concepts like quantity and linearity. For example, items 11 and 12 asked students to identify quantities in a situation and know the difference between quantities that are fixed and quantities that vary. Item 15 was designed to assess students' understanding of what linearity is and how it is manifested in a table. In order to correctly answer this item, students have to use proportional and covariational reasoning. Item 18 asked students to write a linear equation for the situation, which assessed the students' correspondence reasoning ability.

I developed a rubric for the procedural items that guided the grading of each question. Students received full point value for providing the correct answer and showing a correct procedure in relation to the correct answer. Partial point values were awarded for minor mistakes in performing the procedures or for only doing part of the procedure as explained in the rubric (See Appendix G).

I developed a rubric for the conceptual items that guided the grading of each question. For these items, I required evidence of a correct understanding of important concepts associated with linearity for full point value. For example, the rubric for items 11 and 12 required that students demonstrate that they know the difference between a fixed and varying quantity and include the attribute, object, and units of a quantity when identifying a quantity. The rubric for item 15 required that students demonstrate correct proportional, covariational, and correspondence reasoning to correctly answer the question (See Appendix G).

Data Collection

The study qualified for exempt status through IRB (see Appendix H). Students in the control and experimental groups took the unit assessment in class immediately after the five days of instruction on the unit. They had 75 minutes to complete the assessment. For the quantitative data, the twenty questions were graded by assigning the point values for each question as prescribed in the unit assessment grading rubric (see Appendix G). The point values for each question for each student were then entered into a spreadsheet for the control group and the experimental group. These two spreadsheets of raw scores for each question were then used to perform the quantitative analysis (See Quantitative Analysis below). For the qualitative data, the tests of the students who agreed to participate were copied and the explanations in their responses to questions 15 & 17 were used to perform the qualitative analysis (See Qualitative Analysis below). These two items ask the student to explain how a table or graph may be used to identify a linear relationship.

Quantitative Analysis

Raw scores for each question were converted to a scaled score mean and an item by item analysis on the set of scaled score means was conducted to determine where students performed differently between the two groups, where students performed the same between the two groups, and where both groups seemed to have difficulty answering a question. Scaled scores were used because the total possible points a student could earn on individual assessment items varied. The set of scaled score means were then statistically tested using a two tailed t-test to determine if the two population means (one for the control group and one for the experimental group) were the same. They were then statistically tested using a one tailed t-test (if not the same) to determine if the mean of one population was greater than the other.

The two-tailed t-test was performed on three groups of items: the 10 procedural items, the 10 conceptual items, and the 10 procedural and 10 conceptual items combined. This was done to begin answering the research question of whether a quantitative reasoning instructional approach has an impact on student performance on linear equations in two variables tasks. It determined if the instructional approach resulted in a difference in performance on the tasks by the populations for the three groups of items. A statistical t-test confirming that the population means are the same meant there was no difference in performance on the tasks by the populations. A statistical t-test confirming that the population means are not the same meant there was a difference in performance on the tasks by the populations.

The one tailed t-test was performed on any group of items where it was concluded that the population means were not the same. This was done to further answer the research question of whether a quantitative reasoning instructional approach has an impact on linear equations in two variables tasks. It determined if the instructional approach resulted in a difference in performance on the tasks by the experimental group population that was greater than the performance by the control group population. The statistical test was done by testing whether the population mean for the control group was greater than or equal to the population mean for the experimental group. A statistical t-test confirming that the population mean for the control group was greater than or equal to the population mean for the experimental group meant the performance of the tasks by the experimental group population was not greater than the control group population. A statistical t-test confirming that the population mean for the control group was not greater than or equal to the population mean for the experimental group meant that the performance of the tasks by the experimental group population was greater than the control group population.

Qualitative Analysis

Analysis of the students' thinking about linearity was conducted by photocopying the unit assessment taken by the control group and experimental group and doing a qualitative analysis of the students' responses to items 15 and 17. I chose these items because they asked students to explain their thinking on how a table or a graph can be used to determine if a situation is or is not linear. This allowed me to infer the students' definition of linearity and the type of reasoning they employed to define linearity.

I started the qualitative analysis by reading each student's response and then placing the responses into categories of similar explanations. This resulted in a total of ten initial categories for item 15 and a total of eleven initial categories for item 17. These initial categories of similar explanations were then examined to determine how they related to covariational, proportional, and correspondence reasoning. Based on this analysis and the combining of initial categories that originally separated incorrect from correct answers, five final categories of student thinking about linearity emerged.

The first of the final categories emerged from the initial categories in which students explained that the table or graph showed the pool filled at a constant rate. Examples of this type of response were submitted by control student 10 who wrote "The table tells you it is a linear relationship by being consistent. For every 3 seconds, 1 gallon is added" and by experimental student 14 who wrote "It is linear b/c for every gallon added the seconds go up by 3, the rate is always the same." It was determined that these explanations used covariational reasoning (attending to how two quantities change together) and proportional reasoning (recognition that the rate at which the quantities were changing together was constant). Hence, the first final category is titled *Covariational and Proportional Reasoning.*

The second of the final categories emerged from the initial categories of explanations in which students explained that the table or graph showed there was a rule that related the two quantities. An example of this type of response was submitted by control student 8 who wrote "It is a constant growth that does not vary. The table shows this because no matter what you plug in for s, g will always be 1/3 of that." It was determined that these type of responses involved covariational reasoning (attending to how two quantities change together) and correspondence reasoning (how the quantities related within a row by a functional rule). Hence, the second final category is titled *Covariational and Functional Reasoning.*

The third of the final categories emerged from the initial categories in which students explained that the table or graph showed as one quantity changed the other quantity changed but with no recognition of a constant change. Examples of this type of response were submitted by control student 7 who wrote “The table shows it is linear because as the seconds increase, so too does the value of gallons” and by experimental student 7 who wrote “It is a linear relationship because as the time increases the number of gallons in the pool also increases.” It was determined that these types of responses involved a beginning level of covariational reasoning that did not explicitly attend to a constant rate of change (proportional reasoning). Hence, the third final category is titled *Beginning Covariational Reasoning.*

The fourth of the final categories emerged from the merging of two initial categories of explanations. One of these initial categories included responses in which students explained that the situation was linear because the table or graph was constant but with no explanation of what feature associated with the table or graph was constant. Examples of this type of response were submitted by control student 20 who wrote “It is a linear relationship because there are consistent measurements” and by experimental student 43 who wrote “it is a linear equation because is constant.” The other initial category included responses in which students explained that the situation was linear because the graph was a straight line. Examples of this type of response were submitted by control student 5 who wrote “It shows it is a linear relationship because the line is a straight line" and by experimental student 2 who wrote "it is a linear because if we graph the results its going to be a straight line." Because these two groups of explanations did not reference specific quantities in their explanations and did not show any evidence of covariational, proportional, or correspondence reasoning they were merged together. Hence, the fourth final category is titled *Non-Quantitative Reasoning*.

The fifth of the final categories include responses by students that were difficult to categorize in any of the first four categories. An example of this type response was submitted by control student 6 who wrote "Because it has numbers that equal." Hence, the fifth final category is titled *Other*.

The responses in these five categories were then analyzed and compared between the control group and the experimental group to identify themes in the students' thinking about linearity.

CHAPTER FOUR: RESULTS

In this chapter, I report on the results of both the quantitative and the qualitative data analysis. The quantitative data analysis consists of an item-by-item analysis and statistical analysis of the scaled score means for the control and experimental groups on the unit assessment. The qualitative data analysis consists of an explanation of the themes of the five categorizations of student responses to items 15 and 17 on the unit assessment and how responses by the control group vs. the experimental group on these items were distributed in the categories. Items 15 and 17 assess a student's ability to use a table or a graph to determine linearity.

Quantitative Data Analysis Results

This section starts with an item-by-item analysis of the scaled score means of the 20 items on the unit assessment to determine the following: where performance between the control and experimental group was different, where performance between the control and experimental group was the same, and where performance by the control and experimental group were both low. I then performed a statistical analysis of the scaled score means of different groups of items by performing a t-test to determine any differences in performance, and if any differences, where the differences were greater.

Item By Item Analysis

Table 2 contains the results for each of the 10 procedural items on the unit assessment, including the scaled score means and standard deviations for the control group, experimental group, and control and experimental group combined. For example, the second row of the table indicates that item 1 on the unit assessment (plotting an ordered pair on the coordinate plane) had a scaled score mean of 0.95 with standard deviation of 0.12 for the control group, a scaled score mean of 0.86 with standard deviation of 0.26 for the experimental group, and a scaled score mean of 0.90 with standard deviation of 0.21 for the control and experimental groups combined.

Table 2 Procedural Item Scaled Score Means and Standard Deviations

Item Number(Description)	**Control** Mean(SD)	**Exper.** Mean(SD)	**Comb.** Mean(SD)
1(Plot an ordered pair on the coordinate plane)	0.95(0.12)	0.86(0.26)	0.90(0.21)
2(Find unknown coordinate of equation)	0.90(0.20)	0.93(0.16)	0.92(0.18)
3(Determine if ordered pair is solution)	0.98(0.09)	0.99(0.04)	0.99(0.07)
4(Find the slope of a line through two points)	0.92(0.19)	0.79(0.32)	0.85(0.27)
5(Graph linear equation using intercepts)	0.73(0.32)	0.71(0.30)	0.72(0.31)
6(Find slope of line parallel and perpendicular)	0.91(0.25)	0.78(0.34)	0.84(0.31)
7(Determine slope and y-intercept of equation)	0.79(0.32)	0.71(0.33)	0.75(0.32)
8(Graph a horizontal line)	0.86(0.29)	0.77(0.35)	0.82(0.32)
9(Determine if parallel, perpendicular, or neither)	0.62(0.44)	0.56(0.42)	0.59(0.43)
10(Graph line given point and slope)	0.80(0.34)	0.65(0.38)	0.72(0.37)

The control group had at least a 10% greater scaled score mean than the experimental group on three of the 10 procedural items (4, 6, and 10). The greatest difference was a 15% greater scaled score mean on item 10. This item asked students to graph a line given a point on the line and the slope of the line. The other two items (4 and 6) also involved the concept of slope. They asked students to find the slope of a line parallel and perpendicular to a given line and find the slope of a line through two given points.

Table 3 shows the 10 conceptual items on the unit assessment and the scaled score mean and standard deviation for the control group, experimental group, and control and experimental group combined. For example, the second row of the table tells us that item 11 on the unit assessment (identifying fixed quantities in a situation) had a scaled score mean of 0.54 with standard deviation of 0.30 for the control group, a scaled score mean of 0.85 with standard deviation of 0.27 for the experimental group, and a scaled score mean of 0.70 with standard deviation of 0.32 for the control and experimental group combined.

Table 3 Conceptual Item Scaled Score Means and Standard Deviations

Item Number(Description)	**Control** Mean(SD)	**Exper.** Mean(SD)	**Comb.** Mean(SD)
11(Identify fixed quantities in a situation)	0.54(0.30)	0.85(0.27)	0.70(0.32)
12(Identify variable quantities in a situation)	0.43(0.27)	0.54(0.26)	0.49(0.27)
13(Create table of values for a situation)	0.50(0.39)	0.70(0.31)	0.61(0.37)
14(Determine if a situation is linear)	0.94(0.20)	0.89(0.32)	0.90(0.30)
15(How does table tell you situation is linear?)	0.43(0.35)	0.58(0.36)	0.51(0.36)
16(Create a graph that models situation)	0.58(0.33)	0.73(0.27)	0.66(0.31)
17(How does graph tell you situation is linear?)	0.36(0.36)	0.59(0.30)	0.48(0.35)
18(Construct linear equation for situation)	0.48(0.32)	0.64(0.17)	0.56(0.26)
19(Explain numbers and variable in equation)	0.40(0.27)	0.59(0.22)	0.50(0.26)
20(Identify and explain quantity)	0.21(0.23)	0.64(0.36)	0.43(0.38)

The experimental group had at least a 10% greater scaled score mean than the control group on 9 out of 10 of the conceptual items (11-13, and 15-20). The greatest differences were a 43% greater scaled score mean on item 20, a 33% greater scaled score mean on item 11, and a 23% greater scaled score mean on item 17. Item 20 asked students to identify a quantity in the situation and explain how the quantity was expressed in the table, graph, and equation. Item 11 asked students to identify fixed quantities in the situation and item 17 asked students to explain how their graph tells them that a situation is linear, which requires proportional, covariational, and/or correspondence reasoning.

Further, items 20 and 17 were the first and second most difficult conceptual items for students respectively (Question 20 had a combined scaled score mean of 0.43 and Question 17 had a combined scaled score mean of 0.48). Student performance on conceptual item 12 was also poor with a combined scaled score mean of 0.49. This item asked students to identify varying quantities in the situation.

Because the differences between the mean scaled scores for the control group vs. the experimental group for the three items above were large (items 11, 17, and 20), the distribution of the scaled scores for these items is examined in Table 4. It compares the minimum scaled score, quartile one scaled score, median scaled score, quartile three scaled score, and maximum scaled score of the control group vs. the experimental group for each of the three items. For example, the fourth row tells us that the median scaled score on item 11 for the control group was 0.33 and for the experimental group was 1.0.

Table 4 Conceptual Items 11, 17, and 20 Distribution of Scaled Score Means

Measure	Control Q11	Exper. Q11	Control Q17	Exper. Q17	Control Q20	Exper. Q20
Min.	0	0	0	0	0	0
Q1	0.33	0.66	0.25	0.25	0	0.25
Median	0.33	1	0.25	0.5	0.25	0.75
Q3	0.67	1	0.5	1	0.25	1
Max.	1	1	1	1	0.75	1

On item 11, at least 50% of the experimental students had a scaled score mean that was 1.0 compared to less than 33% of the control students having a scaled score mean of 1.0. On item 20, at least 50% of the experimental students had a scaled score mean that was 0.75 or greater compared to less than 33% of the control students having a scaled score mean of 0.75 or greater. For each item, the median scaled score mean was greater for the experimental group compared to the control group. The greatest difference was the difference in medians for item 11, which had a difference of 0.67.

The control group and the experimental group performed about the same (performance difference less than 10%) on 7 out of 10 procedural items (items 1-3, 5, and 7-9) and 1 out of 10 conceptual items (item 14). One of the procedural items where the performance was about the same was item 9, which was also the most difficult procedural item for students with a combined scaled score mean of 0.60. This item asked students to determine if two lines are parallel, perpendicular, or neither. Performance was also about the same for procedural items 5 and 7, which were the third and fourth most difficult procedural items for students respectively. They asked students to graph a linear equation using intercepts and determine the slope and y-intercept from a given equation.

Analysis of the scaled score means of the items on the unit assessment also showed concepts and skills that were difficult for students. For the procedural items, three out of the top four most difficult items involved the concept of slope and two out of the four most difficult items involved graphing. For the conceptual items, three out of five of the most difficult conceptual items involved explaining concepts from a table, graph, and equation. Of note, the graphing item in the conceptual items (16) was also difficult, having a scaled score mean similar to the scaled score mean for the graphing items in the procedural items (0.66 vs. 0.72).

Statistical Analysis

The following table states the null hypotheses and alternate hypotheses that were tested for the control vs. experimental group along with the result of the t-test for the 10 procedural items, the 10 conceptual items, and the 20 procedural and conceptual items combined. For example, row two of the table tested the null hypothesis that the mean for the control group is equal to the mean of the experimental group and gave a result of 0.2054367 for the 10 procedural items, a result of 0.0175837 for the 10 conceptual items, and a result of 0.3389684 for the 10 procedural items and 10 conceptual items combined. The alternate hypothesis for this row is that the mean for the control group is not equal to the mean for the experimental group. Statistically significant results are marked with an asterisk.

Table 5 Unit Assessment T-test Results

Control (c) vs. Experimental (e) Hypotheses	Procedural Items Assessed	Conceptual Items Assessed	Combined Items Assessed
$H_0: \mu_c = \mu_e$ $H_A: \mu_c \neq \mu_e$	0.2054367	0.0175837*	0.3389684
$H_0: \mu_c \geq \mu_e$ $H_A: \mu_c < \mu_e$	N/A	0.0087919*	N/A

*Statistically Significant

The two-tailed t-test on the scaled score means of the 10 procedural items returned a probability of 0.2054367, which is greater than 0.05. Therefore, the hypothesis that the population means are the same is not rejected. It is concluded that there was no difference in performance on the 10 procedural items for the populations.

The two-tailed t-test on the scaled score means of the 10 conceptual items returned a probability of 0.0175837, which is less than 0.05. Therefore, the hypothesis that the population means are the same is rejected. It is concluded that there was a difference in performance on the 10 conceptual items for the populations.

The two-tailed t-test on the scaled score means of the 10 procedural and 10 conceptual items together returned a probability of 0.3389684, which is greater than 0.05. Therefore, the hypothesis that the population means are the same is not rejected. It is concluded that there was no difference in performance on the 10 procedural and 10 conceptual items together for the populations.

The one-tailed t-test on the scaled score means of the 10 conceptual items returned a probability of 0.0087919, which is less than 0.05. Therefore, the hypothesis that the population mean for the control group is greater than or equal to the population mean for the experimental group on the 10 conceptual items is rejected. It is concluded that performance on the 10 conceptual items for the experimental group population is greater than the performance on the 10 conceptual items for the control group population.

Qualitative Data Analysis Results

As explained in the methodology chapter, the categorization and merging of initial categories of explanation by students on question 15 and 17 according to their types of reasoning led to five categories of student thinking about linearity: *1. Covariational Reasoning and Proportional Reasoning* if a student explained that the table or graph showed there was a constant rate or slope (for every three seconds the gallons increased by one), *2. Covariational Reasoning and Functional Reasoning* if a student explained that the table or graph showed there was a rule relating the two quantities, *3. Beginning Covariational Reasoning* if a student only explained that one quantity changed as the other changed in the table or graph without recognizing that the change was constant, *4. Non-Quantitative Reasoning* if a student only explained that the table or graph was constant or that the graph was a straight line without reasoning about the two quantities, and *5. Other* if the student's explanation could not be placed in any of the first four categories.

Table 6 shows the counts for student responses in the five categories of student thinking about linearity on items 15 and 17 of the unit assessment. These items asked students to explain how a table or graph tells you the situation is or is not linear. For example, the first row indicates there were 14 responses in the control group and 24 responses in the experimental group on item 15 that demonstrated use of covariational and proportional reasoning. This row also indicates there were 12 responses in the control group and 21 responses in the experimental group on item 17 that demonstrated use of covariational and proportional reasoning.

There are a total of 31 responses from the control group and 35 responses from the experimental group (out of 71 participants) on item 15, and a total of 31 responses from the control group and 36 responses from the experimental group (out of 71 participants) on item 17 (one student in the experimental group did not respond to item 15 but did respond to item 17). Some of the responses were placed in more than one category (See Overlapping Responses below).

Table 6 Qualitative Analysis Category Student Response Counts

Qualitative Analysis Category	**Item 15**		**Item 17**	
	Control	**Exper.**	**Control**	**Exper.**
1. Covariational & Proportional Reasoning	14	24	12	21
2. Covariational & Functional Reasoning	3	1	1	0
3. Beginning Covariational Reasoning	8	3	7	6
4. Non-Quantitative Reasoning	4	3	8	8
5. Other	3	4	6	1
Totals	32 (31*)	35(35*)	34(31*)	36(36*)

*Denotes the number of student responses

Overlapping Responses

One of the 31 responses in the control group on item 15, and three of the 31 responses in the control group on item 17, were placed in more than one category. For item 15, the response, “It is a constant growth that does not vary. The table shows this because no matter what you plug in for s, g will always be $\frac{1}{3}$ of that, and when graphed it will create a constant line" (control student 28) was placed in category 1 and 2 because it had evidence of both proportional reasoning and functional reasoning. For item 17, control student 10 and control student 31 responses were placed in categories 1 and 4 because they contained a proportional reasoning explanation and the explanation that the situation was linear because the graph is a straight line. The response on item 17, "It is linear. Each point form part of the line, growing exponentially as it rises" (control student 26), was placed in category 3 and category 4 because the explanation had evidence of beginning covariational reasoning and non-quantitative reasoning.

In the remainder of this section, I explain the defining characteristics of each category with examples of student responses and discuss the distribution of the control group vs. the experimental group in these five categories for the two assessment items.

1. Covariational Reasoning and Proportional Reasoning

The covariational reasoning and proportional reasoning category consisted of student responses that showed evidence of attendance to how the two quantities in the situation varied together (covariational reasoning) and recognition that they varied together at a constant rate (proportional reasoning). Evidence of an understanding of a constant rate was identified by use of the words "constant" or "same" and the word "rate" or the stating of a specific rate. For example, the response, "The table shows that it is a linear relationship because the rate remains constant and does not change" (control student 14), was placed in this category because they clearly explained that the rate was constant. The response, "b/c it is all increasing at the same rate of 1gallon every 3 seconds" (experimental student 19), was also in this category. However, use of the word "constant" or "same" was not necessary for a response to be in this category. For example, the response, “For every 6 seconds, 2 gallons of water fill up” (control student 9), was also in this category because the phrase “for every” implies same or constant and a specific rate was given.

Fourteen of the 31 responses in the control group and twenty-four of the 35 responses in the experimental group on item 15 were placed in the covariational reasoning and proportional reasoning category. Twelve of the 31 responses in the control group and twenty-one of the 36 responses in the experimental group on item 17 were placed in the covariational reasoning and proportional reasoning category.

2. Covariational Reasoning and Functional Reasoning

The covariational reasoning and functional reasoning category consisted of student explanations that showed evidence of attendance to how the two quantities in the situation varied together in the situation (covariational reasoning) and attempted to come up with a functional rule (linear equation) that related the two quantities (correspondence reasoning). For example, the response, “It is a linear relationship because it can be expressed as a linear equation $y = \frac{1}{3}x + 0$” (control student 25), was placed in this category. Not all of the student responses that were placed in this category included a correct equation but they all showed evidence of trying to explain linearity in terms of the two quantities satisfying a linear equation. For example, the response, “It’s a linear relationship because it would read as $3g + 5 \leq 100$ 3 seconds a gallon is added, and it started with 5 gallons and it can only have up to 100 gallons” (control student 2), was also placed in this category.

Three of the 31 responses in the control group and one of the 35 responses in the experimental group on item 15 were placed in the covariational reasoning and functional reasoning category. One of the 31 responses in the control group and none of the 36 responses in the experimental group on item 17 were placed in the covariational reasoning and functional reasoning category.

Although the student responses to items 15 and 17 that were placed in the covariational reasoning and functional reasoning category did not have the correct equation for the situation, there were five control students and four experimental students who wrote the correct equation for the situation on item 18 of the unit assessment. This item asked students to write an equation that modeled the situation. However, overall performance on item 18 was poor with the control group having a scaled score mean of 0.48 and the experimental group having a scaled score mean of 0.64.

The poor performance by students on item 18 may have been related to performance on item 12. This item asked students to identify variable quantities in the situation. The quantitative analysis on this item showed that students had difficulty identifying the variable quantities (scaled score mean was 0.43 for the control group and 0.54 for the experimental group). Further, the qualitative analysis showed that students have a misconception that a rate is a varying quantity. Ten of the 31 responses in the control group and twenty-two of the 36 in the experimental group claimed that a rate is a varying quantity. This apparent confusion about the varying quantities appeared to lead to difficulty in using the correct varying quantities and fixed quantities when constructing a linear equation for the situation (item 18).

3. Beginning Covariational Reasoning

The beginning covariational reasoning category consisted of student explanations that showed evidence of attendance to quantities changing together in the situation but failing to recognize that they were changing together at a constant rate (proportional reasoning). Their responses typically only explained that as one quantity increased the other quantity increased. For example, the response on item 15, "The table shows it is linear because as the seconds increase, so too does the value of gallons" (control student 7), and response on item 17, "Amount of gallons goes up as time goes up" (experimental student 29), were placed in this category. But the responses in this category were not just limited to explanations that the situation was an increasing situation. They also included any response with evidence of attending to both quantities changing, as in the response on question 15, "They go together" (control student 3).

Eight of the 31 responses in the control group and three of the 35 responses in the experimental group on item 15 were placed in the beginning covariational reasoning category. Seven of the 31 responses in the control group and six of the 36 responses in the experimental group on item 17 were placed in the beginning covariational reasoning category.

4. Non-Quantitative Reasoning

The non-quantitative reasoning category consisted of student explanations that had incomplete elements of covariatonal and proportional reasoning mainly because they failed to refer to the rate or specific quantities of the situation that are necessary to explain why the situation is linear. For example, several student responses in this category (three for item 15 and three for item 17) just stated that the table or graph was constant without explaining what feature of the situation was constant (the rate or slope). The response, "It is linear because it is costenet [sic]" (experimental student 13), is an example of this type of response. A larger group of student responses in this category (four for item 15 and thirteen for question 17) explained that the situation was linear only because its graph is a straight line. For example, the response on item 17, “It shows it is a linear relationship because the line is a straight line” (control student 5), and the response on item 17, “Yes because it is a consistent straight line” (experimental student 11), were placed in this category. These responses were considered non-quantitative because they only referred to a characteristic of the graph without explaining how that characteristic shows that the situation is linear (how the characteristic shows a constant rate with specific quantities).

Four of the 31 responses in the control group and three of the 35 responses in the experimental group on item 15 were placed in the non-quantitative reasoning category. Eight of the 31 responses in the control group and eight of the 36 responses in the experimental group on item 17 were placed in the non-quantitative reasoning category.

5. Other

The other category consisted of student responses that were difficult to place in the first four categories. They were difficult to place in the first four categories because they showed no indication of covariational, proportional, or correspondence reasoning. For example, the response on item 17, “Because they are lines that touch at some point” (experimental student 24), was placed in this category.

Three of the 31 responses in the control group and four of the 35 responses in the experimental group on item 15 were placed in the other category. Six of the 31 responses in the control group and one of the 36 responses in the experimental group on item 17 were placed in the other category.

CHAPTER FIVE: DISCUSSION

In the methodology chapter, the research question for this study was stated as: What is the impact of a quantitative reasoning instructional approach to teaching linear equations in two variables on student performance on linear equations in two variables tasks and students' thinking about linearity? In this chapter, I will discuss how the results of the quantitative and qualitative analysis described in Chapter 4 contribute to answering this question. I begin with a discussion of the impact of the quantitative reasoning instructional approach on performance on linear equations in two variables tasks and conclude with a discussion of the impact on student's thinking about linearity.

Performance on Linear Equations in Two Variables Tasks

The quantitative analysis of student scores for the control and experimental groups on the unit assessment resulted in three key findings about the impact of a quantitative reasoning instructional approach on performance on linear equations in two variables tasks. First, the quantitative instructional approach did not impact performance on the procedural items of the unit assessment. Second, the quantitative instructional approach did impact performance on the conceptual items of the unit assessment. Third, the level of performance on the procedural items was the same for the control group and the experimental group even though the experimental group received significantly less procedural instruction time. This section will discuss each of these three findings.

Quantitative Reasoning Instructional Approach and Performance on Procedural Items

The statistical analysis presented in the results chapter of the scores of the control group vs. the experimental group on the unit assessment resulted in the conclusion that there was no difference in performance on the 10 procedural items ($p = 0.2054367$, see page 43). Therefore, the quantitative reasoning instructional approach, with attention to developing covariational reasoning, proportional reasoning, and correspondence reasoning, resulted in similar student performance on linear equations in two variables tasks that require students to use a procedure to accomplish the task.

Quantitative Reasoning Instructional Approach and Performance on Conceptual Items

The statistical analysis in the results chapter of the scores of the control group versus the experimental group on the unit assessment resulted in the conclusion that there was a difference in performance on the 10 conceptual items ($p = 0.0175837$, see page 43). Further statistical analysis also revealed that the performance by the experimental group was greater than the performance by the control group on the 10 conceptual items ($p = 0.0087919$, see page 43). Therefore, the quantitative reasoning instructional approach, with attention to developing covariational reasoning, proportional reasoning, and functional reasoning, resulted in a difference in student performance on tasks that require students to understand and explain concepts about linearity.

Procedural Item Performance and Procedural Instructional Time

Students in the control group and the experimental group performed about the same on the 10 procedural items with the control group having five days of procedural instruction time and the experimental group only having one day of procedural instruction time. This implies that a certain level of procedural competency can be achieved without a lot of procedural instructional time. Therefore, perhaps more instructional time in the classroom can be spent on developing conceptual understanding and higher order thinking skills and less time spent on teaching procedures without sacrificing student performance on procedural items.

Further, the quantitative analysis of student scores showed better performance on the conceptual items after only four days of quantitative reasoning instruction. This raises a question about whether using the quantitative reasoning instructional approach during the entire course would lead to better performance in the course overall.

Student Thinking About Linearity

The qualitative analysis of student responses for the control and experimental group on the unit assessment resulted in four key findings about student thinking about linearity. First, students in the experimental group had a larger number of responses than the control group that showed covariational reasoning and proportional reasoning ability. Second, students in the experimental group had about the same number of responses as the control group that showed covariational and functional reasoning ability (ability to write an equation that models the situation). Third, student responses from both groups revealed a misconception that an increasing relationship between two quantities implies linearity. Fourth, student responses from both groups revealed a misconception that a constant rate (like one gallon every three seconds) is a varying quantity. In this section, I will discuss each of these findings.

Covariational and Proportional Reasoning Responses

The qualitative analysis showed that fourteen of the 31 responses in the control group (45%) and 24 of the 35 responses in the experimental group (69%) on item 15 were placed in the covariational reasoning and proportional reasoning category (a difference of 24%). It also showed that twelve of the 31 responses in the control group (39%) and twenty-one of the 36 responses in the experimental group (58%) on item 17 were placed in the covariational reasoning and proportional reasoning category (a difference of 19%). Since students in both the control group and the experimental group started the unit with the same level of ability (due to being placed in the course by the same placement test or by passing the same prerequisite Prealgebra course), this is evidence that the quantitative reasoning instructional approach had a positive impact on the covariational and proportional reasoning ability of the students.

Covariational and Functional Reasoning Responses

The qualitative analysis showed that 3 of the 31 responses in the control group (10%) and one of the 35 responses in the experimental group (3%) on item 15 were placed in the covariational reasoning and functional reasoning category. It also showed that one of the 31 responses in the control group (3%) and none of the 36 responses in the experimental group (0%) on item 17 were placed in the covariational reasoning and functional reasoning category. Further, the qualitative analysis revealed that only five students in the control group and four students in the experimental group wrote the correct equation that modeled the situation on the unit assessment (item 18). This data, along with the scaled score means for item 18 (0.48 for the control group and 0.64 for the experimental group), confirm that the students' functional reasoning ability was weak and the quantitative reasoning instructional approach did not seem to contribute to the students' functional reasoning ability.

Misconception: Covariation Implies Linearity

The qualitative analysis showed that eight of the 31 responses in the control group (26%) and three of the 35 responses in the experimental group (9%) on item 15 were placed in the beginning covariational reasoning category. It also showed that seven of the 31 responses in the control group (23%) and six of the 36 responses in the experimental group (17%) on item 17 were placed in the beginning covariational reasoning category. The beginning covariational reasoning category contained responses that claimed that quantities that change together are linear relationships. Most students claimed that quantities that increase together are linear relationships. Even though the counts for this category seem to indicate that the quantitative reasoning instructional approach may have somewhat dispelled this misconception, this is a misconception that must be addressed in teaching linearity.

Misconception: Constant Rate Is a Varying Quantity

The quantitative analysis for item 12 of the unit assessment (identify a varying quantity in the situation) shows that the control students had a scaled score mean of 0.43 and the experimental students had a scaled score mean of 0.54. Hence, both groups performed poorly on this item. During the grading of this item, I noticed that many students responded to this item by identifying the constant rate (1 gallon every 3 seconds) as a varying quantity (ten of the 31 responses in the control group and twenty-two of the 36 in the experimental group). This revealed a misconception by community college students who think that a constant rate is a varying quantity. This could be explained by students who have not conceptualized a constant rate as a composed unit (Lobato et al., 2010). This is a misconception that must be addressed when teaching linearity and could be a reason why students struggled to write an equation for the situation on the unit assessment (item 18).

CHAPTER SIX: CONCLUSION

In this chapter, I address the limitations of the study, the implications for teaching, and some possible topics of future research that arose as a result of the study.

Limitations of the Study

There are several limitations to this study. First, the instructor who used the quantitative reasoning instructional approach with the experimental group had no prior experience with the instructional approach and thus was using the instructional approach for the first time. A more experienced instructor may have been able to better manage resistance by the students to the approach and may have been able to better instruct the students using the innovative curricula. The use of an inexperienced instructor may have resulted in poorer performance on the unit assessment by students in the experimental group.

Second, scoring of the items on the unit assessment was done by the author only. This may have resulted in a lack of consistency in applying the rubric and a lack of checks and balances to make sure the rubric was applied as intended.

Third, it was discovered that the point value for some of the descriptions in the rubric did not correctly reflect the level of understanding of the descriptions. For example on item 15, the rubric indicated that a point value of 1 should be given to a response that explained that the relationship is linear because the rate is constant with no explanation from the data in the table. But an explanation that the rate is constant is evidence of proportional reasoning. Therefore, this type of response should have received more than one point but may not have because of the way the rubric was written.

Fourth, students in the control and experimental group did not take a pre-test with procedural and conceptual items similar to the unit assessment before the unit began. Therefore, the level of mathematical ability of the students on the items on the unit assessment at the beginning of the unit may have been different for the two groups. It could be argued that this may have been the reason why the experimental group performed better on the conceptual items. However, the mathematical ability of the control and experimental group must have been similar since both groups were placed in the course based on their scores on the same placement test or passing the same prerequisite Prealgebra course. This gives some degree of confidence that the students' mathematical ability for the control and experimental group at the beginning of the unit were similar.

Implications for Teaching

The following are implications for teaching that can be drawn from the study:

1. A quantitative reasoning instructional approach to teaching linear equations in two variables can improve community college students' proportional reasoning ability, which can lead to better performance on conceptual tasks. The statistical analysis concluded that the performance by the experimental group was better than the performance by the control group on conceptual tasks ($p = 0.0087919$).

2. More time can be spent teaching conceptual understanding and higher order thinking skills and less time on teaching procedural skills without sacrificing community college student proficiency on procedural tasks. The experimental group was instructed

one day on procedural tasks and the control group was instructed 5 days on procedural tasks and yet the statistical analysis showed that both groups performed the same on the procedural tasks of the unit assessment.

3. If a quantitative reasoning instructional approach were applied consistently over an entire beginning algebra course, performance in the course overall may have improved. In this experiment, student performance was significantly better on the conceptual items of the unit assessment after only four days of using the quantitative reasoning instructional approach. If the quantitative reasoning instructional approach had been used during the entire course, performance in the course overall would have most likely improved.

4. Beginning algebra community college students may begin the course with a misconception that quantities that change together are in a linear relationship. This study showed that 26% of the control group and 9% of the experimental group on item 15 had this misconception, and 23% of the control group and 17% of the experimental group on item 17 had this misconception. Further, the study showed that many of these students believed that quantities that increase together constitute a linear relationship. They failed to recognize that quantities in a linear relationship must change together at a constant rate. This misconception could be addressed by giving students many different types of relationships between quantities and asking them talk or write about whether they are increasing, decreasing, and/or constant.

5. Beginning algebra community college students may begin the course with a misconception that a constant rate is a variable quantity. This study showed that 32% of the control group and 61% of the experimental group had this misconception on item 12

of the unit assessment. This misconception could be addressed by having students practice identifying fixed and variable quantities for different situations and having them explain why a constant rate is not a variable quantity.

Future Research

This study led to the following questions that could be topics of future research:

1. What is the relationship between the learning objectives of procedural fluency and conceptual understanding in teaching mathematics? How much time should be spent addressing these two learning objectives and in what order should they be taught? Which learning objective is more important and why? In this study, procedural learning objectives were used for the control group and conceptual learning objectives were used for the experimental group. The increased emphasis on conceptual learning of the experimental group did not seem to have an impact on their performance on procedural tasks. And conversely, the increased emphasis on procedural learning of the control group did not impact their performance on conceptual tasks. Why? Is it because the two learning objectives are not related? Was it because the learning objectives were taught separately? Would performance increase if the two learning objectives were more closely integrated together in instruction? Similar questions have been investigated by other researchers like Baroody (2003) and more recently Ross and Wilson (2012). A closer examination of their findings along with the findings of this study may lead to future research that may answer these questions.

2. How can instructors support community college students in performing better on tasks that require functional reasoning ability (ability to write an equation that represents a linear situation)? The quantitative reasoning instructional approach for this

study attempted to increase performance on these types of tasks by having students create tables for linear real world situations and having them reason covariationally about the data. It was thought that this would increase performance on writing linear equations for a situation (like item 18 of assessment), but the results of this study showed that it didn't. Why did the tasks for the experimental group not improve performance on writing a linear equation for the situation? How should the tasks be modified to improve performance? An answer may lie in Cooney, Beckmann, & Lloyd's (2010) discussion on how a covariational perspective and emphasis on rate of change can support writing a general rule for a linear situation (see pages 23-33). Further research in this area may result in better instructional methods for supporting students' ability to write equations that model linear situations.

3. How can instructors support community college students in understanding the difference between an increasing relationship between two quantities and a relationship that is linear? This study revealed that some community college students have a misconception that a linear relationship is either a relationship where two quantities change together or two quantities increase together. What tasks can dispel these ideas? Perhaps tasks that require students to identify the different patterns of covariation between two quantities in a situation and the types of relationships they classify may help students avoid developing this misconception (Cooney et. al, 2010). Research on tasks that attempt to dispel this misconception could be done to determine their effectiveness.

4. How can instructors support community college students in understanding that a constant rate is a fixed quantity and not a variable quantity? This study involved an experimental instructional approach that started with instructing students about quantities

and the difference between fixed and varying quantities. However, the assessment of these concepts showed that many community college students believed that a constant rate is a varying quantity. How can instruction be improved to dispel this misconception? What tasks can be given to community college students to clear up this confusion? The task of identifying fixed and varying quantities in a situation was adapted from Carlson's et al. (2013) Precalculus curriculum for this study. A closer examination of this curriculum and other research on student thinking about fixed and varying quantities (Lloyd, Herbel-Eisenmann, & Star, 2011) could lead to further research on tasks that address this misconception.

Although this study had some limitations, I discovered some important truths about teaching linear equations in two variables and am motivated to find answers to some of the questions that arose from this study. I discovered that a different instructional approach can improve community college students' proportional reasoning ability. I learned that more time can be spent on conceptual understanding without sacrificing procedural understanding. I want to find ways to successfully integrate the instruction of procedural and conceptual understanding. I want to know how to better support community college students' ability to write equations that model a linear situation and how to dispel their misconceptions about linearity and variable quantities.

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APPENDIX A

Skills Based Approach Unit Objectives Outline

(From Trigsted et al., 2014)

Learning Objectives	Procedural Assessment Item (See Appendix F)
The Rectangular Coordinate System • Read Line Graphs • Identify Points in the Rectangular Coordinate System • Plot Ordered Pairs in the Rectangular Coordinate System • Create Scatter Plots	1
Graphing Linear Equations in Two Variables • Determine If an Ordered Pair is a Solution to an Equation • Determine the Unknown Coordinate of an Ordered Pair Solution • Graph Linear Equations by Plotting Points • Find x and y intercepts • Graph Linear Equations Using Intercepts • Use Linear Equations to Model Data • Graph Horizontal and Vertical Lines	2, 3, 5, 8
Slope • Find the Slope of a Line Given Two Points • Find the Slopes of Horizontal and Vertical Lines • Graph a Line Using the Slope and a Point • Find and Use the Slopes of Parallel and Perpendicular Lines • Use Slope in Applications	4, 6, 9, 10
Equations of Lines • Determine the Slope and y-intercept from a Linear Equation • Use the Slope-Intercept Form to Graph a Linear Equation • Write the Equation of a Line Given Its Slope and y-intercept • Write the Equation of a Line Given Its Slope and a Point on the Line • Write the Equation of a Line Given Two Points	7

APPENDIX B

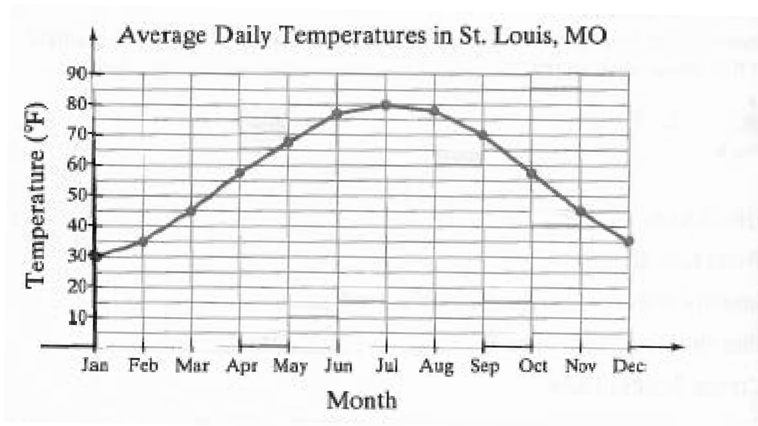
Skills Based Approach Day 1 Lesson Plan

12.1 The Rectangular Coordinate System

(From Trigsted et. al, 2014)

Objective 1: Read Line Graphs

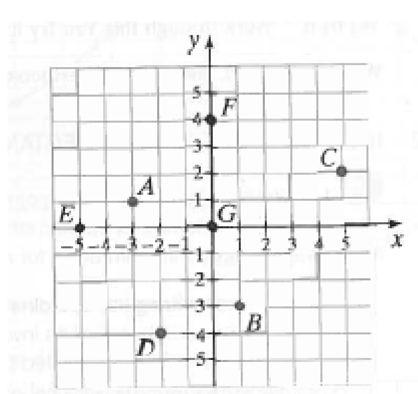
(Explain the fact that graphs show the relationship between two variables. Show the following line graph and discuss the questions with the class.)



a. What is the average temperature in February?
b. What is the average daily temperature in November?
c. In what month is the average daily temperature 70 degrees F?
d. Which month has the highest average daily temperature?
e. In what months are the average daily temperatures above 65 degrees F?

Objective 2: Identify Points in the Rectangular Coordinate System

(Explain how to identify some of the points shown in the following coordinate plane and then have students practice identifying the rest of the points.)



Objective 3: Plot Ordered Pairs in the Rectangular Coordinate System

(Explain how to plot an ordered pair in the coordinate plane and then have students practice plotting the following ordered pairs in the coordinate plane.)

a. (2,4)
b. (4,-5)
c. (0,-2)
d. (-3,-4)

Objective 4: Create Scatter Plots

(Explain how to create a scatter plot using the data in the first table, then have students practice creating a scatter plot for the second table.)

Year	Number of Plants
2000	54
2001	56
2002	61
2003	68
2004	72
2005	81
2006	95
2007	110
2008	139
2009	170
2010	189

Year	Tuition & Fees (dollars)
2005	5,611
2006	5,936
2007	6,284
2008	6,736
2009	7,180
2010	7,566
2011	8,043

12.2 Graphing Linear Equations in Two Variables

Objective 1: Determine if an Ordered Pair is a Solution to an Equation

(Write the following problem on the board or show on projector screen.)

Determine if each ordered pair is a solution to the equation $x + 2y = 8$.

a. (-2,5)
b. (2,6)
c. (0,4)

(Do a. on the board to show students how to do the problem.)

(Have students practice by doing b. and c. in their notebooks in class and share their answer with a neighbor.)

Objective 2: Determine the Unknown Coordinate of an Ordered Pair Solution

(Write the following problem on the board or show on projector screen.)

Find the unknown coordinate so that each ordered pair satisfies $2x – 3y = 15$.

a. (6, ?)
b. (?, 7)
c. (-5/2,?)

(Do a. on the board to show students how to do the problem.)

(Have students practice by doing b. and c. in their notebooks in class and share their answer with a neighbor.)

Objective 3: Graph Linear Equations by Plotting Points

(Write the following problem on the board or show on projector screen.)

Graph $3x - y = 2$ by plotting points.

(Do this problem on the board. Show students how to find ordered pair solutions by picking an arbitrary x or y and finding the corresponding y or x. Display these solutions in a table of values. Graph the ordered pairs from the table of values and draw a line through the points.)

(Have students practice by doing the following problem and share their answer with a neighbor.)

Graph $y = 3x - 5$ by plotting points.

Objective 4: Find x and y Intercepts

(Explain what an x- and y-intercept is on the graph of a linear equation in two variables. Explain that to find an x-intercept let $y = 0$ and solve for x. Explain that to find a y-intercept let $x = 0$ and solve for y. Then show the following two problems on the board or projector screen.)

a. $2x + y = 4$
b. $4x = 3y + 8$

(Do a. on the board to show students how to do the problem.)

(Have students practice by doing b. in their notebooks and share their answer with a neighbor.)

Objective 5: Graph Linear Equations Using Intercepts

(Demonstrate how to graph the following equation using the x- and y-intercept.)

$3x - 2y = 6$

(Have students practice graphing the following equation using the x- and y-intercept.)

$5x + 3y = 15$

Objective 6: Use Linear Equations to Model Data

(Do the following problem with the class as a group.)

The number of U.S. drive-in theaters can be modeled by the linear equation $y = -7.5x + 435$, where x is the number of years after 2000.

a. Sketch the graph of the equation for the year 2000 and beyond.
b. Find the missing coordinates for the ordered pair solution (?,390)
c. Interpret the point from part (b).
d. Find and interpret the y-intercept.
e. What does the x-intercept represent in this problem?

Objective 7: Graph Horizontal and Vertical Lines

(Explain to class how to graph the equations $x = 4$ and $y = -6$ and then have them practice graphing the following two equations. Explain that $x = 4$ may be written as $1(x) + 0(y) = 4$ and $y = -6$ may be written as $0(x) + 1(y) = -6$.)

a. $x = -4$
b. $y = 3$

APPENDIX C

Quantitative Reasoning Approach Unit Objectives Outline

Learning Objectives	Conceptual Assessment Item (See Appendix F)
• Identify fixed quantities in a real world situation.	11
• Identify varying quantities in a real world situation.	12
• Construct tables, graphs, and equations that represent a linear relationship between two quantities in a real world situation. Use proportional (correspondence) reasoning to construct equations.	13, 16, 18
• Identify and explain a linear relationship between two quantities in a real world situation represented by tables, graphs, and equations. Use covariational or proportional (correspondence) reasoning.	14, 15, 17
• Use variables to represent quantities in a real world situation that is linear.	18, 19
• Identify and explain the rate of change (slope) in tables, graphs, and equations that represent a linear relationship between two quantities in a real world situation. Use covariational reasoning.	15, 17, 19, 20

APPENDIX D

Quantitative Reasoning Approach Day 1 Lesson Plan

Problem 1: Identifying Quantities

[Adapted from *Module 2 & Investigation 1, Quantities and Co-variation of Quantities* (Carlson et al., 2013)]

(Write on the board or show on the screen)

A quantity is a person's conception of a measureable attribute of an object or event. It consists of four things:

1. An object or event
2. An attribute of the object or event
3. An appropriate unit or dimension of the attribute
4. A process for assigning a numerical value to the attribute

Example of a Quantity:

1. Object: The floor of the classroom
2. Attribute: The distance from one wall to the parallel wall
3. Unit: Feet
4. Process: Take a tape measure and measure the number of feet from one wall to the other.

(Ask the following question for class discussion.)

What is another quantity associated with the floor of the classroom?

(Allow students to come up with other quantities associated with the floor of the classroom. Write the quantities they come up with on the board and list the four components of each quantity they give. Be specific in the descriptions. Other quantities may be the area, perimeter, and other distances.)

What are attributes associated with the floor that are not quantities?

(Discuss with students why certain attributes are and are not quantities. Then say the following.)

Now suppose I stand at this wall and walk across the floor to the wall at the other end of the floor.

(Ask question for class discussion.)

What are some of the quantities involved in this situation?

(Again allow students to come up with quantities involved in the situation and list them on the board with the four components of each. Make sure the list includes the following quantities. Help students isolate the speed of person from other attributes.)

1. Number of steps
2. Time
3. Size of steps
4. Distance from wall or distance traveled
5. Speed of person
6. Speed of feet
7. Distance between walls

(Explain the following.)

The distance from wall or distance traveled varies and is therefore called a variable quantity. The quantity describing the distance from one wall to the other does not vary and is therefore called a fixed quantity whose value is constant.

The speed of person may be difficult for students and may require giving them some hints. Spend time on discussing the process (Moore & Carlson, 2012). It is important because it is the foundation for understanding the rate of change of the situation (Jacobson, 2014). Have students verbally describe how the speed would be measured. This task is a ratio-as-measure task and is therefore important in the students' development of a robust understanding of slope which will be discussed later in the unit (Simon & Blume, 1994)

Problem 2: Modeling a Linear Relationship
[Adapted from Connected Mathematics 2, Moving Straight Ahead (Lappan et al, 2009)]

(Now do the following.)

Have students with a watch or timer use the process they came up with above to actually determine my walking speed as I walk across the floor.

(Write on the board)

1. My walking rate is ________________ feet in __________________ seconds.

(Have the students break up into groups and work together to answer the following questions that will be written down on a handout for them to fill out. Move around the classroom and help the students minimally.)

2. Give two different distance and time values that compose a rate that is the same as the walking rate in 1. above and explain why it is the same.

   a. __________________ feet in __________________ seconds

   Why is it the same?

   b. __________________ feet in __________________ seconds

   Why is it the same?

3. For the walking rate in 1. above how many feet do I walk in 1 second?

4. For the walking rate in 1. above how many seconds does it take to walk 1 foot?

Assume I walk at the constant rate for number 1. above.

1. How long will it take me to walk 600 feet?

2. How far could I walk in 30 seconds? 10 minutes? 1 hour?

3. Describe in words the distance in feet I could walk in a given number of seconds.

4. Write an equation that represents the distance d in feet that I could walk in t seconds if I maintain this pace.

5. Use the equation to predict the distance I could walk in 45 seconds.

Problem 3: Representing Linear Relationships

[Adapted from Connected Mathematics 2, Moving Straight Ahead (Lappan et al, 2009)]

(Answer the questions for the walking rates of the people in the table)

Name	Walking Rate
Alana	1 meter per second
Gilberto	2 meters per second
Leanne	2.5 meters per second

A. 1. Make a table showing the distance walked by each student for the first ten seconds. How does the walking rate affect the data?

2. Graph the time and distance on the same coordinate axes. Use a different color for each student's data. How does the walking rate affect the graph?

3. Write an equation that gives the relationship between the time t and the distance d walked for each student. How is the walking rate represented in the equation?

B. For each student:

1. If t increases by 1 second, by how much does the distance change? How is this change represented in a table? In a graph?

2. If t increases by 5 seconds, by how much does the distance change? How is this change represented in a table? In a graph?

3. What is the walking rate per minute? The walking rate per hour?

C. The following representations are based on data for the walking rates of four people. Are any of these relationships linear? Explain why or why not.

George's Walking Rate	
Time (seconds)	Distance (meters)
0	0
1	2
2	9
3	11
4	20
5	25

Elizabeth's Walking Rate	
Time (seconds)	Distance (meters)
0	0
2	3
4	6
6	9
8	12
10	15

Billie's Walking Rate
$D = 2.25t$

D represents distance (in meters)
t represents time (in seconds)

Bob’s Walking Rate
t = 100/r

t represents time (in seconds)
r represents walking rate (in meters/second)

APPENDIX E

Quantitative Reasoning Approach Day 2 & 3 Lesson Plan Outline

Day 2 Outline

Problem 1: Using Linear Relationships

1. Construct a table, graph, and equation for three real world situations that are linear (constant linear, linear without a y-intercept, linear with a y-intercept)
2. Explore the patterns of change in the three situations.
3. Use equations to answer questions about the situations.
4. Explore meaning of the y-intercept in the real world situation that is linear with a y-intercept.

Problem 2: Negative Rates of Change

1. Construct a table, graph, and equation for a real world situation that has a negative rate of change.
2. Explain how the table, graph, and equation tell you that the relationship is linear.

Problem 3: Connecting Graphs, Tables, and Equations

1. Find solutions to a problem involving a real world situation using a table or graph.
2. Connect solutions in graphs and tables to solutions of equations.
3. Find missing coordinates of ordered pairs of real world situations and write questions than can be answered by finding the missing coordinate.

<u>Day 3 Outline</u>

Problem 1: Understanding Slope of a Line Through Modeling

1. Draw a set of stairs that has a particular tread width (run) and riser height (rise).
2. Construct a graph in the coordinate plane that models the set of stairs. Make the graph a line that passes through the origin and has y-values that change by the riser height for each x-value that changes by the tread width.
3. Write an equation for the line that models the set of stairs.
4. Explain how each part of the equation relates to the set of stairs modeled.

Problem 2: Finding Slope of a Line

1. Find the slope from a graph, table, and equation that represent a linear situation.
2. Write an equation for the graph and table.
3. Explain how to find two more points on a line that passes through two given points.

Problem 3: Exploring Patterns with Lines

1. Explore lines that are parallel to discover patterns in their equations and graphs.
2. Explore lines that are perpendicular to discover patterns in their equations and graphs.
3. Write equations for lines that form a parallelogram and right triangle.
4. Explain how you can decide if two lines are parallel or perpendicular.

APPENDIX F

Unit Assessment Items

Items 1-10 adapted from Trigsted et al., 2014.

1. Plot the ordered pair in a coordinate plane. In which quadrant or on which axis does the point lie?

   (6,-2)

2. Find the unknown coordinate so that the ordered pair satisfies the given equation.

   3x – 2y = 1, (-5, ?)

3. Determine if the given ordered pair is a solution to the equation.

   4x + 2y = 20, (4,2)

4. Find the slope of the line containing the given points. Simplify if possible.

   (4,-6) and (0,5)

5. Graph the linear equation using intercepts.

   2x + y = 2

6. For the given slope of line l_1, (a) find the slope of a line l_2 parallel to l_1, and (b) find the slope of a line l_3 perpendicular to l_1.

   m = -1/6 (slope of line l_1)

7. Determine the slope and y-intercept for the given equation.

   9x + y = 10

8. Graph the given line.

   y = -2

9. Determine if the two lines are parallel, perpendicular, coinciding, or only intersecting.

   $3x – 4y = -18$

   $8x + 6y = -7$

10. Graph the line given the slope and a point on the line.

    $m = 1$; (-3,3)

Items 11-20 adapted from Lappan et al., 2009 & Carlson et al., 2013.

Kevin has 5 gallons of water in his swimming pool that holds 100 gallons. He begins to fill the swimming pool with a hose at a rate of 1 gallon of water every 3 seconds.

11. Identify one fixed quantity (quantity that does not change) and state the units of measurement.
12. Identify two varying quantities (quantities that do change) and state the units of measurement.
13. Construct a table that shows the amount of water in Kevin's swimming pool from the time 0 to 30 seconds. Provide at least four rows in the table and be sure to include a row for 0 and 30 seconds. A row for every second is not required.
14. Assume that Kevin continues to add water at the same rate. Is the relationship between the time in seconds and the number of gallons of water in Kevin's swimming pool a linear relationship (Yes or No)?
15. If it is a linear relationship, explain how the table tells you it is a linear relationship. If it is not a linear relationship, explain how the table tells you that it is not a linear relationship.

16. Construct a graph that models this situation on a coordinate plane. Put time on the x-axis and gallons of water on the y-axis.
17. If it is a linear relationship, explain how the graph tells you it is a linear relationship. If it is not a linear relationship, explain how the graph tells you that it is not a linear relationship.
18. Construct an equation that represents this relationship.
19. Explain what information the numbers and variable(s) in the equation represent.
20. Identify a quantity that is found in the table, graph, and equation for this relationship and explain how it is expressed in the table, graph, and equation. For example, how is the rate of change expressed in the table, graph, and equation?

APPENDIX G

Unit Assessment Grading Rubric

Points	Description
1. Plot the ordered pair in the coordinate plane. In which quadrant or on which axis does the point lie? (6,-2) (3 points total)	
3	x-coordinate correct, y-coordinate correct, and quadrant correct
2	2 out of 3 above correct.
1	1 out of 3 above correct.
0	0 out of 3 above correct.
2. Find the unknown coordinate so that the ordered pair satisfies the given equation. 3x – 2y = 1, (-5, ?) (4 points total)	
4	The variable x is replaced with -5 and the equation is correctly solved for y.
3	The variable x is replaced with -5 and the equation is correctly solved for y except the sign is incorrect.
2	The variable x is replaced with -5 and the equation is partially correctly solved for y, or The variable y is replaced with -5 and the equation is correctly solved for x.
1	The variable x is replaced with -5 but no correct steps are taken to solve the equation for y.
0	The variable x is not replaced by -5 and no correct steps are taken to solve the equation.
3. Determine if the given ordered pair is a solution to the equation. 4x + 2y = 20. (4, 2) (4 points total)	
4	The variable x and y are replaced with 4 and 2 respectively and the left hand side is simplified to show that 20 = 20, or (4, 2) is shown to be on graph of the equation.
3	The variable x and y are replaced with 4 and 2 respectively and the left hand side is simplified incorrectly with a minor mistake.
2	Variable x is replaced with 2 and variable y replaced with 4 (reversed) and simplification is correct.
1	The variable x and y are replaced with 4 and 2 respectively but none of the simplification is correct.
0	The variable x and y are not replaced with 4 and 2 and no simplification is done.

4. Find the slope of the line containing the given points. Simplify if possible. (4,-6) and (0,5) (4 points total)	
4	Coordinates are correctly placed in slope formula and the ratio is correctly simplified, or Ordered pairs are plotted in coordinate plan and slope triangle is used to find the ratio of the change of y-coordinates to the change of x-coordinates.
3	Coordinates are correctly placed in slope formula and correctly simplified except for an incorrect sign, or Slope is found by graphing with an incorrect sign.
2	Slope formula is used, coordinates are correctly placed in formula, but simplification of formula has a mistake.
1	Slope formula is used but coordinates are incorrectly placed in the formula.
0	Process for calculating slope is incorrect
5. Graph the linear equation using intercepts. 2x + y = 2 (4 points total)	
4	x-intercept is correct, y-intercept is correct, x-intercept is correctly graphed, and y-intercept is correctly graphed
3	3 out of 4 above correct.
2	2 out of 4 above correct.
1	1 out of 4 above correct.
0	0 out of 4 above correct.
6. For the given slope of line l_1, (a) find the slope of line l_2 parallel to l_1, and (b) find the slope of a line l_3 perpendicular to l_1. m = -1/6 (slope of line l_1) (4 points total)	
4	(a) and (b) are correct
3	(a) is correct and (b) is correct with incorrect sign, or (a) is correct with incorrect sign and (b) is correct
2	Both are correct with incorrect signs or only one is correct.
1	(a) is incorrect and (b) is correct with incorrect sign
0	Both are incorrect.

7. Determine the slope and y-intercept for the given equation. 9x + y = 10 (4 points total)	
4	Slope is correct, y-intercept is correct, and work is shown to find these by writing equation in slope – intercept form or graphing the equation.
3	Answers are correct and work is shown but one incorrect sign.
2	2 out of 3 above correct.
1	1 out of 3 above correct.
0	0 out of 3 above correct.
8. Graph the given line. y = -2 (3 points total)	
3	Line is horizontal line and correctly graphed by showing at least two points on the graph.
2	Line is horizontal line but not through (0,-2)
1	Only one point is correctly graphed, line is not horizontal
0	Line is not horizontal and no correct points are graphed.
9. Determine if the two lines are parallel, perpendicular, coinciding, or only intersecting. 3x – 4y = -18; 8x + 6y = -7 (4 points total)	
4	Slope of first line is correctly determined, slope of second line is correctly determined, and determination of relationship is correct.
3	Minor mistake in determining one of the slopes
2	Slopes are correctly determined but determination of relationship is incorrect. Determination of relationship correct but no work shown.
1	Only one of the slopes is correct and determination of relationship is incorrect.
0	Slopes are incorrect and determination of relationship is incorrect.
10. Graph the line given the slope and a point on the line. m = 1; (-3,3) (4 points total)	
4	Point is graphed correctly, slope is used correctly to determine second point, and line is drawn through two points.
3	Slope is used correctly to determine second point but first point is graphed incorrectly or line is not drawn through two points.
2	Point is graphed correctly and slope is partially used correctly to determine second point.
1	Only point is graphed correctly.
0	Point is not graphed correctly and second point is not correctly determined from slope.

Points	Description
11. Identify one fixed quantity (quantity that does not change) and state the units of measurement. (3 points total)	
3	Quantity is correct and includes attribute and object. Units are correct.
2	Quantity is correct but does not include attribute and/or object and units are correct, or Quantity is correct and includes attribute and object but units are incorrect.
1	Quantity is incorrect and units are correct, or Quantity is partially correct and units are incorrect.
0	Quantity is incorrect and units are incorrect.
12. Identify two varying quantities (quantities that do change) and state the units of measurement. (6 points total, 3 points for each quantity)	
3	Quantity is correct and includes attribute and object. Units are correct.
2	Quantity is correct but does not include attribute and/or object and units are correct, or Quantity is correct and includes attribute and object but units are incorrect.
1	Quantity is incorrect and units are correct, or Quantity is partially correct and units are incorrect.
0	Quantity is incorrect and units are incorrect.
13. Construct a table that shows the amount of water in Kevin's swimming pool from time 0 to 30 seconds. Provide at least four rows in the table and be sure to include a row for 0 and 30 seconds. A row for every second is not required. (4 points total)	
4	Table begins with row of 0 seconds and 5 gallons and ends with row of 30 seconds and 15 gallons. At least four rows and each row has correct corresponding number of gallons.
3	Table does not begin with row of time 0 and 5 gallons but each row has correct corresponding number of gallons, or Table does begin with row of time 0 and 5 gallons and has one row with an incorrect corresponding number of gallons.
2	Table does not begin with row of time 0 and 5 gallons and has one row with an incorrect corresponding number of gallons, or Table does begin with row of time 0 and 5 gallons and has two rows with an incorrect corresponding number of gallons.
1	Table only has one row with the correct corresponding number of gallons.
0	None of the rows of the table have a correct corresponding number of gallons.
14. Assume that Kevin continues to add water at the same rate. Is the relationship between the time in seconds and the number of gallons of water in Kevin's swimming pool a linear relationship? (1 point total)	
1	Yes or linear.
0	No or not linear.

15. If it is a linear relationship, explain how the table tells you it is a linear

	relationship. If it is not a linear relationship, explain how the table tells you that it is not a linear relationship. (4 points total)
4	Explanation shows correct covariational reasoning by stating that the change in gallons between two rows in the table relative to (divided by) the change in time between the same two rows in the table is (or is not) constant, and gives an example from the table, or Explanation shows correct correspondence reasoning by stating that the number of gallons in each row is (or is not) equal to 1/3 of the time plus 5, i.e. gallons = 1/3(time) + 5, and gives an example from the table.
3	Covariational reasoning or correspondence reasoning explanation is correct but does not give an example from the table, or Covariational reasoning or correspondence reasoning explanation is weak but an example is given from the table.
2	Covariational reasoning or correspondence reasoning is weak and explanation does not use data from table.
1	Only states that the relationship is linear because the rate at which the pool is filled is constant. No explanation from data in the table is used and no evidence of covariational or correspondence reasoning.
0	Explanation is incorrect and no example is given from data in the table.
16.	**Construct a graph that models this situation on a coordinate plane. Put time on the x-axis and gallons of water on the y-axis. (4 points total)**
4	Ordered pairs from table are correct, correctly graphed, and time is on the x-axis and gallons of water is on the y-axis.
3	Ordered pairs from table are correct and correctly graphed but time is not on x-axis and gallons of water is not on y-axis.
2	Graphing is correct and axes are correct but data from table is incorrect, or Graphing is partially correct and axes are correct.
1	Only one ordered pair from table is graphed correctly and axes are correct, or Only axes are correct.
0	Graphing is incorrect and axes are incorrect.
17.	**If it is a linear relationship, explain how the graph tells you it is a linear relationship. If it is not a linear relationship, explain how the graph tells you that it is not a linear relationship. (4 points total)**
4	Explanation states that for any two points on the graph the change in the y-coordinates divided by the change in the x-coordinates is constant (or not) and an example from points on the graph is provided. The graph should show the points on the graph and a slope triangle that shows the value for the change in y-coordinates and change in x-coordinates.
3	Explanation is correct but no example is provided and shown on the graph, or Explanation is weak but an example is provided and shown on the graph.
2	Explanation is weak and no example is provided and shown on the graph.
1	Only explanation is that the graph is a straight line.
0	Explanation is incorrect and no example is provided from the graph.

18. Construct an equation that represents the relationship. (4 points total)	
4	Equation has 1) variables for gallons and time in right place, 2) correct slope, 3) slope multiplied by the variable for time, and 4) y-intercept in correct place, g = (1/3)t +5
3	3 out of 4 above correct.
2	2 out of 4 above correct.
1	1 out of 4 above correct.
0	0 out of 4 above correct.
19. Explain what information the numbers and variables in the equation represent. (4 points total)	
4	One variable represents the number of gallons in the pool. One variable represents the time elapsed after beginning to fill the pool. The slope (1/3) represents the rate at which water is being put into the pool. 1/3 of the time is the number of gallons in addition to 5 gallons. The y-intercept represents the gallons of water in the pool before Kevin started filling it.
3	3 out of 4 above correct.
2	2 out of 4 above correct.
1	1 out of 4 above correct.
0	0 out of 4 above correct.
20. Identify a quantity that is found in the table, graph, and equation for this relationship and explain how it is expressed in the table, graph, and equation. For example, how is the rate of change expressed in the table, graph, and equation? (4 points total)	
4	An appropriate quantity is identified and explained with examples from the table, graph, and equation.
3	Quantity is appropriate but explanation is weak and example is provided from table, graph, and equation, or Quantity is appropriate and explanation is correct but examples from table, graph, and equation are incomplete.
2	Only quantity and explanation are correct or examples provided.
1	Only quantity is given with no explanation or examples.
0	No quantity, explanation, or examples provided.

APPENDIX H

IRB Exemption Approval Letters

BOISE STATE UNIVERSITY
RESEARCH AND ECONOMIC DEVELOPMENT

Date: July 21, 2014

To: Paul Belue cc: Margaret Kinzel

From: Office of Research Compliance (ORC)

Subject: SB-IRB Notification of Exemption - 014-SB14-107
The Impact of a Quantitative Reasoning Instructional Approach of Linear Equations in Two Variables On Student Achievement

The Boise State University ORC has reviewed your protocol application and has determined that your research is exempt from further IRB review and supervision under 45 CFR 46.101(b).

Protocol Number: 014-SB14-107
Approved: 7/21/2014 Application Received: 7/17/2014
Review: Exempt
Category:

This exemption covers any research and data collected under your protocol as of the date of approval indicated above, unless terminated in writing by you, the Principal Investigator, or the Boise State University IRB. All amendments or changes (including personnel changes) to your approved protocol **must** be brought to the attention of the Office of Research Compliance for review and approval before they occur, as these modifications may change your exempt status. Complete and submit a Modification Form indicating any changes to your project.

Annual renewals are not required for exempt protocols. When the research project is completed, please notify our office by submitting a Final Report. The exempt status expires when the research project is completed (closed) or when the review category changes as described above.

All forms are available on the ORC website at http://goo.gl/D2FYTV

Please direct any questions or concerns to ORC at 426-5401 or humansubjects@boisestate.edu.

Thank you and good luck with your research.

Office of Research Compliance

1910 University Drive Boise, Idaho 83725-1139
Phone (208) 426-5401 orc@boisestate.edu
This letter is an electronic communication from Boise State University

College of Western Idaho

School of Social Sciences
and Public Affairs
Ada County Center
1360 S. Eagle Flight Way
Boise, ID 83709

Mary Rohlfing
Assistant Dean
Phone (208) 562-2646
Fax (208) 562-2490

August 11, 2014

Paul Thomas Belue
6210 N. Saguro Hills Avenue
Meridian, ID 83646
paulbelue@cwidaho.cc

Dear Paul:

Thank you for submitting your IRB review application for your research, "The Impact of Quantitative Reasoning Instructional Approach of Linear Equations in Two Variables On Student Achievement."

In reviewing your application, the committee has determined that your study qualifies for "exempt" status.

All the best,

Regards,

Dr. Mary E. Rohlfing
Assistant Dean, School of Social Sciences and Public Affairs

Mailing address:
College of Western Idaho
MS 5000
PO Box 3010
Nampa, ID 83653

cc: Melissa Bechaver
CWI, MS 1000