STUDENT UNDERSTANDING OF FUNCTION AND SUCCESS IN CALCULUS

by

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ABSTRACT

The purpose of this study was to determine if there is a relationship between student success in calculus and student understanding of function. Student understanding of function was measured using two questionnaires, one of which is a modification of an existing measure based on APOS theory. The other I developed with items from the concept image literature. The participants of this study were 116 high school students who were enrolled in a first-year calculus course. The results of the questionnaires were aligned to course exam scores to determine connections between function understanding and rate of success in calculus.

A major finding of this study is that students can be successful in a first-year calculus course without demonstrating a process level understanding of function at the beginning of the course. In general, a positive correlation between understanding of function and success in calculus was found.

An item-by-item analysis of the two questionnaires revealed that students demonstrated competence, relative to their measured understanding of function, with items that are typically presented in standard mathematics courses taken prior to calculus, such as when provided a function as an algebraic rule and asked to calculate the value of the function. Also, students tended to justify decisions for solutions based on criteria not necessarily related to the definition of function. This however, appeared to have little impact on the level of success a student was able to achieve in calculus.

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CHAPTER ONE: INTRODUCTION

Statement of the Problem

As a calculus teacher, I expect students to come to me with conceptual understanding of functions. For example, students should be able to solve equations, graph and evaluate functions, transfer between forms of functions and compute basic sums, differences, products and quotients of functions. However, this is far from the truth. As I introduce topics to students in calculus, I find myself reviewing topics from previous courses. For this reason, there is often much frustration for the teacher as well as the student. My frustration stems from having to find time to review topics that are not necessarily part of the curriculum. The students become frustrated due to their struggles in trying to learn new topics without having complete understanding of foundational material. Identifying the area in which a student struggles and correcting the way a student understands a topic should be a goal at each level of mathematics.

In a first-year calculus course, functions play a critical role. With an incomplete understanding, a student will tend to struggle. Students need to be able to shift from one representation of function to another as called for by the problem at hand. Every year, this deficiency is almost immediately recognized in calculus as we begin the year. The curriculum that I teach starts with limits where students are trying to identify a y-value that a function approaches as the x-value gets arbitrarily close to a particular quantity. With this topic students are immediately tested on their ability to shift between the different representations of function, namely equation, graphical, tabular, and description of a contextual situation. There are times when students are asked to use a graphical representation of function to evaluate a limit. If the representation of function does not fit into what students understand about what a function should look like, for example, the graph of a piece-wise defined function, some students produce an incorrect limit. Other times students have to calculate limits given a function defined by a rule or equation. Here students are comfortable evaluating limits by substitution, but struggle when the limit could not be evaluated in this way. Many students appear to understand these concepts as I introduce them through examples, however, once they are asked to perform them on their own, they are unsure which technique or procedure they should implement. This issue is not only faced with limits, but throughout the calculus curriculum. A possible cause for student difficulty with functions is described in the following quote: "If algebraic and procedural methods were more connected to conceptual learning, students would be better equipped to apply their algebraic techniques appropriately in solving novel problems and tasks" (Oehrtman, Carlson, & Thompson, 2008, p. 28). That is, the difficulties described above stem from the fact that students are being introduced to procedures rather than the concept.

Aside from the use of the appropriate procedure or representation, what students hold as a definition of function can play a critical role in their success when interpreting or working with functions. Students frequently identify a polynomial in the form, $f(x) = ax^n + bx^{n-1} + ... + c$, where *a*, *b*, and *c* are rational numbers, as a function but when it comes to other functions like trigonometric, radical, piece-wise or rational functions, their definitions possibly lead to incorrect interpretations of the problem. An example of this occurs when thinking about the domain, or values for which an expression is defined, such as $f(x) = \sqrt{x-1}$. Students frequently have difficulty identifying the domain as all real numbers x, such that $x \geq 1$. This is an important skill to have in calculus, as students should only interpret a function using differentiation, integration and other calculus techniques for values in the domain of that function.

Research Questions

My observations as a calculus instructor at the high school level have prompted me to more carefully investigate how student understanding of function may be related to student success (or lack thereof) in a first-year calculus course. Based on the research literature student difficulties with understanding functions can be due to several factors throughout the development in the courses that the students have taken prior to calculus. Students need to know how to work with a function in order to solve the problem at hand (see for example Breidenbach, Dubinsky, Hawks, & Nichols, 1992, and Oehrtman et al., 2008). They also need to be able to make connections between different representations of functions, that is: tabular, graphical, equation form and word form (see for example Carlson, n.d., Dubinsky & Wilson, 2013 & Davis, 2007). Researchers have also examined issues related to the development of functional understanding and the importance that functions play in calculus (see for example Monk, 1994; Carlson, 1997; & Oehrtman et al., 2008). For example, Oehrtman et al. (2008) suggest that students need a strong understanding of functions in order to be successful in calculus.

In this study I aimed to determine if students with a deep understanding of function as they enter calculus tended to perform better in calculus than those students with a lesser understanding of function at the beginning of the course. The following research questions were used to guide my work.

1. Is there a correlation between student understanding of function and performance in calculus?
2. What patterns in student understanding of function are related (or unrelated) to student performance in calculus?

CHAPTER 2: REVIEW OF THE LITERATURE

As students enter high school mathematics, functions play a vital role in students' studies. By the time a student reaches more advanced mathematics classes, like calculus, they must be comfortable working with functions and be able to apply the definition of function. In this chapter, I define what it means for a student to understand function, student ideas that inhibit understanding of function, and what teachers can do to develop student understanding of function. First, I provide an overview of the ideas presented in this chapter.

Overview

In a first-year calculus course students are introduced to the concepts of limits, differentiation, and integration. However, as Monk (1994) states, "the concept that the subject is built out of, the one that lies behind such notions as limit, derivative, and integral is that of function" (p. 21). In students' learning of calculus they work with a variety of functions, which include: polynomial functions, rational functions, trigonometric functions, piece-wise functions, transcendental functions, and radical functions. This leads to an interesting question. What role does understanding of functions play in a student's success in calculus?

According to the Idaho Common Core State Standards (ICCSS; Common Core State Standards Initiative, n.d.) the term function is introduced as early as the eighth grade. Students have also been exposed to functions in courses prior to calculus, like precalculus. Therefore, one might expect a student who enters higher level math courses in high school, like calculus, to know and apply the definition of function, know the purpose that a function serves, and know how to manipulate and interpret functions. Formally, a function in mathematics is defined as "a correspondence between two nonempty sets that assigns to every element of the first set (the domain) exactly one element in the second set (the codomain)" (Vinner & Dreyfus, 1989, p. 357). This definition of function is known as the Dirichlet-Bourbaki concept (Vinner & Dreyfus, 1989) and allows for functions that are discontinuous, functions defined on split domains and functions defined by graphs or ordered pairs, and so on.

In general, students enter the classroom every year with different levels of understanding of function. Mason (2008) states that teaching cannot force, necessitate or guarantee learning, but teaching can make learning more likely and more effective if it makes use of learners' powers and dispositions, and exposes them to significant and fruitful ways of thinking and perceiving (p. 271). In the case of functions, a student introduced to functions as simply a tool to manipulate numbers may be limited to this idea throughout his or her mathematical studies. The student who is introduced to function in its many representations and purposes may have a better understanding.

Because of its relevance to so many other mathematical topics and its role in college-level mathematics, function constitutes one of the most important topics in secondary school mathematics (Cooney, Beckman, & Lloyd, 2010, p. 7). As noted in the National Research Council's 1989 report, *Everybody Counts,* "if undergraduate mathematics does nothing else, it should help students develop function sense" (p. 51). From the early introduction in middle school to college-level mathematics courses, students are using, analyzing and interpreting functions. Cooney et al. (2010) list a few examples of textbook definitions of function. These include:

- A function is a relationship between input and output. In a function, the output depends on the input. There is exactly one output for each input.
- A function is a relation in which each element of the domain is paired with exactly one element of the range.
- A function is a set of ordered pairs (or number pairs) that satisfies this condition: There are no two ordered pairs with the same input and different outputs.
- A real-valued function f defined on a set D of real numbers is a rule that assigns to each number x in D exactly one real number, denoted by $f(x)$.

The variety of definitions found in textbooks is due to the mathematics level of the student. Students need to be introduced to the concept of function in line with the formal definition without being overburdened with notation and vocabulary. While students are introduced to a variety of formal definitions of function, each student possesses his or her own concept image of function (Vinner, 1992) and students must be able to decipher and make sense of these formal definitions. Concept image as defined by Vinner (1992) is a nonverbal entity associated in our mind with the concept name. A student's concept image of function is shaped possibly by the various definitions that they encounter, examples that teachers use in class, or class assignments. With their concept image, students create their own definition of function, which at times is incorrect and therefore leads students to make incorrect interpretations (Vinner & Dreyfus, 1989).

As students make progress through the mathematics curriculum, they create and refine their concept image based on problems they are faced with in the particular class. Breidenbach et al. (1992) and Carlson (n.d.) suggest that early in the mathematics curriculum the aim should be to move students from viewing functions as a physical manipulation of objects, to interiorization of objects and finally encapsulation of the process in its totality. Figure 1 illustrates the process through which a student develops concepts, like function, in mathematics. In the case of function, students first act on them at the action level where a function is a tool to manipulate a number.

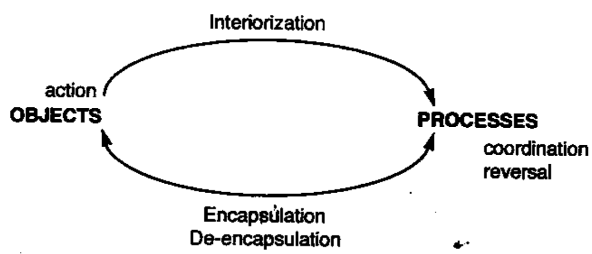

Figure 1. APOS Constructions for mathematical understanding (Asiala et al., 1996)

The student is then able to interiorize this and move to the process level. At this level a student understands that a function will produce a value, but is not concerned with producing a specific output. From this point the student can view functions at the object level where they have the ability to perform operations on functions. Finally, they are able to view functions at the appropriate level as called for by the problem. Here they are determined to be at the schema level. The terms action, process, object and schema form the APOS framework and will be discussed further later in this chapter (Asiala et al., 1996).

Student difficulty with functions occurs for various reasons. Ronda (2009) and Clement (2001) state that representations of functions are a stumbling point for students. These representations include equation, graphical, tabular, and descriptions of contextual situations. Being introduced to concepts using different representations has been shown to help students make connections between the different representations of functions (Habre & Abboud, 2006). When introduced to concepts with different representations students have made connections between the representations. Introducing concepts with different representations of functions also improve students' concept image (Vinner, 1992) and definition of function.

Because functions play a vital role in today's mathematics curriculum and are also so extremely useful in capturing aspects of real-life phenomena, students must develop a deep understanding of function to ensure success in mathematics (Oehrtman et al., 2008; Davis, 2007; Breidenbach, et al., 1992; Carlson, 1997).

What Does It Mean for Students to Understand Function?

Students need to be able to apply tools and techniques to demonstrate their understanding of mathematical topics. For my study, I chose to focus on two aspects of function understanding: 1) how students work with functions given in various forms, and 2) how students apply the definition of function. Whether it be reciting a definition, being able to evaluate a function for particular values, generating various representations of a real-life situation, or different ways of working with functions, each of these contribute to determining a student's level of understanding. For me, student understanding of functions depends on the APOS level at which a student can work with functions and how well the student can apply the definition of function.

Sierpinska (1992) states that understanding a concept, like functions, is achieved when one is able to use, identify, apply, generalize and create extensions with its use. In a study conducted by Ronda (2009), 444 students from Philippine schools in grades eight through ten participated. Each student completed tasks that identified how well a student understood functions in equation form. Each task allowed for the identification of growth points, which included: equations are procedures for generating values, equations are representations of relationships, equations describe properties of relationships, and functions are objects that can be manipulated and transformed. Ronda concluded that the framework, or growth points, would be a way to guide teachers and their instruction to aid in student understanding of functions.

APOS Theory of Understanding as Working with Functions

APOS refers to the phases that a student goes through in the process of understanding functions, action-process-object-schema (APOS; Mahir, 2010). Several researchers describe student understanding of functions using APOS theory (Martinez-Planell, Gaismann, 2012; Mahir, 2010; Thompson, 1994). Piaget's mental constructions for learning mathematics served as a guide in the development of APOS (Asiala et al., 1996). APOS theory emerged in an attempt to interpret the type of thinking that occurs as individuals develop understanding of mathematics, in general. A small group of researchers have "been using APOS theory within a broader research and curriculum development framework" (Dubinsky & McDonald, 2001, p. 4). See the list below for a description of each phase. The phases below are ordered from the lowest level of ability of working with functions to the highest.

- Action (A) View: In the action view functions are regarded as static. A function is tied to a specific rule, formula or computation and requires the completion of specific computations and/or steps (Oehrtman et al., 2008). A student at the Action level is unlikely able to solve a situational problem that involves a function without being given a formula (Moore, 2012).
- Process (P) View: At the Process View functions are regarded as dynamic. A function is a generalized input-output process that defines a mapping of a set of input values to a set of output values (Oehrtman et al., 2008). When working with functions, students are at the process level if the function is not restricted to numbers for the domain and ranges of the function, or if the student can imagine certain operations with functions with no explicit formula (Breidenbach et al., 1992). To better develop the understanding of the function as a process teachers can apply reverse-path-development (Eisenberg, 1992). Here the process is discussed in both directions. For example, a student given two functions, $f(\mathrm{x})$ and $g(\mathrm{x})$, can produce the composition, $f(g(\mathrm{x}))$. At the same time if given $f(g(\mathrm{x}))$ and $f(\mathrm{x})$ the student is able to identify $g(\mathrm{x})$.
- Object (O) View: At the Object View function is regarded as an object that can be manipulated and changed much like any number. An *object* is constructed from a process when the individual becomes aware of the process as a totality and realizes that transformations can act on it (Dubinsky & MacDonald, 2001).
- Schema (S) View: At the schema view the student is able to transition between the three previous views (action, process or object) as deemed necessary by the problem. For example, when working through problems in a calculus course,

> they may need to use the action view of a function to determine values and in the same problem treat the function as an object in order to manipulate the function into a usable form.

Determining the level at which a student is able to work with function is one way to determine the level of understanding a student has of function (Dubinsky & MacDonald, 2001).

APOS theory has been used to develop several different studies. For example, the development of the Precalculus Concept Assessment (PCA), a validated instrument which measures a student’s understanding of functions based on the APOS perspective (Carlson, n.d.) measures a student’s ability to work with functions at the process level of the APOS framework. Dubinsky and Wilson’s (2013) analysis on the effectiveness of the Algebra Project and student understanding of function also applied APOS Theory. Finally, Oehrtman et al. (2008) state the importance of the process view to understand calculus and also describe how to foster the process view. The APOS framework will be used in this study to analyze the aspect of student understanding of function related to how students work with functions.

Understanding in Terms of the Definition of Function

Definitions also play a vital role in student understanding of functions. Students build the strongest definitions through experiences and involvement. Definitions should only be introduced once examples of student experiences of function have been applied and extended (Vinner & Dreyfus, 1989). That is, students should not be introduced to definitions until they have developed an appropriate concept image. Class discussion on the definition of functions may have a greater impact on student understanding and a willingness to accept definitions if the students are involved in the creation of definitions (Edwards & Ward, 2008). Also, Vinner (1992) claims that appropriate pedagogies should be implemented before suggesting definitions to the students. In other words, educators need to allow students the opportunity to, for lack of a better word, “play” with a concept before they burden them with definitions. By the time students reach a first-year calculus course, they should be able to apply the Dirichlet-Bourbaki definition of a function.

Thompson (1994), Sierpinska (1992), Carlson (1997) and Clement (2001) all state that students tend to believe that a function is a mathematical statement with an equal sign. From prior experiences with functions, students believe that the function must be continuous and cannot be constant or defined over split domains (Dubinsky & Wilson, 2013; Sierpinska, 1992; Clement, 2001). Dubinsky and Wilson (2013) continue discussing student difficulty with functions. Included in the discussion is the one-to-one property. Here students think that each element of the domain must be mapped to a different element of the range. This causes difficulty in understanding the idea of the constant function, a function where every element in the domain is mapped to the same element in the range. Also mentioned is the vertical line test, a test used on graphs of functions where a vertical line is drawn to help determine whether a graph represents a function. Because of these difficulties students’ concept images of function often have flaws. In order for students to have a deep understanding, they must be able to interpret features from different representations of functions as well as learn and understand the formal definition of function (Carlson, 1997).

Other Aspects of Functions that Inhibit Understanding

Difficulty with interpreting graphs of functions also plays a role in student understanding of functions. For example, students sometimes think an oscillating velocity versus time graph means that the object is travelling over some sort of oscillating, “hilly”, terrain rather than simply speeding up or slowing down (Monk, 1992; Oehrtman et al., 2008). Another example is seen on the Precalculus Concept Assessment (PCA). Figure 2 illustrates the velocities of two cars travelling in the same direction. Carlson (n.d.) remarks that there are students who interpret the graph as a collision between the two cars occurring at 1 hour.

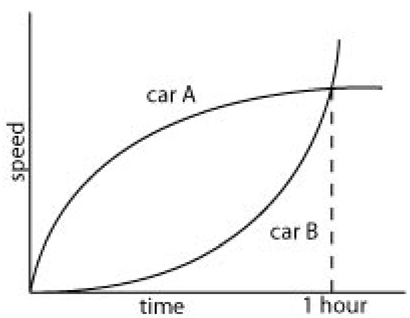

Figure 2. PCA question modeling the velocity of two cars from t = 0 to t = 1 hr.

Creating connections between various representations of functions is another important aspect of understanding functions (Thompson, 1994; Davis, 2007; Carlson, Oehrtman, & Engelke, 2010; Akkus, Hand, & Seymour, 2008) and related to another difficulty students experience (Dubinsky & Wilson, 2013). Students have great difficulty making the connections between various representations of functions, these include: equations, graphs, tables and word forms (Thompson, 1994; Davis, 2007; Carlson et al., 2010; Akkus, Hand, & Seymour, 2008). That is, if presented with the graph of a function and asked to find the input value that produces a particular output value, a student may demonstrate difficulty. Here the student is demonstrating difficulty connecting the information given by the graph to solve a problem given in equation form.

However, representations of functions play a vital role in understanding functions. Eisenberg (1992) and Mahir (2010) stress the visualization of a graphical representation of function. What does the picture look like? If students are able to visualize a function, concepts like slope, rates of change, and area would make much more sense to students (Oehrtman et al., 2008). In other words, being able to use a mental image of the function to understand where a function is increasing or decreasing, or where the function is constant will aid in the student's interpretation.

Developing Understanding of Function

Students depend upon prior knowledge, or their concept image, to interpret meanings of mathematical concepts (Vinner & Dreyfus, 1989). If this concept image is faulty, then it is reasonable to expect limitations associated with a student's ability to apply the definition. Teachers must be able to identify holes in student understanding and aid in filing those holes (Ronda, 2009). Unfortunately, teachers frequently introduce topics with an assumption that students enter a class with required knowledge of a topic. This is often not true (Lobato, 2008). Introducing topics in mathematics like functions at an appropriate time in a student's understanding of mathematical concepts and definitions will create more success in understanding for students (Sierpinska, 1992).

Connections that students make with mathematical concepts to previously developed ideas also aids in the understanding of the concepts. To aid in student understanding, Oehrtman et al. (2008), motivated by the APOS theory, state that educators need to allow students to explain the behavior of functions using appropriate terminology such as: input, output, dependent and independent variables. An example of this in the early stages of developing the function concept could have students develop functions that calculate the amount of money in their bank account with a job that pays a particular amount of money. To begin moving away from just an action view, the discussion could then progress to determining the independent and dependent variables. This could then be extended and students could be asked to determine how long it would take them to pay for an item they would like to buy and also interpret the graph of their function in the context of the problem. The researchers above also mention that teachers need to be flexible in allowing students to use multiple representations. Representations of functions should allow students to see that a function is not only a rule but rather can also be represented as a statement or a graph. In the earlier example, students could model the amount of money in their bank account graphically, verbally, algebraically or in a table and observe the effects of changing aspects of the function. In this way, students might begin to develop a process view of functions. A final remark made by the researchers is that discussing with students not only how a function manipulates a number, but also what is causing this change will support the development of understanding of function. Davis (2007) and Mahir (2010) also make the argument that the connection to real-life scenarios plays an important part for student success in the understanding of functions. Creating a reason and connection for students not only makes a lesson meaningful to them but it develops understanding.

Student difficulty with functions may stem from various points in their educational progress. Kinzel (2006) conducted a study to determine if students’ abilities to shift between the object view and process view was a reason for their difficulty to work with algebraic expressions. In particular, the study involved six participants enrolled in a mathematics education class and Kinzel interviewed the students using questions that either involved using the process view or the object view. The process view of an algebraic expression means an algebraic expression is being used to manipulate a number, whereas the object view is treating the expression as an object or quantity. The researcher's conclusion claimed that successful students are able to seamlessly shift between the two views.

Following a sequence of learning steps, like those of APOS, creating connections to real life and making lessons meaningful for students are all techniques that aid in the development of understanding. Sajka (2003) states what we usually write and do in a mathematics lesson is very important for the student (p. 247). For example, when students are consistently introduced to functions as a rule or equation, they tend to have a difficult time understanding that a table or a graph of distinct points can also represent a function (Sajka, 2003). Hence, lessons that allow for connection to real-life and also to different representations, allow for flexibility, and take into account students' prior knowledge are lessons that would be the most beneficial in enhancing understanding.

Functions in Calculus

Prior to calculus, students are introduced to procedures for evaluating functions and solving equations. Therefore, they are seeing functions as a static entity. In other words, although they are able to view functions in the way best suited for the problem, they are primarily instructed on functions at the action view of the APOS framework (Carlson et al., 2010). If students have developed a deep understanding of functions prior to attending a high school calculus course or even college, they will be better prepared for these courses (Akkus et al., 2008). In calculus, functions play a vital role. Some of the problems that students are faced with include slope of a function and the equation of tangent lines. Because of this, development of the function concept in earlier mathematics courses is vital for student success (Davis, 2007). Further, the ability to apply the definition of function is a vital part of the background of any student hoping to understand calculus (Breindenbach et al., 1992; Carlson, 1997). Much of the success that students achieve in calculus classes appears to stem from their understanding of functions.

Almost every aspect of a first-year calculus course depends on students' understanding of functions. Finding limits of functions forces students to interpret behavior of a function near a particular *x*-coordinate. Students must possess a dynamic view of function, and they must view functions at least at the process level (Carlson, 1997). Carlson (n.d.) suggests that a deep understanding of the concept of function is a vital part of the background of any student hoping to comprehend calculus. Being able to visualize, make the connection between the equation form and the graphical form of a function, plays a key role in understanding limits of functions. Habre and Abboud (2006) also state that one fundamental change that calculus witnessed in recent years is an increased emphasis on visualization. Once students begin exploring the concept of derivative, they then explore deeper into their understanding of function. Here they are calculating rates of change, slopes of tangent lines, looking at related rates and use the derivative concept to describe the graphical representation of a function. Understanding functions and being able to connect their different representations as well as being able to analyze and make use of particular parts of functions is necessary for success with derivatives.

The concept of function is central to a student's ability to describe relationships of change between variables, explain parameter changes, and interpret and analyze graphs (Clement, 2001, p. 745). The final topic of a typical first-year calculus course is the integral. Here students use integration to calculate areas of regions between two functions, find volumes of solids created by revolving a region around an axis, and apply integration to solve physics problems. Creating the connection of what role functions play in each of these topics is a struggle for many students. With a strong mathematical background and understanding of function, students are typically successful understanding these concepts. Students with weak understanding of function struggle to make sense of much of what happens in calculus (Oehrtman et al., 2008).

The role of functions in first-year calculus is tremendous. Ensuring that students entering calculus or any college level mathematics course have a strong understanding of functions is vital to their success. Every aspect of first-year calculus involves students' understanding of function. Whether it be looking at limits, finding derivatives or using the integral to calculate areas of regions, functions play a role somewhere in the process.

Summary

As students move on to a higher level of mathematics classes, the role of functions and understanding of functions becomes more important. Understanding of functions in my study is determined, in part, by a student's ability to work with functions as measured by the APOS framework. The ability to apply the definition of function was used to determine another aspect of that student's understanding of function. Calculus students must enter the classroom being able to connect between the different representations of functions, interpret and make sense of the definition of function, manipulate functions, and understand how functions apply to phenomena in the world around. Previous research suggests that functions must be taught using a variety of representations and to allow students to create meanings and connections to the world around them. Students must develop a strong definition of function and be able to navigate through the different phases of APOS when working with functions. Doing this will ensure readiness when beginning to analyze functions in Calculus.

CHAPTER THREE: METHODOLOGY

The purpose of this study was to determine whether or not there is a relationship between student understanding of functions and their performance in calculus. Data were collected via two questionnaires given at the beginning of the course, along with the first semester calculus cumulative exam grades. In this chapter, I describe the research design, the participants, and the methods for data collection and analysis.

Research Design

This is a descriptive research study. Data were gathered from two questionnaires along with students' semester grades from a first-year calculus course. The questions that guided this research are:

1. Is there a correlation between student understanding of function and performance in calculus?
2. What patterns in student understanding of function are related (or unrelated) to student performance in calculus?

Two questionnaires (see Appendix A and D) were designed for this study to determine student understanding of function. Questionnaire 1 is a modification of the Precalculus Concept Assessment (Precalculus Assessment, n.d.) while I created Questionnaire 2 inspired by work by Vinner and Dreyfus (1989). I used Questionnaire 1, *Measurement of Ability to Work with Functions* to determine students' level of understanding based on the APOS framework. I used Questionnaire 2, *Measuring Student*

Concept Image (Definition) of Function to measure a student's ability to apply the definition of function. Individual student responses to the questionnaires were combined to form a composite score to represent a student's overall level of understanding of function. A student's performance in calculus was measured using the cumulative percentage that they achieved on the exams taken in the first semester of the first-year calculus course.

Participants and Setting

Participants of this study were members of two different high school calculus courses taught at a large suburban high school. Of the 116 students who participated, 60 were enrolled in AP Calculus AB and 56 were enrolled in Introduction to Calculus. The participants were chosen for convenience and ease of data collection as the researcher had access to all who might participate in the research. All the students, in either group, are considered higher achieving math students and thus are expected to have a deep understanding of function.

Both calculus courses, AP Calculus AB as well as Introduction to Calculus, offer college credit. Material that is covered in each course is relatively similar. However, AP Calculus AB goes deeper into calculus concepts. For example, AP calculus students are responsible for knowing how to integrate and differentiate polynomial, rational, radical, trigonometric and other transcendental functions. However, in the Introduction to Calculus course, students learn to differentiate and integrate polynomial, radical and rational functions along with a few transcendental functions like exponential and logarithmic functions.

From AP Calculus AB, students progress on to Calculus 2 or AP Statistics. Introduction to Calculus students have AP Calculus AB or AP Statistics as options the following year. Students in Introduction to Calculus are primarily seniors. Very few of the students taking Introduction to Calculus take AP Calculus the following year. Students enrolled in AP Calculus or Introduction to Calculus are typically eleventh or twelfth graders. Occasionally a tenth grade student may be enrolled in AP Calculus AB.

Data Collection

Two questionnaires were administered at the beginning of the school year in which the students were enrolled in the either of the first-year calculus courses. The research design and questionnaires were approved through IRB (Appendix F) and consent was collected through a form (Appendix G) sent to all students' parents electronically. Data from students who did not submit a consent form were not included in the study. The questionnaires were administered by either a teacher or an administrator at the high school. Questionnaire 1 was administered during the first two days of the school year; Questionnaire 2 was administered one week following Questionnaire 1. Exam grades were collected at the end of the semester for all sections of both courses. Codes were assigned to each student by the co-principal investigator (my advisor) to maintain confidentiality.

Questionnaire 1

Questionnaire 1 had 22 items (2 items had two parts) and was based on the Precalculus Concept Assessment (Precalculus Assessment, n.d.), which was modified for this research project. The PCA was chosen as a model because it is a validated instrument and a good measurement of preparation for calculus (Carlson et al., 2010). The assessment was modified from a multiple choice format to primarily an open-ended format to discourage guessing as well as to see student thought since interviews were not conducted. Also one of the 23 items was omitted as the wording of the question was determined to be poor and there were other items that measured the same skill. The PCA taxonomy (see https://mathed.asu.edu/instruments/PCA/pcataxonomy.shtml) is a listing of reasoning, conceptual and analytic abilities that the assessment measures. I aligned each item with the PCA taxonomy (See Appendix B) and used the categories of the taxonomy to identify patterns in student understanding of working with functions.

Once students completed the questionnaire, each item was assigned a point value based on a rubric that I developed (Appendix C). Point values for each item were determined based on a combination of answers provided for the PCA and answers given on the pilot version of Questionnaire 1 given at the end of the prior school year. I used the discussion for each answer provided on the PCA website as well as answers provided through the pilot version of the questionnaire to determine the point values for each item. The example below shows how the rubric was applied to possible student responses from item 11b.

Item 11b:

x	$f(x)$	$g(x)$
-2	0	5
-1	6	3
0	4	2
1	-1	1
2	3	-1
3	-2	0

Given the table above determine $g^{-1}(-1)$.

Score of 4: A process level of ability to work with functions would have to be shown. Here a student would have to correctly interpret what is being asked and produce the correct value, which is 2.

Score of 3: A student would have to show that they are at least approaching the process level. The student shows the ability to correctly interpret the procedure involved in the problem, however makes a mistake, for example finds $f^{-1}(-1)$ rather than $g^{-1}(-1)$. Here a student would produce an answer of 1.

Score of 2: A student demonstrates that he or she is at the action level. The student shows the ability to use some of the given information to find a solution. Here a student sees that the input is -1 and finds the output of *g* for this input, disregarding the inverse notation. The answer that would be provided here is 3.

Score of 1: Here a student does not demonstrate the ability to interpret the problem nor use any of the given information to find a solution. A student may simply pick a number from the table, not listed in the above scores, as a solution.

Once the questionnaire was completed, the co-principal investigator and I scored ten completed questionnaires. Both researchers then compared the scoring on each item and discrepancies in scoring were discussed and corrected. Once the rubric and scoring were agreed upon, the two researchers divided the completed questionnaires into two parts, and each of the researchers scored a particular set of items (myself: items 1-13, co-principal investigator: items 14-22). The co-principal investigator then recorded all data into an Excel spreadsheet.

Points were added and the total score determined whether a student is working with functions at the pre-action, action, pre-process or process level of the APOS framework (Table 1). The score ranges were determined by first taking the highest possible score of 96 (24 × 4) and by allowing for a few scores of three, the cut-off of 90 was determined. The low-end was then determined in much the same way. I included the condition of no fours since scoring a four even on one question would show some ability of working with functions. The two middle ranges were created to ensure that scores of all 3's or all 2's would fall into the appropriate ranges of pre-process or action respectively.

Table 1. Score ranges for Questionnaire 1

Total Score Range			
90-96	60-89	38-59	24-37 (with no 4's)
Student is at the **Process Level** of working with functions	Student is at the **Pre-Process Level** of working with functions	Student is at the **Action Level** of working with functions	Student is at the **Pre-Action Level** of working with functions

The questionnaire only allowed the researcher to determine that a student is at most at the process level. This was determined to be acceptable as research indicates that, first, being at the process level is important for a student to be successful in calculus (Carlson, 1997). Second, it is very difficult to measure and distinguish the difference between the object and process levels (Asiala et al., 1996).

Questionnaire 2

Questionnaire 2 consisted of eight items that I created based on work by Vinner and Dreyfus (1989) and its intent was to determine how well a student can apply the definition of function. The first five items of the questionnaire gave students a model (i.e. representation) and asked students to determine whether the model is or is not a function and then give a reason for their choice. The items included in the questionnaire were based on items from the literature (Vinner & Dreyfus, 1989, Mahir, 2010). The models that students were given in these first five items include: equation form, table form, graph form, and contextual form. The next two items asked students to define a function based on given criteria (Vinner & Dreyfus, 1989). One item asked students to define a constant function, the other asked for a piece-wise defined function. The final item asked students to give their definition of function. I developed a rubric (Appendix E) to assign points (3, 2, or 1) to student responses for each question. In order to score a three, the student must have demonstrated a complete ability to use the definition of function to justify his or her reasoning. A score of two was given when the student showed partial ability to use the definition of function. A score of one was assigned if the student incorrectly answered the item or if the justification showed no understanding of the definition of function.

An example of the rubric in use for item 3 is listed below. The sample responses below show how item 3 from the questionnaire was scored.

> **Item 3**: Is “The amount of money earned at a job.” a function? Give a reason for your answer.
>
> **Score of 3 response**: “It models a function.” Reason: “At different points in time you earn a certain amount of money.”
>
> **Score of 2 Response**: “It models a function”. Reason: “Every *x* has 1 *y*.”

Score of 1 Response: "It does not model a function." Or models a function with a reason like "steady slope".

To discuss the rationale for the scoring above, I make the following arguments. The student whose answer earned the full 3 points showed the understanding that a function must have one corresponding output for each input. Here the student showed this understanding and also related the input and output to the context of the problem. The student who earned 2 points on this item showed that they have a limited ability to apply the definition of function. This student's response indicates that the student thinks that *x* is always an input and *y* is the associated output and does not show the ability to apply the definition to a particular context. A score of 1 was earned for an incorrect answer or a justification that shows an incomplete ability to apply the definition of function.

Scores from each of the items on Questionnaire 2 were totaled and a strength of definition score was assigned to each student (see Table 2). The ranges below were determined in much the same way as those for Questionnaire 1. I started with a perfect score (all 3s) and allowing for 2s, the Strong definition range was created. A score with all ones created the low end (Weak), then allowing for a majority of 1s with some 2s created the range. The remaining values were then assigned to the Average range ensuring that scoring all twos fell in that range. A student falling into each of the categories would earn scores primarily showing that ability level (strong = 3, average = 2, weak = 1).

Table 2. Score ranges for Questionnaire 2

Strength	**Strong**	**Average**	**Weak**
Score range	21-24	13-20	8-12

Performance in Calculus

Students were organized in a successful category or unsuccessful category based on the percentage of the total cumulative score they earned on all the exams taken in the first semester. Table 3 shows the percentage necessary to be considered successful in each of the respective classes involved in this study. The exam scores were chosen because I felt that I had control over the scores rather than students being able to get help on take-home assignments like homework. Also, at times the overall grade, which includes homework, quizzes and exams may be inflated due to factors beyond my control. The responses of students with similar grades were then compared and it was noted as to whether any patterns emerged. For example, did low achieving students have a weak definition of function?

Table 3. Success in calculus based on course

Course	Successful Student Exam Percentage Score	Unsuccessful Student Exam Percentage Score
AP Calculus AB	> 88% of possible exam points	≤ 88% of possible exam points
Introduction to Calculus	> 91% of possible exam points	≤ 91% of possible exam points

I decided to use different exam percentages based on the class that the student was enrolled in primarily because of the material that is covered in each course. In AP Calculus AB, the students immediately begin covering calculus topics from day one. There is no review of any algebraic concepts. Whereas, the curriculum for Introduction to Calculus begins with an intensive review of algebraic concepts such as: domain, functions, solving equations, and simplifying expressions. This review is covered in a chapter and a half, therefore one and a half of the exams cover review topics on which the majority of students should be successful. Also, the material in the Introduction to Calculus course is not as in depth and thorough as it is in the AP Calculus course.

Data Analysis

To determine student understanding of function an item-by-item analysis was done on both questionnaires as well as a comparison of the overall scores on each of the questionnaires. I then used the responses to the questionnaires to calculate a composite score to represent a student's overall understanding of function. To find the composite score I found the sum of the scores of the two questionnaires and then aligned the scores by student. The data for students who did not answer both questionnaires were not included.

Table 4. Composite score ranges

111-120	73-110	50-72	32-49
Strong understanding of function	Average understanding of function	Fair understanding of function	Weak understanding of function.

The ranges in Table 4 were determined by combining the ranges from Tables 2 and 3. The upper range combines the ranges for a student who is at the process level (score of 90-96 on Questionnaire 1) from Table 2 with the range for having a strong definition of function (score of 21-24 on Questionnaire 2) from Table 3. The lowest range is a combination of ranges for students who are at the pre-action level (score of 24-37 on Questionnaire 1) with the range for a weak definition (score of 8-12 on Questionnaire 2) of function. The 75-111 range combines the range for students in the pre-process level (score of 60-89 on Questionnaire 1) and the range for having an average definition of function (score of 13-20 on Questionnaire 2). The range of scores between the average understanding and weak understanding (50-72) were assigned to the fair understanding.

I used Excel to perform a linear regression between the composite score and the student's semester exam grade to determine whether there was a correlation between student understanding of function and performance in calculus. I then performed linear regressions by course (AP Calculus versus Introduction to Calculus), using the composite score and the scores on each questionnaire as independent variables. This allowed me to answer question 1: Is there a correlation between student understanding of function and performance in calculus? To answer question 2: What patterns in student understanding of function are related (or unrelated) to student performance in calculus, an Excel spreadsheet was created. First, each student composite score was aligned with the student's first semester cumulative Calculus exam grade. I then identified patterns in the responses through an item-by-item analysis, a total questionnaire score analysis and finally a composite score analysis. Through this comparison, it was noted whether there was a pattern in responses that relates to student scores on the two questionnaires and their level of success in the introductory calculus course. All data were stored in an Excel spreadsheet so that the analysis was efficiently and accurately performed.

CHAPTER FOUR: RESULTS

In this chapter, I report the results of my analysis of students' responses to the two questionnaires, characteristics of the composite score groups, students' exam scores, and the regression analysis.

Questionnaire 1

Questionnaire 1 was used to determine at what level a student entering a first-year calculus course was able to work with functions. The mean score for this questionnaire was 60.8. This score is at the lower range of the pre-process ability with functions based on the rubric for this questionnaire. It was found that of the 116 students who participated in the study, one student performed at the process level, 61 students performed at the pre-process level, 52 students performed at the action level and 2 students performed at the pre-action level.

To determine whether there were any areas where student performance tended to be stronger (or weaker), a table was created to show the percentage of students that were successful on items in relation to each of the PCA taxonomy categories. The table below (Table 5) shows the percentage of students who scored an average score greater than 2.5 on all items in the section of the taxonomy as well as the percentage of students who scored an average of 2.5 or less on all items in the section of the taxonomy. The score of 2.5 was chosen as it was a convenient division point on an assessment that produced scores between one and four points. Looking at Table 5, the row that corresponds to items related to R1 (View function as a process) shows that 48% of the students scored an average of more than 2.5 points on all the items related to this category and 52% of the students earned an average score of 2.5 or less on these items.

Table 5: Percentage of students scoring an average greater than 2.5 or less than or equal to 2.5 on all items corresponding to sections of the PCA taxonomy

	Percentage of Average Scores > 2.5	Percentage of Average scores ≤ 2.5
PCA Taxonomy category		
R1 (Function as a process)	48%	52%
R2 (Covariational reasoning)	54%	46%
R3 (Proportional reasoning)	35%	65%
C1 (Evaluate and interpret functions)	61%	39%
C2 (Represent function situations)	47%	53%
C3 (Perform function operations)	50%	50%
C4 (Inverse functions)	47%	53%
C5 (Interpret and represent function behaviors)	50%	50%
C6 (Rate of Change)	53%	47%

As a whole group, student performance was roughly equally split between the groupings of scores that had an average greater than 2.5 and scores with an average less than or equal to 2.5 on all categories of the taxonomy with the exception of categories R3 and C1. Category R3 was aimed at measuring the students' abilities to engage in proportional reasoning. The larger percentage of students had average scores less than or equal to 2.5 in this category. Category C1 measured student ability to evaluate and interpret function information given the function's formula, graph and table. In this category students performed well with 61% of students earning an average score greater than 2.5 on all items in C1.

The following table (Table 6) shows student performance for each item on Questionnaire 1. The first column shows the item number along with the appropriate taxonomy category. The next four columns show the percentage of students that earned each of the respective scores based on the rubric used in scoring the questionnaire.

Table 6: Item breakdown of Questionnaire 1

Item (taxonomy category)	**% of 4s**	**% of 3s**	**% of 2s**	**% of 1s**
1 (R1, C1, C3, C5)	58%	9%	8%	25%
2 (R1, C1, C3, C4)	47%	1%	24%	28%
3 (R3, C2)	46%	16%	10%	28%
4 (R2, C2)	34%	4%	30%	31%
5a (R1, C1, C3, C4)	43%	3%	18%	35%
5b (R1, C1, C3, C4)	46%	14%	4%	36%
6 (R1, C5)	38%	24%	17%	22%
7 (R1, R2, C1, C5, C6)	5%	28%	47%	20%
8 (R1, C1, C3, C4)	28%	9%	30%	33%
9 (R2, R3, C2, C5)	12%	18%	31%	39%
10 (R2, C5, C6)	6%	2%	64%	29%
11a (R1,C1, C3, C4)	32%	26%	12%	30%
11b (R1, C1, C3, C4)	11%	10%	26%	52%
12 (R1, C4)	12%	2%	23%	62%
13 (R2, C1, C2, C6)	42%	42%	4%	12%
14 (R1, C3, C4)	73%	3%	1%	23%
15 (R2, C2, C6)	17%	7%	50%	26%
16 (R2, C1)	64%	11%	5%	20%
17 (R2, C1, C5, C6)	55%	36%	1%	8%
18 (R1, R2, C3)	6%	18%	51%	25%

19 (R1, C1, C5)	6%	30%	18%	45%
20 (R2, C1, C5)	18%	17%	56%	9%
21 (R2, C1, C6)	65%	20%	1%	14%
22 (R2, C1, C5)	43%	15%	10%	32%

Items on this questionnaire were determined to be items on which students showed a strong performance if the overall percentage of a score of 4 was greater than 60%. Therefore, items 14, 16 and 21 would be items on which students showed strong performance. Item 14 had students evaluate the composition of two functions given the function rules. Items 16 and 21 were items in taxonomy categories R2 and C1. These items both had students interpreting the values of a function for particular input values. Item 1 was not included as a strong performance item, first because the percentage of scores of 4 did not exceed 60%. The item asked students to evaluate a given function at an input of $(x + a)$. Although students who scored a 2, 3, and 4 all showed the ability to correctly begin the process of evaluating $f(x + a)$, students who earned the lower scores either did not simplify the expression or made algebraic or arithmetic mistakes in the simplification process. Students who did not simplify were counted as earning a score of 4, but after discussion with the co-primary investigator, it was determined that we could not determine whether the student had the ability to simplify the expression correctly. Therefore, there is a possibility that the number of 4's could be inflated. Finally, items 13 and 17 were not included as strong performance items, as a score of 3, based on the rubric, would not necessarily show a strong ability to interpret the rate of change of a function, which was the aim of both items. Both items allowed students the ability to choose one of two possible answers counted as a score of 3 of the five total possible choices. What this means is that a student had a 3 out of 5 chance of picking a response and earning a score of either 4 or 3 showing a relative high ability with function.

Items 10 and 12 were items on which students showed weak performance. The performance on these items was considered weak as more than 80% of the students involved in the study received a score of 1 or 2 on these items. Item 10 had students calculate the average velocity of a car over a period of time given the position function of the car. Students primarily responded to this item by calculating the position of the car at one of the given time values (score of 2) or were not able to respond to the item at all (score of 1). Item 12, had students find the inverse function of a function given as a rule. For this item, students either did not produce a function at all or students produced a function that was the reciprocal of the given rule. Item 18 was not included as an item with weak performance as it was determined that many of the students misinterpreted the question.

Additionally, items 10 and 16 (both in category R2) and items 12 and 14 (both in category R1 and C4) showed the trend that one of the items was considered an item on which the students showed strong performance and the second item students showed weak performance. For items 12 and 14, recall item 12 was considered as a weak performance item while item 14 was considered a strong performance item. Although they were both measuring abilities in the same taxonomy category, the amount of success shown on the item could have been due to familiarity with the notation. For items 10 and 16, students were demonstrating their ability to apply covariational reasoning to interpret function behavior. That is how does a change in one variable affect changes in the other variable of an equation. In both items students were presented with an equation. Item 16 had students directly using the equation to interpret its behavior. While for item 10, students had to understand how to use the given position function to find information about velocity. The indirect use of the given equation and also unfamiliarity with the measured concept was a possible cause of the poor performance on item 10.

Finally, items 14 and 16 had a very large percentage with a score of 4 as well as a large percentage with a score of 1, with less than 20% of the students with scores of 2 or 3. This shows that students either demonstrated complete ability to answer the item correctly, or demonstrated very little if any ability with the item. Students who scored a 4 on item 14 showed the ability to produce the appropriate composition of two functions either by simply producing the correct answer or by showing the process that they used to produce the correct solution. The students earning a score of 1 either were unable to produce a solution or produced some form of product of functions, for example some students wrote $g(h(2)) = g(x) \cdot h(x) \cdot 2$. A score of 4 was earned on item 16 if students correctly chose the option that describes the behavior of a rational function.

Questionnaire 2

Questionnaire 2 was intended to measure a student's ability to apply the definition of function upon entering a first-year calculus course. Based on the summative score on the questionnaire, each student was assigned a strength of ability to apply the definition ranking: strong, average, or weak. The mean score for this questionnaire was 12.9. This score is just below the low end of the range of an average ability to apply the definition of function. Of the 116 students involved in the study, 1 student entered calculus with a strong ability to apply the definition, 59 students entered with an average ability to apply the definition and 56 students entered calculus with a weak ability to apply the definition of function. The table below (Table 7) shows the breakdown of the percentage of students who scored each of the possible scores of 3, 2, or 1 with a score of 3 being the highest on each item of Questionnaire 2.

Table 7: Item breakdown of Questionnaire 2

Item	% of 3s	% of 2s	% of 1s
1	7%	28%	65%
2	23%	53%	24%
3	6%	17%	78%
4	20%	35%	45%
5	57%	14%	29%
6	9%	22%	70%
7	13%	30%	57%
8	2%	35%	62%

As a group, the students demonstrated a lack of ability to explain their reasoning as to how they determined whether the representations in items 1 through 5 were or were not functions. Many students used reasons such as “passes the vertical line test” to justify their conclusion. With more than 70% of the students scoring a 1, items 3 and 6 were items on which the students showed weak performance. Item 3 asked students to determine whether a scenario described in words was a function and then justify their reasoning. Item 6 asked students to create a piece-wise function that satisfied particular parameters. Although item 1 had a high percentage of 1’s, I felt that students did not pay close attention to the “±” in the relation $f(x) = \pm\sqrt{x}$. Therefore, I did not judge this item as one on which the students performed poorly. Because only 2% of the students involved in the study scored a 3 on item 8, it can be concluded that students were not able to state a precise definition of function.

The table below (Table 8) shows the results of the two questionnaires and the number of students in each category.

Table 8: Student results on the two questionnaires

Questionnaire 1		Questionnaire 2	
Total Students (N)	116	Total Students (N)	116
Process	1	Strong Definition	1
Pre-Process	61	Average Definition	59
Action	52	Weak Definition	56
Pre-Action	2		

From the table it can be concluded that the participants of this study entered calculus performing primarily at the pre-process or action level and with a primarily average or weak ability to apply the definition of function. The student who performed at the process level was not the same student with the strong ability to apply the definition of function.

Composite Group Results

Once both questionnaires were administered, each student was assigned a composite score. This composite score separated each student into one of three subgroups: average, fair, or weak. Each subgroup was based on a student's understanding of function as measured by the student's composite score on the two questionnaires. A student's composite score on the two questionnaires was used to determine the level of understanding of function for each student. As discussed in the literature, a student's ability to work with functions as well as their ability to apply the definition of function may determine how well they understand functions.

After each questionnaire was analyzed separately, the composite scores were then analyzed. Students were placed into groups and I tried to determine if there was a particular characteristic of each group. Table 9 shows the percentage (number) of students that scored in each of the composite categories.

Table 9: Composite score category percentages (based on scores on two questionnaires)

Total N = 116	Strong (111-120 Composite Score)	Average (73-110 composite score)	Fair (50-72 composite score)	Weak (32-49 composite Score)
Percentage (number) in each composite category	0% (0)	52.6% (61)	44% (51)	3.4% (4)

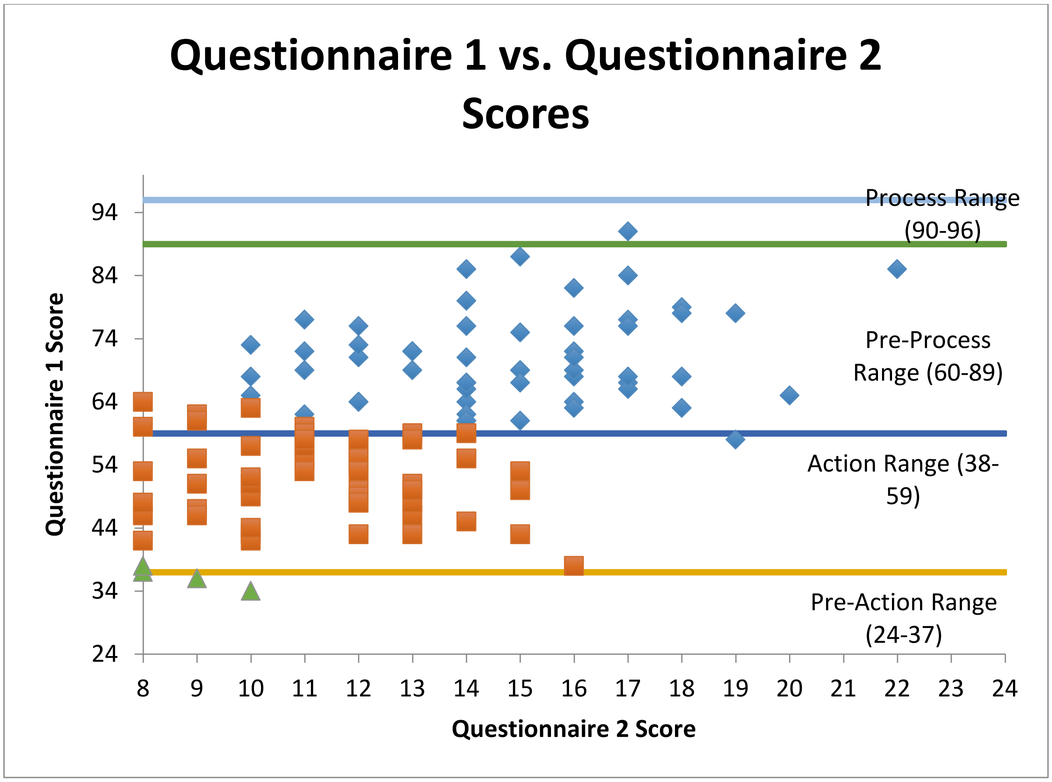

Figure 3: Graph of Student Results of Questionnaire 1 vs. Questionnaire 2

Figure 3 above is a scatter plot of student scores on the two questionnaires. The horizontal axis is the range of scores for Questionnaire 2 and the vertical axis is the range of scores for Questionnaire 1. The student scores within each subgroup are represented by a different shape in the plot (weak understanding group are triangles, fair understanding group are squares, and average understanding group are rhombi). The horizontal lines show the upper value of the range for each of the levels from Questionnaire 1 (Pre-action, action, pre-process, and process). From the graph, one could observe that a student scoring a 10 on Questionnaire 2 scored in the range of 34 to 73 on Questionnaire 1. A student scoring a 10 on Questionnaire 2 could be considered as have a pre-action, action or pre-process ability to work with function as the points associated with a Questionnaire 2 score of 10 fall into each of the Questionnaire 1 ranges in the graph.

The table below (Table 10) shows the performance of each subgroup on Questionnaire 1 items that are included in the taxonomy categories. The percentage of students of each subgroup that are considered successful (an average score greater than 2.5 on all items in the category) as well as the percentage of students who are considered not successful (average score of 2.5 or lower on all items in the category are listed for each subgroup in Table 10. For example the first two cells in the row for R1 show that 85% of the students of the average understanding subgroup showed success on the items in the category, while 15% of the students in the average understanding subgroup did not show success with items in the category.

Table 10: Subgroup performance on Questionnaire 1 taxonomy categories

	Student Group Based on Composite Questionnaire Score					
	Average (N=61)		Fair (N=51)		Weak (N=4)	
Taxonomy Category	% of students with average score >2.5	% of students with average score ≤ 2.5	% of students with average score >2.5	% of students with average score ≤ 2.5	% of students with average score >2.5	% of students with average score ≤ 2.5
R1	85%	15%	10%	90%	0%	100%
R2	77%	23%	33%	67%	0%	100%
R3	51%	49%	20%	80%	0%	100%
C1	93%	7%	29%	71%	0%	100%
C2	67%	33%	27%	73%	0%	100%
C3	84%	16%	16%	84%	0%	100%
C4	80%	20%	12%	88%	0%	100%
C5	75%	25%	25%	75%	0%	100%
C6	84%	16%	41%	59%	0%	100%

The Average Understanding Group

A student in this study was placed in the average understanding subgroup if his or her composite score fell in the range of 75 and 111 on the two questionnaires. Of the 116 students involved in the study, it was determined that 61 were in this subgroup.

Results from Questionnaire 1 determined that, in this subgroup, one student had the ability to work with functions at the process level, 56 students worked with functions at the pre-process level and four students worked with functions at the action level. Therefore, students in this subgroup primarily have the ability to work with functions at the pre-process level. The results of student performance in this subgroup in each of the PCA taxonomy categories are shown in the table above.

A student in the average understanding subgroup had the ability to answer the majority of the items in each category with a good level of success. The values in Table 10 show the percentage of students who answered questions in each category correctly with an average score of more than 2.5 for all the items in that category. Much like the group of students in this study as a whole, the average subgroup performed in a similar manner on PCA taxonomy categories R3 (engage in proportional reasoning) and C1 (evaluate and interpret function information given the function's formula, graph and table). Of the students in the average subgroup, 93% were able to score an average of more than 2.5 on all the items in category C1. This was this subgroups strongest category. Although category R3 had 51% of the students show the ability to score an average score of more than 2.5 on items in this category, the percentage was the lowest of all the categories for this subgroup.

A student in the average understanding subgroup entered calculus primarily with an average ability to apply the definition of function. Of the 61 students in this subgroup, one student scored a strong ability to apply the definition of function on Questionnaire 2. Forty-seven students scored an average ability to apply the definition of function while 13 students scored at the weak ability level. The students in the average subgroup demonstrated the ability to correctly answer (an average group score greater than 2) two items on questionnaire two. These two items included interpreting a table and an input-output mapping diagram as a function (items 2 and 5 on Questionnaire 2).

The Fair Understanding Group

A student with a fair understanding of function scored a composite score between 50 and 72 on the two questionnaires. Fifty-one of the 116 students involved in the study fell into this subgroup. Of these students 5 worked with functions at the pre-process level while 46 of the students worked with functions at the action level. The group's performance by PCA taxonomy category is in Table 10.

To discuss any possible trends in student performance on the two questionnaires, I will again focus on this subgroup's performance in categories R3 and C1 of the PCA taxonomy. Students of this subgroup had the third highest percentage of students show the ability to score over an average of 2.5 points on all the items in category C1. Although there was a greater percentage of students who scored less than 2.5, this remains consistent with the whole group as a category on which the groups are performing well, the percentage of students in the subgroup that scored an average score of more than 2.5 point on all items in the category was among the greatest. Similarly, category R3 had the same performance in this subgroup as it had in the whole group. It was considered as a category on which the students showed poor performance since 80% of the fair subgroup scored an average score less than 2.5 on all items in this category. In addition, students in this subgroup showed poor performance in category R1 (View function as a process) as only 10% of the students in the subgroup were successful with items in this category.

Results of Questionnaire 2 showed that students in this subgroup primarily had a weak ability to apply the definition of function. Twelve students began the year with an average ability to apply the definition of function while 39 were measured as having a weak ability to apply the definition of function based on results of Questionnaire 2. Here again students performed the best on items 2 and 5. The average score of the entire group was the highest on these two items. The average scores on the items were 1.8 and 2 for items 2 and 5 respectively.

The Weak Understanding Group

A student was placed into the weak understanding subgroup if the composite score on the two questionnaires was between 32 and 49. This subgroup contained four students. Of the four students, two of the students demonstrated the ability to work with functions at the action level while the remaining two students performed at the pre-action level. All four students were measured as having a weak ability to apply the definition of function.

No student in this subgroup was able to answer all the questions in any of the PCA taxonomy categories with an average of greater than 2.5. The students in this subgroup showed the greatest performance on items 1, 6, 13, and 21 of Questionnaire 1 with average scores of 2.25, 2.5, 3, and 2.5 respectively. Items 1, 13, and 21 are all items in taxonomy category C1, the category on which the whole group is showing the greatest performance. Because of the overall performance of the weak understanding subgroup, the trend with category R3 was maintained in this subgroup as one in which the students demonstrated low performance.

By analyzing answers on Questionnaire 2 for this subgroup, it should be noted that students received primarily a score of one (the lowest possible score) on the items. One student received a score of 3 on item 5 (the input-output mapping diagram). Also, only one student scored a 2 on item 2. Item 2 presented students with a relation in the form of a table and asked them to determine whether the relation represented a function.

Determining Success in Calculus

To determine which students were successful, recall that I used the students' cumulative exam scores for the first semester. This means that the scores earned on all exams in the first semester were averaged. The average of all the scores was used to determine whether or not the student was successful. Table 11 shows the percentage of students that were considered as successful in the two courses. For example, the second row and second column of Table 11 shows the percentage of successful students (36%) along with the number of successful students (22) who were enrolled in AP Calculus AB.

Table 11: Student breakdown by calculus course.

Calculus Class	% Successful	% Not successful
AP Calculus AB (N=60)	36% (N=22)	64% (N=38)
Introduction to Calculus (N=56)	30% (N=17)	70% (N=39)

Although one might consider that any grade higher than an 80% would be successful in high school calculus, with these students being some of the higher achieving students, I felt that higher percentages should be required in order for the student to be considered successful. For reference, the mean exam score for the AP Calculus AB students was 84.7%, while the mean exam score for the Introduction to Calculus students was 80.7%. Table 12 contains other basic statistics associated with the exam scores for each course. As could be seen in Table 12, the values in the columns labeled AP Calculus and Introduction to Calculus are somewhat similar. However, when comparing the standard deviation of the exam scores for the two courses, notice that the scores of Introduction to Calculus have a greater distribution than those of AP Calculus.

Table 12: Basic Statistics on Exam Scores

Statistic	AP Calculus	Introduction to Calculus
Mean	84.7	80.7
Standard Deviation	9.4	13.8
Min (Course Exam %)	57.7	50.6
Max (Course Exam %)	101.4	101.2
N	60	56

The figures below show the distribution of exam score percentages for the two classes involved in this study, AP Calculus AB (Figure 3) and Introduction to Calculus (Figure 4).

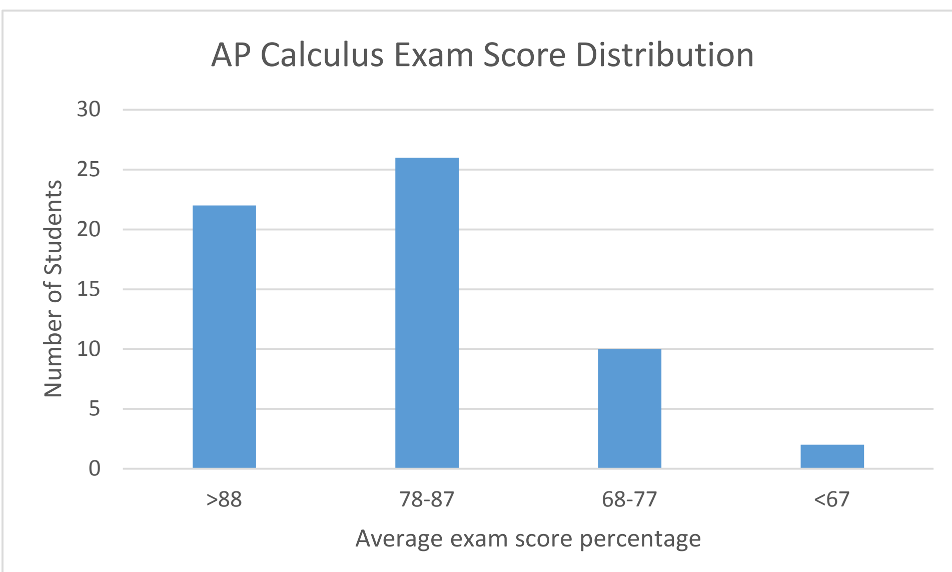

Figure 4: Distribution of exam scores for students enrolled in AP Calculus AB

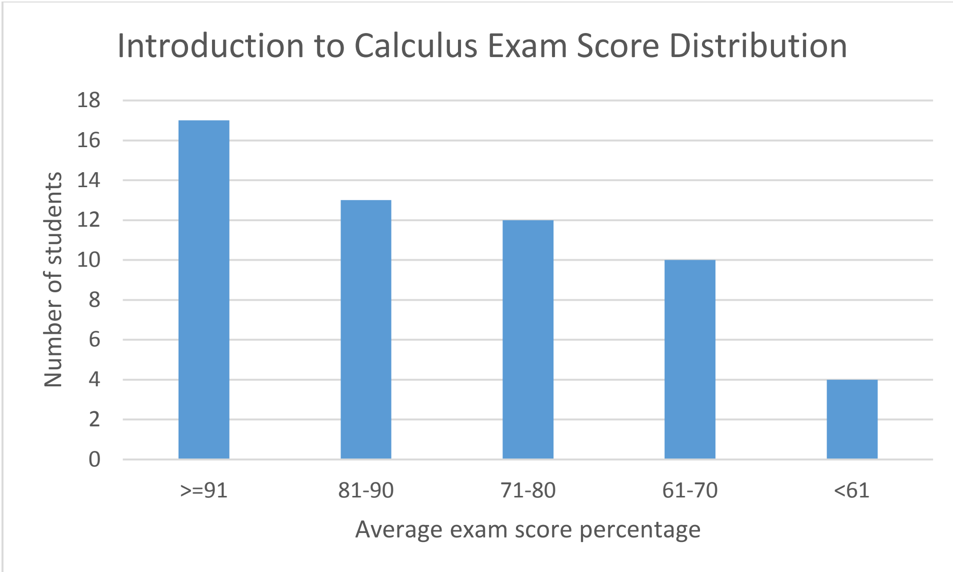

Figure 5: Distribution of exam scores for students enrolled in Introduction to Calculus

Once students were determined as being in either the success group or the not successful group, I then tried to determine if there were any trends among the scores in each of the groups. The results of this initial analysis are shown in Table 13 below. The percentages in the table describe the percent of students in that category that were considered to have an average, fair, or weak understanding of function based on results of the two questionnaires. For example, looking at the first row and third column of Table 13, one would see that 81.8% (18 students), enrolled in AP Calculus AB, who were considered successful in the course scored as having an average understanding of function based on results of the two questionnaires. The other 18.2% (4 students) of the successful AP students demonstrated a fair understanding of function.

Table 13: Successful and not successful student percentages at each level of understanding of function.

		% average	% fair	% weak
AP Calculus AB (N=60)	Successful (exam average > 88%) N=22	81.8% (N=18)	18.2% (N=4)	0% (N=0)
	Not successful (exam average ≤ 88%) N=38	73.7% (N=28)	26.3% (N=10)	0% (N=0)
Introduction to Calculus (N=56)	Successful (exam average > 91%) N=17	64.7% (N=11)	29.4% (N=5)	5.9% (N=1)
	Not successful (exam average ≤ 91%) N= 39	10.3% (N=4)	82.1% (N=32)	7.7% (N=3)

It is fairly surprising that in the AP Calculus AB class that the percentage of students that were considered as having an average versus fair understanding of function were fairly close in both the successful and not successful categories. In other words, the rate of success for a student enrolled in AP Calculus was not determined by the level of understanding the student had with function. However, of the 28 students enrolled in AP Calculus AB with an average understanding of function that were not considered successful in the course, sixteen of these students scored an average exam score between 78% and 87% during the first semester.

The Introduction to Calculus class was much more telling. The majority of the students that were considered successful started the year with an average understanding of function. There was also a fair percentage of students that were considered successful that started the year with a fair understanding of function. The group of students who were not considered to be successful in the Introduction to Calculus course had a very heavy percentage of students that had a fair understanding of function at the beginning of the course. Very few students who were considered not successful in Introduction to Calculus had an average understanding of function. A student's rate of success in Introduction to Calculus appears to be more closely related to their measured understanding of function than it was in AP Calculus AB. That is a student with an average understanding of function had a greater rate of success in the course over a student with a fair or weak understanding of function.

Recall that I assigned each student a semester exam grade based on the cumulative total of all the exams taken the first semester of the respective introductory calculus course. I used Excel to perform a linear regression between the composite score and the student's semester exam grade to determine whether there was a correlation between student understanding of function and performance in calculus. I found that student exam scores can be predicted by student composite scores with following equation: Exam Score=56.5+0.356*Composite Score ($R=0.426$; $S=10.77$; $p=0$).

I also performed linear regressions by course (AP Calculus versus Introduction to Calculus), using the composite score and the scores on each questionnaire as independent variables. These results are illustrated in Table 14. The column labeled R (Exam score vs. Composite Score) shows the correlation for each course involved in the study when taking the composite score as the independent variable. The asterisk (*) indicates that the exam score could be predicted by the corresponding independent variable for students in a particular course at the $p < 0.05$ level. The p-value for each regression score is listed in each cell in parentheses.

Table 14: Correlation between average exam scores and questionnaire scores by course.

	R (Exam score vs. Composite score)	R (Exam score vs. Questionnaire 1 score)	R (Exam score vs. Questionnaire 2 score)
AP Calculus AB	.254 (p=0.050)	.266* (p=0.0398)	.089 (p=0.499)
Introduction to Calculus	.537* (p=0.00002)	.512* (p=0.00005)	.361* (p=0.0063)

The scores for Questionnaire 1 show statistical significance at the $p < 0.05$ level for both the AP Calculus AB and the Introduction to Calculus courses. That means that one could reasonably predict the exam score for a student knowing the score a student earned on Questionnaire 1. This result was also true for the composite score and the Questionnaire 2 score of the Introduction to Calculus student scores. The predictability of the results is however limited as the standard error in all cases is quite large.

The composite scores and the Questionnaire 2 scores did not show statistical significance at the $p < 0.05$ level as predictors of exam scores for the students enrolled in AP Calculus AB. That means, that knowing either a student’s composite score or Questionnaire 2 score, one would not be able to determine that student’s semester exam grade in the course. The linear regression models for each of the statistically significant results are shown in Table 15 below. Residual plots were checked for each to be sure there were no patterns in the residuals.

Table 15: Linear Regression models for statistically significant regression data.

Course (Independent Variable)	**Regression Model**	**Standard Error**
AP Calculus AB (Questionnaire 1 Score)	Exam %= 68.5802 + 0.2392 * Q1 Score	9.12
Introduction to Calculus (Composite Score)	Exam %= 38.2339 + 0.6484 * Comp. Score	11.76
Introduction to Calculus (Questionnaire 1 Score)	Exam % = 43.1584 + 0.6972 * Q1 Score	11.97
Introduction to Calculus (Questionnaire 2 Score)	Exam % = 58.5322 + 1.9003 * Q2 Score	13.00

CHAPTER FIVE: DISCUSSION

Students entering calculus at the high school level would, in most people's eyes, already be considered successful. However, in this study, I aimed to determine whether understanding of function plays a role in student success as well as if there were any patterns in student understanding that contributed to the student's success in calculus. In this chapter, I use the results of my analysis to answer my research questions and discuss some possible limitations of this study.

Is There a Correlation Between Student Understanding of Function and Performance in Calculus?

The data indicate that student performance on Questionnaire 1 serves as a reasonable predictor of student success in calculus in a limited way. The results showed that exam scores could be predicted based on scores from Questionnaire 1 for both AP Calculus AB as well as Introduction to Calculus. In other words, a student's ability to work with functions would translate into roughly the appropriate level of success in calculus (working with functions at the process level would translate into a high rate of success while working with functions at the pre-action levels would translate into a lower rate of success). Questionnaire 2 can serve as a limited predictor for students enrolled in Introduction to Calculus, however, not for AP Calculus AB. While the composite score for the entire group appeared to be a predictor of exam score, this result did not carry through when examined by course. That is, the composite score could be used to predict whether a student entering Introduction to Calculus could be successful, however, the composite score is not a good predictor of exam score (or success) for a student entering AP Calculus AB.

While analyzing the results of Questionnaire 1, I discovered interesting trends in performance for the students that were involved in the study. From this questionnaire, I found that the students entered calculus with the ability to work with functions at either the action or pre-process level of the APOS framework. The results of Questionnaire 2 showed that students entered calculus with difficulty defining function in line with the Dirichlet-Bourbaki definition. The participants of the study began the calculus course with either an average or weak ability to apply the definition of function.

The composite scores showed that students entered their respective first-year calculus courses with either an average or a fair understanding of function based on composite scores. The most significant finding related to success is that students can be successful in a first year calculus course without demonstrating a process level understanding of function at the beginning of the course. Further, it is possible for students to be successful in calculus with an average or fair understanding of function although generally much less likely for students with a fair understanding compared to those with an average understanding of function.

For students enrolled in the AP Calculus AB course, the percentage of students with an average understanding of function compared to those with a fair understanding was almost equal in the successful versus not successful categories. However, these students do all come from Honors Pre-Calculus which is taught by one teacher at the school where this study was conducted. Therefore, material prior to calculus should have been presented to the students in similar manners. Therefore student performance should be somewhat more consistent for the students entering AP Calculus AB.

The difference between the students entering Introduction to Calculus as opposed to those entering AP Calculus AB is that the students entering Introduction to Calculus have the possibility of either taking Honors Pre-Calculus or Math Analysis prior to the calculus course. Because of this, students frequently enter the class with varying degrees of mathematical understanding and ability. These two courses were also taught by three different teachers the year prior to this study. As mentioned earlier, a student who was enrolled in Introduction to Calculus and was measured as having an average understanding of function, had a high rate of success in the course. Whereas, a high percentage of students enrolled in the same class with a fair or weak understanding of function were not considered successful in the course.

What Patterns in Student Understanding of Function Are Related (or Unrelated) to Student Performance in Calculus?

The results of the two questionnaires showed that students entered their respective first-year calculus courses with a varied understanding of function. As a group, the students showed the ability to correctly solve problems with which they should have been familiar. These items included ones where the student was given the algebraic rule and asked to calculate the value of the function. Here students are working with functions at the action view of function. When presented with a problem in context, or in a form with which the student is not familiar, these were items where students tended to demonstrate weaker performance. Overall, students demonstrated difficulty stating the definition of function precisely, which likely contributed to the difficulty students had using the definition to justify decisions

Students had the ability to answer items on Questionnaire 1 that were familiar from previous mathematics courses. The PCA taxonomy category that included items which are typically presented in a standard mathematics text are in category C1, the category which, recall from the results chapter, was the category where the students showed a strong performance. These items involved giving the students the function as a rule and finding the value of the function at a particular input. Items 1 and 14 were these kinds of items. However, if students were presented with a function as a table or graph and asked to perform the same task, the percentage of 3s and 4s was not as high as when the students were provided with the rule. It is interesting to note that 75% of the students involved in the study were able to start correctly when determining $f(x + a)$ on item number 1 (students with scores of 2, 3, or 4). However, when we began the study of the derivative, this concept seemed difficult to the students. This could be because the idea is combined with many other concepts and not just simply evaluating a function for a particular input.

Items 16, 17, 21, and 22 are items that are covered near the end of typical courses taken prior to a student entering calculus. Recall from the results chapter, Table 6 showed a large percentage of students showing the ability to score either a 3 or 4 on three of the four items. These items should be the most familiar since they were introduced at the end of the course typically taken prior to calculus. Many of the other items on which the students showed success are repeatedly covered throughout the high school mathematics curriculum. Therefore, it is not a surprise that students were successful with these items.

Two items on which the students showed the ability to earn scores of 3s and 4s, items 13 and 17, were multiple-choice items. Unlike the “choose all that apply” items, 20, 21, and 22, it was difficult to tell the ability a student had with function from these items as I felt that some students could have guessed the correct choice and earned a 4 or 3 on that item without having true ability to work with functions in the way intended by the item.

Item 5 on Questionnaire 2 showed a fair amount of success by students as a majority of students (57%), earned a score of 3. This item asked students to determine whether an arrow diagram mapping of domain elements to range elements was a function. This is typically how functions are presented to students in courses prior to calculus. Therefore, I feel that the students’ familiarity with the form of the item was a primary cause for their success.

When students were asked to work with functions in an unfamiliar way, the performance on the items was not as high as when they were provided with the function as an algebraic rule, even for items within the same taxonomy category. The students’ difficulty came when asked to use functions, whether in equation, graphical, or tabular form, in ways that were unfamiliar to them. For example, item 2 asked students to find the input of a function that produced a particular output given the graph of a function. A second example, students were considered very successful on item 14, where they were asked to calculate the composition of two functions for a particular input given each function as an algebraic rule. However, when asked to perform a similar task on item 11a given the table, the level of success decreased. This mirrors much discussion in the literature that students have difficulty connecting the different forms of function: rule, table, or graph (Thompson, 1994; Davis, 2007; Carlson, et al., 2010; Akkus et al., 2008).

Next, students demonstrated that they had difficulty calculating inverse functions. Item 11b presented students with a table and asked students to identify the value of the inverse function at a given input. Here students either did not provide an answer at all, provided an answer, which showed no connection to the problem, or simply found the function value rather than the inverse function value. Item 12 was another problem asking students to calculate the inverse of a given function. This time however, students were provided with the rule and asked to find the inverse function as a rule. On this item again, the incorrect responses either involved no solution at all or students interpreted the notation f^{-1} as the reciprocal of the function f.

Also, items 7 and 10 proved as items with which students had difficulty. Items 7 and 10 had students connecting position and velocity information. Item 7 presented students with the graph of the speed of two cars with respect to time. The students were asked to identify the position of one car relative to the other at a particular time. Students used the fact that the two graphs intersected to conclude that the cars were at the same position at the time in question. Here students disregarded the information that they were being presented with in the graph. Much like what has been mentioned in the literature, for these students, the shape of the graph models the “trip” of each car (Monk, 1992, Oehrtman et al., 2008). On item 10, students were given the position of a car relative to time as an algebraic rule. Here students were asked to calculate the average velocity over a given period of time. The common solutions that students produced were, first the position of the car at either or both of the given times. Here students simply evaluated the given function at the given values, relying on a concept that was familiar from previous mathematics courses. Second, students found the sum of the position at the beginning of the time interval and the position at the end of the time interval and divided the sum by two. In other words, some students calculated the average of the two positions. It was not surprising to me that these two items proved as a challenge to students, as these are concepts that are covered in calculus and physics. Many students entering calculus are concurrently enrolled in physics.

However, much like Questionnaire 1, items on Questionnaire 2 that were not familiar to students, such as functions which were not continuous, functions written as a verbal statement, and piece-wise functions all gave the students difficulty. This again is consistent with the literature (Dubinsky & Wilson, 2013; Sierpinska, 1992; Clement, 2001). Students also tended to use very simple explanations to explain their choices. These included “the vertical line test” or continuity as how they determined whether a relation was or was not a function. This was also mentioned in the literature (Dubinsky & Wilson, 2013).

Limitations

Although some of the results of the study proved to be statistically significant, there are some limitations that need to be discussed.

First, the administration of the questionnaires may not have been done in a way to truly measure student ability. Being administered at the beginning of the school year, there is a possibility that students may have simply forgotten some basic concepts on which possibly a little review may have sparked their memory. Also, since the results of the questionnaires were not connected to the students’ grades, some students may have not taken the process of answering the items as seriously as they would have if the results were part of their grade.

Next, the results of some of the items on both questionnaires may be skewed. For example, as discussed earlier, some of the items being multiple choice could have encouraged guessing. Therefore, a student may have earned full points on the item without truly having the ability to understand why he or she was making the choice they did. Also, some items presented problems in grading. An example of this occurred on item 18. Students were given a function S(m) which was an employee's salary per month after m months on the job. Students were asked to explain what the function S(m + 12) represented. To earn full points, I was looking for solutions like S(m +12) as a horizontal shift left 12 units of the graph of S(m) or the salary after m + 12 months on the job. Many students here provided answers like a "raise". As such, it was difficult to interpret what this meant in the mind of a student. Therefore, the score received for an answer such as this may not have correlated to the student's true understanding. Therefore, it may be beneficial in the future to conduct interviews on particular items.

Frequently, in a calculus class at the high school level, students are high achieving and motivated students. Therefore, one may wonder whether the student's success was truly due to the fact that they have a strong understanding of function or whether their motivation to perform at a high level determined their success. Also is it possible that some of the students with a strong definition of function or at the process level may not perform very well in calculus due to a lack of effort that they are putting into the course. Finally, the school at which the research was conducted is in an upper-middle to upper class area. Therefore, the question arises as to whether the same results would be obtained in a more rural or lower income area. The purpose of the study was, in part, to begin the process of discussing what effect level of understanding of functions plays on success in upper level high school mathematics classes like calculus.

CHAPTER SIX: CONCLUSIONS

In this chapter, I share some implications of the study from the perspective of a teacher and some ideas for future research.

Implications as a Teacher

The results of this study have confirmed beliefs that I had about the relationship between how well a student understands functions and how well the student performs in class. Based on the results of Questionnaire 1, students who have a higher ability to work with functions based on the APOS framework tended to be more successful in their respective first-year calculus course. The concept of function is introduced to students beginning in the eighth grade. Students entering calculus should have the ability to work with functions in a variety of ways given the exposure to the concept through their studies. That being said, as teachers, we must present students with functions in a variety of contexts and representations. We, the teachers, must also allow for our students to make connections between ideas presented throughout the course. For example, when making the connection between the equation of a function and its graph, we probably need to get away from the "replace $f(x)$ with y" mentality and, instead, help our students understand the meaning of the $f(x)$ notation. This would help students move away from the action level of APOS and begin viewing functions at the process level. Supporting students in understanding at all levels of the APOS framework and applying the definition of function must also be an area of improvement in mathematics courses.

The results of my study provide an excellent opportunity to discuss with teachers of all mathematics classes to the importance of introducing students to functions using a variety of representations. The results of the study showed students had difficulty interpreting graphs, tables and verbal phrases in the context of function. Therefore much more emphasis needs to be placed on these kinds of problems in the course of a student's studies of mathematics. Also, getting away from exclusively using x as the input and y as the output would be of benefit to students and building their confidence when working with functions. Finally, to possibly assist in student understanding of function, as teachers we need to ensure that all students are defining function in ways consistent with the Dirichlet-Bourbaki definition of function. This study showed that students primarily used the vertical line test or continuity to justify their reasoning as to why a relation was a function. Taking a close look at how students work with and define functions will inform my teaching so that I might provide better support for students enrolled in first-year calculus courses in the future.

It is recommended that much more emphasis be placed on functions in classes prior to calculus in order for all students entering calculus to be successful. This focus must be on all representations of functions: equation, graphical, tabular, and description of a contextual situation. Involving teachers in the discussion of concepts that students struggle with in calculus would benefit not only the calculus teacher (and students) but also the teachers (and students) of courses prior to calculus.

Future Research

As a researcher, it would be interesting to see whether putting weight on the questionnaires would affect the results of the study. I feel that some students may not have taken the questionnaires seriously and therefore the results were not a true measure of their ability. It would also be interesting to conduct a similar study on students entering calculus that have gone through curriculum that aligns with the Common Core State Standards in Mathematics (CCSSM; Common Core State Standards Initiative, n.d.) I am curious to know whether or not these students would show better ability with functions than those that came from the standard track of study in mathematics.

It would also be nice to simply focus on a smaller group of students. This way interviews could be conducted and I could more accurately discuss what the student understands or has the ability to do. Many times in this study, I found myself trying to read into the solutions that students provided and possibly misinterpreted what they were trying to present to me.

There is a possibility that success on the exams that were part of the calculus courses involved in this study did not heavily rely on a dynamic view of function. For this reason, a follow-up study could be conducted to perform an analysis of the course exams. An alignment between the APOS theory and the exams would reveal the level of demand of function understanding required.

According to Carlson (1997), students must possess a dynamic view of function, and they must view functions at least at the process level. The results of this study proved otherwise. Again it is reasonable to wonder whether this is a case of students not taking the questionnaires seriously or whether or not what they produced a true assessment of their ability. Regardless, the results of this study do provide evidence that students with a pre-process view and limited ability to apply the definition of function do have a chance at success in a first-year calculus course.

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APPENDIX A

Questionnaire 1: Modified PCA (Precalculus Assessment, n.d.)

Questionnaire 1

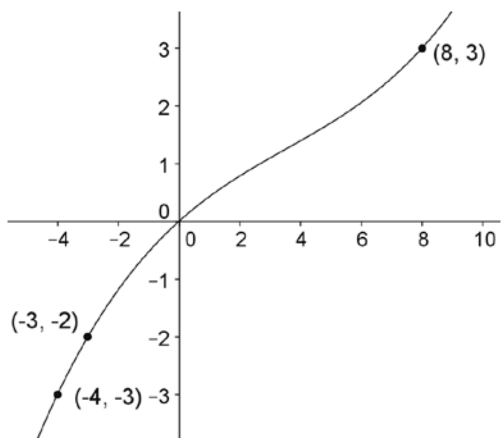
Measurement of Ability to Work with Functions

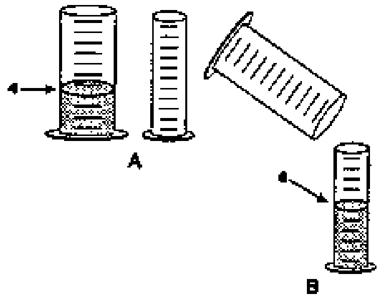
Name____________________________________

Answer each of the following to the best of your abilities. Show all work that lead to your conclusion or include a description of how you arrived at your conclusion.

1. Given the function, *f*, defined by $f(x) = 3x^2 + 2x - 4$, find $f(x+a)$.

2. Use the graph to solve $f(x) = -3$ for *x*.

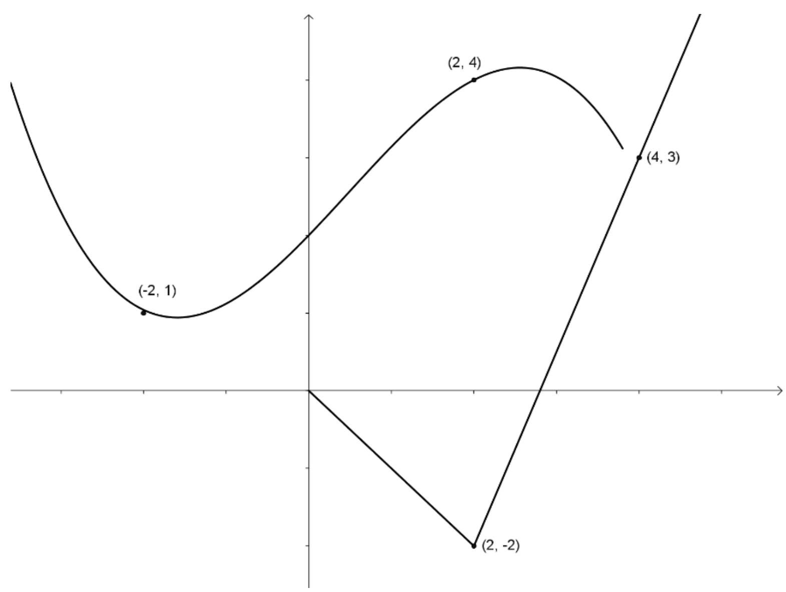




To the left are drawings of a wide and a narrow cylinder.
The cylinders have equally spaced marks on them. Water is poured into the wide cylinder up to the fourth mark (see A). This water rises to the sixth mark when poured into the narrow cylinder (see B). Both cylinders are emptied and water is poured into the narrow cylinder up to the 11^{th} mark. How high would this water rise if it were poured into the empty wide cylinder?

4. Write a formula which defines the area of a square, A, in terms of its perimeter, P.

5.

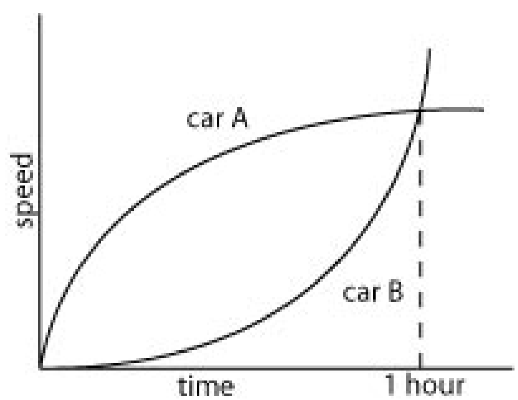


a. Use the graphs of f and g above to find $g(f(2))$.

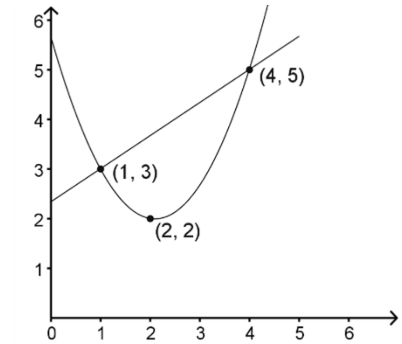
b. Evaluate $f(2) - g(0)$ using the graphs of f and g above.

6. The model for the number of bacteria in a culture has been updated from $P(t) = 7(2)^t$ to $P(t) = 7(3)^t$ where t is measured in days. What implications can you draw from this new model? (Choose one answer below)
    a. The final number of bacteria is three times as much of the initial value.
    b. The initial number of bacteria is 3.
    c. The number of bacteria triples every day.
    d. The growth rate of bacteria in the culture is 30% per day.
    e. None of these

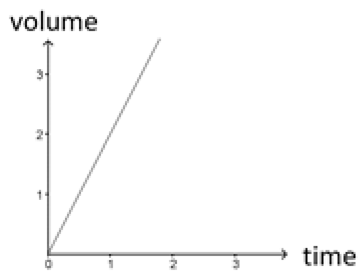
7. The graph below shows the speed of two cars during a one-hour period. Assume the cars start at the same point and at the same time and are traveling in the same direction.



What is the relationship between the position of car A and car B at t = 1 hour?

8. 

Use the graphs of f and g above to find the values of x for which $g(\mathrm{x}) > f(\mathrm{x})$.

9. 

A hose is used to fill an empty wading pool. The graph above shows volume (in gallons) in the pool as a function of time (in minutes). Define a formula for computing the time, t, as a function of the volume, v.

10. The distance, s (in feet) traveled by a car moving in a straight line is given by the function, $s(t) = t^2 + t$, where t is measured in seconds. Find the average velocity, in feet per second, for the time period from $t = 1$ to $t = 4$.

x	$f(x)$	$g(x)$
-2	0	5
-1	6	3
0	4	2
1	-1	1
2	3	-1
3	-2	0

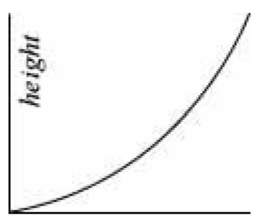
11.

a. Given the table above, determine $f(g(3))$.

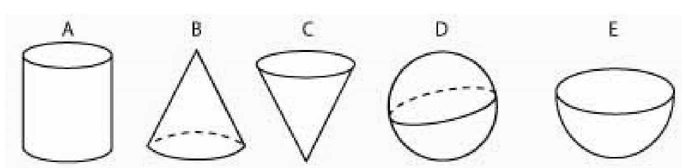
b. Given the table above determine $g^{-1}(-1)$.

12. Given that f is defined by $f(t) = 100t$, define a formula for f^{-1}.

13.



The above graph represents the height of water as a function of volume as water is poured into a container. Which container is represented by this graph?



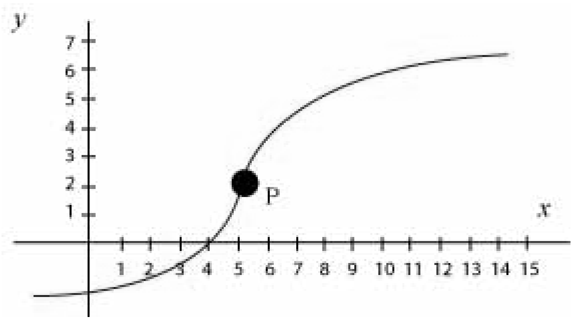
14. Given the function $h(x) = 3x - 1$ and $g(x) = x^2$, evaluate $g(h(2))$.

15. A ball is thrown into a lake, creating a circular ripple that travels outward at a speed of 5 cm per second. Express the area, A, of the circle in terms of the number of seconds, s, that have passed since the ball hit the lake.

16. The wildlife game commission poured 5 cans of fish (each can contained approximately 100 fish) into a farmer's lake. The function N defined by $N(t) = \dfrac{600t + 5}{0.5t + 1}$ represents the approximate number of fish in the lake as a function of time (in years). Which of the following best describes how the number of fish in the lake changes over time?

A. The number of fish gets larger each year, but does not exceed 500.
B. The number of fish gets larger each year, but does not exceed 1200.
C. The number of fish gets smaller each year, but does not get smaller than 500.
D. The number of fish gets larger each year, but does not exceed 600.
E. The number of fish gets smaller each year, but does not get smaller than 1200.

17. Using the graph below, explain the behavior of function f on the interval from x = 5 to x = 12.



A. Increasing at an increasing rate.
B. Increasing at a decreasing rate.
C. Increasing at a constant rate.
D. Decreasing at a decreasing rate.
E. Decreasing at an increasing rate.

18. If $S(m)$ represents the salary (per month) of an employee after m months on the job, what would the function $R(m) = S(m + 12)$ represent?

19. What is the domain of the following function? $f(x) = \dfrac{\sqrt{x+2}}{x-1}$

20. A baseball card increases its value according to the function, $b(t) = \frac{5}{2}t + 100$ where b gives the value of the card (in dollars) and t is the time (in years) since the card was purchased. Which of the following describe what $\frac{5}{2}$ conveys about the situation?

I. The cards value increases by \$5 every 2 years.

II. Every year the cards value is 2.5 times greater than the previous year.

III. The cards value increases by $\frac{5}{2}$ dollars every year.

(More than one choice may describe the situation. Choose all that apply)

21. A function f is defined by the following graph. Which of the following best describes the behavior of f?

I. As the value of x increases, the value of f increases.

II. As the value of x increases, the value of f approaches 0.

III. As the value of x approaches 0, the value of f approaches 0.

(More than one choice may be correct. Choose all that apply.)

22. Which of the following best describes the function *f* defined by, $f(x) = \dfrac{x^2}{x-2}$?

I. As the value of *x* gets very large, the value of *f* approaches 2.

II. As the value of *x* gets very large, the value of *f* increases.

III. As the value of *x* approaches 2, the value of *f* approaches 0.

(More than one choice may be correct. Choose all that apply.)

APPENDIX B

The Precalculus Assessment (PCA) Taxonomy Aligned With Questionnaire 1 Items

(Precalculus Assessment, n.d.)

The Precalculus Assessment (PCA) Taxonomy Aligned With Questionnaire 1 Items

Items in parentheses are items from Questionnaire 1 that require the specific ability

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Reasoning Abilities

R1 View function as a process.

- View a function's formula, graph, and table as defining relationships that accept input and produces output.

(Items: 1, 2, 5a, 5b, 6, 7, 8, 11a, 11b, 12, 14, 18, 19)

R2 Apply covariational reasoning.

- Coordinate two varying quantities that change in tandem while attending to how the quantities change in relation to each other.

(Items: 4, 7, 9, 10, 13, 15, 16, 17, 18, 20, 21, 22)

R3 Engage in proportional reasoning.

- Vary the measures of two quantities and recognize that they are proportionally related when the measures of the two quantities are always in the same ratio.
- Recognize that when two quantities' measures are always in a constant ratio then the measure of one is always the same multiple of the measure of the other.

(Items: 3, 9)

Conceptual and Analytic Abilities

C1 Evaluate and interpret function information given the function's formula, graph and table.

(Items: 1, 2, 5a, 5b, 7, 8, 11a, 11b, 13, 16, 17, 19, 20, 21, 22)

C2 Represent contextual function situations using algebraic notation.

- Identify, define and relate variable quantities as functional relationships.

(Items: 3, 4, 9, 13, 15)

C3 Perform function operations and interpret their meaning (table, graph, formula)

C3E Evaluate a function value and interpret its meaning. (Items: 2)

C1D Interpret domain restrictions inherent in the function. (Items: 19)

C3A Understand and use function arithmetic. (Items: 1, 5b, 8)

C3C Understand and use function composition. (Items: 5a, 11a, 14)

C3I Understand and use function inverse. (Items: 11b)

C3T Understand and use function translations. (Items: 18)

C4 Understand the meaning of an inverse function and how to reverse the function process (table, graph, formula)

C4E Solve equations that involve functional relationships and interpret their meaning.

(Items: 2, 5a, 5b, 11a, 14)

C4IN Solve inequalities that involve functional relationships and interpret their meaning. (Items: 8)

C4IF Determine inverse functions and interpret their meaning. (Items: 11b, 12)

C5 Interpret and represent function behaviors for various function types.

C5L Linear (Items: 9, 20)

C5P Polynomial (Items: 1, 7, 10, 17)

C5R Rational (Items: 16, 19, 22)

C5E Exponential (Items: 6, 7)

C5L Logarithm (Items: 7)

C6 Interpret and represent rate of change information for a function (table, formula, graph)

C6D Understand and represent the meaning of the slope of a linear function as additive growth. (Items: 20)

C6C Interpret and represent how the input and output variables change in tandem.

(Items: 7, 13, 15, 16, 21)

C6A Determine and understand average rate-of-change. (Items: 10)

C6I Interpret and represent rate-of-change information on intervals of the domain.

(Items: 13, 17)

C6E Understand exponential functions and multiplicative growth. (Items: 7)

APPENDIX C

RUBRIC for Questionnaire 1

RUBRIC for Questionnaire 1

Item 1

4	3	2	1
Student correctly evaluates and simplifies $f(x + a)$	Evaluates $f(x + a)$ with arithmetic errors.	Evaluates $f(x + a)$ with some algebraic mistakes (ie. Distributive property)	Evaluate $f(x) + a$ or other

Item 2

4	3	2	1
Produces -4 as the solution	Produces 8 as the solution. (understood concept but did not see the negative sign)	Produces -2 as the solution. (Reversed input output values)	Produces -3 as a solution or unable to produce a solution

Item 3

4	3	2	1
Correctly uses proportional reasoning to determine the solution (22/3)	Correct setup for proportional reasoning with minor arithmetic error (i.e. Multiplies by the wrong scale factor)	Approximates the solution to be between two values.	Uses additive reasoning. (ie. Since pouring the big into the little, went up 2 marks, little in big should go down 2 marks) or no solution given

Item 4

4	3	2	1
Produces the formula: $A = \frac{P^2}{16}$, $A = \left(\frac{P}{4}\right)^2$	Produces the formula: $P = \sqrt{16A}$, $A = \frac{1}{4}P^2$	Other formulas involving area and perimeter	Formula for either area or perimeter or unable to produce a formula.

Item 5a

4	3	2	1
Finds $g(f(2)) = 1$	Finds $g(f(2)) = 3$ (Reversed composition) or $f(g(2)) = 3$	Finds $g(2) = 4$ or $f(2) = -2$ (Did not perform composition)	Unable to produce a solution.

Item 5b

4	3	2	1
Finds $f(2) - g(0) = -4$	Finds $f(2) - g(0) = 0$ (subtraction done incorrectly) or reversed functions $g(2) - f(0) = 4$.	Finds $f(2) - g(0) = 2$ (subtracts the inputs)	Any other solution or no solution

Item 6

4	3	2	1
Chooses C	Chooses D or A	Chooses B	Chooses E

Item 7

4	3	2	1
States that the position of car A is ahead of car B	States that car A and car B are traveling at the same speed, but unable to determine position.	States that car A and car B are at the same position or that the cars traveled the same distance.	No solution given.

Item 8

4	3	2	1
Correctly assigns the solution as (1, 4) using appropriate notation	Includes 1 and 4 in solution.	Solution is only one x in the interval (1, 4) or interval for $f(x) > g(x)$	Solution is not an x-value in (1, 4) or no solution given

Item 9

4	3	2	1
Defines a linear formula in the form $t = av$ where $a < 1$.	Defines a linear formula in the form $t = av$ where $a \geq 1$.	Defines a linear formula in the form $v = at$	No formula provided

Item 10

4	3	2	1
Successfully calculates $\frac{s(4)-s(1)}{4-1}$	Calculates $\frac{s(4)-s(1)}{4-1}$ with some errors or $\frac{s(4)+s(1)}{4-1}$	Calculate $s(4)$ or $s(1)$	No Solution

Item 11a

4	3	2	1
Finds $f(g(3)) = 4$	Finds $g(f(3)) = 5$	Finds g(3)=0 or f(3)=-2	Unable to find a solution

Item 11b

4	3	2	1
Finds $g^{-1}(-1) = 2$	Finds $f^{-1}(-1) = 1$	Finds $g(-1) = 3$	Any other value given.

Item 12

4	3	2	1
Produces $f^{-1}(t) = \frac{1}{100}t$	Produces $t = 100y$	Produces $f^{-1} = \frac{1}{100t}$	Any other solution or no solution given

Item 13

4	3	2	1
Choice B	Choice C or E	Choice D	Choice A

Item 14

4	3	2	1
Finds $g(h(2)) = 25$	Finds $g(h(2))$ but has errors in arithmetic	Finds $h(g(2))$ correctly or with some errors in arithmetic	Does not produce a solution or unable to follow logic of how solution is produced.

Item 15

4	3	2	1
Produces $A = 25\pi s^2$	Produce $A = C\pi s^2$ where $C \neq 25$	Produce some other formula relating *A* and *s*.	Unable to relate *A* and *s*.

Item 16

4	3	2	1
Choice B	Choice D or E	Choice A	Choice C

Item 17

4	3	2	1
Choice B	Choice A or E	Choice D	Choice C

Item 18

4	3	2	1
Interprets $S(m + 12)$ as a horizontal shift 12 units left of $S(m)$ or as the salary after $m + 12$ months on the job.	Interprets $S(m + 12)$ as a vertical shift up 12 units of $S(m)$ or 12 dollars more than the salary of $S(m)$.	Comes to any other conclusion that involves $S(m)$ and 12.	Unable to determine

Item 19

4	3	2	1
Determines the domain as the set of all $x \geq -2$ and $x \neq 1$	Determines the domain as the set of all $x \geq -2$ or $x \neq 1$	Determines the domain as $x = -2$ and/or $x = 1$	Makes no mention of either $x = -2$ or $x = 1$ playing a role in determination of domain (ie. Domain is the set of real numbers)

Item 20

4	3	2	1
Choice I and III	Choice I or III	Choice II or choice II and III or choice I and II	Chooses all 3 choices

Item 21

4	3	2	1
Chooses only choice II.	Chooses choice II along with one other choice	Chooses all 3 choices or choice I and II or choice I and III	Does not choose choice II.

Item 22

4	3	2	1
Choice II	Choice II and III	Choice I, II, and III	Choice I or III only

Score Breakdown

Total Score Range			
90-96	60-89	38-59	24-37 (with no 4’s)
Student is at the **Process Level** of working with functions	Student is at the **Pre-Process Level** of working with functions	Student is at the **Action Level** of working with functions	Student is at the **Pre-Action Level** of working with functions

APPENDIX D

Questionnaire 2

Questionnaire 2

Measuring Student Concept Image (Definition) of Function

Please determine if each of the models below represents a function. Please also give your reasoning as to why it is or is not.

1. $f(x) = \pm\sqrt{x}$ for $x \geq 0$. function not a function (circle one)

Reason____

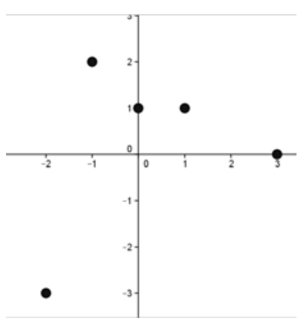
2. function not a function (circle one)

x	$f(x)$
-2	3
0	3
-1	3
3	3

Reason____

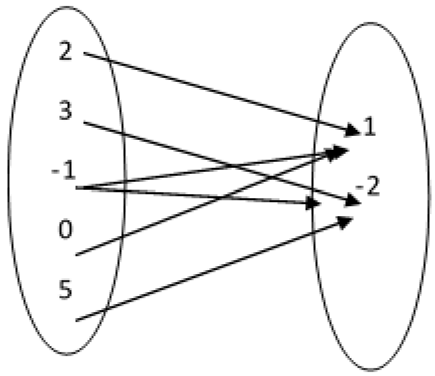
3. The amount of money earned at a job. function not a function (circle one)

Reason____

4. 

function not a function (circle one)

Reason____

5. 

function not a function (circle one)

Reason____

6. If possible, create an example of a function which assigns to every number different from 0 its square and to 0 it assigns 1. If not possible, explain why not. (Vinner & Dreyfus, 1989)

____

7. If possible, create an example of a function all of whose values are equal to each other. If not possible, explain why not. (Vinner & Dreyfus, 1989)

____

8. Define what a function is:____

____

APPENDIX E

RUBRIC for Questionnaire 2

RUBRIC for Questionnaire 2

Item 1

3	2	1
Identifies model as a function since every input (y) produces a single output (x) OR identifies model as not a function since every input (x) produces 2 outputs (y)	Identifies model is not a function and justification shows incomplete understanding of the definition of function (i.e. Vertical line test)	No reason or superficial reason given.

Item 2

3	2	1
Identifies model as a function since every input (x) produces a single output *f*(x)	Identifies model as a function but justifies with continuity or vertical line test	Identifies model as not a function or no justification given if states model is a function.

Item 3

3	2	1
Identifies model as a function since every person earns a specific amount of money or identifies as not a function since many people can earn the same amount.	Identifies model as a function but does not use people and money as input and output but rather uses x and y, but makes some connection to context	Identifies model as not a function or identifies model as a function but no reason or superficial reason given

Item 4

3	2	1
Identifies model as a function since every input produces a single output	Identifies model as a function but mentions continuity or vertical line test	Identifies model as not a function. Or no justification given.

Item 5

3	2	1
Identifies model as not a function since input -1 produces two outputs	Identifies model as a function with the reason that every input has one output. (Assumed student did not see the two arrows from -1)	No reason or superficial reason given.

Item 6

3	2	1
Able to produce a piece-wise function that satisfies the given conditions.	Produces a function that satisfies one of the given conditions	Unable to produce a function

Item 7

3	2	1
Able to produce a function that satisfies the condition	Produces a function (i.e. $y = x$) and explanation shows understanding of the definition of function	Unable to produce a function and/or explanation does not show complete understanding about function definition

Item 8

3	2	1
Defines function as a relation in which every input is mapped to exactly one output.	Defines a function as a rule, uses *x* as input and *y* as output, definition shows a general understanding of function, but is missing some information	Unable to define or makes mention of the vertical line test or continuity.

Strength of Definition

Strength	**Strong**	**Average**	**Weak**
Score range	22-24	15-21	below 15

APPENDIX F

IRB Approval Form

BOISE STATE UNIVERSITY
RESEARCH AND ECONOMIC DEVELOPMENT

Date: May 13, 2014

To: Daniel Drlik cc: Laurie Cavey

From: Social & Behavioral Insitutional Review Board (SB-IRB)
c/o Office of Research Compliance (ORC)

Subject: SB-IRB Notification of Approval - Original - 020-SB14-076
Student Understanding of Functions and Success in Calculus

The Boise State University IRB has approved your protocol submission. Your protocol is in compliance with this institution's Federal Wide Assurance (#0000097) and the DHHS Regulations for the Protection of Human Subjects (45 CFR 46).

Protocol Number:	**020-SB14-076**	Received:	5/2/2014	Review:	Expedited
Expires:	**5/12/2015**	Approved:	5/13/2014	Category:	6, 7

Your approved protocol is effective until 5/12/2015. To remain open, your protocol must be renewed on an annual basis and cannot be renewed beyond 5/12/2017. For the activities to continue beyond 5/12/2017, a new protocol application must be submitted.

ORC will notify you of the protocol's upcoming expiration roughly 30 days prior to 5/12/2015. You, as the PI, have the primary responsibility to ensure any forms are submitted in a timely manner for the approved activities to continue. If the protocol is not renewed before 5/12/2015, the protocol will be closed. If you wish to continue the activities after the protocol is closed, you must submit a new protocol application for SB-IRB review and approval.

You must notify the SB-IRB of any additions or changes to your approved protocol using a Modification Form. The SB-IRB must review and approve the modifications before they can begin. When your activities are complete or discontinued, please submit a Final Report. An executive summary or other documents with the results of the research may be included.

All forms are available on the ORC website at http://goo.gl/D2FYTV

Please direct any questions or concerns to ORC at 426-5401 or humansubjects@boisestate.edu.

Thank you and good luck with your research.

Mary E. Pritchard

Dr. Mary Pritchard
Chair
Boise State University Social & Behavioral Insitutional Review Board

1910 University Drive Boise, Idaho 83725-1139
Phone (208) 426-5401 orc@boisestate.edu

This letter is an electronic communication from Boise State University

APPENDIX G

Informed Consent/Assent Form

Informed Consent/Assent

Study Title: Student understanding of functions and success in calculus
Principal Investigator: Mr. Daniel Drlik **Co-Investigator:** Dr. Laurie Cavey

Dear Parent/Guardian and Calculus Student:

My name is Daniel Drlik and I am a student in the Mathematics Education Master's Program at Boise State University. I am asking for your permission to include your child in my research. This consent will give you the information you will need to understand why this research study is being done and why your child is being invited to participate. It will also describe what your child will need to do to participate as well as any known risks, inconveniences or discomforts that your child may have while participating. I encourage you to ask questions at any time. If you decide to allow your child to participate, you will be asked to sign this form and it will be a record of your agreement to participate. You will be given a copy of this to keep.

- **Purpose and Background**

As a calculus teacher at the high school level, I often wonder why particular students are successful while others are not. To answer this question, I have designed two questionnaires which will be administered at the end of the year prior to entering calculus and the other at the beginning of the year in which you are taking calculus. Your child is being invited to participate because he/she will be a student in either my AP Calculus AB or my Introduction to Calculus during the 2014-2015 school year.

- **Procedures**

Your child will be asked to complete two questionnaires at the beginning of the school year in which they are enrolled in a Calculus course. Each of these questionnaires are designed to measure understanding of functions. I am asking for your permission to analyze these questionnaires for my research study. The results of these questionnaires will not be used as part of your child's grade for either class. Your child's participation will not require him/her to do anything above and beyond what he/she would be doing in class anyway. If you choose not to allow your child to participate, he/she will still complete the tasks, but I will not use the results of his/her questionnaires in my research report. All data will be coded by Dr. Cavey, the co-investigator, to ensure confidentiality prior to Mr. Drlik's analysis.

- **Risks/Discomforts**

There are minimal risks associated with this study, as your child is not being asked to do anything that is not already part of either of his/her math courses. If, at any time, you do not wish for your child's data to be analyzed for this research, you may withdraw your child's participation.

- **Extent of Confidentiality**

Reasonable efforts will be made to keep the personal information in your child's research record private and confidential. Any identifiable information obtained in connection with this study will remain confidential and will be disclosed only with your permission or as required by law. The members of the research team and the Boise State University Office of Research Compliance (ORC) may access the data. The ORC monitors research studies to protect the rights and welfare of research participants.

Your child's name will not be used in any written reports or publications which result from this research, unless you have given explicit permission for us to do this. Data will be kept for three years (per federal regulations) after the study is complete and then destroyed.

- **Benefits**

There will be no direct benefit to you or your child from participating in this study. However, the information gained from this research may help education professionals better understand what students understand in secondary mathematics classes.

- **Payment**

There will be no payment to you or your child as a result of taking part in this study.

- **Questions**

If you have any questions or concerns about participation in this study, you should first talk with the principal investigator at (208) 350-4340 or drlik.daniel@meridianschools.org.

If you have questions about your or your child's rights as a research participant, you may contact the Boise State University Institutional Review Board (IRB), which is concerned with the protection of volunteers in research projects. You may reach the board office between 8:00 AM and 5:00 PM, Monday through Friday, by calling (208) 426-5401 or by writing: Institutional Review Board, Office of Research Compliance, Boise State University, 1910 University Dr., Boise, ID 83725-1138.

- **Participation in Research is Voluntary**

You do not have to give your child permission to be in this study if you do not want to. If you volunteer your child to be in this study, you may withdraw from it at any time without consequences of any kind or loss of benefits to which you are otherwise entitled.

Documentation of Consent

I and my child have read this form and decided that I will allow my child to participate in the project described above. Its general purposes, the particulars of involvement and possible risks have been explained to my satisfaction. I understand I can withdraw my child at any time.

______________________________ ______________________________

Printed Name of Child **Signature** of Child

Printed Name of Parent/Guardian
Date

Signature of Parent/Guardian

Signature of Person Obtaining Consent

Date