

The Sortability of Graphs and Matrices under Context Directed Swaps

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INTRODUCTION: CILIATES

Ciliates, unicellular organisms, sort the genome of their micronucleus to assemble a transcriptionally functional macronucleus. The study of sorting algorithms helps in understanding the mechanisms by which this process occurs.

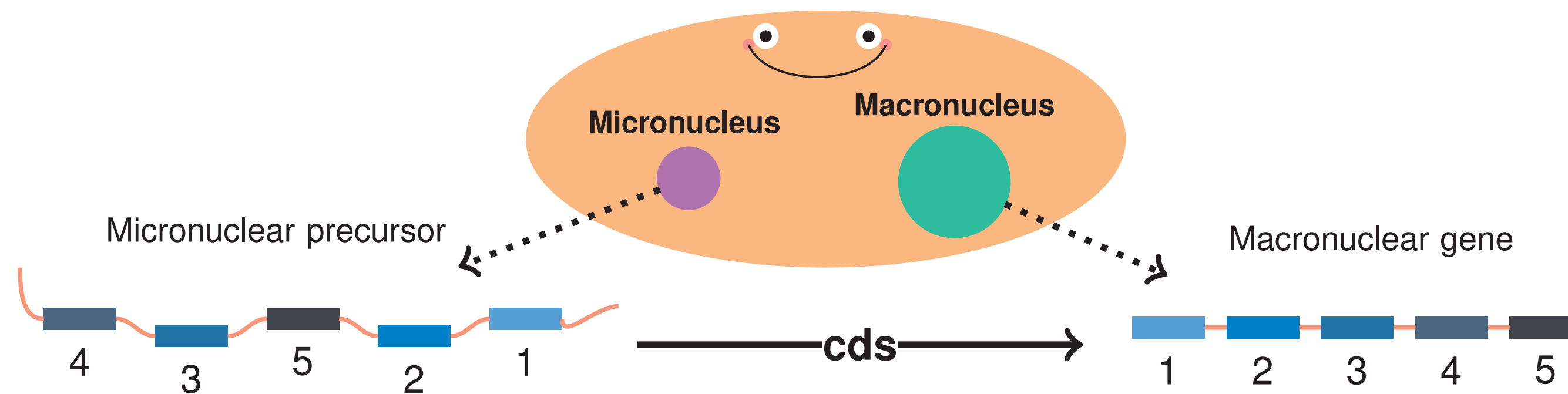


Figure 1: Context Directed Swaps in Ciliate Nuclei

Our research focus was to identify when permutations may be sorted by context directed swaps. We generalize the results of [1] through a novel graph theoretical and linear algebraic framework.

CDS AND BREAKPOINT GRAPH

In the positive permutation π , assign to each entry a a left pointer $(a-1, a)$ and a right pointer $(a, a+1)$. Let

$$\pi = \left[\{\alpha_1\}_p \{ \alpha_2\}_q \{ \alpha_3\} \{ \alpha_4\}_p \{ \alpha_5\}_q \right],$$

where each α_i is a block of the permutation and $p = (x, x+1)$ and $q = (y, y+1)$ are some pointers in the permutation. As in [1], let the **context directed swap** on π with context p and q , $\text{cds}_{\{p,q\}}(\pi)$, be

$$\left[\{\alpha_1\}_p \{ \alpha_4\}_p \{ \alpha_3\} \{ \alpha_2\}_q \{ \alpha_5\}_q \right].$$

The **breakpoint graph** of π , $\text{BG}(\pi)$, has the pointers of π as vertices and an edge between each pair of identical pointers.

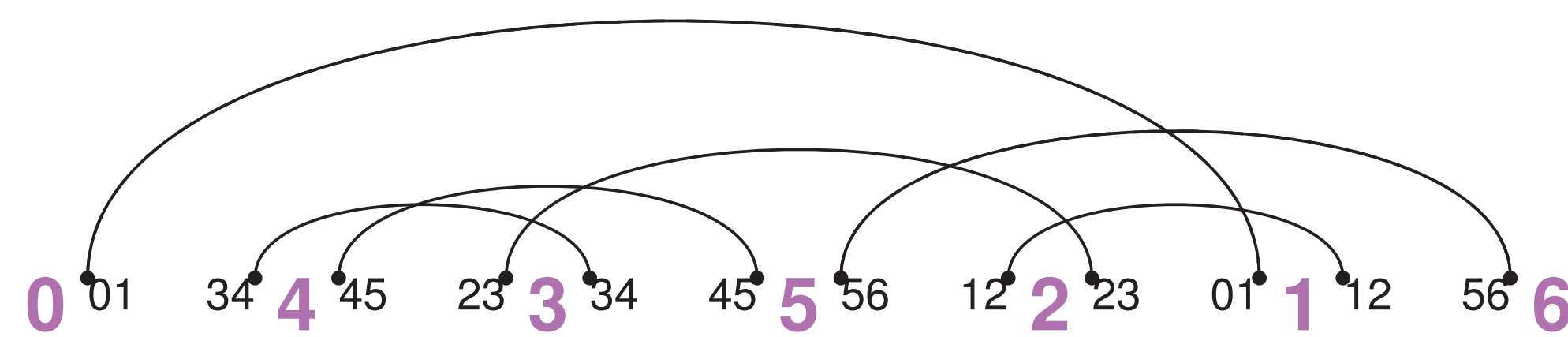


Figure 2: $\text{BG}(\pi)$ of $\pi = [4, 3, 5, 2, 1]$

The $(0, 1)$ and $(n, n+1)$ pointers are called the **roots** and are not used in valid contexts with which **cds** can be performed.

FUTURE WORK

- Further analyze and classify unsortable permutations, graphs, and matrices.
- Extend the overlap graph to include additional relationships.

CDS ON THE OVERLAP GRAPH (GCDS)

The overlap graph of a permutation π , $\text{OG}(\pi)$, has as vertices the edges of $\text{BG}(\pi)$, each labeled by the pointer supporting the corresponding edge in $\text{BG}(\pi)$. Two vertices of $\text{OG}(\pi)$ are adjacent if the respective edges of $\text{BG}(\pi)$ intersect.

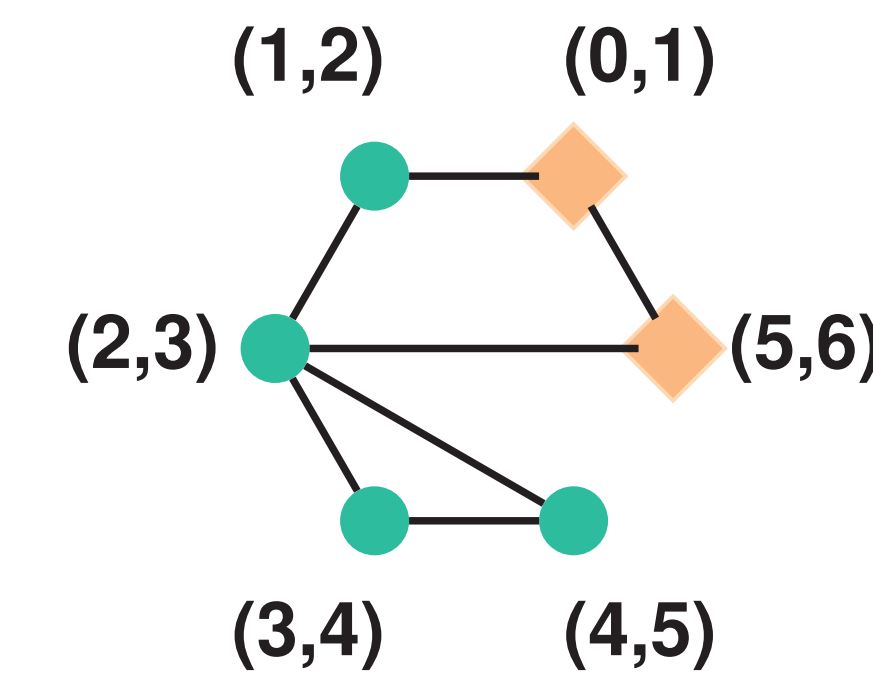


Figure 3: $\text{OG}(\pi)$ of $\pi = [4, 3, 5, 2, 1]$

For vertices x and y , let $f_x(y)$ be 1 if x and y are adjacent and 0 otherwise. For any adjacent vertices p and q , let $\text{gcds}_{\{p,q\}}(\text{OG}(\pi))$ be the graph $\text{OG}'(\pi)$ with the same vertices as $\text{OG}(\pi)$, where for any vertices u and v , $\text{OG}'(\pi)$ includes the edge $\{u, v\}$ if and only if

$$f_p(u)f_q(v) + f_p(v)f_q(u) + f_u(v) \equiv 1 \pmod{2}.$$

A graph is **gcds-sortable** if some sequence of **gcds** operations removes all of its edges.

CDS ON THE ADJACENCY MATRIX (MCDS)

The **adjacency matrix** of $\text{OG}(\pi)$ is the matrix A_π over \mathbb{F}_2 such that, for $1 \leq i, j \leq (n+1)$, $A_\pi(i, j) = 1$ if $\{(i-1, i), (j-1, j)\}$ is an edge in $\text{OG}(\pi)$ and $A_\pi(i, j) = 0$ otherwise.

$$A_\pi = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Figure 4: Adjacency Matrix of $\pi = [4, 3, 5, 2, 1]$

For any entry p and q , define the **cds** operation on matrices as

$$\text{mcds}_{p,q}(A_\pi) = A_\pi + A_\pi I_{pq} A_\pi,$$

where $I_{pq}(i, j) = 1$ if $i = p$ and $j = q$ or $i = q$ and $j = p$ and $I_{pq}(i, j) = 0$ otherwise.

A matrix is **mcds-sortable** if applying some sequence of **mcds** operations to it yields the zero matrix.

REFERENCES

- [1] K.L.M. Adamyk et al., *Sorting Permutations: Games, Genomes and Cycles*, arXiv:1410.235.
- [2] H.Q. Li et al., *Parity Cuts In 2-Rooted Graphs*, preprint.
- [3] J. MacWilliams, *Orthogonal matrices over finite fields*, American Mathematical Monthly 76: 2 (1969), 152–164.

GENERALIZED PARITY CUTS

Extending the definition of parity cut given in [2], let a **generalized parity cut** of a graph be a set S of vertices such that, for each vertex in the graph, the number of edges between that vertex and vertices in S is even.

Unsortable Overlap Graph Sortable Overlap Graph

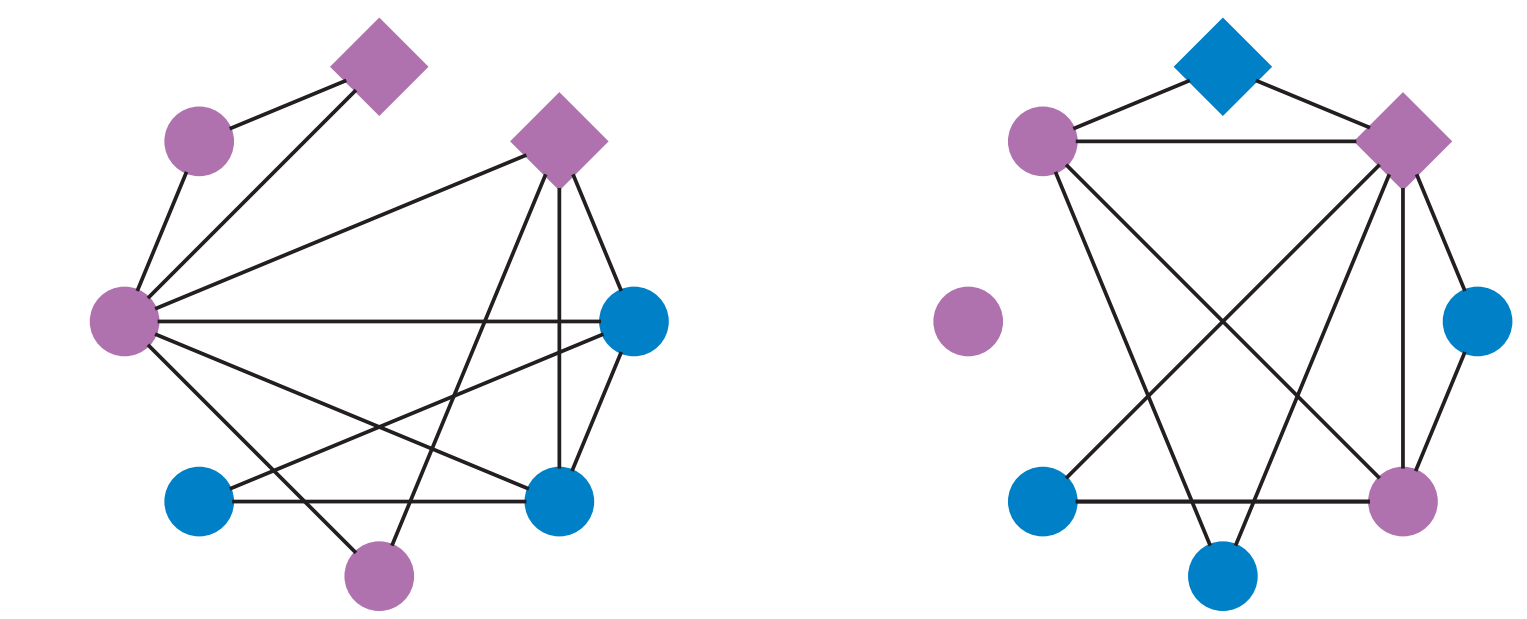


Figure 5: Generalized Parity Cuts (shown in purple)

RESULTS

THEOREM 1. A two-rooted graph with roots x, y is **gcds-sortable** if and only if for each of x, y there is a generalized parity cut containing that root and not the other.

The set of generalized parity cuts of a graph corresponds to the kernel of the adjacency matrix of the graph.

THEOREM 2. A matrix is **mcds-sortable** if and only if its kernel contains an element \vec{x} such that $x_1 = 0$ and $x_n = 1$ and an element \vec{y} such that $y_1 = 1$ and $y_n = 0$.

THEOREM 3. The number of **gcds-sortable** two-rooted graphs on n vertices is

$$\sum_{s=0}^{\lfloor n/2 \rfloor - 1} 2^{s(s+3)} \left(\frac{\prod_{i=0}^{2s-1} (2^{n-2-i} - 1)}{\prod_{i=1}^s (2^{2i} - 1)} \right).$$

Let r_n be the proportion of graphs on n vertices that are **gcds-sortable**. Then, the sequences $\{r_{2n}\}$ and $\{r_{2n+1}\}$ converge, and

$$\lim_{n \rightarrow \infty} r_{2n} \approx 0.2272 \quad \text{and} \quad \lim_{n \rightarrow \infty} r_{2n+1} \approx 0.1061.$$

Let G be a graph on the vertices $(1, 2), (2, 3), \dots, (n-1, n)$ with adjacency matrix A taken over the field \mathbb{F}_2 . Then, the number of ways we can add two roots to G to form a **gcds-sortable** graph is $4^{\text{rank}(A)}$.

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