MONODROMY REPRESENTATION OF THE BRAID GROUP

by

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ABSTRACT

In the mid 1980s, it was realized that solutions to what is known as the Knizhnik-Zamolodchikov equation, or KZ equation, provided a pathway to representations of the braid group B_n on n strands, with early mathematical treatments of the topic by Kohno and Drinfel'd. Such representations are typically referred to as *monodromy representations* of the braid group along solutions of the KZ equation. These linear representations are of great interest within topology, integral to the construction of isotopy invariants of knots and links, such as the well known Jones polynomial. More current discussions of the KZ equation and the associated monodromy representations are available in [6] and [9]. The former provides extensive algebraic background, while assuming a broad knowledge of differential geometry and eschewing certain calculable details of an explicit monodromy representation. The latter is more elementary, while containing nontrivial gaps and irregularities in the presentation. The following is intended to be a complement to both. Chapter 3 provides details of the argument by which solutions of the KZ equation induce representations of the braid group B_n for arbitrary n. Chapter 4 solves the KZ equation in the cases of $n = 2, 3$ and carries out explicit calculation of the monodromy representation on generators of the respective braid groups. From the work of Sections 3.1, 3.2, and 4.2.1, it is observed that the representation property of the KZ representations may be reduced to the uniqueness of solution to a particular initial value problem.

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CHAPTER 1

INTRODUCTION

1.1 Classical Monodromy

The following discussion is intended to hint at the overall concepts used in the paper by pointing out a few analogous notions in a more familiar context, such as functions defined on the complex plane. Consider $f(z) = \log z$ restricted by $\arg z \in (-\pi, \pi]$. Constructing an analytic continuation of f along a loop γ in $\mathbb{C}$ running once (counterclockwise) around the origin yields a multivalued function, simply meaning f does not return to $f(z)$ as the path returns to z. Rather f now differs by $2\pi i$ at z, by virtue of the fact that

$$\int_\gamma \frac{dz}{z} = 2\pi i,$$

which holds for any closed path taken once around the origin in $\mathbb{C}$. The study of such multivalued behavior falls under the heading of monodromy. To further elaborate on our example from another perspective, consider the universal cover $\exp\colon \mathbb{C} \to \mathbb{C} \setminus \{0\}$ of the punctured complex plane with fiber $F_z = \exp^{-1}(z)$, where $\exp^{-1}$ can be considered the (multivalued) log function. Given the aforementioned loop $\gamma\colon I \to \mathbb{C} \setminus \{0\}$ based at z, consider the unique lift $\tilde{\gamma}$, beginning at $\tilde{\gamma}(0) = \tilde{z} \in F_z$ and ending at $\tilde{\gamma}(1) \in F_z$, which we also denote suggestively by $\gamma\tilde{z}$. It may be that

$\tilde{\gamma}(0) \neq \tilde{\gamma}(1)$. It can be shown that such a construction defines the *monodromy action*, a well-defined action of the fundamental group $\pi_1(\mathbb{C} \setminus \{0\})$ on the fiber F_z via $\tilde{z} \mapsto \gamma\tilde{z}$. The induced homomorphism $\pi_1(\mathbb{C} \setminus \{0\}, z) \to \mathrm{Aut}(F_z)$ is the *monodromy* of the covering, capturing the nonsingular behavior latent within the covering. Key to these introductory remarks is observing that the base space $\mathbb{C} \setminus \{0\}$ is homotopy equivalent to a *configuration space* (see [5]) of two points in $\mathbb{C}$. Viewing the base space from this perspective, the preceding construction may be extended to a more general configuration space X of n distinct points in $\mathbb{C}$. In this setting, the KZ equation extends the associated action of the *permutation monodromy* to an action on $V^{\otimes n}$, where V is a Lie algebra representation of a finite dimensional semi-simple Lie algebra. A path γ in X will be lifted along multivalued solutions $W\colon X \to V^{\otimes n}$ of the KZ equation, yielding a map $W(\gamma(0)) \mapsto W(\gamma(1))$, typically refered to as *parallel transport*. Structure unique to the Lie algebra (see [9, Lemma 5.1]) plays an important role in establishing the homotopy invariance of the parallel transport, which in turn induces a homomorphism, or monodromy representation, $\pi_1(X) \to \mathrm{Aut}(V^{\otimes n})$. It is also worth pointing out the similarity between the KZ equation in the case $n = 2$ and the integral formula giving the number of twists in a braid on two strands. Solutions to the KZ equation can in general be presented in integral form, albeit using iterated integrals. By virtue of the KZ equation's higher dimensionality and communication of additional braid group information, the KZ equation can be seen as a far reaching generalization of such a formula.

1.2 Braid Relatives

It would be remiss not to briefly mention the close relationship between braids and links. It is known from a theorem of W. A. Alexander that every oriented link in $\mathbb{R}^3$ is isotopic to the closure of a braid [7, pg. 59]. Isotopic braids close up to isotopic links. The converse does not necessarily hold. Two braids have isotopic closure if and only if they are equivalent under so called Markov moves [7, pg. 68]. Using specific normalizations of traces of the endomorphisms associated to braids by our representations, isotopy invariants of links like the well known Jones polynomial can be defined. See [6] and [9] for further details.

1.3 Useful Topics

It may be worthwhile for the reader, before going further, to briefly mention some of the requisite mathematics discussed in the paper, and possible sources of reference. For discussions of braids and configuration spaces, consult [5] and [7]. Introductory material on Lie algebras, universal enveloping algebras, and their representations may be found in [4]. For background covering complex manifolds and complex valued differential forms, as well as bundle structures, see [11]. Extensions to vector-valued and algebra-valued forms can be found in [2]. We would like to remind the reader that a differential form on a complex manifold M with values in an algebra A assigns to each point of M and tangent vector at that point an element of A, where the form is complex linear on the tangent space.

CHAPTER 2

THE KZ EQUATION

2.1 Braids and Configuration Spaces

Let

$$X_n = \{(z_1, ..., z_n) \in \mathbb{C}^n \mid z_i \neq z_j \text{ for all } i \neq j\},$$

be the *configuration space* of (ordered) n-tuples of distinct points in $\mathbb{C}$, or the set of (ordered) *configurations* of n distinct points of $\mathbb{C}$.

In the following, the unit interval $[0, 1]$ is denoted by I. For a C^∞, or *smooth* configuration path $\gamma\colon I \to X_n$ defined by

$$\gamma(t) = (z_1(t), ..., z_n(t)),$$

the set

$$\coprod_{i=1}^{n} \left(\bigcup_{t \in I} (z_i(t), t) \right) \subset \mathbb{C} \times I,$$

i.e., the disjoint union of n strands each diffeomorphic to I, is the (geometric) braid defined by the configuration path. A homotopy of the configuration path yields an isotopy of the corresponding braid.

If γ is a loop in X_n, that is to say, if $z_i(0) = z_i(1)$ for $1 \leq i \leq n$, then the corresponding braid is called a *pure braid*. Then, the *pure braid group* P_n on n

strands is defined by the fundamental group $\pi_1(X_n, b)$ where the base point b is the n-tuple $(1, ..., n) \in X_n$.

The symmetric group S_n acts freely on X_n by permutation of the coordinates. This yields the orbit or quotient space X_n/S_n. The *braid group* B_n on n strands is then the fundamental group $\pi_1(X_n/S_n, q)$, where the base point q is the *set* $\{1, ..., n\} \subset \mathbb{C}$.

Let $s_i \in S_n$ be the simple transposition $(i, i+1)$, and $\sigma_i \in B_n$ a braid generator (see Figure 1). There exists a unique group homomorphism $\pi\colon B_n \to S_n$ defined by $\pi(\sigma_i) = s_i$ for $1 \leq i \leq n-1$. As the simple transpositions generate S_n, the map is well defined and surjective. We have $P_n = \operatorname{Ker} \pi$, and the two fundamental groups are then related by the short exact sequence

$$1 \to P_n \xrightarrow{\iota} B_n \xrightarrow{\pi} S_n \to 1,$$

where $\iota\colon P_n \to B_n$ is the inclusion.

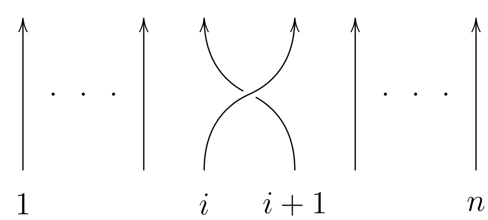

Figure 2.1: The braid generator σ_i of B_n

2.2 The Algebra $U(\mathfrak{g})^{\otimes n}$

Let $\mathfrak{g}$ be a finite dimensional $\mathbb{C}$-vector space with a bilinear map $[\ ,\ ]\colon \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the *Lie bracket*, satisfying the following two conditions for all $x, y, z \in \mathfrak{g}$:

(1) (*antisymmetry, equivalent to alternating* $[x, x] = 0$)

$$[x, y] = -[y, x]$$

(2) (*Jacobi identity*)

$$[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$$

Then $\mathfrak{g}$ quipped with such a Lie bracket is a *Lie algebra.* The *adjoint representation* $\mathrm{ad}\colon \mathfrak{g} \to \mathrm{End}(\mathfrak{g})$ is defined by $\mathrm{ad}_x(y) = [x, y]$. The symmetric bilinear *Killing form* $B\colon \mathfrak{g} \otimes \mathfrak{g} \to \mathbb{C}$ is defined by $B(x, y) = \mathrm{trace}(\mathrm{ad}_x \mathrm{ad}_y)$. If the Killing form is non-degenerate, $\mathfrak{g}$ is called a *semi-simple* Lie algebra.

As $\mathfrak{g}$ is a *complex* vector space $\mathfrak{g}$ has orthonormal basis $\{I_\mu\}$ with respect to the Killing form. See [4, Lemma 16.14].

The Lie algebra $\mathfrak{g}$ embeds into the *universal enveloping algebra* $U(\mathfrak{g})$ of $\mathfrak{g}$, defined to be the associative algebra over $\mathbb{C}$ with multiplicative unit 1, generated by a basis $\{X_i\}$ of $\mathfrak{g}$ satisfying the relations $X_i X_j - X_j X_i = [X_i, X_j]$. It is known that this algebra does not depend on the choice of basis $\{X_i\}$. See [4, exercise 15.8] .

The Lie algebra action of $\mathfrak{g}$ on a finite dimensional vector space V extends to an action of $U(\mathfrak{g})$ on V. See [4, Definition 7.2 and Lemma 15.10]. This action can then be extended to a component-wise action of $U(\mathfrak{g})^{\otimes n}$ on $V^{\otimes n}$ by defining

$$(x_1 \otimes \cdots \otimes x_n)(v_1 \otimes \cdots \otimes v_n) = x_1 v_1 \otimes \cdots \otimes x_n v_n,$$

for $x_i \in U(\mathfrak{g})$ and $v_i \in V$. In particular, this defines the action (via linear extension) of such elements as

$$\tau_{ij} = \sum_{\mu} 1 \otimes \cdots \otimes 1 \otimes I_\mu \otimes 1 \otimes \cdots \otimes 1 \otimes I_\mu \otimes 1 \otimes \cdots \otimes 1 \in U(\mathfrak{g})^{\otimes n}$$

on $V^{\otimes n}$, where I_μ appears in the ith and jth factors of the tensor product $U(\mathfrak{g})^{\otimes n}$, with $1 \leq i \neq j \leq n$.

2.3 The 1-form τ_n

Define on X_n the differential system

$$dW = \frac{\hbar}{2\pi\sqrt{-1}} \sum_{1 \leq i < j \leq n} \tau_{ij} \frac{dz_i - dz_j}{z_i - z_j} W, \tag{2.1}$$

with complex parameter $\hbar$, referred to as the *Knizhnik-Zamolodchikov equation*, or *KZ equation*. A solution is a smooth function $W\colon X_n \to V^{\otimes n}$ satisfying (2.1). The right hand side of (2.1) contains the 1-form

$$\tau_n = \frac{\hbar}{2\pi\sqrt{-1}} \sum_{1 \leq i < j \leq n} \tau_{ij} \frac{dz_i - dz_j}{z_i - z_j}$$

on X_n, taking values in $U(\mathfrak{g})^{\otimes n}$. This 1-form defines a connection (see [8]) $d - \tau_n$ on the trivial vector bundle $X_n \times V^{\otimes n}$. An arbitrary connection Γ is said to be *flat* if it satisfies $d\Gamma - \Gamma \wedge \Gamma = 0$.

Remark. For $\alpha \in \Omega^1(X_n, U(\mathfrak{g})^{\otimes n})$, and $v, w \in T_p X_n$ for some $p \in X_n$ the standard wedge product gives $(\alpha \wedge \alpha)(v, w) = \alpha(v)\alpha(w) - \alpha(w)\alpha(v)$. See [2, pgs. 1-11] for standard facts on forms. As α is operator or matrix-valued by virtue of the action of $U(\mathfrak{g})^{\otimes n}$ on $V^{\otimes n}$, $\alpha(v)$ and $\alpha(w)$ do not necessarily commute in the operator algebra. Consequently, $\alpha \wedge \alpha$ may be nonzero.

In particular, it is the flatness of $d - \tau_n$ that permits construction of the monodromy representations of $\pi_1(X_n) = P_n$, and ultimately B_n, the full braid group. In general, a representation can be constructed from any given flat connection on X_n. It is this more general setting that we address first.

CHAPTER 3

MONODROMY REPRESENTATION OF THE BRAID GROUP

This chapter is devoted to the construction and verification of certain properties of the so called parallel transport map, properties that allow us to subsequently define a map on homotopy classes $[\gamma]$ of paths γ in X_n and X_n/S_n. Assigning homotopy classes of paths to a corresponding parallel transport map induces the desired monodromy representations. Proposition 3.1.1 will be used in demonstrating Lemma 3.1.2 of Section 3.1. The proof of Proposition 3.1.1 is addressed in Section 3.2. The statement of Proposition 3.1.1 first is a matter of expositional choice.

3.1 Parallel Transport and Path Composition

Proposition 3.1.1. *Given a smooth manifold X and a non-commutative algebra A acting on a complex vector space V', let $\alpha \in \Omega^1(X, A)$. If the connection $d - \alpha$ is flat, then for functions $W\colon X \to V'$, the differential equation $(d - \alpha)W = 0$ (note the KZ equation is of this form) with arbitrary initial condition $v = W(x_0)$ has a unique local solution W in a neighborhood of any point $x_0 \in X$.*

The following lemma requires defining the parallel transport map on the fibers V' of the trivial bundle $X \times V' \to X$. This definition requires the following constructions. Consider a path $\gamma\colon I \to X$ connecting the point x_0 with an arbitrary point x in a neighborhood of x_0, i.e., $\gamma(0) = x_0$ and $\gamma(1) = x$. A 1-form on the unit interval I can be obtained from α via the associated pullback $\gamma^*\alpha$. This defines a function $w_\gamma\colon I \to A$, by $\gamma^*\alpha_t = w_\gamma(t)dt$, with $w_\gamma(t)dt \in \Omega^1(I, A)$. Note that $w_\gamma(t) = \alpha(\gamma_* \frac{\partial}{\partial t})_t$.

Consider the differential equation

$$\frac{df}{dt} = w_\gamma f, \tag{3.1}$$

for functions $f\colon I \to V'$. Let $f^v_\gamma(t)$ denote the solution with $f^v_\gamma(0) = W(x_0) = v$. Also define $f^v_\gamma(1) = W(x)$. We can now define the *parallel transport* map $\mu_\gamma\colon V'_{\gamma(0)} \to V'_{\gamma(1)}$ as follows:

$$\mu_\gamma(v) = f^v_\gamma(1),$$

or

$$\mu_\gamma\colon f^v_\gamma(0) \mapsto f^v_\gamma(1),$$

where $\gamma(0) = x_0$, and $\gamma(1) = x$. Also $V'_{\gamma(t)}$ indicates the fiber over any point $\gamma(t) \in X$ with $V'_{\gamma(t)}$ equal to V'.

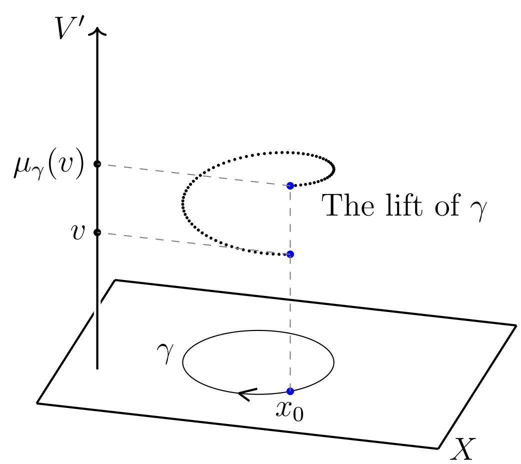

Figure 3.1: The graph of W

Lemma 3.1.2. *Given a path $\gamma\colon I \to X$,*

(a) The map μ_γ is linear, i.e.,

$$\mu_\gamma(v+w) = \mu_\gamma(v) + \mu_\gamma(w),$$

and

$$\mu_\gamma(\lambda v) = \lambda\mu_\gamma(v).$$

for $\lambda \in \mathbb{C}$.

(b) Given $\eta, \gamma\colon I \to X$ such that $\eta(1) = \gamma(0)$, then

$$\mu_{\gamma\eta} = \mu_\gamma \circ \mu_\eta.$$

(c) If $\gamma \simeq \eta$ then $\mu_\gamma = \mu_\eta$. In particular, if $\gamma^{-1}\gamma \simeq *$ then

$$\mu_{\gamma^{-1}\gamma} = \mu_* = id_{V'}.$$

The map μ_γ is also sometimes referred to as the *holonomy operator* along γ.

Proof. (a) It needs to be shown that $f_\gamma^{\lambda v}(1) + f_\gamma^{\lambda w}(1) = \lambda f_\gamma^{v+w}(1)$. Given that (3.1) is a homogeneous differential equation, the solution space is a linear space. Given two solutions $f_\gamma^{\lambda v}$ and $f_\gamma^{\lambda w}$ of (3.1), their sum is then a solution as well, with initial condition $\lambda(v+w)$. This solution is precisely λf_γ^{v+w}. Thus μ_γ is linear (by construction).

(b) For two paths $\eta, \gamma\colon I \to X$, define the composite path

$$\gamma\eta(t) = \begin{cases} \eta(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ \gamma(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases}$$

The aim is to show

$$f^v_{\gamma\eta}(t) = \begin{cases} f^v_\eta(2t) & \text{if } 0 \le t \le \frac{1}{2}, \\ f_\gamma^{f^v_\eta(1)}(2t-1) & \text{if } \frac{1}{2} \le t \le 1. \end{cases} \tag{3.2}$$

Then

$$\mu_{\gamma\eta}(v) = f^v_{\gamma\eta}(1) = f_\gamma^{f^v_\eta(1)}(1) = \mu_\gamma(f^v_\eta(1)) = \mu_\gamma(\mu_\eta(v))$$

for all $v \in V'$, showing the parallel transport map is well defined with respect to path composition.

For $t \in [0, \frac{1}{2}]$,

$$\begin{aligned} w_{\gamma\eta}(t)dt =& (\gamma\eta)^*\alpha_t \\ =& \eta^*\alpha_{2t} \\ =& w_\eta(2t)d(2t) \\ =& 2w_\eta(2t)dt. \end{aligned}$$

Thus

$$w_{\gamma\eta}(t) = 2w_\eta(2t). \tag{3.3}$$

Since $f^v_{\gamma\eta}(t)$ solves

$$\left.\frac{df}{dt}\right|_t = w_{\gamma\eta}(t)f(t), \qquad \text{with } f(0) = v,$$

from (3.3) it also solves

$$\left.\frac{df}{dt}\right|_t = 2w_\eta(2t)f(t), \qquad \text{with } f(0) = v. \tag{3.4}$$

Let $g(t) = f_\eta^v(2t)$. Then

$$\begin{aligned}\left.\frac{dg}{dt}\right|_t =& 2\left.\frac{df_\eta^v}{dt}\right|_{2t}\\ =& 2w_\eta(2t)f_\eta^v(2t)\\ =& 2w_\eta(2t)g(t).\end{aligned}$$

As $g(0) = v$, g solves (3.4). Thus $f_{\gamma\eta}^v(t)$ and $f_\eta^v(2t)$ agree on $[0, \frac{1}{2}]$.

For $t \in [\frac{1}{2}, 1]$,

$$\begin{aligned}w_{\gamma\eta}(t)dt =& (\gamma\eta)^*\alpha_t\\ =& \gamma^*\alpha_{2t-1}\\ =& w_\gamma(2t-1)d(2t-1)\\ =& 2w_\gamma(2t-1)dt.\end{aligned}$$

Thus

$$w_{\gamma\eta}(t) = 2w_\gamma(2t-1). \tag{3.5}$$

Since $f_{\gamma\eta}^v(t)$ solves

$$\left.\frac{df}{dt}\right|_t = w_{\gamma\eta}(t)f(t), \qquad \text{with } f\left(\frac{1}{2}\right) = f_\eta^v(1),$$

from (3.5) it also solves

$$\left.\frac{df}{dt}\right|_t = 2w_\gamma(2t-1)f(t), \qquad \text{with } f\left(\frac{1}{2}\right) = f_\eta^v(1). \tag{3.6}$$

Let $g(t) = f_\gamma^{f_\eta^v(1)}(2t-1)$. Then

$$\begin{aligned}\left.\frac{dg}{dt}\right|_t =& 2\left.\frac{df_\gamma^{f_\eta^v(1)}}{dt}\right|_{2t-1}\\ =& 2w_\gamma(2t-1)f_\gamma^{f_\eta^v(1)}(2t-1)\\ =& 2w_\gamma(2t-1)g(t).\end{aligned}$$

As $g\left(\frac{1}{2}\right) = f_\eta^v(1)$, g solves (3.6). Thus $f_{\gamma\eta}^v(t)$ and $f_\gamma^{f_\eta^v(1)}(2t-1)$ agree on $[\frac{1}{2}, 1]$. This proves (3.2) and thus (b) holds.

(c) This follows from Proposition 1. □

3.2 Homotopy Invariance of Parallel Transport

This section provides a proof of Proposition 3.1.1, which is a standard result. See e.g. [8, Chapter 9].

Using the same constructions as the Lemma 1, consider a path $\phi\colon I \to X$ connecting the point x_0 with an arbitrary point x in a neighborhood of x_0, i.e., $\phi(0) = x_0$ and $\phi(1) = x$. For arbitrary v in the fiber V'_{x_0}, define the map

$$f_\phi(t)\colon v \mapsto f_\phi^v(t).$$

Then $f_\phi(t) \in \mathrm{Aut}(V')$. Thus the automorphism $f_\phi(t)$ corresponds to a family of solutions satisfying (3.1), effectively indexed by the initial condition v. Thus

$$\frac{df(t)}{dt} = \alpha\left(\phi_*\frac{\partial}{\partial t}\right)_t f(t) \tag{3.7}$$

has solution $f_\phi(t)$, with $f_\phi(0) = id_{V'}$.

Let θ_0 and θ_1 be two paths connecting x_0 and x lying within a neighborhood of x_0. As local paths they are homotopic. Hence, between them there exists a homotopy $\psi\colon I \times I \to X$ such that $\theta_0([0,1]) = \psi((\{0\} \times [0,1]) \cup ([0,1] \times \{1\}))$ and $\theta_1([0,1]) = \psi(([0,1] \times \{0\}) \cup (\{1\} \times [0,1]))$. The associated pullback

$$\psi^*\alpha = p(x,y)dx + q(x,y)dy,$$

defines a one form $\psi^*\alpha \in \Omega^1(I \times I, A)$. Define the differential equations

$$\frac{\partial P(x,y)}{\partial x} = p(x,y)P(x,y), \qquad P(0,y) = id_{V'} \tag{3.8}$$

$$\frac{\partial Q(x,y)}{\partial y} = q(x,y)Q(x,y), \qquad Q(x,0) = id_{V'}. \tag{3.9}$$

Let $P(x,y)$ and $Q(x,y)$ denote the respective solutions. Define three paths $I \to I \times I$ as follows:

$$\iota\colon t \mapsto (t,0)$$

$$\gamma_T\colon t \mapsto (T,t)$$

$$\rho\colon t \mapsto (t,1).$$

Define the composite path $\tau_T\colon [0,2] \to I \times I$ by

$$\tau_T(t) = \begin{cases} \iota(t) & \text{if} \quad 0 \le t \le T, \\ \gamma_T(t-T) & \text{if} \quad T \le t \le T+1, \\ \rho(t-1) & \text{if} \quad T+1 \le t \le 2. \end{cases} \tag{3.10}$$

Then

$$f_{\psi\circ\rho}(0) = f_{\psi\circ\gamma_T}(0) = f_{\psi\circ\iota}(0) = id_{V'}.$$

Notice that Im $\theta_0 = \text{Im}\,(\psi \circ \tau_0)$ and Im $\theta_1 = \text{Im}\,(\psi \circ \tau_1)$.

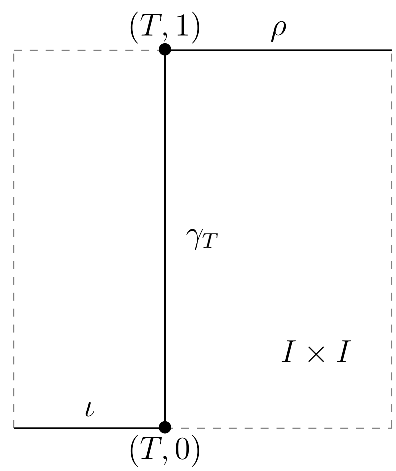

Figure 3.2: The path τ_T on the unit square

The aim is to show that given $\psi \circ \tau_T \colon [0,2] \to X$,

$$f_{\psi\circ\tau_T}(2)$$

does not depend on T.

From previous notation is the equality $\mu_\phi = f_\phi(1)$. Used throughout is the compatibility of path composition with composition of the corresponding automorphisms. That is to say, given appropriately compatible paths $\gamma, \eta \colon I \to X$,

$$\mu_{\gamma\eta} = \mu_\gamma \circ \mu_\eta,$$

which was shown previously. Alternatively,

$$f_{\gamma\eta}(a+b) = f_\gamma(a) \circ f_\eta(b),$$

with $\gamma\colon [0,a] \to X$ and $\eta\colon [0,b] \to X$.

For the path $\psi \circ \iota$ in X, the associated pullback is

$$\begin{aligned}(\psi \circ i)^*\alpha &= i^*(\psi^*\alpha) \\ &= (p \circ i)i^*dx + (q \circ i)i^*dy.\end{aligned}$$

As i has unit speed, $i_*\frac{\partial}{\partial t} = \frac{\partial}{\partial x}$. Thus $i^*dx = dt$ and $i^*dy = 0$. This gives

$$\begin{aligned}\frac{df_{\psi\circ\iota}(t)}{dt} &= (p \circ i)(t) f_{\psi\circ\iota}(t) \\ &= p(t,0) f_{\psi\circ\iota}(t), \qquad f_{\psi\circ\iota}(0) = id_{V'}.\end{aligned}$$

From (3.8), it's clear that $P(t,0)$ satisfies this identical differential equation with respect to t. In particular, $f_{\psi\circ\iota}(T) = P(T,0)$.

In a similar fashion, it can be shown that $f_{\psi\circ\gamma_T}(t) = Q(T,t)$, and $f_{\psi\circ\rho}(t) = P(t,1)$.

Lemma 3.2.1.

$$f_{\phi^{-1}}(1) = f_\phi(1)^{-1}$$

Proof. Define the inverse path ϕ^{-1}

$$\phi^{-1}(t) = \phi(1-t).$$

Since

$$\left.\frac{d\phi^{-1}}{dt}\right|_t = -\left.\frac{d\phi}{dt}\right|_{1-t}$$

and α is linear on the tangent bundle TX, it follows that

$$w_{\phi^{-1}}(t) = -w_\phi(1-t).$$

Let $g(t) = f_\phi(1-t)$. Then

$$\left.\frac{dg}{dt}\right|_t = -\left.\frac{df_\phi}{dt}\right|_{1-t} \tag{3.11}$$

$$= -w_\phi(1-t) \circ f_\phi(1-t) \tag{3.12}$$

$$= w_{\phi^{-1}}(t) \circ f_\phi(1-t). \tag{3.13}$$

Further, define $h(t) = f_\phi(1-t) \circ f_\phi(1)^{-1}$. From (3.11-3.13), it follows that

$$\begin{aligned}
\left.\frac{dh}{dt}\right|_t &= -\left.\frac{df_\phi}{dt}\right|_{1-t} \circ f_\phi(1)^{-1} \\
&= -w_\phi(1-t) \circ f_\phi(1-t) \circ f_\phi(1)^{-1} \\
&= w_{\phi^{-1}}(t) \circ f_\phi(1-t) \circ f_\phi(1)^{-1} \\
&= w_{\phi^{-1}}(t) \circ h(t).
\end{aligned}$$

As $h(0) = id_{V'}$, it's clear that h satisfies (3.7) with respect to ϕ^{-1}. Thus $f_{\phi^{-1}}(t) = f_\phi(1-t) \circ f_\phi(1)^{-1}$. In particular, Lemma 3.2.1 holds. □

Resuming the proof of Proposition 3.1.1, as $f_{\psi\circ\rho}(T) = P(T, 1)$, Lemma 2 gives

$$\begin{aligned} f_{(\psi\circ\rho)^{-1}}(T) =& f_{\psi\circ\rho}(T)^{-1} \\ =& P(T,1)^{-1}. \end{aligned}$$

Then

$$f_{\psi\circ\rho}(1)f_{\psi\circ\rho}(T)^{-1}f_{\psi\circ\gamma_T}(1)f_{\psi\circ\iota}(T) = P(1,1)P(T,1)^{-1}Q(T,1)P(T,0). \tag{3.14}$$

The invariance of the right side of (3.14) with respect to T is shown in Appendix C. (Note it is only there that the flatness of $d-\alpha$ is used.) It remains to show (3.13) is equal to $f_{\psi\circ\tau_T}(2)$.

From (3.10), it follows that

$$w_{\psi\circ\tau_T}(t) = \begin{cases} w_{\psi\circ\iota}(t) & \text{if} \quad 0 \le t \le T, \\ w_{\psi\circ\gamma_T}(t-T) & \text{if} \quad T \le t \le T+1, \\ w_{\psi\circ\rho}(t-1) & \text{if} \quad T+1 \le t \le 2. \end{cases}$$

It then follows from (3.7) that

$$f_{\psi\circ\tau_T}(t) = \begin{cases} f_{\psi\circ\iota}(t) & \text{if} \quad 0 \le t \le T, \\ f_{\psi\circ\gamma_T}(t-T)f_{\psi\circ\iota}(T) & \text{if} \quad T \le t \le T+1, \\ f_{\psi\circ\rho}(t-1)f_{\psi\circ\gamma_T}(1)f_{\psi\circ\iota}(T) & \text{if} \quad T+1 \le t \le 2. \end{cases}$$

In particular,

$$f_{\psi\circ\tau_T}(2) = f_{(\psi\circ\rho)|_{[T,1]}}(1)f_{(\psi\circ\gamma_T)|_{[0,1]}}(1)f_{(\psi\circ\iota)|_{[0,T]}}(T).$$

As

$$f_{(\psi\circ\rho)|_{[T,1]}}(1) f_{(\psi\circ\rho)|_{[0,T]}}(T) = f_{(\psi\circ\rho)|_{[0,1]}}(1),$$

it is immediate that

$$f_{(\psi\circ\rho)|_{[T,1]}}(1) = f_{(\psi\circ\rho)|_{[0,1]}}(1) f_{(\psi\circ\rho)|_{[0,T]}}(T)^{-1}.$$

This equates (3.14) and $f_{\psi\circ\tau_T}(2)$. This establishes the proposition.

For verification that the connection $d - \tau_n$ is flat, and consequently Proposition 3.1.1 is applicable to the KZ equation, see [6, pg. 452] or [9, pg. 106]. Consequently, for a closed path $\gamma\colon I \to X_n$, Lemma 1 defines the parallel transport map $\mu_\gamma\colon V_{\gamma(0)}^{\otimes n} \to V_{\gamma(1)}^{\otimes n}$. As a homotopy class of γ corresponds to a pure braid in X_n, this combines with the established properties of μ_γ to induce the *monodromy representation* of the pure braid group via the assignment $[\gamma] \mapsto \mu_\gamma$. In short, this assignment induces a homomorphism

$$P_n \to \mathrm{Aut}(V^{\otimes n}).$$

Theorem 3.2.2. *The assignment above defines a homomorphism.*

$$\mathfrak{m}\colon B_n \to Aut(V^{\otimes n}).$$

Proof. Consider the left action given by

$$s(v_1 \otimes \cdots \otimes v_n) = v_{s^{-1}(1)} \otimes \cdots \otimes v_{s^{-1}(n)}$$

for $s \in S_n$ and $v_1, ..., v_n \in V$. It is then possible to define a right action of S_n on the trivial vector bundle $X_n \times V^{\otimes n}$ given by

$$(z_1, ..., z_n; v)s = (z_{s(1)}, ..., z_{s(n)}; s^{-1}v) \tag{3.15}$$

for $s \in S_n$, $(z_1, ..., z_n) \in X_n$, and $v \in V^{\otimes n}$. The resulting quotient space $(X_n \times V^{\otimes n})/S_n$ is a (nontrivial) vector bundle over X_n/S_n, where the composition

$$X_n \times V^{\otimes n} \xrightarrow{p} X_n \xrightarrow{q} X_n/S_n$$

is constant on the equivalence classes of $(X_n \times V^{\otimes n})/S_n$, where p is the projection and q is the quotient map. For a closed path $\gamma \colon I \to X_n/S_n$ based at $[x_0] = [(z_1, ..., z_n)]$, a given $s \in S_n$ determines a lift $\tilde{\gamma} \colon I \to X_n$ with $\tilde{\gamma}(0) = x_0$ and $\tilde{\gamma}(1) = sx_0 = (z_{s(1)}, ..., z_{s(n)})$. (The action of S_n on $V^{\otimes n}$ is employed in the next section.) The same construction of parallel transport $\mu_{\tilde{\gamma}} \colon f^v_{\tilde{\gamma}}(0) \mapsto f^v_{\tilde{\gamma}}(1)$ defines a linear map on the fiber (still identified with $V^{\otimes n}$) of the nontrivial bundle $(X_n \times V^{\otimes n})/S_n$. Alternatively, it is clear that the KZ equation is invariant under the action of S_n, and consequently $d - \tau_n$ descends to a (flat) connection on X_n/S_n. From either perspective, as a closed path in X_n/S_n corresponds to an arbitrary braid in B_n, the assignment $[\tilde{\gamma}] \mapsto \mu_{\tilde{\gamma}}$ induces a homomorphism $B_n \to \mathrm{Aut}(V^{\otimes n})$. A more explicit definition will be provided for the case $n = 3$ in Section 4.1.1. $\square$

CHAPTER 4

SOLVING THE KZ EQUATION

The purpose of this chapter is to solve the KZ equation in the cases $n = 2, 3$ and explicitly compute the monodromy representation for generators of the braid groups B_2 and B_3. This in effect computes the representation in general, as automorphisms are multiplied, or composed, just as braid generators are multiplied. As a representation is a group homomorphism, in particular the braid relations will persist in $\mathrm{Aut}(V^{\otimes n})$. To that end, we compute the monodromy along paths that (up to homotopy) correspond to the braid generators.

4.1 The Case $n = 2$

When $n = 2$, the KZ equation (2.1) reduces to

$$dW = \frac{\hbar\tau_{12}}{2\pi\sqrt{-1}}\frac{dz_1 - dz_2}{z_1 - z_2}W.$$

Placing $\mathrm{Arg}(z_2 - z_1) \in (-\pi, \pi]$, the solution

$$W(z_1, z_2) = (z_2 - z_1)^{\hbar\tau_{12}/2\pi\sqrt{-1}}v$$

is single-valued, with $W(0, 1) = v \in V \otimes V$. See Appendix A for verification of the solution wherein we define z^A as $\exp(A \log z)$ and briefly discuss Was a multivalued (non)function. The braid generator σ of B_2 can be represented by the path $\gamma(t) = (z_1(t), z_2(t))$ for $t \in I$, where

$$z_1(t) = \frac{1}{2}(3 - e^{\pi\sqrt{-1}t}) \quad \text{and} \quad z_2(t) = \frac{1}{2}(3 + e^{\pi\sqrt{-1}t}).$$

This gives

$$W(\gamma(t)) = e^{\hbar\tau_{12}t/2}v.$$

Note that $\gamma(t)$ defines a loop in X_2/S_2, starting and ending at the point [1,2]. Restated, the simple transposition s_1, which generates S_2, determines the lift $\gamma(t) \in X_2$ of $[\gamma(t)] \in X_2/S_2$. As (3.15) explicitly states, the aforementioned right action of S_n on the trivial vector bundle $X_n \times V^{\otimes n}$ means within the nontrivial vector bundle $(X_n \times V^{\otimes n})/S_n$ is the equality

$$(2, 1; v_1 \otimes v_2) = (1, 2; v_2 \otimes v_1). \tag{4.1}$$

Subsequent to the monodromy action of σ on $v_1 \otimes v_2 \in V \otimes V$ given by

$$v_1 \otimes v_2 \mapsto e^{\hbar\tau_{12}/2}(v_1 \otimes v_2),$$

realized by taking t from 0 to 1, (4.1) calls for a permutation $P\colon V \otimes V \to V \otimes V$ of the resulting entries. The representation for σ is then given by

$$\mathfrak{m}(\sigma)(v_1 \otimes v_2) = P\left(e^{\hbar\tau_{12}/2}(v_1 \otimes v_2)\right).$$

4.2 The Case $n = 3$

When $n = 3$, the KZ equation (2.1) appears as

$$dW = \overline{h}\big(\tau_{12} d\log(z_1 - z_2) + \tau_{13} d\log(z_1 - z_3) + \tau_{23} d\log(z_2 - z_3)\big)W, \tag{4.2}$$

with $\overline{h} = \hbar/2\pi\sqrt{-1}$. For $\tilde{G}\colon X_3 \to V^{\otimes 3}$, define

$$W(z_1, z_2, z_3) = (z_3 - z_1)^{\overline{h}(\tau_{12}+\tau_{13}+\tau_{23})}\tilde{G}(z_1, z_2, z_3) \tag{4.3}$$

to affect a change of variable. Exterior differentiation of (4.3) combines with (4.2) to give

$$d\tilde{G} = \overline{h}\Big(\tau_{12} d\log\frac{z_2 - z_1}{z_3 - z_1} + \tau_{23} d\log\big(\frac{z_2 - z_1}{z_3 - z_1} - 1\big)\Big)\tilde{G}. \tag{4.4}$$

Making the change of variable $z = (z_2 - z_1)/(z_3 - z_1)$ defines $\tilde{G}(z_1, z_2, z_3) = G(z)$ and from (4.4) results the linear differential equation

$$dG = \frac{1}{2\pi\sqrt{-1}}\Big(A d\log z + B d\log(z - 1)\Big)G, \tag{4.5}$$

where $G(z)$ belongs to $\mathbb{C}\langle\langle A, B\rangle\rangle$, the ring of formal series in non-commuting variables A and B. Appendix B verifies $G(z)$ is a solution of the above if and only if $W(z_1, z_2, z_3)$ is a solution of the KZ system (4.2), with $A = \hbar\tau_{12}$ and $B = \hbar\tau_{23}$.

Lemma 4.2.1. *There exist unique solutions G_0 and G_1 of (4.5) such that*

$$G_0(z) = f(z) z^{A/2\pi\sqrt{-1}} \tag{4.6}$$

$$G_1(z) = g(1-z)(1-z)^{B/2\pi\sqrt{-1}}, \tag{4.7}$$

where f and g are respective analytic continuations in respective neighborhoods of 0 and 1 in $\mathbb{C}$, with $f(0) = g(0) = 1 \in \mathbb{C}\langle\langle A, B\rangle\rangle$. Both $z^{A/2\pi\sqrt{-1}}$ and $(1-z)^{B/2\pi\sqrt{-1}}$ are well defined on $\mathbb{C} \setminus (\,]-\infty, 0] \cup [1, \infty[\,)$.

Proof. Consider the second formula of the lemma. Let $\overline{A} = A/2\pi\sqrt{-1}$ and $\overline{B} = B/2\pi\sqrt{-1}$. Exterior differentiation of (4.7) gives

$$\begin{aligned} \frac{dG_1}{dz} &= -\frac{dg}{dz}\bigg|_{1-z}(1-z)^{\overline{B}} - g(1-z)\frac{\overline{B}(1-z)^{\overline{B}}}{(1-z)} \\ &= -\frac{dg}{dz}\bigg|_{1-z}(1-z)^{\overline{B}} + g(1-z)\frac{\overline{B}(1-z)^{\overline{B}}}{(z-1)} \\ &= \left(-\frac{dg}{dz}\bigg|_{1-z} + g(1-z)\frac{\overline{B}}{(z-1)}\right)(1-z)^{\overline{B}}. \end{aligned} \tag{4.8}$$

As G_1 is of a single variable z, (4.5) can be expressed as

$$\frac{dG_1}{dz} = \left(\frac{\overline{A}}{z} + \frac{\overline{B}}{z-1}\right) G_1. \tag{4.9}$$

Setting (4.8) equal to (4.9) gives

$$\begin{aligned} -\frac{dg}{dz}\bigg|_{1-z} &= \frac{\overline{A}}{z}g(1-z) + \frac{\overline{B}g(1-z)}{z-1} - \frac{g(1-z)\overline{B}}{z-1} \\ &= \frac{\overline{A}}{z}g(1-z) + \frac{1}{z-1}[\overline{B}, g(1-z)] \\ &= \frac{\overline{A}}{z}g(1-z) - \frac{1}{1-z}[\overline{B}, g(1-z)]. \end{aligned} \tag{4.10}$$

We construct a formal power series solution to (4.7) first by letting

$$g(1-z) = 1 + \sum_{k=1}^{\infty} g_k(1-z)^k,$$

which gives

$$-\frac{dg}{dz}\bigg|_{1-z} = \sum_{k=1}^{\infty} kg_k(1-z)^{k-1}. \tag{4.11}$$

Equating (4.10) and (4.11) gives

$$\begin{aligned} \sum_{k=1}^{\infty} kg_k(1-z)^{k-1} &= \frac{\overline{A}}{z}g(1-z) - \frac{1}{1-z}[\overline{B}, g(1-z)] \\ &= \frac{\overline{A}}{z}\left(1 + \sum_{i=1}^{\infty} g_i(1-z)^i\right) - \frac{1}{1-z}\left[\overline{B}, 1 + \sum_{m=1}^{\infty} g_m(1-z)^m\right] \\ &= \sum_{n=0}^{\infty} \overline{A}(1-z)^n \left(1 + \sum_{i=1}^{\infty} g_i(1-z)^i\right) - \sum_{m=1}^{\infty} [\overline{B}, g_m](1-z)^{m-1}. \end{aligned} \tag{4.12}$$

Equating coefficients of $(1-z)^{k-1}$ in (4.12) gives

$$kg_k = \overline{A}(1 + g_1 + \cdots + g_{k-1}) - [\overline{B}, g_k],$$

or in terms of the operator $k\,\mathrm{id} + \mathrm{ad}(\overline{B})$,

$$(k\,\mathrm{id} + \mathrm{ad}(\overline{B}))(g_k) = \overline{A}(1 + g_1 + \cdots + g_{k-1}). \tag{4.13}$$

Since

$$\begin{aligned}
\left(k\,\mathrm{id} + \mathrm{ad}(\overline{B})\right)^{-1} &= \frac{1}{k\,\mathrm{id} + \mathrm{ad}(\overline{B})} \\
&= \frac{1}{k}\left(\frac{1}{\mathrm{id} - \left(-\frac{\mathrm{ad}(\overline{B})}{k}\right)}\right) \\
&= \frac{1}{k}\sum_{i=0}^{\infty}\left(-\frac{\mathrm{ad}(\overline{B})}{k}\right)^{i} \\
&= \sum_{i=0}^{\infty}\frac{(-1)^i \mathrm{ad}(\overline{B})^i}{k^{i+1}},
\end{aligned}$$

g_k is uniquely determined (iteratively or recursively) by $g_1, \ldots, g_{k-1}$, and the inverse operator $\left(k\,\mathrm{id} + \mathrm{ad}(\overline{B})\right)^{-1}$ applied to (4.13) gives

$$g_k = \sum_{i=0}^{\infty}\frac{(-1)^i}{k^{i+1}}\mathrm{ad}(\overline{B})^i\left(\overline{A}(1 + g_1 + \cdots + g_{k-1})\right).$$

Refer to [10] for details assuring the convergence of $g(1-z)$ in $\mathbb{C}\langle\langle A, B\rangle\rangle$.

A solution for (4.6) can be obtained in a similar fashion. See [6, pg. 464] for details. □

For example,

$$\begin{aligned}
g_1 &= \sum_{i=0}^{\infty}(-1)^i \mathrm{ad}(\overline{B})^i(\overline{A}) \\
&= \overline{A} - [\overline{B}, \overline{A}] + [\overline{B}, [\overline{B}, \overline{A}]] - [\overline{B}, [\overline{B}, [\overline{B}, \overline{A}]]] + \cdots,
\end{aligned}$$

and

$$g_2 = \sum_{i=0}^{\infty} \frac{(-1)^i}{2^{i+1}} \mathrm{ad}(\overline{B})^i(\overline{A} + g_1)$$

$$= \frac{1}{2}(\overline{A} + g_1) - \frac{1}{4}[\overline{B}, \overline{A}(1 + g_1)] + \frac{1}{8}[\overline{B}, [\overline{B}, \overline{A}(1 + g_1)]] \\ -\frac{1}{16}[\overline{B}, [\overline{B}, [\overline{B}, \overline{A}(1 + g_1)]]] + \cdots .$$

As G_0 and G_1 are both non-zero solutions of (4.7), they differ by an invertible element in $\mathbb{C}\langle\langle A, B\rangle\rangle$. We define such an element as the formal series $\Phi_{KZ}(A, B) \in \mathbb{C}\langle\langle A, B\rangle\rangle$, relating the two solutions by

$$G_0 = G_1 \Phi_{KZ}(A, B). \tag{4.14}$$

The element $\Phi_{KZ}(A, B)$ is referred to as the *Drinfel'd associator*, which is of interest beyond the present context. See [3, pg. 837] where $\Phi_{KZ}(A, B)$ is expressed in terms of Riemann's zeta function.

Recalling our change of variable, where $\tilde{G}(z_1, z_2, z_3) = G(z)$, from (4.3) we have induced solutions of the KZ system (4.2) appearing as

$$\begin{aligned} W_0(z_1, z_2, z_3) &= (z_3 - z_1)^{\overline{h}(\tau_{12}+\tau_{13}+\tau_{23})} G_0(z) \\ &= (z_3 - z_1)^{\overline{h}(\tau_{12}+\tau_{13}+\tau_{23})} f(z) z^{\overline{h}\tau_{12}} \\ &= f\left(\frac{z_2 - z_1}{z_3 - z_1}\right) \left(\frac{z_2 - z_1}{z_3 - z_1}\right)^{\overline{h}\tau_{12}} (z_3 - z_1)^{\overline{h}(\tau_{12}+\tau_{13}+\tau_{23})} && (4.15) \\ &= f\left(\frac{z_2 - z_1}{z_3 - z_1}\right) (z_2 - z_1)^{\overline{h}\tau_{12}} (z_3 - z_1)^{\overline{h}(\tau_{13}+\tau_{23})} && (4.16) \end{aligned}$$

and

$$W_1(z_1,z_2,z_3) = (z_3-z_1)^{\hbar(\tau_{12}+\tau_{13}+\tau_{23})}G_1(z)$$

$$= (z_3-z_1)^{\hbar(\tau_{12}+\tau_{13}+\tau_{23})}g(1-z)(1-z)^{\hbar\tau_{23}}$$

$$= g\left(\frac{z_3-z_2}{z_3-z_1}\right)\left(\frac{z_3-z_2}{z_3-z_1}\right)^{\hbar\tau_{23}}(z_3-z_1)^{\hbar(\tau_{12}+\tau_{13}+\tau_{23})} \quad (4.17)$$

$$= g\left(\frac{z_3-z_2}{z_3-z_1}\right)(z_3-z_2)^{\hbar\tau_{23}}(z_3-z_1)^{\hbar(\tau_{12}+\tau_{13})}, \quad (4.18)$$

where (4.15-4.18) follow from the commutativity of $\beta = \tau_{12}+\tau_{13}+\tau_{23}$ with both τ_{12} and τ_{23}. As $W_0 = (z_3-z_1)^{\hbar\beta}G_0(z)$ and $W_1 = (z_3-z_1)^{\hbar\beta}G_1(z)$, from (4.14) we have

$$W_0 = W_1\Phi_{KZ}(\hbar\tau_{12}, \hbar\tau_{23}). \quad (4.19)$$

4.2.1 Explicit Representation for the Case $n = 3$

In order to determine $\mathfrak{m}\colon B_3 \to \mathrm{Aut}(V^{\otimes 3})$, we first calculate the monodromy representation for the pure braid group P_3. Consider the path $\gamma(t) = (0, \varepsilon e^{2\pi\sqrt{-1}t}, 1) \in X_3$, representing the pure braid generator σ_1^2 when t is running from 0 to 1. From (4.16) is

$$W_0(\gamma(t)) = f(\varepsilon e^{2\pi\sqrt{-1}t})(\varepsilon e^{2\pi\sqrt{-1}t})^{\hbar\tau_{12}/2\pi\sqrt{-1}} \quad (4.20)$$

$$= f(\varepsilon e^{2\pi\sqrt{-1}t})\varepsilon^{\hbar\tau_{12}/2\pi\sqrt{-1}}e^{\hbar\tau_{12}t}, \quad (4.21)$$

where (4.21) follows from (4.20) per the discussion in Appendix A. This gives

$$W_0(\gamma(0)) = f(\varepsilon)\varepsilon^{\hbar\tau_{12}/2\pi\sqrt{-1}}$$

and

$$W_0(\gamma(1)) = f(\varepsilon)\varepsilon^{\hbar\tau_{12}/2\pi\sqrt{-1}}e^{\hbar\tau_{12}}.$$

This determines the parallel transport $\mu_\gamma : f(\varepsilon)\varepsilon^{\hbar\tau_{12}/2\pi\sqrt{-1}}v \mapsto f(\varepsilon)\varepsilon^{\hbar\tau_{12}/2\pi\sqrt{-1}}e^{\hbar\tau_{12}}v$ given $v \in V^{\otimes 3}$. Consequently,

$$\mathfrak{m}(\sigma_1^2) = f(\varepsilon)\varepsilon^{\hbar\tau_{12}/2\pi\sqrt{-1}}e^{\hbar\tau_{12}}\left[\varepsilon^{\hbar\tau_{12}/2\pi\sqrt{-1}}\right]^{-1}[f(\varepsilon)]^{-1}. \tag{4.22}$$

Let the pure braid generator σ_2^2 be represented by a composition of paths beginning with the real valued path γ_1 from $(0, \varepsilon, 1)$ to $(0, 1-\varepsilon, 1)$, followed by $\gamma_2 = (0, 1-\varepsilon e^{2\pi\sqrt{-1}t}, 1)$, and finished by γ_3, a real-valued path from $(0, 1-\varepsilon, 1)$ back to the base point $(0, \varepsilon, 1)$.

Parametrizing $\gamma_1(t) = (0, (1-2\varepsilon)t + \varepsilon, 1)$ for $t \in [0, 1]$ from (4.16) is

$$W_0(\gamma_1(t)) = f((1-2\varepsilon)t + \varepsilon)((1-2\varepsilon)t + \varepsilon)^{\hbar\tau_{12}/2\pi\sqrt{-1}}.$$

This gives

$$\begin{aligned} W_0(\gamma_1(0)) &= f(\varepsilon)\exp\left[\frac{\hbar\tau_{12}}{2\pi\sqrt{-1}}\log(\varepsilon)\right] \\ &= f(\varepsilon)\varepsilon^{\hbar\tau_{12}/2\pi\sqrt{-1}}, \end{aligned}$$

and similarly,

$$W_0(\gamma_1(1)) = f(1-\varepsilon)(1-\varepsilon)^{\hbar\tau_{12}/2\pi\sqrt{-1}}.$$

From (4.14) is the equality

$$f(1-\varepsilon)(1-\varepsilon)^{\hbar\tau_{12}/2\pi\sqrt{-1}} = g(\varepsilon)(\varepsilon)^{\hbar\tau_{23}/2\pi\sqrt{-1}}\Phi_{KZ}(\hbar\tau_{12}, \hbar\tau_{23}).$$

The monodromy along the path γ_1 is then

$$g(\varepsilon)(\varepsilon)^{\hbar\tau_{23}/2\pi\sqrt{-1}}\Phi_{KZ}(\hbar\tau_{12},\hbar\tau_{23})\left[\varepsilon^{\hbar\tau_{12}/2\pi\sqrt{-1}}\right]^{-1}[f(\varepsilon)]^{-1}.$$

To compute the monodromy along γ_2 from (4.16) is

$$\begin{aligned} W_0(\gamma_2(t)) &= f(1-\varepsilon e^{2\pi\sqrt{-1}t})(1-\varepsilon e^{2\pi\sqrt{-1}t})^{\hbar\tau_{12}/2\pi\sqrt{-1}} \\ &= g(\varepsilon e^{2\pi\sqrt{-1}t})(\varepsilon e^{2\pi\sqrt{-1}t})^{\hbar\tau_{23}/2\pi\sqrt{-1}}\Phi_{KZ} \\ &= g(\varepsilon e^{2\pi\sqrt{-1}t})\varepsilon^{\hbar\tau_{23}/2\pi\sqrt{-1}}e^{\hbar\tau_{23}t}\Phi_{KZ}, \end{aligned} \tag{4.23}$$

where $\Phi_{KZ} = \Phi_{KZ}(\hbar\tau_{12},\hbar\tau_{23})$, and (4.23) follows from (4.19). This gives

$$W_0(\gamma_2(0)) = g(\varepsilon)\varepsilon^{\hbar\tau_{23}/2\pi\sqrt{-1}}\Phi_{KZ}$$

and

$$W_0(\gamma_2(1)) = g(\varepsilon)\varepsilon^{\hbar\tau_{23}/2\pi\sqrt{-1}}e^{\hbar\tau_{23}}\Phi_{KZ}.$$

The monodromy along the path γ_2 is then

$$g(\varepsilon)\varepsilon^{\hbar\tau_{23}/2\pi\sqrt{-1}}e^{\hbar\tau_{23}}\left[\varepsilon^{\hbar\tau_{23}/2\pi\sqrt{-1}}\right]^{-1}[g(\varepsilon)]^{-1}.$$

Since γ_3 and γ_1 are inverse paths, and as $\mathfrak{m}(g) \in \mathrm{Aut}(V^{\otimes 3})$ for $g \in B_3$, the monodromy along γ_3 is the inverse of the monodromy along γ_1. The monodromy along the composite path $\gamma_3\gamma_2\gamma_1$ is then

$$\mathfrak{m}(\sigma_2^2) = f(\varepsilon)\varepsilon^{\hbar\tau_{12}/2\pi\sqrt{-1}}\Phi_{KZ}^{-1}e^{\hbar\tau_{23}}\Phi_{KZ}\left[\varepsilon^{\hbar\tau_{12}/2\pi\sqrt{-1}}\right]^{-1}[f(\varepsilon)]^{-1}. \tag{4.24}$$

Definition 4.2.2. Two linear representations $\rho_1, \rho_2 \colon G \to \mathrm{Aut}(V)$ of a group G are *equivalent* if and only if there exists $f \in \mathrm{Aut}(V)$ such that $\rho_1(g) = f^{-1}\rho_2(g)f$ for all $g \in G$.

We redefine the representation given by (4.22) and (4.24) with a global conjugation in $\mathrm{Aut}(V^{\otimes 3})$. Let $\mathfrak{m} \colon P_3 \to \mathrm{Aut}(V^{\otimes 3})$ be given by

$$\mathfrak{m}(\sigma_1^2) = e^{\hbar\tau_{12}}$$
$$\mathfrak{m}(\sigma_2^2) = \Phi_{KZ}^{-1} e^{\hbar\tau_{23}} \Phi_{KZ},$$

where $\mathfrak{m}$ is more precisely regarded as the simplest representative from an equivalence class of representations, monodromy representations being unique up to global conjugation, where a change of base point in X_3 affects a conjugation within $\mathrm{Aut}(V^{\otimes 3})$. It is worth noting that, just as $\mathrm{tr} \colon \mathrm{Aut}(W) \to \mathbb{C}$ is invariant under similarity, the isotopy invariants derived from monodromy representations via (modified) traces are invariant under conjugation as well.

Consider the pure braid generator σ_1^2 represented by the loop $\gamma \colon [0, 2] \to X_3$ with basepoint $(0, \varepsilon, 1)$, such that $[\gamma(t)] = [\gamma(1 + t)] \in X_3/S_3$ for $t \in [0, 1]$. For $t \in [0, 1]$, γ is a path in X_3 from $(0, \varepsilon, 1)$ to $(\varepsilon, 0, 1)$, which projects to a loop in the quotient space X_3/S_3 with basepoint $[0, \varepsilon, 1]$. Likewise, for $t \in [1, 2]$, γ is a path in X_3 from $(\varepsilon, 0, 1)$ to $(0, \varepsilon, 1)$, which again projects to the same loop in the quotient space X_3/S_3. Running the complete path γ amounts to running through the projected loop twice in the same direction within the quotient space X_3/S_3. We already know the monodromy of this path (in the ordered space) is $e^{\hbar\tau_{12}}$. Thus $e^{\hbar\tau_{12}}$ should simply be the square of the monodromy given by running along either path which projects to a single navigation of the loop in X_3/S_3. Thus

$$\left[\text{monodromy along } \gamma|_{[0,1]}\right]^2 = \left[\text{monodromy along } \gamma|_{[1,2]}\right]^2 = e^{\hbar\tau_{12}}.$$

It is also clear that the braid generator σ_1 is equally well represented by γ for $t \in [0,1]$ as well as for $t \in [1,2]$, i.e., by a path in X_3 which projects to the loop in X_3/S_3. Thus the monodromy action of σ_1 on $v_1 \otimes v_2 \otimes v_3 \in V^{\otimes 3}$ is given by $v_1 \otimes v_2 \otimes v_3 \mapsto e^{\hbar\tau_{12}/2}(v_1 \otimes v_2 \otimes v_3)$, not taking into account the permutation $P_{12}\colon v_1 \otimes v_2 \otimes v_3 \mapsto v_2 \otimes v_1 \otimes v_3$ of the tensor factors induced by lifting.

Proposition 4.2.3. *Define $\mathfrak{m}(\gamma) = s_i \mathfrak{m}(\tilde{\gamma})$, where $\tilde{\gamma} \in X_3$ is the lift of the loop $\gamma \in X_3/S_3$ induced by the permutation $s_i = \pi(\sigma_i)$ where σ_i is the braid generator represented by the loop $\gamma \in X_3/S_3$. This defines a representation*

$$\mathfrak{m}\colon B_3 \mapsto Aut(V^{\otimes 3}).$$

Sketch of proof. Let γ and η be loops in X_3/S_3 with the same basepoint, with $\tilde{\gamma}, \tilde{\eta} \in X_3$ their respective lifts induced by the respective permutations $s_1, s_2 \in S_3$. The lift of the composite path $\eta\gamma$ is given by $\widetilde{\eta\gamma} = s_1(\tilde{\eta})\tilde{\gamma}$ induced by the permutation $s_2 s_1$. Using the definition,

$$\begin{aligned}
\mathfrak{m}(\eta\gamma) &= s_2 s_1 \mathfrak{m}(s_1(\tilde{\eta})\tilde{\gamma}) \\
&= s_2 s_1 \mathfrak{m}(s_1(\tilde{\eta}))\mathfrak{m}(\tilde{\gamma}) \\
&= s_2 s_1 s_1^{-1} \mathfrak{m}(\tilde{\eta}) s_1 \mathfrak{m}(\tilde{\gamma}) \\
&= s_2 \mathfrak{m}(\tilde{\eta}) s_1 \mathfrak{m}(\tilde{\gamma}) \\
&= \mathfrak{m}(\eta)\mathfrak{m}(\gamma),
\end{aligned}$$

where the second equality follows from the fact that we have a homomorphism for arbitrary paths in X_3. The third equality follows from a comparison of respective monodromies for a path in X_3/S_3 and its lift in X_3. Using the symmetry of the KZ equation with respect to permutations, the argument uses the same overall constructions as in the pure braid case, as the claim deals exclusively with a path in X_3. Specifically, a 1-form $(s_1(\tilde{\eta}))^*\alpha$ defines an ODE on the unit interval as in (3.7). $\square$

The full monodromy action of σ_1 is then given by $v_1 \otimes v_2 \otimes v_3 \mapsto P_{12}(e^{\hbar\tau_{12}/2}(v_1 \otimes v_2 \otimes v_3))$.

The analogous argument applies in the case of the braid generator σ_2 and the corresponding permutation $P_{23}\colon v_1 \otimes v_2 \otimes v_3 \mapsto v_1 \otimes v_3 \otimes v_2$. The monodromy representation $\mathfrak{m}\colon B_3 \to \mathrm{Aut}(V^{\otimes 3})$ is then defined by

$$\mathfrak{m}(\sigma_1) = P_{12}e^{\hbar\tau_{12}/2}$$
$$\mathfrak{m}(\sigma_2) = P_{23}\Phi_{KZ}^{-1}e^{\hbar\tau_{23}/2}\Phi_{KZ}.$$

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APPENDIX A

THE SOLUTION W FOR $N = 2$ AND MULTIVALUEDNESS

Here we verify the solution W in the $n = 2$ case, and briefly discuss restrictions on W. We define for $z \in \mathbb{C}$ and operator (or matrix) A

$$z^A := \exp(A \log z) \tag{A.1}$$

and

$$e^A := \sum \frac{1}{k!} A^k.$$

Together these define

$$\exp(A \log z) = \sum_{k=0}^{\infty} \frac{(\log z)^k}{k!} A^k.$$

It follows that

$$\begin{aligned} d(z^A) = d(\exp(A \log z)) &= d\left(\sum_{k=0}^{\infty} \frac{(\log z)^k}{k!} A^k\right) \\ &= \sum_{k=0}^{\infty} \frac{A^k}{k!} \frac{d}{dz} (\log z)^k dz \\ &= A\left(\sum_{k=1}^{\infty} \frac{A^{k-1}}{(k-1)!} (\log z)^{k-1}\right) \frac{dz}{z} \\ &= A \exp(A \log z) \frac{dz}{z} \\ &= A z^A d \log z. \end{aligned} \tag{A.2}$$

From this result we verify that

$$W(z_1, z_2) = (z_2 - z_1)^{\hbar \tau_{12}/2\pi\sqrt{-1}} v \tag{A.3}$$

solves

$$dW = \frac{\hbar\tau_{12}}{2\pi\sqrt{-1}}\frac{dz_1 - dz_2}{z_1 - z_2}W,$$

where $\mathrm{Arg}(z_2 - z_1) \in (-\pi, \pi]$, and $W(0, 1) = v \in V \otimes V$. Applying (A.2) to (A.3) gives

$$\begin{aligned} dW &= d(z_2 - z_1)^{\hbar\tau_{12}/2\pi\sqrt{-1}} \\ &= \frac{\hbar\tau_{12}}{2\pi\sqrt{-1}}(z_2 - z_1)^{\hbar\tau_{12}/2\pi\sqrt{-1}} d\log(z_2 - z_1) \\ &= \frac{\hbar\tau_{12}}{2\pi\sqrt{-1}}\frac{dz_1 - dz_2}{z_1 - z_2}W \end{aligned}$$

which verifies the solution.

Restricting Arg of W is acknowledging the log of a complex number is not a function. It is multivalued. See [1, pgs. 46-52] for discussion and examples of multivalued functions. Similarly, in general

$$(\varepsilon e^{2\pi\sqrt{-1}t})^{\hbar\tau_{12}/2\pi\sqrt{-1}} \neq \varepsilon^{\hbar\tau_{12}/2\pi\sqrt{-1}} e^{\hbar\tau_{12}t}.$$

The result of (A.2) requires choosing a branch of the log, in which the above equality follows from

$$\begin{aligned} (\varepsilon e^{2\pi\sqrt{-1}t})^{\hbar\tau_{12}/2\pi\sqrt{-1}} &= \exp\left[\frac{\hbar\tau_{12}}{2\pi\sqrt{-1}}\log(\varepsilon e^{2\pi\sqrt{-1}t})\right] \\ &= \exp\left[\frac{\hbar\tau_{12}}{2\pi\sqrt{-1}}(\log\varepsilon + \log(e^{2\pi\sqrt{-1}t}))\right] \\ &= \exp\left[\frac{\hbar\tau_{12}}{2\pi\sqrt{-1}}[\log\varepsilon + 2\pi\sqrt{-1}t]\right] \\ &= \exp\left[\frac{\hbar\log\varepsilon}{2\pi\sqrt{-1}}\tau_{12} + \hbar\tau_{12}t\right] \qquad (A.4) \\ &= \exp\left[\frac{\hbar\log\varepsilon}{2\pi\sqrt{-1}}\tau_{12}\right]\exp\left[\hbar\tau_{12}t\right] \qquad (A.5) \\ &= \exp\left[\log\varepsilon^{\frac{\hbar\tau_{12}}{2\pi\sqrt{-1}}}\right]\exp\left[\hbar\tau_{12}t\right] \\ &= \varepsilon^{\hbar\tau_{12}/2\pi\sqrt{-1}} e^{\hbar\tau_{12}t}. \end{aligned}$$

Here (A.5) follows from (A.4) as $\lambda_1\tau_{12}$ and $\lambda_2\tau_{12}$ commute for $\lambda_1, \lambda_2 \in \mathbb{C}$.

APPENDIX B

COMPATIBILITY OF SOLUTIONS UNDER CHANGE OF VARIABLE FOR $N = 3$

We want to verify that $G(z)$ is a solution of (4.5) if and only if $W(z_1, z_2, z_3)$ is a solution of the KZ system (4.2), with $A = \hbar\tau_{12}$ and $B = \hbar\tau_{23}$. Let $\beta = \tau_{12} + \tau_{13} + \tau_{23}$. As G is dependent on a single variable z, (4.5) may be expressed as

$$\begin{aligned} \frac{dG}{dz} &= \left(\frac{\overline{A}}{z} + \frac{\overline{B}}{z-1}\right) G \\ &= \overline{h}\left(\frac{\tau_{12}}{z} + \frac{\tau_{23}}{z-1}\right) G(z). \end{aligned} \tag{B.1}$$

Since

$$\begin{aligned} dW &= \overline{h}\big(\tau_{12} d\log(z_1 - z_2) + \tau_{13} d\log(z_1 - z_3) + \tau_{23} d\log(z_2 - z_3)\big) W \\ &= \sum_{i=1}^{3} \frac{\partial W}{\partial z_i}\, dz_i, \end{aligned} \tag{B.2}$$

assuming W is a solution to the KZ system (B.2) means, in particular, that

$$\begin{aligned} \frac{\partial W}{\partial z_2} &= \overline{h}\left(\frac{\tau_{12}}{z_2 - z_1} + \frac{\tau_{23}}{z_2 - z_3}\right) W \\ &= \overline{h}\left(\frac{\tau_{12}}{z_2 - z_1} + \frac{\tau_{23}}{z_2 - z_3}\right) (z_3 - z_1)^{\overline{h}\beta} G(z), \end{aligned} \tag{B.3}$$

where the second equality follows from $W(z_1, z_2, z_3) = (z_3 - z_1)^{\overline{h}\beta} G(z)$. Also from this comes

$$\frac{\partial W}{\partial z_2} = (z_3 - z_1)^{\overline{h}\beta} \frac{G'(z)}{z_3 - z_1}, \tag{B.4}$$

and equating (B.3) and (B.4) gives

$$\begin{aligned} 0 &= (z_3 - z_1)^{\overline{h}\beta} \frac{G'(z)}{z_3 - z_1} - \overline{h} \left(\frac{\tau_{12}}{z_2 - z_1} + \frac{\tau_{23}}{z_2 - z_3} \right) (z_3 - z_1)^{\overline{h}\beta} G(z) \\ &= (z_3 - z_1)^{\overline{h}\beta} \left[\frac{G'(z)}{z_3 - z_1} - \overline{h} \left(\frac{\tau_{12}}{z_2 - z_1} + \frac{\tau_{23}}{z_2 - z_3} \right) G(z) \right] \qquad \text{(B.5)} \\ &= (z_3 - z_1)^{\overline{h}\beta} \left[G'(z) - \overline{h} \left(\tau_{12} \frac{z_3 - z_1}{z_2 - z_1} + \tau_{23} \frac{z_3 - z_1}{z_2 - z_3} \right) G(z) \right] \\ &= (z_3 - z_1)^{\overline{h}\beta} \left[G'(z) - \overline{h} \left(\frac{\tau_{12}}{z} + \frac{\tau_{23}}{z - 1} \right) G(z) \right], \end{aligned}$$

where (B.5) follows from the commutativity of β with both τ_{12} and τ_{23}. Thus G satisfies (B.1). The converse is left to the reader. It may be helpful to use $G(z) = \left[(z_3 - z_1)^{\overline{h}\beta} \right]^{-1} W(z_1, z_2, z_3)$.

APPENDIX C

T INVARIANCE

The aim is to show $f_{\psi\circ\tau_T}(2)$ does not depend on T, or

$$\frac{d}{dT}\left[P(1,1)P(T,1)^{-1}Q(T,1)P(T,0).\right] = 0 \tag{C.1}$$

Proof. All differentiation is done component-wise, where $P, Q, p, q \in \mathrm{End}(V')$ have matrix representations. Since

$$\frac{\partial P(x,y)^{-1}}{\partial x} = -P(x,y)^{-1}\frac{\partial P(x,y)}{\partial x}P(x,y)^{-1},$$

in particular, from (8) it follows that

$$\begin{aligned}\frac{dP(T,1)^{-1}}{dT} &= -P(T,1)^{-1}\frac{dP(T,1)}{dT}P(T,1)^{-1}\\ &= -P(T,1)^{-1}p(T,1).\end{aligned} \tag{C.2}$$

Recall

$$\psi^*\alpha = p(x,y)dx + q(x,y)dy$$

defines the one-form $\psi^*\alpha \in \Omega^1(I \times I, A)$. The connection $d-\alpha$ is flat, meaning $d\alpha - \alpha\wedge\alpha = 0$. Consequently

$$\psi^*(d\alpha - \alpha\wedge\alpha) = \left(q(x,y)(p(x,y) - p(x,y)q(x,y) + \frac{\partial q(x,y)}{\partial x} - \frac{\partial p(x,y)}{\partial y}\right)dxdy = 0,$$

From the above, we have

$$-qp = \frac{\partial q}{\partial x} - \frac{\partial p}{\partial y} - pq. \tag{C.3}$$

Also note

$$\frac{\partial^2 Q(x,y)}{\partial x\partial y} = \frac{\partial^2 Q(x,y)}{\partial y\partial x}. \tag{C.4}$$

Define

$$f(x,y) = \frac{\partial Q(x,y)}{\partial x} - p(x,y)Q(x,y) + Q(x,y)p(x,0).$$

Differentiating both sides with respect to y, and using (3.9), (C.3), and (C.4), it can be shown that

$$\frac{\partial f}{\partial y} = qf.$$

It follows from (3.9) that $f(x,0) = 0$. Thus, $f(x,y) = 0$. Thus,

$$\frac{\partial Q(x,y)}{\partial x} = p(x,y)Q(x,y) - Q(x,y)p(x,0),$$

and in particular,

$$\frac{dQ(T,1)}{dT} = p(T,1)Q(T,1) - Q(T,1)p(T,0). \tag{C.5}$$

The equalities (3.8), (3.9), (C.2), and (C.5) are sufficient to show (C.1). $\square$