REGULAR HOMOTOPY OF CLOSED CURVES ON SURFACES

by

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ABSTRACT

The use of rotation numbers in the classification of regular closed curves in the plane up to regular homotopy sparked the investigation of winding numbers to classify regular closed curves on other surfaces. Chillingworth [1] defined winding numbers for regular closed curves on particular surfaces and used them to classify orientation preserving regular closed curves that are based at a fixed point and direction. We define geometrically a group structure of the set of equivalence classes of regular closed curves based at a fixed point and direction. We prove this group structure coincides with the one introduced by Smale [9] via a weak homotopy equivalence. The set of equivalence classes of orientation preserving regular closed curves is a subgroup. This thesis investigates the relationship between this subgroup and the winding number of each element. Specifically, it is proven that this subgroup is isomorphic to the direct product of the integers with the group of orientation preserving closed curves up to homotopy where the isomorphism sends an equivalence class to its winding number and corresponding homotopy class. Using this result, we describe the subgroup for several surfaces by depicting representatives of generators.

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CHAPTER 1

INTRODUCTION

Whitney [11] launched the investigation of rotation numbers of regular closed curves in the plane. Geometrically, the rotation number is the net angle the tangent vector rotates through as the curve is traversed. He used rotation numbers to classify regular closed curves in the plane up to regular homotopy and gave a simple method of calculating the rotation number of a given regular closed curve in the plane. These results allow us to easily find representatives of each regular homotopy class.

It later became the goal of Reinhart [8] and Chillingworth [1] to extend these results by defining a winding number for regular closed curves on other surfaces. Using his definition of winding number, Chillingworth classified orientation preserving regular closed curves on a surface with a continuous, nonzero vector field up to regular homotopy.

In this thesis, we describe the geometric group structure of the set of regular homotopy classes of regular closed curves on surfaces. Using the definition of winding number given by Chillingworth, we focus our attention on surfaces with continuous, nonzero vector fields. For these surfaces, we prove the function that maps the set of regular homotopy classes of orientation preserving regular closed curves into the integers by the winding number is a homomorphism. Using this homomorphism, we provide a different proof of Chillingworth's classification theorem. Lastly, we prove that the set of regular homotopy classes of orientation preserving regular closed curves is isomorphic to the direct product of the homotopy classes of orientation preserving closed curves with the integers. With this result, we are able to describe the generators of the group of regular homotopy classes of orientation preserving regular closed curves on several surfaces.

CHAPTER 2

GROUP STRUCTURE OF $\pi_R(M, v_{x_0})$

2.1 Definitions

We begin with definitions of the main structures used throughout this thesis. The first two definitions are those given by Guillemin and Pollack [3, p. 1–11].

Definition 2.1.1. (*Local Diffeomorphism of $x \in M \subset \mathbb{R}^m$*). A local diffeomorphism of x is a smooth bijection $\xi : V \to U$ where V is an open subset of the half-space $\mathbb{H}^k$, $U \subset M$ is open where M is given the subspace topology, $x \in U$, and ξ^{-1} is smooth. If for each $x \in M$ there exists a local diffeomorphism $\xi_x : V_x \to U_x$ where $V_x \subset \mathbb{H}^k$, then we say M is locally diffeomorphic to $\mathbb{H}^k$.

Definition 2.1.2. (*Tangent Space at $x \in M \subset \mathbb{R}^m$*). Let $\xi : V \to U$ be a local diffeomorphism of x where $\xi(v) = x$. Then, the tangent space at $x \in M$, or the fiber over x, is $d\xi_v(\mathbb{R}^k)$ where $d\xi_v : \mathbb{R}^k \to \mathbb{R}^m$ is the usual derivative mapping defined by

$$d\xi_v(y) = \lim_{h \to 0} \frac{\xi(v + yh) - \xi(v)}{h}$$

for each $y \in \mathbb{R}^k$. We label this TM_x.

Definition 2.1.3. (*Riemannian Manifold*). $M \subset \mathbb{R}^m$ is a k-dimensional Riemannian manifold if it is locally diffeomorphic to $\mathbb{H}^k$ and each tangent space is assigned an inner product that varies smoothly over M. Here, we assign the inner products by restricting the dot product on $\mathbb{R}^m$ to each tangent space. Although some structures defined in this thesis use the inner products, the results presented are independent of the choice of inner products.

Equivalently, we could have defined a Riemannian manifold to be a second countable, Hausdorff topological space that is locally homeomorphic to $\mathbb{H}^k$ and is equipped with a smooth atlas and a smoothly varying choice of inner products on the tangent bundle. Then, by the Nash embedding theorems, we can choose an embedding of the topological space into some $\mathbb{R}^m$ that preserves the inner product on each tangent space. As a result of the Whitney embedding theorem, any two embeddings are isotopic if we choose $m \geq 2k + 1$.

Definition 2.1.4. (*Surface*). A surface is a connected 2-dimensional Riemannian manifold.

Compact surfaces are classified using three criteria: orientability, the number of boundary components, and the Euler characteristic. So two compact surfaces are diffeomorphic if and only if they are both orientable or both non-orientable, they have the same number of boundary components, and the same Euler characteristic (Hirsch [6, p. 207]).

Definition 2.1.5. (*Derivate Mapping of a Function Between Manifolds*). Let $f : N \to M$ where $N \subset \mathbb{R}^n$ and $M \subset \mathbb{R}^m$ are l-dimensional and k-dimensional Riemannian manifolds respectively. For $x \in N$, there exists a local diffeomorphism $\xi : V \to U$ where $\xi(0) = x$. Similarly, for $f(x) \in M$, there exists a local diffeomorphism $\zeta : V' \to U'$ where $\zeta(0) = f(x)$. For V small enough, we have that

$$\begin{tikzcd} N \arrow[r, "f"] & M \\ V \arrow[u, "\xi"] \arrow[r, "j = \zeta^{-1} \circ f \circ \xi"'] & V' \arrow[u, "\zeta"'] \end{tikzcd}$$

is a commutative diagram. Then, $df_x : TN_x \to TM_{f(x)}$ is a linear transformation of tangent spaces defined by $df_x = d\zeta_0 \circ dj_0 \circ d\xi_0^{-1}$.

Theorem 2.1.6. (Guillemin and Pollack [3, p. 11]). *Let N, M, L be Riemannian manifolds and $f : N \to M$, $g : M \to L$. For each $x \in N$,*

$$d(g \circ f)_x = dg_{f(x)} \circ df_x.$$

Definition 2.1.7. (*Tangent Bundle of M*). The tangent bundle of M, denoted TM, is

$$\{v_x = (x, \widehat{v_x}) \in M \times \mathbb{R}^m : \widehat{v_x} \in TM_x\}.$$

As a subset of $\mathbb{R}^m \times \mathbb{R}^m$, the tangent bundle of M is given the subspace topology.

Definition 2.1.8. (*Projection of TM onto M*). The projection of the tangent bundle onto M is a function $P : TM \to M$ where for each $v_x \in TM$, $P(v_x) = x$. Note $P^{-1}(\{x\}) = TM_x$ where we identify $\{x\} \times TM_x \subset TM$ with TM_x.

Definition 2.1.9. (*Spherical Tangent Bundle of M*). We denote the spherical tangent bundle of M as STM. It is

$$\{v_x = (x, \widehat{v_x}) \in TM : ||\widehat{v_x}||_x = 1\}$$

where $|| \ ||_x : TM_x \to \mathbb{R}$ is the norm determined by the inner product assigned to TM_x. As a subset of $\mathbb{R}^m \times \mathbb{R}^m$, the spherical tangent bundle of M is given the subspace topology.

Throughout the thesis, v_{x_0} is a fixed point of STM.

Definition 2.1.10. (*Projection of STM onto M*). The projection of the spherical tangent bundle of M onto M is $p : STM \to M$ where $p = P\big|_{STM}$.

Definition 2.1.11. (*Fiber over x in STM*). The fiber over x in STM is

$$STM_x = \{v_x \in TM : ||\widehat{v_x}||_x = 1\}.$$

Equivalently, $STM_x = p^{-1}(\{x\})$.

The spherical tangent bundle of M is clearly dependent upon the choice of inner products on the tangent spaces. Suppose the tangent spaces of M are assigned different inner products that vary smoothly over M. We denote the spherical tangent bundle of M taken with respect to these inner products as STM'. STM' is bundle isomorphic to STM. That is, there exists a continuous function $\kappa : STM' \to STM$ such that

$$\kappa\big|_{STM'_x} : STM'_x \to STM_x$$

is an isomorphism for each $x \in M$. This function is defined by

$$(x, \widehat{v_x}) \mapsto \left(x, \frac{\widehat{v_x}}{||\widehat{v_x}||_x}\right).$$

It is this bundle isomorphism that allows our results to be independent of the choice of inner products.

Definition 2.1.12. (*Closed Curve*). A closed curve is a continuous function $f : S^1 \to M$. S^1 is thought of as the unit circle in $\mathbb{R}^2$ oriented counterclockwise.

Definition 2.1.13. (*Closed Curve Based at x_0 or v_{x_0}*). A closed curve f is based at $x_0 \in M$ if $f\big((1,0)\big) = x_0$. f is based at v_{x_0} if $f\big((1,0)\big) = x_0$ and $df_{(1,0)}\big((0,1)\big) = \widehat{v_{x_0}}$. We use Ω_{x_0} to represent the set of closed curves on M based at x_0.

Definition 2.1.14. (*Regular Closed Curve*). A closed curve f is regular if $f' : S^1 \to \bigcup_{t \in S^1} TM_{f(t)}$ defined by $f'(t) = df_t\big((0,1)\big)$ is continuous and $f'(t) \neq 0$ for each $t \in S^1$. The topology of $\bigcup_{t \in S^1} TM_{f(t)} \subset \mathbb{R}^m$ is the subspace topology. Equivalently, f is a closed curve with continuously varying, nonzero tangent at each point $t \in S^1$. We let $\Gamma_{v_{x_0}}$ be the set of all regular closed curves based at v_{x_0}.

Definition 2.1.15. (*Homotopy*). Consider continuous functions $f, g : W \to Z$ where W, Z are topological spaces. A homotopy between f and g is a continuous function $H : W \times I \to Z$ such that $H(-,0) = f$ and $H(-,1) = g$. When convenient, we also notate homotopies as the family of maps $h_s : W \to Z$ where $h_s = H(-,s)$ for each $s \in I$. A homotopy is based at $z_0 \in Z$ if for some fixed $w_0 \in W$, $h_s(w_0) = z_0$ for each $s \in I$. Let $\simeq$ rel $\{z_0\}$ denote a homotopy between two functions where the homotopy is based at z_0.

Definition 2.1.16. (*Regular Homotopy Between Regular Closed Curves f and g*). A regular homotopy between f and g is a homotopy $H : S^1 \times I \to M$

between f and g such that $h_s : S^1 \to M$ is a regular closed curve for each $s \in I$. A regular homotopy is based at v_{x_0} if $h_s\big((1,0)\big) = x_0$ and $h'_s\big((1,0)\big) = \widehat{v_{x_0}}$ for each $s \in I$. Let $\simeq_R$ rel $\{v_{x_0}\}$ denote a regular homotopy between two regular closed curves where the homotopy is based at v_{x_0}.

If any part of the image of a regular closed curve is contained within the boundary of M, we can apply a regular homotopy to the function to obtain a regularly homotopic function that is contained in the interior of M. Thus, we assume the image of a regular closed curve is contained within the interior of M as this will simplify later arguments.

In order to define the composition of closed curves based at x_0, we use an alternate definition of a closed curve based at x_0 that is defined on I. Let

$$\dot{\Omega}_{x_0} = \{f : I \to M \big| f \text{ is continuous and } f(0) = f(1) = x_0\}$$

and $q : I \to S^1$ where $q(t) = \big(\cos(\tau(t)), \sin(\tau(t))\big)$ where $\tau : [0,1] \to [0, 2\pi]$ is a bijection defined by

$$\tau(t) = (-4\pi + 2)t^3 + (6\pi - 3)t^2 + t.$$

τ is defined such that $(f \circ q)'$ is continuous, nonzero, and

$$(f \circ q)'(0) = (f \circ q)'(1) = \widehat{v_{x_0}}$$

where f is a regular closed curve based at v_{x_0}. This behavior is needed to make precise an alternate definition of a regular closed curve based at v_{x_0}.

Since q is continuous and $q(0) = q(1) = (1, 0)$, $f \circ q \in \dot{\Omega}_{x_0}$ for each $f \in \Omega_{x_0}$. Hence, we define $Q_\Omega : \Omega_{x_0} \to \dot{\Omega}_{x_0}$ by $Q_\Omega(f) = f \circ q$. We claim Q_Ω defines a one-to-one correspondence between Ω_{x_0} and $\dot{\Omega}_{x_0}$. First, it is clear that for each $f, g \in \Omega_{x_0}$, $f \circ q = g \circ q$ implies $f = g$. So Q_Ω is one-to-one. To show that Q_Ω is onto, choose $h \in \dot{\Omega}_{x_0}$. Then, $h \circ \dot{q} \in \Omega_{x_0}$ where $\dot{q} : S^1 \to I$ is defined by $\dot{q} = \left(q\big|_{[0,1)}\right)^{-1}$. Therefore, when convenient, we use elements in $\dot{\Omega}_{x_0}$ as closed curves based at x_0.

Similarly, we have a one-to-one correspondence between $\Gamma_{v_{x_0}}$ and $\dot{\Gamma}_{v_{x_0}}$ where $\dot{\Gamma}_{v_{x_0}}$ is the set of functions $f : I \to M$ such that $f \in \dot{\Omega}_{x_0}, f' : I \to \bigcup_{t \in I} TM_{f(t)}$ defined by $f'(t) = df_t(1)$ is continuous and nonzero, and $f'(0) = f'(1) = \widehat{v_{x_0}}$. Here, the one-to-one correspondence is given by $Q_\Gamma : \Gamma_{v_{x_0}} \to \dot{\Gamma}_{v_{x_0}}$ defined by $Q_\Gamma(f) = f \circ q$. Thus, we also use elements of $\dot{\Gamma}_{v_{x_0}}$ as regular closed curves based at v_{x_0}.

Definition 2.1.17. (*Composition of Closed Curves Defined on I*). Consider closed curves $f, g : I \to M$ based at $x_0 \in M$. That is, $f(0) = f(1) = x_0$ and $g(0) = g(1) = x_0$. The composition of f and g is $f \cdot g : I \to M$ where

$$(f \cdot g)(t) = \begin{cases} f(2t) & \text{if} \quad 0 \leq t \leq \frac{1}{2} \\ g(2t-1) & \text{if} \quad \frac{1}{2} \leq t \leq 1 \end{cases}$$

for each $t \in I$. Since f and g are based at x_0, $f \cdot g$ is well-defined at $t = \frac{1}{2}$ and based at x_0.

Definition 2.1.18. (*Reverse of a Closed Curve Defined on I*). We define the reverse of a closed curve $f : I \to M$ to be $\overline{f} : I \to M$ such that for each $t \in I$,

$$\overline{f}(t) = f(1-t).$$

For closed curves $f, g : I \to M$ based at x_0, we let $[f]$, $[g]$ represent the based homotopy classes of each function respectively. Then, $[f], [g] \in \pi_1(M, x_0)$ and we define $[f] \cdot [g] = [f \cdot g]$. $\pi_1(M, x_0)$ is a group under the operation $\cdot$ where the homotopy class of the function $c_{x_0} : I \to M$ defined by $c_{x_0}(t) = x_0$ is the identity of the group and $[f]^{-1} = [\overline{f}]$.

Definition 2.1.19. ($\pi_R(M, v_{x_0})$). We define $\pi_R(M, v_{x_0})$ as the set of equivalence classes of $\Gamma_{v_{x_0}}$ under regular homotopy based at v_{x_0}. For each $f \in \Gamma_{v_{x_0}}$, we use $[f]_R$ to represent the based regular homotopy class of f.

$\pi_R(M, v_{x_0})$ is equivalent to $\pi_0(\Gamma_{v_{x_0}})$, the set of path components of $\Gamma_{v_{x_0}}$ when $\Gamma_{v_{x_0}}$ is assigned the following topology. Consider any metric $\overline{d} : TM \times TM \to \mathbb{R}^+$ on TM where $\mathbb{R}^+$ is the set of non-negative real numbers. For $f, g \in \Gamma_{v_{x_0}}$, define

$$d(f, g) = \max\Big\{\overline{d}\Big(\big(f(t), f'(t)\big), \big(g(t), g'(t)\big)\Big) : t \in S^1\Big\}.$$

Then, d is a metric and $\Gamma_{v_{x_0}}$ is given the topology induced by d (Smale [9]).

For M of dimension 1, 2, or 3, $\pi_R(M, v_{x_0})$ does not depend on the smooth structure of M. This is because, in these dimensions, two manifolds that are homeomorphic are also diffeomorphic. Clearly the equivalence classes that constitute $\pi_R(M, v_{x_0})$ are preserved under a diffeomorphism of M. Thus, $\pi_R(M, v_{x_0})$ only depends on the topology of M.

2.2 An Alternate View of $\pi_R(M, v_{x_0})$

Smale [9] defined a specific weak homotopy equivalence to prove $\pi_R(M, v_{x_0})$ is in one-to-one correspondence with $\pi_1(STM, v_{x_0})$. To define a weak homotopy equivalence, we need to define the higher homotopy groups of a topological space Z with respect to a basepoint z_0. For each non-negative integer n, $\pi_n(Z, z_0)$ is a partition of the set of continuous functions from S^n into Z that are based at z_0. The set is partitioned using the equivalence relation of based homotopy. That is, $f, g : S^n \to Z$ are equivalent if and only if there exists a homotopy between f and g that is based at z_0. We use $[f]$ to represent the based homotopy class of f.

For $n \neq 0$, $\pi_n(Z, z_0)$ is the nth homotopy group and for $n > 1$, $\pi_n(Z, z_0)$ is abelian. Hatcher [4, p. 340] describes the group structure of these homotopy groups. $\pi_0(Z, z_0)$ is not necessarily a group but is the set of path components of Z.

Definition 2.2.1. (*Weak Homotopy Equivalence*). For topological spaces W and Z, a weak homotopy equivalence is a continuous function $\phi : Z \to W$ that induces isomorphisms of all homotopy groups. That is, $\phi_* : \pi_n(Z, z_0) \to \pi_n(W, \phi(z_0))$ defined by $\phi_*([f]) = [\phi \circ f]$ for each $[f] \in \pi_n(Z, z_0)$ is a bijection for each n and an isomorphism for $n \neq 0$.

With Smale's weak homotopy equivalence, some aspects of the study of regular closed curves on M classified up to regular homotopy were simplified to the study of closed curves on STM classified up to homotopy. Since $\pi_1(STM, v_{x_0})$ is group, a group structure can be induced on $\pi_R(M, v_{x_0})$ using the weak homotopy equivalence. We give a geometric description of this induced group structure in the next section.

To define the function Smale proved to be a weak homotopy equivalence, let $\Omega_{v_{x_0}}$ be the set of closed curves on STM based at v_{x_0}. $\Omega_{v_{x_0}}$ is given the compact open topology (Hu [7, p. 73]). Let $\phi : \Gamma_{v_{x_0}} \to \Omega_{v_{x_0}}$ be defined as follows. For each $f \in \Gamma_{v_{x_0}}$, let $\phi(f) : S^1 \to STM$ be defined by

$$\phi(f)(t) = \left(f(t), \frac{f'(t)}{||f'(t)||_{f(t)}} \right).$$

Theorem 2.2.2. (Smale [9]). *ϕ is a weak homotopy equivalence between $\Gamma_{v_{x_0}}$ and $\Omega_{v_{x_0}}$.*

In particular, taking $n = 0$, the induced function $\phi_* : \pi_0(\Gamma_{v_{x_0}}) \to \pi_0(\Omega_{v_{x_0}})$ defined by $\phi_*([f]) = [\phi(f)]$ for each $[f] \in \pi_0(\Gamma_{v_{x_0}})$ is a bijection. Since $\pi_0(\Gamma_{v_{x_0}}) = \pi_R(M, v_{x_0})$ and $\pi_0(\Omega_{v_{x_0}}) = \pi_1(STM, v_{x_0})$, Theorem 2.2.3 follows from Theorem 2.2.2.

Theorem 2.2.3. (Smale [9]). *$\phi_* : \pi_R(M, v_{x_0}) \to \pi_1(STM, v_{x_0})$ defined by $\phi_*([f]_R) = [\phi(f)]$ is a bijection.*

Accordingly, regular closed curves f and g are regularly homotopic based at v_{x_0} if and only if $\phi(f)$ and $\phi(g)$ are homotopic based at v_{x_0}.

For the remainder of the thesis, we use M to denote a surface. f and g will be used to denote regular closed curves on M based at $v_{x_0} = (x_0, \widehat{v_{x_0}}) \in STM$ unless otherwise stated.

2.3 Group Structure of $\pi_R(M, v_{x_0})$

We would like to use the function defined by $([f]_R, [g]_R) \mapsto [f \cdot g]_R$ as the group operation on $\pi_R(M, v_{x_0})$. However, $f \cdot g : I \to M$ is not based at v_{x_0} but at $(x_0, 2\widehat{v_{x_0}})$. Thus, we define an orientation-preserving reparameterization, r, of I such that $r'(0) = r'(1) = \frac{1}{2}$. We prove both $f \circ r$ and $g \circ r$ are regular closed curves based at $(x_0, \frac{1}{2}\widehat{v_{x_0}})$ and their composition $(f \circ r) \cdot (g \circ r)$ is based at v_{x_0}.

Let $r : I \to I$ be defined by $r(t) = -t^3 + \frac{3}{2}t^2 + \frac{1}{2}t$. Since $r'(t) = -3t^2 + 3t + \frac{1}{2}$, we use the quadratic formula to check that $r'(t) \neq 0$ for each $t \in I$. Since $r(0) = 0$, $r(1) = 1$, and $r'(t) \neq 0$ for each $t \in I$, r is an increasing bijection. Therefore, r is a reparameterization of the unit interval. Lastly, we check that $r'(0) = \frac{1}{2}$ and $r'(1) = \frac{1}{2}$.

Lemma 2.3.1. *$f \circ r$ is a regular closed curve based at $\left(x_0, \frac{1}{2}\widehat{v_{x_0}}\right) \in TM$.*

Proof. First, we show $f \circ r$ is a closed curve. From the definition of r,

$$(f \circ r)(0) = f(0) = x_0 = f(1) = (f \circ r)(1).$$

Since both f and r are continuous, $f \circ r$ is also continuous. Thus, $f \circ r$ is a closed curve.

We next prove $(f \circ r)'$ is continuous and nonzero. From the chain rule in Theorem 2.1.6, $(f \circ r)'(t) = \big(r'(t)\big)\big(f' \circ r\big)(t)$. Then, $(f \circ r)'$ is continuous because r, r', and f' are continuous. Since f is regular, $f' \circ r$ is nonzero on I and we have already seen r' is nonzero on I. Consequently, $(f \circ r)'$ is never zero so $f \circ r$ is a regular closed curve.

We have shown $(f \circ r)(0) = (f \circ r)(1) = x_0$. Then, because

$$(f \circ r)'(0) = r'(0) f'\big(r(0)\big) = \frac{1}{2} f'(0) = \frac{1}{2} \widehat{v_{x_0}}$$

and

$$(f \circ r)'(1) = r'(1) f'\big(r(1)\big) = \frac{1}{2} f'(1) = \frac{1}{2} \widehat{v_{x_0}},$$

$f \circ r$ is based at $(x_0, \frac{1}{2}\widehat{v_{x_0}})$.

□

Lemma 2.3.2. *$(f \circ r) \cdot (g \circ r)$ is a regular closed curve based at v_{x_0}.*

Proof. Because $f \circ r$ and $g \circ r$ are closed curves such that $(f \circ r)(0) = (f \circ r)(1) = x_0$ and $(g \circ r)(0) = (g \circ r)(1) = x_0$, $(f \circ r) \cdot (g \circ r)$ is a closed curve such that $\big((f \circ r) \cdot (g \circ r)\big)(0) = \big((f \circ r) \cdot (g \circ r)\big)(1) = x_0$. From the chain rule,

$$\big((f \circ r) \cdot (g \circ r)\big)'(t) = \begin{cases} 2r'(2t) f'\big(r(2t)\big) & \text{if } \ 0 \leq t \leq \frac{1}{2} \\ 2r'(2t-1) g'\big(r(2t-1)\big) & \text{if } \ \frac{1}{2} \leq t \leq 1 \end{cases}.$$

It is clear from the previous lemma that the derivative is everywhere nonzero. Since $(f \circ r)'$ and $(g \circ r)'$ are continuous, we now only need to check $\big((f \circ r) \cdot (g \circ r)\big)'$ is well-defined at $t = \frac{1}{2}$ and $\big((f \circ r) \cdot (g \circ r)\big)'(0) = \big((f \circ r) \cdot (g \circ r)\big)'(1)$ to prove $(f \circ r) \cdot (g \circ r)$ is a regular closed curve. Taking $t = \frac{1}{2}$, $2r'\big(2(\frac{1}{2})\big) f'\Big(r\big(2(\frac{1}{2})\big)\Big) = \widehat{v_{x_0}}$ and $2r'\big(2(\frac{1}{2}) - 1\big) g'\Big(r\big(2(\frac{1}{2}) - 1\big)\Big) = \widehat{v_{x_0}}$. Similarly,

$$2r'\big(2(0)\big) f'\Big(r\big(2(0)\big)\Big) = \widehat{v_{x_0}} = 2r'\big(2(1) - 1\big) g'\Big(r\big(2(1) - 1\big)\Big).$$

Thus, $(f \circ r) \cdot (g \circ r)$ is a regular closed curve. Since

$$\big((f\circ r)\cdot(g\circ r)\big)(0)=\big((f\circ r)\cdot(g\circ r)\big)(1)=x_0$$

and

$$\big((f\circ r)\cdot(g\circ r)\big)'(0)=\widehat{v_{x_0}}=\big((f\circ r)\cdot(g\circ r)\big)'(1),$$

$(f\circ r)\cdot(g\circ r)$ is based at v_{x_0}. $\square$

Now consider the operation on $\pi_R(M,v_{x_0})$ defined by

$$[f]_R\cdot[g]_R=[(f\circ r)\cdot(g\circ r)]_R.$$

This will be used as the group operation on $\pi_R(M,v_{x_0})$. To check this is a well-defined function from $\pi_R(M,v_{x_0})\times\pi_R(M,v_{x_0})$ to $\pi_R(M,v_{x_0})$, let $f\simeq_R \widehat{f}$ rel $\{v_{x_0}\}$ and $h_{f_s}:I\to M$ be a regular homotopy between f and $\widehat{f}$ based at v_{x_0}. Similarly, let $g\simeq_R\widehat{g}$ rel $\{v_{x_0}\}$ and $h_{g_s}:I\to M$ be a regular homotopy between g and $\widehat{g}$ based at v_{x_0}. Obviously, $(h_{f_s}\circ r)\cdot(h_{g_s}\circ r)$ is a homotopy. It follows from Lemma 2.3.2 that this is a regular homotopy based at v_{x_0}. Thus, $[(f\circ r)\cdot(g\circ r)]_R=\big[(\widehat{f}\circ r)\cdot(\widehat{g}\circ r)\big]_R$.

Theorem 2.3.3. $\pi_R(M,v_{x_0})$ *is a group under the operation defined by*

$$[f]_R\cdot[g]_R=[(f\circ r)\cdot(g\circ r)]_R.$$

Proof. We begin by showing that the operation is associative. For regular closed curves f,g,j based at v_{x_0}, we compare

$$\big([f]_R\cdot[g]_R\big)\cdot[j]_R=\Big[\Big(\big((f\circ r)\cdot(g\circ r)\big)\circ r\Big)\cdot(j\circ r)\Big]_R$$

and

$$[f]_R \cdot \big([g]_R \cdot [j]_R\big) = \Big[(f \circ r) \cdot \Big(\big((g \circ r) \cdot (j \circ r)\big) \circ r\Big)\Big]_R.$$

By definition,

$$\left(\left(\big((f \circ r) \cdot (g \circ r)\big) \circ r\right) \cdot (j \circ r)\right)(t) = \begin{cases} (f \circ r)\big(2r(2t)\big) & \text{if} \quad 0 \leq t \leq \frac{1}{4} \\ (g \circ r)\big(2r(2t) - 1\big) & \text{if} \quad \frac{1}{4} \leq t \leq \frac{1}{2} \\ (j \circ r)\big(2t - 1\big) & \text{if} \quad \frac{1}{2} \leq t \leq 1 \end{cases}$$

and

$$\left((f \circ r) \cdot \left(\big((g \circ r) \cdot (j \circ r)\big) \circ r\right)\right)(t) = \begin{cases} (f \circ r)\big(2t\big) & \text{if} \quad 0 \leq t \leq \frac{1}{2} \\ (g \circ r)\big(2r(2t - 1)\big) & \text{if} \quad \frac{1}{2} \leq t \leq \frac{3}{4} \\ (j \circ r)\big(2r(2t - 1) - 1\big) & \text{if} \quad \frac{3}{4} \leq t \leq 1 \end{cases}.$$

Let $\varphi : I \to I$ such that

$$\varphi(t) = \begin{cases} \frac{1}{2}r^{-1}(t) & \text{if} \quad 0 \leq t \leq \frac{1}{2} \\ \frac{1}{2}r^{-1}\big(r(2t - 1) + \frac{1}{2}\big) & \text{if} \quad \frac{1}{2} \leq t \leq \frac{3}{4} \\ r(2t - 1) & \text{if} \quad \frac{3}{4} \leq t \leq 1 \end{cases}$$

where $r^{-1} : I \to I$ is the inverse function of r. We claim

$$h_s(t) = \left(\left(\big((f \circ r) \cdot (g \circ r)\big) \circ r\right) \cdot (j \circ r)\right)\big((1 - s)\varphi(t) + st\big)$$

is a regular homotopy from $(f \circ r) \cdot \left(\big((g \circ r) \cdot (j \circ r)\big) \circ r\right)$ to $\left(\big((f \circ r) \cdot (g \circ r)\big) \circ r\right) \cdot (j \circ r)$ based at v_{x_0}.

First, φ is a bijection that maps $\big[0, \frac{1}{2}\big]$ to $\big[0, \frac{1}{4}\big]$, $\big[\frac{1}{2}, \frac{3}{4}\big]$ to $\big[\frac{1}{4}, \frac{1}{2}\big]$, and $\big[\frac{3}{4}, 1\big]$

to $\left[\frac{1}{2}, 1\right]$. It follows that

$$h_0 = \bigg(\Big(\big((f \circ r) \cdot (g \circ r)\big) \circ r\Big) \cdot (j \circ r)\bigg)(\varphi) = (f \circ r) \cdot \Big(\big((g \circ r) \cdot (j \circ r)\big) \circ r\Big)$$

and $h_1 = \Big(\big((f \circ r) \cdot (g \circ r)\big) \circ r\Big) \cdot (j \circ r)$. It is clear that h_s is a homotopy between the two functions based at x_0.

To check that h_s is a regular homotopy, we first show φ' is continuous and nonzero. From the inverse function theorem, $(r^{-1})'(t) = \dfrac{1}{r'(t)}$. Using this, we check $\varphi'(t)$ is well-defined at $t = \frac{1}{2}, \frac{3}{4}$ and $\varphi'(0) = \varphi'(1)$. In fact, $\varphi'\left(\frac{1}{2}\right) = \frac{2}{5}$, $\varphi'\left(\frac{3}{4}\right) = \frac{5}{2}$, and $\varphi'(0) = \varphi'(1) = 1$. Now it is clear that φ' is continuous. Since r' is nonzero, it follows that $(r^{-1})'$ and in turn φ' are nonzero.

As a result of Theorem 2.1.6,

$$h_s'(t) = \big((1-s)\varphi(t) + st\big)' \bigg(\Big(\big((f \circ r) \cdot (g \circ r)\big) \circ r\Big) \cdot (j \circ r)\bigg)' \big((1-s)\varphi(t) + st\big).$$

It is clear that h_s' is continuous and

$$\bigg(\Big(\big((f \circ r) \cdot (g \circ r)\big) \circ r\Big) \cdot (j \circ r)\bigg)' \big((1-s)\varphi(t) + st\big)$$

is nonzero. Thus, $h_s'(t) = 0$ only if $\big((1-s)\varphi(t) + st\big)' = 0$. However, this implies $(1-s)\varphi'(t) + s = 0$ so $\varphi'(t) = \dfrac{-s}{1-s}$. But $\varphi'(t)$ is always positive. Therefore, $h_s'(t) \neq 0$ for each $t \in I$ and h_s is regular for each $s \in I$. Since $\varphi'(0) = \varphi'(1) = 1$, it follows that $h_s'(0) = h_s'(1) = \widehat{v_{x_0}}$ and the regular homotopy is based at v_{x_0}. Thus, we have proven the operation is associative.

We next prove there exists an identity element in $\pi_R(M, v_{x_0})$. In a neighborhood of x_0 that is diffeomorphic to a disk, we place the regular closed curve e that is based at v_{x_0} and pictured in Figure 2.1. To show $[e]_R$ is the identity,

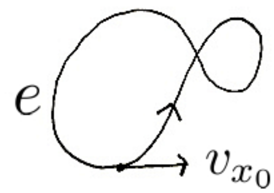

Figure 2.1: The regular closed curve e representing the identity of $\pi_R(M, v_{x_0})$.

choose any $[f]_R \in \pi_R(M, v_{x_0})$. We can apply a regular homotopy to f to ensure f does not intersect itself in a small neighborhood around x_0. Thus, we assume

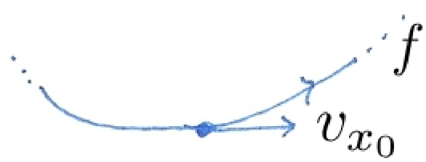

Figure 2.2: f is assumed to not intersect itself in a neighborhood of x_0 as depicted.

f has the behavior depicted in Figure 2.2. This is done to simplify later figures.

In Figure 2.3, $[e]_R \cdot [f]_R$ is represented by the first curve and shown to be equal to $[f]_R$. Similarly, $[f]_R \cdot [e]_R$ is represented by the first curve and shown to be equal to $[f]_R$ in Figure 2.4. Hence, $[e]_R$ is the identity of $\pi_R(M, v_{x_0})$.

To prove each element has an inverse, we will use the closed curves γ and δ defined in a neighborhood of x_0. See Figure 2.5. These are not regular closed curves since $\gamma'(0) = \widehat{v_{x_0}}$ and $\delta'(0) = -\widehat{v_{x_0}}$ but $\gamma'(1) = -\widehat{v_{x_0}}$ and $\delta'(1) = \widehat{v_{x_0}}$.

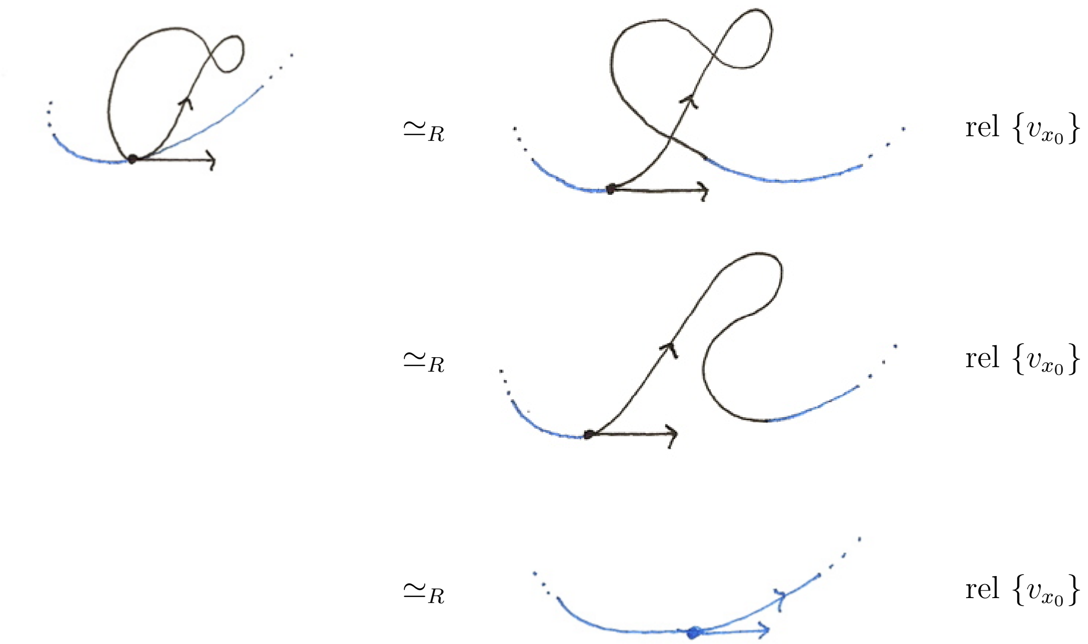

Figure 2.3: Regular homotopy based at v_{x_0} between $(e \circ r) \cdot (f \circ r)$ and f.

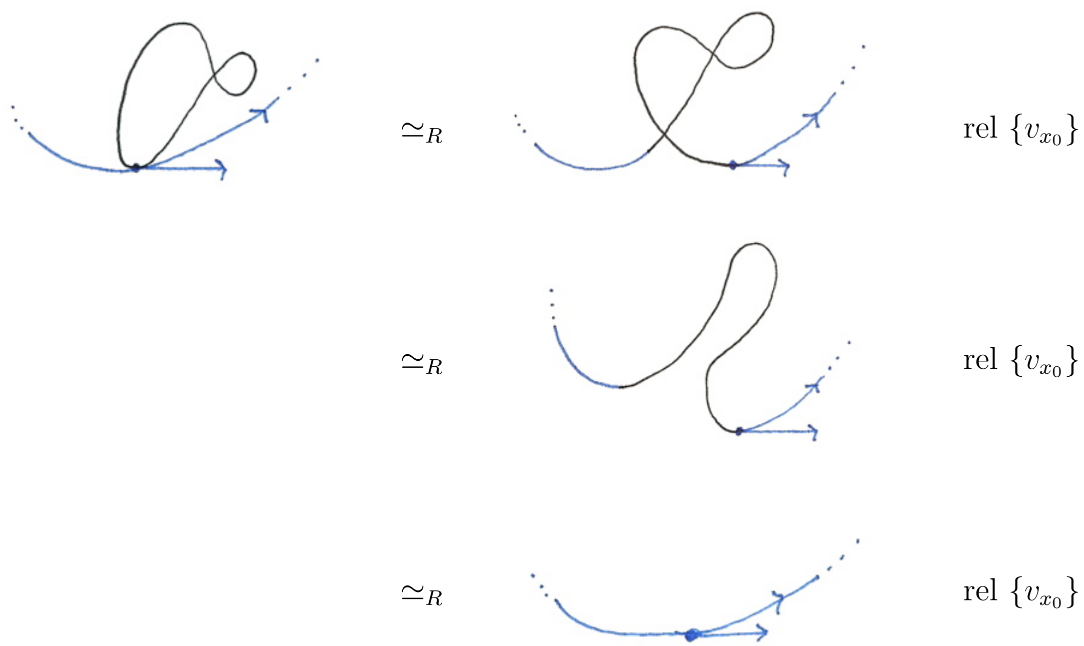

Figure 2.4: Regular homotopy based at v_{x_0} between $(f \circ r) \cdot (e \circ r)$ and f.

Figure 2.5: Closed curves γ and δ. We let $-v_{x_0} = (x_0, -\widehat{v_{x_0}})$.

Using arguments similar to ones used in previous lemmas, it can be shown that $\left[(\gamma \circ r) \cdot \left(((\overline{f} \circ r) \cdot (\delta \circ r)) \circ r\right)\right]_R$ is well-defined. We prove this is the inverse of $[f]_R$. Since $\overline{f}'(0) = -\widehat{v_{x_0}}$, we can apply a regular homotopy to $\overline{f}$ and assume $\overline{f}$ has the behavior of Figure 2.6.

Figure 2.6: $\overline{f}$ is assumed to not intersect itself in a neighborhood of x_0 as depicted.

$(\gamma \circ r) \cdot \left(((\overline{f} \circ r) \cdot (\delta \circ r)) \circ r\right)$ is the first regular closed curve in Figure 2.7. Consequently, Figure 2.8 shows $\left[(\gamma \circ r) \cdot \left(((\overline{f} \circ r) \cdot (\delta \circ r)) \circ r\right)\right]_R \cdot [f]_R = [e]_R$.

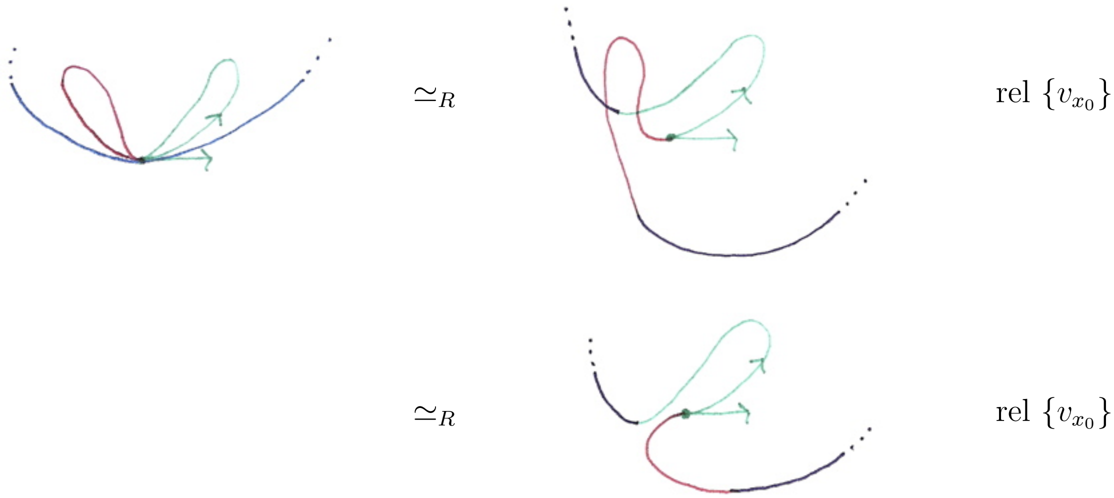

Figure 2.7: A regular homotopy of $(\gamma \circ r) \cdot \Big(\big((\overline{f} \circ r) \cdot (\delta \circ r)\big) \circ r\Big)$.

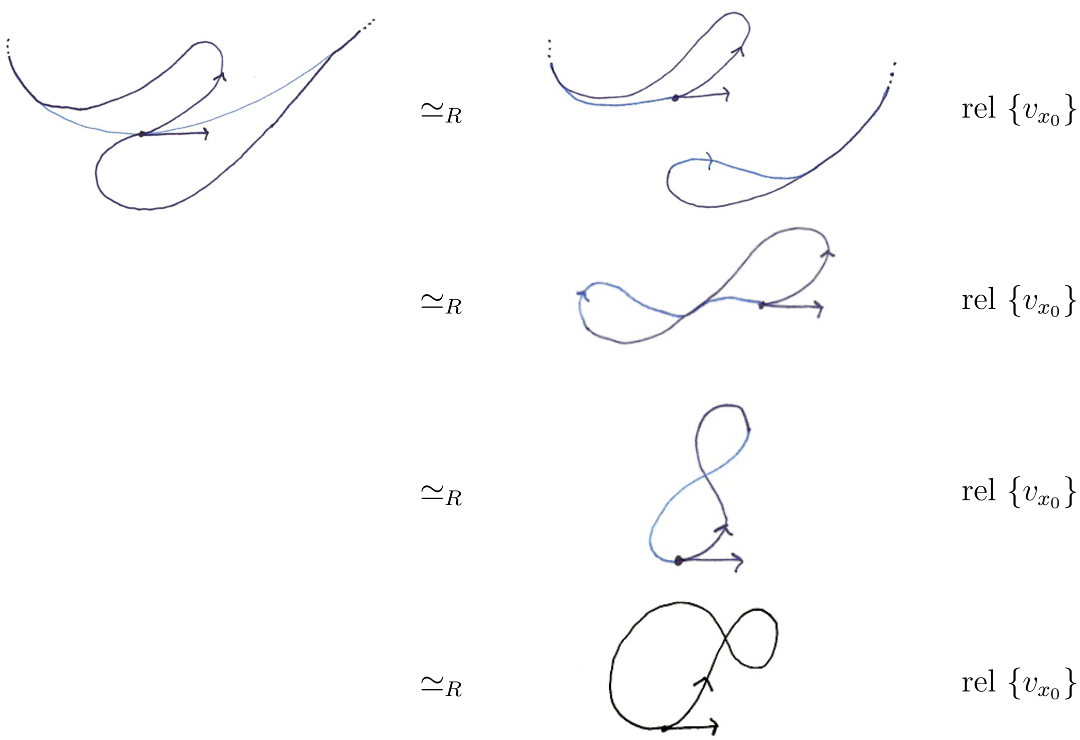

Figure 2.8: Regular homotopy showing $\Big[(\gamma \circ r) \cdot \Big(\big((\overline{f} \circ r) \cdot (\delta \circ r)\big) \circ r\Big)\Big]_R \cdot [f]_R = [e]_R$.

The following homotopy from the series of homotopies in Figure 2.8 requires explanation:

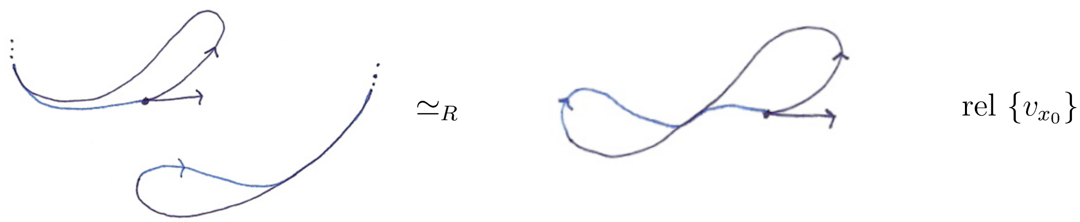

Figure 2.9: Regular homotopy from Figure 2.8.

Here, the first regular closed curve is $\overline{f} \cdot f$ except at the ends of f and $\overline{f}$ there are loops used to connect the ends and preserve regularity. Obviously, this is homotopic to a constant map so the regular closed curve is in a neighborhood that is diffeomorphic to a disk. When the line connecting the two loops is contracted, the orientation of both loops is preserved because the disk is orientable. Therefore, the regular homotopy of Figure 2.9 holds. Similarly, $[f]_R \cdot \Big[(\gamma \circ r) \cdot \Big(\big((\overline{f} \circ r) \cdot (\delta \circ r)\big) \circ r\Big)\Big]_R = [e]_R$. This is shown in Figure 2.10. Therefore, $\pi_R(M, v_{x_0})$ is a group. $\square$

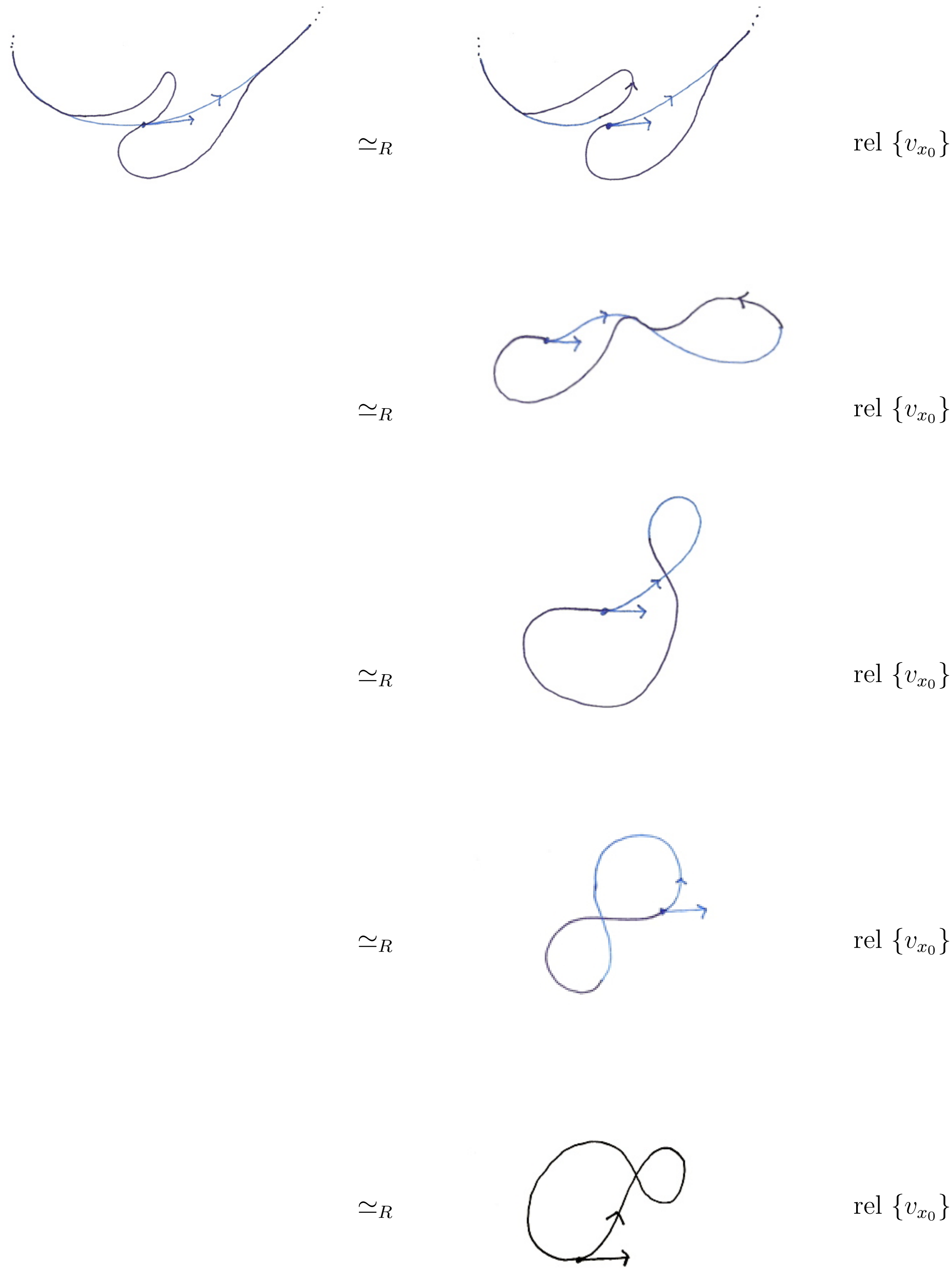

Figure 2.10: Regular homotopy showing $[f]_R \cdot \left[(\gamma \circ r) \cdot \Big(\big((\overline{f} \circ r) \cdot (\delta \circ r)\big) \circ r\Big)\right]_R = [e]_R$.

Corollary 2.3.4. *Smale's bijection* $\phi_* : \pi_R(M, v_{x_0}) \to \pi_1(STM, v_{x_0})$ *is a group isomorphism.*

Proof. First note that $\phi_*\big([f]_R \cdot [g]_R\big) = [\phi\big((f \circ r) \cdot (g \circ r)\big)]$. Since

$$\phi\big((f\circ r)\cdot(g\circ r)\big)(t) = \begin{cases} \left((f \circ r)(2t), \dfrac{f'\big(r(2t)\big)}{||f'\big(r(2t)\big)||_{(f\circ r)(2t)}}\right) & \text{if} \quad 0 \le t \le \frac{1}{2} \\ \\ \left((g \circ r)(2t-1), \dfrac{g'\big(r(2t-1)\big)}{||g'\big(r(2t-1)\big)||_{(g\circ r)(2t-1)}}\right) & \text{if} \quad \frac{1}{2} \le t \le 1 \end{cases}$$

and

$$\big(\phi(f) \cdot \phi(g)\big)(t) = \begin{cases} \left(f(2t), \dfrac{f'(2t)}{||f'(2t)||_{f(2t)}}\right) & \text{if} \quad 0 \le t \le \frac{1}{2} \\ \\ \left(g(2t-1), \dfrac{g'(2t-1)}{||g'(2t-1)||_{g(2t-1)}}\right) & \text{if} \quad \frac{1}{2} \le t \le 1 \end{cases},$$

$\phi\big((f\circ r)\cdot(g\circ r)\big)$ is just a reparameterization of $\phi(f)\cdot\phi(g)$. So $\big[\phi\big((f\circ r)\cdot(g\circ r)\big)\big] = \big[\phi(f) \cdot \phi(g)\big]$. Consequently,

$$\begin{aligned} \phi_*\big([f]_R \cdot [g]_R\big) &= \big[\phi\big((f \circ r) \cdot (g \circ r)\big)\big] \\ &= \big[\phi\big(f\big) \cdot \phi\big(g\big)\big] \\ &= \big[\phi\big(f\big)\big] \cdot \big[\phi\big(g\big)\big] \\ &= \phi_*\big([f]_R\big) \cdot \phi_*\big([g]_R\big) \end{aligned}$$

so ϕ_* is a homomorphism. Since ϕ_* is also a bijection, the result follows. $\square$

2.4 Regular Homotopy Classes within a Homotopy Class

Regular homotopy refines homotopy. In this section, it is proven that within each homotopy class there are infinitely many regular homotopy classes for all surfaces other than S^2 and $\mathbb{RP}^2$. We begin by proving a few results for exact sequences.

Definition 2.4.1. (*Section of a Surjective Map*). For sets B, C and surjection $v : B \to C$, a section of v is a function $s : C \to B$ such that $v \circ s$ is the identity function on C.

Note that for any surjection there exists a section. This is because for each $c \in C$, we can choose some $b \in B$ such that $v(b) = c$. Then, s can be defined such that $s(c) = b$ so $v(s(c)) = v(b) = c$.

Lemma 2.4.2. *Consider groups A, B, C with the exact sequence*

$$1 \longrightarrow A \xrightarrow{k} B \xrightarrow{v} C \longrightarrow 1.$$

Then, for each section s of v, $\widetilde{s} : A \times C \to B$ defined by $\widetilde{s}(a, c) = k(a)s(c)$ is a bijection.

Proof. We first show $\widetilde{s}$ is one-to-one. Suppose there exists $(a, c), (a', c') \in A \times C$ such that $\widetilde{s}(a, c) = \widetilde{s}(a', c')$. Consequently,

$$\begin{aligned} k(a)s(c) &= k(a')s(c') \\ k(a')^{-1}k(a) &= s(c')s(c)^{-1} \\ k\big((a')^{-1}a\big) &= s(c')s(c)^{-1}. \end{aligned}$$

Applying v to both sides of the equation, $v\Big(k\big((a')^{-1}a\big)\Big) = v\Big(s(c')s(c)^{-1}\Big)$. As a result of exactness, im $k = \ker v$ so $v\Big(k\big((a')^{-1}a\big)\Big) = 1$. Then,

$$\begin{aligned} 1 &= v\Big(s(c')s(c)^{-1}\Big) \\ &= v\big(s(c')\big)v\big(s(c)\big)^{-1} \\ &= c'c^{-1} \end{aligned}$$

so $c = c'$. Then, because $k(a)s(c) = k(a')s(c')$,

$$\begin{aligned} k(a)s(c) &= k(a')s(c) \\ k(a) &= k(a') \\ a &= a', \end{aligned}$$

where the last equality holds because k is injective. Hence, $(a, c) = (a', c')$ and $\widetilde{s}$ is injective.

Next, we prove $\widetilde{s}$ is surjective. Consider $b \in B$. Since $v(b) \in C$, we let $c = v(b)$. So $s(c)^{-1} \in B$ and the product $bs(c)^{-1} \in B$. Then,

$$v\big(bs(c)^{-1}\big) = v(b)v\big(s(c)^{-1}\big) = v(b)\Big(v\big(s(c)\big)\Big)^{-1} = v(b)(c)^{-1} = v(b)\big(v(b)\big)^{-1} = 1.$$

Hence, $bs(c)^{-1} \in \ker v$. Again since im $k = \ker v$, there exists $a \in A$ such that $k(a) = bs(c)^{-1}$. Therefore,

$$\widetilde{s}(a, c) = k(a)s(c) = bs(c)^{-1}s(c) = b$$

and $\widetilde{s}$ is surjective. $\square$

The next theorem holds for any surface M other than S^2 and $\mathbb{RP}^2$ because $\pi_2(M, x_0) \cong 1$ for $M \neq S^2, \mathbb{RP}^2$. To see this, we first assume that the surface has no boundary since omitting any boundary circles results in a surface of the same homotopy type and without boundary. If we let $\overline{\pi} : \overline{M} \to M$ be the universal cover of M, then $\pi_2(M, x_0) \cong \pi_2(\overline{M}, \overline{x_0})$ where $\overline{\pi}(\overline{x_0}) = x_0$ (Hatcher [4, p. 342]). Thus, we focus our attention on calculating $\pi_2(\overline{M}, \overline{x_0})$.

If $\overline{M}$ is not compact, then the second homology group of $\overline{M}$ is trivial (Vick [10, p. 152]). By the Hurewicz theorem, the second homology group of $\overline{M}$ is isomorphic to $\pi_2(\overline{M}, \overline{x_0})$. Thus, in the case where $\overline{M}$ is not compact, $\pi_2(\overline{M}, \overline{x_0}) \cong 1$. If $\overline{M}$ is compact, then $\overline{M} = S^2$ and $\pi_2(\overline{M}, \overline{x_0}) \cong \mathbb{Z}$. However, if $\overline{M}$ is compact, then M is compact and M is either S^2 or $\mathbb{RP}^2$ since they are the only compact surfaces without boundary that have S^2 as a universal cover. Therefore, if $M \neq S^2, \mathbb{RP}^2$, then $\pi_2(M, x_0) \cong 1$.

Theorem 2.4.3. *For any surface M other than S^2 and $\mathbb{RP}^2$, there exists a one-to-one correspondence between $\pi_1(M, x_0) \times \mathbb{Z}$ and $\pi_R(M, v_{x_0})$.*

Proof. By Theorem 2.2.3, it suffices to show there exists a bijection between $\pi_1(M, x_0) \times \mathbb{Z}$ and $\pi_1(STM, v_{x_0})$. From the homotopy sequence of the fibration $p : STM \to M$ based at v_{x_0}, the sequence

$$\pi_2(M, x_0) \to \pi_1(STM_{x_0}, v_{x_0}) \xrightarrow{i_*} \pi_1(STM, v_{x_0}) \xrightarrow{p_*} \pi_1(M, x_0) \to \pi_0(STM_{x_0}, v_{x_0})$$

is exact where $i : STM_{x_0} \to STM$ is the inclusion of the fiber and i_*, p_* are the induced homotopies of i, p respectively (Hu [7, p. 152]). Since $M \neq S^2, \mathbb{RP}^2$, $\pi_2(M, x_0) \cong 1$. Because STM_{x_0} is isomorphic to S^1, $\pi_1(STM_{x_0}, v_{x_0}) \cong \mathbb{Z}$ and $\pi_0(STM_{x_0}, v_{x_0}) \cong 1$. Then, it follows from the previous lemma that there is a bijection between $\pi_1(M, x_0) \times \mathbb{Z}$ and $\pi_1(STM, v_{x_0})$. $\square$

Consider $[f]_R \in \pi_R(M, v_{x_0})$. For a section s of p_*, Theorem 2.4.3 implies there exists $n \in \pi_1(STM_{x_0}, v_{x_0})$ and $\alpha \in \pi_1(M, x_0)$ such that $\widetilde{s}(n, \alpha) = [\phi(f)]$. Moreover,

$$[f] = p_*\big([\phi(f)]\big) = p_*\big(\widetilde{s}(n, \alpha)\big) = p_*\big(i_*(n) \cdot s(\alpha)\big) = p_*\big(i_*(n)\big) \cdot p_*\big(s(\alpha)\big) = [c_{x_0}] \cdot \alpha = \alpha.$$

Therefore, the bijection maps $(\alpha, n) \in \pi_1(M, x_0) \times \mathbb{Z}$ to a regular homotopy class with homotopy class α. Since this is true for each $n \in \pi_1(STM_{x_0}, v_{x_0}) \cong \mathbb{Z}$, within each homotopy class there are infinitely many regular homotopy classes.

CHAPTER 3

WINDING NUMBERS

3.1 Orientations

In order to define the winding number and prove results related to it, the concept of orientation is needed for vector spaces, linear transformations, closed curves, and fibers of the spherical tangent bundle. In this section, we define these orientations.

Definition 3.1.1. (*Orientation of a Real, 2-Dimensional Vector Space*). We first define an equivalence relation $\sim$ on the set of ordered bases of a real, 2-dimensional vector space V. Let $(a_1, a_2) \sim (b_1, b_2)$ if and only if $\det A > 0$ where A is the change of basis matrix. Then, there are two equivalence classes in the set of ordered bases. Assigning one equivalence class $+$ and the other $-$ is an orientation of V.

We will refer to the standard orientation on $\mathbb{R}^2$. This is the orientation where the equivalence class of the standard basis $(e_1, e_2) = \big((1,0),(0,1)\big)$ is assigned $+$.

Definition 3.1.2. (*Orientation Preserving and Reversing Linear Transformations*). For oriented real, 2-dimensional vector spaces V, W and a linear function $T : V \to W$, T is orientation preserving if the basis $(T(v_1), T(v_2))$ is in the $+$

equivalence class of ordered bases on W where (v_1, v_2) is an ordered basis of V that is in the $+$ equivalence class. T is orientation reversing if $(T(v_1), T(v_2))$ is in the $-$ equivalence class of ordered bases on W.

The last two definitions are well-defined (Guillemin and Pollack [3, p. 95]).

Definition 3.1.3. (*Orientation Double Covering*). The orientation double covering of M is the covering space $\pi : \widetilde{M} \to M$ where

$$\widetilde{M} = \{(x, *) : x \in M \text{ and } * \text{ is an orientation of } TM_x\}$$

and π is defined by $\pi(x, *) = x$. To describe the topology of $\widetilde{M}$, we describe an element of the basis of the topology. For $B \subset \widetilde{M}$ to be a basis element, it has to meet the following requirements. First, the restriction $\pi|_B$ must be injective and $\pi(B) \subset M$ must be open. Suppose for each $x \in \pi(B)$ the vector space TM_x is given the orientation $*$ where $(x, *) \in B$. Then, there must exist a local diffeomorphism $\xi : V \to U$ of each x such that $U \subset \pi(B)$ and $d\xi_v : \mathbb{R}^2 \to TM_{\xi(v)}$ is orientation preserving for each $v \in V$ where $\mathbb{R}^2$ is given the standard orientation.

To define orientation preserving and orientation reversing closed curves, we again view closed curves as functions defined on I. Then, for each closed curve $f : I \to M$, there exists a lift $\widetilde{f}$ of f (Hatcher [4, p. 61]). That is a continuous function $\widetilde{f} : I \to \widetilde{M}$ such that $\pi \circ \widetilde{f} = f$.

Let o and $-o$ be the two possible orientations of a real, 2-dimensional vector space. Since the image of $0 \in I$ under any lift of a closed curve based at x_0 is only one of two possibilities, (x_0, o) or $(x_0, -o)$, it follows from the unique lifting property that there are only two possible lifts of any closed curve (Hatcher [4, p. 62]). Again as a result of the unique lifting property, if for some lift $\widetilde{f}$ of f, $\widetilde{f}(0) = \widetilde{f}(1)$, then, without loss of generality, $\widetilde{f}(0) = \widetilde{f}(1) = (x_0, o)$ and $\widetilde{f}_*(0) = \widetilde{f}_*(1) = (x_0, -o)$ where $\widetilde{f}_*$ is the second lift of f. Similarly, if for some lift $\widetilde{f}(0) \neq \widetilde{f}(1)$, then, without loss of generality, $\widetilde{f}(0) = (x_0, o)$, $\widetilde{f}(1) = (x_0, -o)$ and $\widetilde{f}_*(0) = (x_0, -o)$, $\widetilde{f}_*(1) = (x_0, o)$.

Definition 3.1.4. (*Orientation Preserving Closed Curve*). A closed curve $f : I \to M$ is orientation preserving if $\widetilde{f}(0) = \widetilde{f}(1)$ where $\widetilde{f}$ is a lift of f.

Definition 3.1.5. (*Orientation Reversing Closed Curve*). A closed curve is orientation reversing if $\widetilde{f}(0) \neq \widetilde{f}(1)$ where $\widetilde{f}$ is a lift of f. In view of the last paragraph, this can be equivalently stated as a closed curve is orientation reversing if and only if it is not orientation preserving.

Let $h_s : I \to M$ be a homotopy where $h_0 = f$ and $h_s(0) = h_s(1) = x_0$ so that each stage of the homotopy is a closed curve based at x_0. Let $\widetilde{f}$ be a lift of f. By the homotopy lifting property, there exists a unique homotopy $\widetilde{h_s} : I \to \widetilde{M}$ that is a lift of h_s where $\widetilde{h_0} = \widetilde{f}$. This lifted homotopy fixes the end points so $\widetilde{h_s}(0) = \widetilde{f}(0)$ and $\widetilde{h_s}(1) = \widetilde{f}(1)$ (Hatcher [4, p. 61]). Since being an orientation preserving or orientation reversing closed curves is only dependent on the end points of a lift of the closed curve, any homotopy of an orientation preserving closed curve is orientation preserving and any homotopy of an orientation reversing closed curve is orientation reversing. Therefore, we can define $\pi_1^{or}(M, x_0)$ to be the set of homotopy classes that contain orientation preserving closed curves and $\pi_1^{rev}(M, x_0)$ to be the set of homotopy classes that contain orientation reversing closed curves.

Recall the isomorphism $\phi_* : \pi_R(M, v_{x_0}) \to \pi_1(STM, v_{x_0})$ and the homomorphism $p_* : \pi_1(STM, v_{x_0}) \to \pi_1(M, x_0)$ induced by the projection $p : STM \to M$. Consider the homomorphism $p_* \circ \phi_* : \pi_R(M, v_{x_0}) \to \pi_1(M, x_0)$. For each $[f]_R \in \pi_R(M, v_{x_0})$, $\big(p_* \circ \phi_*\big)([f]_R) = [f]$. We define $\pi_R^{or}(M, v_{x_0}) = \big(p_* \circ \phi_*\big)^{-1}\big(\pi_1^{or}(M, x_0)\big)$ and $\pi_R^{rev}(M, v_{x_0}) = \big(p_* \circ \phi_*\big)^{-1}\big(\pi_1^{rev}(M, x_0)\big)$.

Lemma 3.1.6. *$\pi_1^{or}(M, x_0)$ is a subgroup of $\pi_1(M, x_0)$ and $\pi_R^{or}(M, v_{x_0})$ is a subgroup of $\pi_R(M, v_{x_0})$.*

Proof. Let $\rho : \pi_1(M, x_0) \to \mathbb{Z}_2$ be defined by $[f] \mapsto 0$ for $[f] \in \pi_1^{or}(M, x_0)$ and $[f] \mapsto 1$ for $[f] \in \pi_1^{rev}(M, x_0)$. It will first be shown that ρ is a homomorphism.

For $[f], [g] \in \pi_1^{or}(M, x_0)$, we claim $[f] \cdot [g] \in \pi_1^{or}(M, x_0)$. There exists a lift $\widetilde{f}$ of f such that $\widetilde{f}(0) = \widetilde{f}(1) = (x_0, o)$. Similarly, there exists a lift $\widetilde{g}$ of g such that $\widetilde{g}(0) = \widetilde{g}(1) = (x_0, o)$. Then, $\widetilde{f} \cdot \widetilde{g}$ is a lift of $f \cdot g$ and $\big(\widetilde{f} \cdot \widetilde{g}\big)(0) = \big(\widetilde{f} \cdot \widetilde{g}\big)(1) = (x_0, o)$. Therefore, $f \cdot g$ is orientation preserving and $[f] \cdot [g] \in \pi_1^{or}(M, x_0)$. So $\rho([f] \cdot [g]) = 0 = \rho([f]) + \rho([g])$.

Using a similar method, it is easy to show for $[f] \in \pi_1^{or}(M, x_0)$ and $[g] \in \pi_1^{rev}(M, x_0)$, $[f] \cdot [g] \in \pi_1^{rev}(M, x_0)$ and $[g] \cdot [f] \in \pi_1^{rev}(M, x_0)$. So $\rho([f] \cdot [g]) = 1 = \rho([f]) + \rho([g])$ and $\rho([g] \cdot [f]) = 1 = \rho([g]) + \rho([f])$. For $[f], [g] \in \pi_1^{rev}(M, x_0)$, the same method can again be used to show $[f] \cdot [g] \in \pi_1^{or}(M, x_0)$ so $\rho([f] \cdot [g]) = 0 = \rho([f]) + \rho([g])$. Thus, ρ is a homomorphism and $\ker \rho = \pi_1^{or}(M, x_0)$ is a subgroup of $\pi_1(M, x_0)$. $\rho \circ p_* \circ \phi_* : \pi_R(M, v_{x_0}) \to \mathbb{Z}_2$ is a homomorphism with kernel $\pi_R^{or}(M, v_{x_0})$. Thus, $\pi_R^{or}(M, v_{x_0})$ is a subgroup of $\pi_R(M, v_{x_0})$. $\square$

Lastly, we define the orientation of STM_x. This will be needed to define the winding number.

Definition 3.1.7. (*Orientation of STM_x*). Since $STM_x \cong S^1$, we can orient S^1 counterclockwise and use the isomorphism to transport the orientation to STM_x. That is, as S^1 is traversed in the counterclockwise direction, the direction in which the image moves is the assigned orientation of STM_x. Note, this is dependent upon the choice of isomorphism.

3.2 Definition of Winding Number

Now, let $f : S^1 \to M$ be a closed curve based at v_{x_0}. For simplicity, we identify $(1, 0) \in S^1$ as 1. There exists the pullback bundle

$$f^*(STM) = \{(t, v) \in S^1 \times STM : f(t) = p(v)\}.$$

Then, the diagram

$$\begin{tikzcd} f^*(STM) \arrow[r, "\mu_f"] \arrow[d, "p^f"'] & STM \arrow[d, "p"] \\ S^1 \arrow[r, "f"'] & M \end{tikzcd}$$

commutes where $p^f : f^*(STM) \to S^1$ is defined by $(t, v) \mapsto t$ and $\mu_f : f^*(STM) \to STM$ is the isomorphism on each fiber defined by $(t, v) \to v$. Let

$$f^*(STM)_t = \{(t, v) \in \{t\} \times STM : f(t) = p(v)\}.$$

Then, since μ_f is an isomorphism on each fiber, $f^*(STM)_t \cong STM_{f(t)}$.

$f^*(STM)$ is either bundle isomorphic to the torus T or the Klein bottle K depending on whether f is orientation preserving or orientation reversing. That is, when the torus is viewed as a bundle over the base space S^1 and f is orientation preserving, we claim there exists a homeomorphism $\varphi : f^*(STM) \to T$ such that $\varphi\big|_{f^*(STM)_t}$ is an isomorphism for each $t \in S^1$ and $p^f = p_T \circ \varphi$ where p_T is the projection of the torus onto its base space S^1. Similarly, when the Klein bottle is viewed as a bundle over the base space S^1 and f is orientation reversing, we claim there exists a homeomorphism $\varkappa : f^*(STM) \to K$ such that $\varkappa\big|_{f^*(STM)_t}$ is an isomorphism for each $t \in S^1$ and $p^f = p_K \circ \varkappa$ where p_K is the projection of the Klein bottle onto its base space S^1.

Lemma 3.2.1. *If f is orientation preserving, $f^*(STM)$ is bundle isomorphic to the torus. If f is orientation reversing, $f^*(STM)$ is bundle isomorphic to the Klein bottle.*

Proof. Let the pullback of the tangent bundle by f be

$$f^*(TM) = \{(t, v) \in S^1 \times TM : f(t) = p(v)\}.$$

The set of pullbacks of the tangent bundle by closed curves up to bundle isomorphism is classified by $\pi_0(GL_2(\mathbb{R})) \cong \mathbb{Z}_2$. The set consists of the trivial bundle $S^1 \times \mathbb{R}^2$ and a nontrivial bundle (Hatcher [5, p. 25]).

If $f^*(TM)$ corresponds to the trivial element of $\mathbb{Z}_2$, then $f^*(TM)$ is orientable and $f^*(STM) \subset f^*(TM)$ is bundle isomorphic to the torus. Otherwise, $f^*(TM)$ corresponds to the nontrivial element of $\mathbb{Z}_2$, is non-orientable, and $f^*(STM) \subset f^*(TM)$ is bundle isomorphic to the Klein bottle. This classification of $f^*(STM)$ follows immediately from the description of the torus and Klein bottle as quotients of the cylinder.

f is orientation preserving if and only if $\widetilde{f}(0) = \widetilde{f}(1)$ where $\widetilde{f}$ is a lift of f. That is, the orientation of TM_{x_0} at the beginning and end of the lift of f to the orientation covering is the same. Hence, the orientation of TM_{x_0} is extended along the curve. So the lift to the orientation covering precisely places an orientation on each fiber of $f^*(TM)$ and thus defines an orientation of $f^*(TM)$. Then, $f^*(STM)$ is bundle isomorphic to the torus. Therefore, if f is orientation preserving, $f^*(STM)$ is bundle isomorphic to the torus.

f is orientation reversing if and only if $\widetilde{f}(0) \neq \widetilde{f}(1)$ where $\widetilde{f}$ is a lift of f. That is, the orientation of TM_{x_0} at the beginning and end of the lift of f to the orientation covering is not the same. Hence, the orientation of TM_{x_0} cannot be extended along the curve. Consequently, an orientation of $f^*(TM)$ cannot be defined. So $f^*(TM)$ is non-orientable. Thus, $f^*(STM)$ is bundle isomorphic to the Klein bottle when f is orientation reversing. $\square$

We will define a relative winding number between two vector fields along a closed curve.

Definition 3.2.2. (*Vector Field along a Closed Curve*). A vector field along a closed curve f is a map $Y : S^1 \to STM$ such that $p(Y(t)) = f(t)$ for each $t \in S^1$.

We will assume all vector fields Y along a closed curve f are continuous and based at v_{x_0} meaning $Y(1) = v_{x_0}$ where we are identifying $(1, 0) \in S^1$ as 1.

For vector fields Y_1, Y_2 along f, we know $Y_1(t), Y_2(t) \in STM_{f(t)}$ for each $t \in S^1$. Thus, we can define sections Y_1^*, Y_2^* of $f^*(STM)$. For $i = 1, 2$, $Y_i^* : S^1 \to f^*(STM)$ is defined by $Y_i^*(t) = \mu_f^{-1}(Y_i(t))$. Since Y_1, Y_2 are based at v_{x_0}, $Y_1^*(1) = Y_2^*(1)$. Let $e_0 = Y_1^*(1) = Y_2^*(1)$. Then, $[Y_1^*], [Y_2^*] \in \pi_1(f^*(STM), e_0)$.

Consider the homomorphism $p_*^f : \pi_1(f^*(STM), e_0) \to \pi_1(S^1, 1)$ induced by p^f. Since Y_1^* and Y_2^* are sections of $f^*(STM)$, $[Y_2^*] \cdot [Y_1^*]^{-1} \in \ker p_*^f$. From the homotopy sequence of the fibration $p^f : f^*(STM) \to S^1$ based at e_0, there is the exact sequence

$$\pi_2(S^1, 1) \longrightarrow \pi_1(f^*(STM)_1, e_0) \xrightarrow{i_*^f} \pi_1(f^*(STM), e_0) \xrightarrow{p_*^f} \pi_1(S^1, 1)$$

where $i^f : f^*(STM)_1 \to f^*(STM)$ is the inclusion map (Hu [7, p. 152]). As a consequence of exactness and $\pi_2(S^1, 1) \cong 1$, there exists a unique element $n \in \pi_1(f^*(STM)_1, e_0)$ such that $i_*^f(n) = [Y_2^*] \cdot [Y_1^*]^{-1}$.

Fixing an orientation of STM_{x_0}, we orient $f^*(STM)_1$ by transporting the orientation of STM_{x_0} under the isomorphism $(\mu_f)^{-1}\big|_{STM_{x_0}}$. Since $f^*(STM)_1$ is isomorphic to S^1, $\pi_1(f^*(STM)_1, e_0) \cong \mathbb{Z}$. Consider the homotopy class of the function from S^1 to $f^*(STM)_1$ where as S^1 is traversed in the counterclockwise direction, the image traverses $f^*(STM)_1$ once in the direction in which it is oriented. Let the isomorphism from $\pi_1(f^*(STM)_1, e_0)$ to $\mathbb{Z}$ be the one that maps this homotopy class to $1 \in \mathbb{Z}$. So n can be thought of as an integer. The winding number of f relative to Y_1 and Y_2 is defined to be n. We denote this $w(f; Y_1, Y_2) = n$.

Lemma 3.2.3. *For a closed curve f based at v_{x_0} with vector fields $Y_1, Y_2, \widehat{Y}_1, \widehat{Y}_2$ where $Y_1 \simeq \widehat{Y}_1$ rel $\{v_{x_0}\}$ and $Y_2 \simeq \widehat{Y}_2$ rel $\{v_{x_0}\}$, $w(f; Y_1, Y_2) = w(f; \widehat{Y}_1, \widehat{Y}_2)$.*

Proof. Since $Y_i \simeq \widehat{Y}_i$ rel $\{v_{x_0}\}$ for $i = 1, 2$, $[Y_i] = \big[\widehat{Y}_i\big]$. Consequently, $[Y_i^*] = \big[\widehat{Y}_i^{\,*}\big]$ and $[Y_2^*] \cdot [Y_1^*]^{-1} = \big[\widehat{Y}_2^{\,*}\big] \cdot \big[\widehat{Y}_1^{\,*}\big]^{-1}$. Thus, $w(f; Y_1, Y_2) = w(f; \widehat{Y}_1, \widehat{Y}_2)$. $\square$

Now, we assume f is an orientation preserving closed curve on M based at v_{x_0}. In this case, we have a second way of calculating the winding number of f relative to the vector fields Y_1, Y_2 along f. This will allow us to prove some of the main results of the next section. Before beginning this alternate definition, we need a few results about H-spaces.

Definition 3.2.4. (*H-space*). Suppose Z is a topological space and $\chi : Z \times Z \to Z$ is a continuous function. Z has a homotopy unit $a \in Z$ if $\chi(a, a) = a$, the function that maps each $z \in Z$ to $\chi(a, z)$ is homotopic to the identity on Z relative to the basepoint a, and the function that maps each $z \in Z$ to $\chi(z, a)$ is homotopic to the identity on Z relative to the basepoint a. A topological space with a continuous function χ and homotopy unit a is an H-space.

Theorem 3.2.5. (Hu [7, p. 81]). *For an H-space Z with continuous function χ and homotopy unit a, $\pi_1(Z, a)$ is abelian.*

For $[g], [h] \in \pi_1(Z, a)$, define $[g] \times [h] = [g \times h]$ where $g \times h : S^1 \to Z$ is defined by

$$(g \times h)(t) = \chi(g(t), h(t)).$$

From the continuity of χ and the fact that $\chi(a, a) = a$, $[g \times h] \in \pi_1(Z, a)$ so the mapping $\big([g], [h]\big) \mapsto [g \times h]$ is well-defined.

Theorem 3.2.6. (Hu [7, p. 82]). *For an H-space Z with continuous function χ and homotopy unit a, $[g] \cdot [h] = [g] \times [h]$ for each $[g], [h] \in \pi_1(Z, a)$.*

Any topological group under its group operation is an H-space with its identity taken as the homotopy unit (Hu [7, p. 81]). When viewed as a subset of the complex plane, S^1 is a topological group under the usual multiplication of complex numbers and with identity 1. Thus, S^1 is an H-space under multiplication with homotopy unit 1.

Corollary 3.2.7. *For $[h] \in \pi_1(S^1, 1)$, $[h]^{-1} = \left[\frac{1}{h}\right]$ where $\frac{1}{h} : S^1 \to S^1$ is defined by $\left(\frac{1}{h}\right)(t) = \frac{1}{h(t)}$.*

Proof. As a result of Theorem 3.2.6 and the fact that $h \times \frac{1}{h} = c_1$,

$$[h] \cdot \left[\frac{1}{h}\right] = [h] \times \left[\frac{1}{h}\right] = \left[h \times \frac{1}{h}\right] = [c_1] = [h] \cdot [h]^{-1}.$$

Consequently, $[h]^{-1} = \left[\frac{1}{h}\right]$. $\square$

We return to proving there is an alternate way to find the winding number of an orientation preserving curve. Since f is orientation preserving, $f^*(STM)$ is bundle isomorphic to $S^1 \times S^1$ by Lemma 3.2.1. Recall, $\varphi : f^*(STM) \to S^1 \times S^1$ is a bundle isomorphism. So $\varphi\big|_{f^*(STM)_t}$ is an isomorphism between fibers and φ induces the identity map of the base space S^1. That is, $p^f = \widetilde{p} \circ \varphi$ where $\widetilde{p} : S^1 \times S^1 \to S^1$ is defined by $(t, s) \mapsto t$. We can choose φ such that $\varphi(e_0) = (1, 1)$. The orientation of $\{1\} \times S^1 = \varphi(f^*(STM)_1)$ is chosen to be that induced by the orientation of $f^*(STM)_1$ under φ. That is, we orient $\{1\} \times S^1$ such that as $f^*(STM)_1$ is traversed in the direction in which it is oriented, the direction in which the corresponding image under φ moves is the assigned orientation of $\{1\} \times S^1$.

Since Y_1^* is a section of $f^*(STM)$, it is of the form $t \mapsto (t, \alpha(t))$ where $\alpha(t) \in f^*(STM)_t$. Similarly, Y_2^* is of the form $t \mapsto (t, \beta(t))$ where $\beta(t) \in f^*(STM)_t$. Define $\widetilde{Y_i^*} = \varphi \circ Y_i^*$ for $i = 1, 2$. Since φ induces the identity map on the base space, $\widetilde{Y_1^*}$ is defined by $t \mapsto (t, \widetilde{\alpha}(t))$ and $\widetilde{Y_2^*}$ is defined by $t \mapsto (t, \widetilde{\beta}(t))$.

Since $Y_1^*(1) = Y_2^*(1) = e_0$, $(1, \alpha(1)) = (1, \beta(1)) = e_0$. Because $\varphi(e_0) = (1, 1)$, $(1, \widetilde{\alpha}(1)) = (1, \widetilde{\beta}(1)) = (1, 1)$. Thus, $\widetilde{\alpha}(1) = \widetilde{\beta}(1) = 1$ and $\widetilde{\alpha}$, $\widetilde{\beta} : S^1 \to S^1$ have basepoint 1. Define $h : S^1 \to \{1\} \times S^1$ such that

$$h(t) = \left(1, \left(\widetilde{\beta} \cdot \overline{\widetilde{\alpha}} \right)(t)\right)$$

and let $j : S^1 \to \{1\} \times S^1$ be defined by

$$j(t) = \left(1, \left(\widetilde{\beta} \times \frac{1}{\widetilde{\alpha}}\right)(t)\right).$$

Consider the homotopy class represented by the function from S^1 to $\{1\} \times S^1$ where as S^1 is traversed in the counterclockwise direction, the image traverses $\{1\} \times S^1$ once in the direction in which it is oriented. Let the isomorphism from $\pi_1(\{1\} \times S^1, (1, 1))$ to $\mathbb{Z}$ be the one that maps this homotopy class to $1 \in \mathbb{Z}$. Then, let n be the integer associated to $[j] \in \pi_1(\{1\} \times S^1, (1, 1)) \cong \mathbb{Z}$.

Theorem 3.2.8. *$w(f; Y_1, Y_2) = n$ where n is the integer associated to $[j]$.*

Proof. Consider the function $\widetilde{Y_2^*} \cdot \overline{\widetilde{Y_1^*}} : S^1 \to S^1 \times S^1$. The first component of $\widetilde{Y_2^*} \cdot \overline{\widetilde{Y_1^*}}$ wraps around S^1 once in the counterclockwise direction and then once in the clockwise direction. The second component of $\widetilde{Y_2^*} \cdot \overline{\widetilde{Y_1^*}}$ is $\widetilde{\beta} \cdot \overline{\widetilde{\alpha}}$. Consequently, $\widetilde{Y_2^*} \cdot \overline{\widetilde{Y_1^*}} \simeq \widetilde{i^f} \circ h$ rel $\{(1, 1)\}$ where $\widetilde{i^f} : \{1\} \times S^1 \to S^1 \times S^1$ is the inclusion map.

From Corollary 3.2.7 and Theorem 3.2.6, $\widetilde{\beta} \cdot \overline{\widetilde{\alpha}} \simeq \widetilde{\beta} \times \dfrac{1}{\widetilde{\alpha}}$ rel $\{1\}$ in S^1 and it follows that $h \simeq j$ rel $\{(1,1)\}$ in $\{1\} \times S^1$. So $\widetilde{i^f} \circ j \simeq \widetilde{i^f} \circ h$ rel $\{(1,1)\}$ in $S^1 \times S^1$ and by transitivity, $\widetilde{Y_2^*} \cdot \overline{\widetilde{Y_1^*}} \simeq \widetilde{i^f} \circ j$ rel $\{(1,1)\}$ in $S^1 \times S^1$. Then, $\widetilde{i^f}_*([j]) = \big[\widetilde{Y_2^*}\big] \cdot \big[\widetilde{Y_1^*}\big]^{-1}$. Since the diagram

$$\begin{tikzcd}
 & \pi_1(\{1\} \times S^1, (1,1)) \arrow[r, "\widetilde{i^f}_*"] & \pi_1(S^1 \times S^1, (1,1)) \arrow[dr, "\widetilde{p}_*"] & \\
\pi_2(S^2) \arrow[r] & \pi_1(f^*(STM)_1, e_0) \arrow[u, "\left(\varphi|_{f^*(STM)_1}\right)_*"] \arrow[r, "i^f_*"'] & \pi_1(f^*(STM), e_0) \arrow[u, "\varphi_*"'] \arrow[r, "p^f_*"'] & \pi_1(S^1, 1)
\end{tikzcd}$$

is commutative and $\left(\varphi|_{f^*(STM)_1}\right)_*$ is an isomorphism, it follows that

$$\left(\varphi_* \circ i^f_* \circ \left(\varphi|_{f^*(STM)_1}\right)_*^{-1}\right)([j]) = \big[\widetilde{Y_2^*}\big] \cdot \big[\widetilde{Y_1^*}\big]^{-1}.$$

Because φ_* is an isomorphism, $\left(i^f_* \circ \left(\varphi|_{f^*(STM)_1}\right)_*^{-1}\right)([j]) = [Y_2^*] \cdot [Y_1^*]^{-1}$ and by definition, $w(f; Y_1, Y_2)$ is the integer associated to $\left(\varphi|_{f^*(STM)_1}\right)_*^{-1}([j])$. Since the orientation of $\{1\} \times S^1$ was chosen to be the orientation induced by $f^*(STM)_1$ under φ, the integer associated to $\left(\varphi|_{f^*(STM)_1}\right)_*^{-1}([j])$ is that associated to $[j]$. Therefore, $w(f; Y_1, Y_2) = n$ where n is the integer associated to $[j]$. $\square$

3.3 Winding Numbers on a Surface with a Vector Field

In this section, it is again assumed that f is a regular closed curve based at v_{x_0} unless otherwise stated. We will also assume M is a surface with a continuous, nonzero vector field X.

Definition 3.3.1. (*Vector Field*). A vector field is a function $X : M \to TM$ such that $P(X(x)) = x$ for each $x \in M$.

The torus, the Klein bottle, non-compact surfaces, and surfaces with boundary all have a continuous, nonzero vector field. We will assume $X(x_0) = v_{x_0}$. Clearly, given a continuous, nonzero vector field, a homotopy can be used to ensure it demonstrates this behavior at x_0.

Chillingworth [1] defined the winding number of f relative to a continuous, nonzero vector field. This is just a special case of the definition given in the previous section. Recall, for each $x \in M$, $X(x) \in TM \subset M \times \mathbb{R}^m$ and $P(X(x)) = x$ so $X(x) = (x, \widehat{X(x)})$. Then, since X is continuous and nowhere zero, X induces a vector field Y_{X_f} along f where $Y_{X_f} : S^1 \to STM$ is defined by

$$Y_{X_f}(t) = \left(f(t), \frac{\widehat{X(f(t))}}{||\widehat{X(f(t))}||_{f(t)}} \right).$$

A second vector field along f is defined from the tangent vector at each point on the curve. Let $Y_{f'} : S^1 \to STM$ where

$$Y_{f'}(t) = \left(f(t), \frac{f'(t)}{||f'(t)||_{f(t)}} \right).$$

Taking $Y_1 = Y_{X_f}$ and $Y_2 = Y_{f'}$, we have Chillingworth's definition of the winding number for f. Since Y_{X_f} and $Y_{f'}$ only depend on f or X, the winding number of f is relative to X only and we notate it $w(f; X)$.

Since both S^2 and $\mathbb{RP}^2$ do not have continuous, nonzero vector fields, $\pi_2(M, x_0)$ is trivial for all M with a continuous, nonzero vector field. Then, from the exact sequence

$$\pi_2(M, x_0) \to \pi_1(STM_{x_0}, v_{x_0}) \xrightarrow{i_*} \pi_1(STM, v_{x_0}) \xrightarrow{p_*} \pi_1(M, x_0) \to \pi_0(STM_{x_0}, v_{x_0})$$

it follows that i_* is injective. Obviously, $p_*([Y_{f'}] \cdot [Y_{X_f}]^{-1}) = [f] \cdot [f]^{-1} = [c_{x_0}]$. Consequently, there exists a unique element $n \in \pi_1(STM_{x_0}, v_{x_0})$ such that $i_*(n) = [Y_{f'}] \cdot [Y_{X_f}]^{-1}$. Since $STM_{x_0} \cong S^1$, $\pi_1(STM_{x_0}, v_{x_0}) \cong \mathbb{Z}$ so we can think of n as an integer. Consider the homotopy class represented by the function from S^1 to STM_{x_0} where as S^1 is traversed in the counterclockwise direction, the image traverses STM_{x_0} once in the direction in which it is oriented. The isomorphism from $\pi_1(STM_{x_0}, v_{x_0})$ to $\mathbb{Z}$ is chosen to be the isomorphism that takes this homotopy class and maps it to $1 \in \mathbb{Z}$.

Theorem 3.3.2. *$w(f; X) = n$ where n is the unique element in $\pi_1(STM_{x_0}, v_{x_0})$ such that $i_*(n) = [Y_{f'}] \cdot [Y_{X_f}]^{-1}$.*

Proof. Let $w(f; X) = n$. So $(\mu_{f*} \circ i_*^f)(n) = \mu_{f*}\big([Y_{f'}^*] \cdot [Y_{X_f}^*]^{-1}\big) = [Y_{f'}] \cdot [Y_{X_f}]^{-1}$. From the commutativity of the diagram

$$\begin{tikzcd}
\pi_2(S^1, 1) \arrow[r] & \pi_1(f^*(STM)_1, e_0) \arrow[r, "i_*^f"] \arrow[d, "\left(\mu_f|_{f^*(STM)_1}\right)_*"'] & \pi_1(f^*(STM), e_0) \arrow[r, "p_*^f"] \arrow[d, "\mu_{f*}"'] & \pi_1(S^1, 1) \\
\pi_2(M, x_0) \arrow[r] & \pi_1(STM_{x_0}, v_{x_0}) \arrow[r, "i_*"] & \pi_1(STM, v_{x_0}) \arrow[r, "p_*"] & \pi_1(M, x_0)
\end{tikzcd}$$

it follows that $\Big(i_* \circ \big(\mu_f|_{f^*(STM)_1}\big)_*\Big)(n) = [Y_{f'}] \cdot [Y_{X_f}]^{-1}$. Because $\mu_f|_{f^*(STM)_1} : f^*(STM)_1 \to STM_{x_0}$ is an isomorphism and the orientation of $f^*(STM)_1$ was chosen from the fixed orientation of STM_{x_0}, $\big(\mu_f|_{f^*(STM)_1}\big)_*(n) = n$. Therefore, $i_*(n) = [Y_{f'}] \cdot [Y_{X_f}]^{-1}$. $\square$

Definition 3.3.3. (*Vector Field Homotopy*). For two continuous, nonzero vector fields X and $\check{X}$, a vector field homotopy is a homotopy $h_s : M \to TM$ between X and $\check{X}$ such that h_s is a continuous, nonzero vector field for each $s \in I$. We denote this $X \simeq_{VF} \check{X}$. If the homotopy is based at v_{x_0}, that is $h_s(x_0) = v_{x_0}$ for each $s \in I$, we say $X \simeq_{VF} \check{X}$ rel $\{v_{x_0}\}$.

Lemma 3.3.4. *For continuous, nonzero vector fields $X, \check{X}$ on M where $X \simeq_{VF} \check{X}$ rel $\{v_{x_0}\}$, $w(f; X) = w(f; \check{X})$.*

Proof. Letting $h_s : M \to TM$ denote a vector field homotopy between X and $\check{X}$ based at v_{x_0}, we define $\dot{h}_s : S^1 \to STM$ by

$$\dot{h}_s(t) = \left(f(t), \frac{\widehat{h_s(f(t))}}{||\widehat{h_s(f(t))}||_{f(t)}} \right).$$

This is a homotopy between Y_{X_f} and $Y_{\check{X}_f}$ based at v_{x_0}. By Lemma 3.2.3, $w(f; X) = w(f; \check{X})$. □

Lemma 3.3.5. *For regular closed curves f, g where $f \simeq_R g$ rel $\{v_{x_0}\}$, $w(f; X) = w(g; X)$.*

Proof. Let $h_s : S^1 \to M$ denote a regular homotopy between f and g based at v_{x_0}. Then, $\dot{h}_s : S^1 \to STM$ defined by

$$\dot{h}_s(t) = \left(h_s(t), \frac{\widehat{h'_s(t)}}{||\widehat{h'_s(t)}||_{h_s(t)}} \right)$$

is a homotopy between $Y_{f'}$ and $Y_{g'}$. By Lemma 3.2.3, $w(f; X) = w(g; X)$. □

Let $w : \pi_R^{or}(M, v_{x_0}) \to \mathbb{Z}$ such that $[f]_R \mapsto w(f; X)$ for each $[f]_R \in \pi_R^{or}(M, v_{x_0})$. By Lemma 3.3.5, this is a well-defined function.

Theorem 3.3.6. *$w : \pi_R^{or}(M, v_{x_0}) \to \mathbb{Z}$ is a homomorphism.*

Proof. Let $[f]_R, [g]_R \in \pi_R^{or}(M, v_{x_0})$. Here, f and g will be viewed as regular closed curves defined on I instead of S^1. However, we still want the pullback bundles to be subsets of $S^1 \times STM$. So $q : I \to S^1$ defined by $q(t) = \big(\cos(\tau(t)), \sin(\tau(t))\big)$ where $\tau : [0,1] \to [0, 2\pi]$ is the bijection defined by

$$\tau(t) = (-4\pi + 2)t^3 + (6\pi - 3)t^2 + t$$

and $\dot{q} : S^1 \to I$ defined by $\dot{q} = \left(q\big|_{[0,1)}\right)^{-1}$ are used to identify S^1 and I. Then,

$$f^*(STM) = \{(t, v) \in S^1 \times STM : f(\dot{q}(t)) = p(v)\}$$

and

$$g^*(STM) = \{(t, v) \in S^1 \times STM : g(\dot{q}(t)) = p(v)\}.$$

Because f and g are based at v_{x_0}, $f^*(STM)_1 = g^*(STM)_1$. We only consider regular closed curves based at v_{x_0}, so we will denote the fiber over $\{1\}$ in the pullback bundles as E_0. Since f, g are orientation preserving, $f^*(STM)$ and $g^*(STM)$ are bundle isomorphic to $S^1 \times S^1$. Let $\varphi_f : f^*(STM) \to S^1 \times S^1$ and $\varphi_g : g^*(STM) \to S^1 \times S^1$ be the respective bundle isomorphisms chosen such that $\varphi_f\big|_{E_0} = \varphi_g\big|_{E_0}$ and $\varphi_f(e_0) = \varphi_g(e_0) = (1, 1)$. Then, the diagrams

$$\begin{tikzcd}
\{1\} \times S^1 \arrow[r, "\widetilde{i^f}"] & S^1 \times S^1 & \\
E_0 \arrow[u, "\varphi_f|_{E_0}"] \arrow[r, "i^f"'] & f^*(STM) \arrow[u, "\varphi_f"'] \arrow[r, "\mu_f"'] & STM
\end{tikzcd}$$

$$\begin{tikzcd}
\{1\} \times S^1 \arrow[r, "\widetilde{i^g}"] & S^1 \times S^1 & \\
E_0 \arrow[u, "\varphi_g\big|_{E_0}"] \arrow[r, "i^g"'] & g^*(STM) \arrow[u, "\varphi_g"'] \arrow[r, "\mu_g"'] & STM
\end{tikzcd}$$

are commutative.

Since $[f]_R \cdot [g]_R = [(f \circ r) \cdot (g \circ r)]_R$, a similar diagram is constructed for $(f \circ r) \cdot (g \circ r)$. For simplicity, $f \circ r$ and $g \circ r$ are denoted $\ddot{f}$ and $\ddot{g}$ respectively. Again, it is worth noting that $\ddot{f} \cdot \ddot{g}$ is defined on I rather than S^1 so

$$(\ddot{f} \cdot \ddot{g})^*(STM) = \{(t, v) \in S^1 \times STM : (\ddot{f} \cdot \ddot{g})(\dot{q}(t)) = p(v)\}.$$

We will define a bundle isomorphism $\varphi_{\ddot{f} \cdot \ddot{g}} : (\ddot{f} \cdot \ddot{g})^*(STM) \to S^1 \times S^1$ using both φ_f and φ_g. In order to do this, we relate the fibers of $(\ddot{f} \cdot \ddot{g})^*(STM)$ with the fibers of $f^*(STM)$ and $g^*(STM)$. We identify how φ_f and φ_g are defined on the fibers of $f^*(STM)$ and $g^*(STM)$ and use this to define $\varphi_{\ddot{f} \cdot \ddot{g}}$.

For $t \in S^1$ where $0 \leq \dot{q}(t) \leq \frac{1}{2}$,

$$(\ddot{f} \cdot \ddot{g})^*(STM)_t = \{(t, v) \in \{t\} \times STM : f\big(r(2\dot{q}(t))\big) = p(v)\}$$

so

$$\mu_{\ddot{f} \cdot \ddot{g}}\big((\ddot{f} \cdot \ddot{g})^*(STM)_t\big) = STM_{f(r(2\dot{q}(t)))} = \mu_f\big(f^*(STM)_{q(r(2\dot{q}(t)))}\big).$$

Similarly, for $t \in S^1$ such that $\frac{1}{2} \leq \dot{q}(t) < 1$,

$$(\ddot{f} \cdot \ddot{g})^*(STM)_t = \{(t, v) \in \{t\} \times STM : g\big(r(2\dot{q}(t) - 1)\big) = p(v)\}$$

so

$$\mu_{\ddot{f}\cdot\ddot{g}}\big((\ddot{f}\cdot\ddot{g})^*(STM)_t\big) = STM_{g(r(2\dot{q}(t)-1))} = \mu_g\big(g^*(STM)_{q(r(2\dot{q}(t)-1))}\big).$$

Recall $p^f = \widetilde{p}\circ\varphi_f$ and $p^g = \widetilde{p}\circ\varphi_g$ where $\widetilde{p} : S^1 \times S^1 \to S^1$ is defined by $(t,s) \mapsto t$. Thus, there exists $\eta_{f,t} : STM_{f(\dot{q}(t))} \to S^1$ and $\eta_{g,t} : STM_{g(\dot{q}(t))} \to S^1$ such that $\varphi_f(t,v) = (t, \eta_{f,t}(v))$ and $\varphi_g(t,v) = (t,\eta_{g,t}(v))$. So $\eta_{f,q(r(2\dot{q}(t)))} : STM_{f(r(2\dot{q}(t)))} \to S^1$ and $\eta_{g,q(r(2\dot{q}(t)-1))} : STM_{g(r(2\dot{q}(t)-1))} \to S^1$. Let $\varphi_{\ddot{f}\cdot\ddot{g}} : (\ddot{f}\cdot\ddot{g})^*(STM) \to S^1 \times S^1$ be defined by

$$\varphi_{\ddot{f}\cdot\ddot{g}}(t,v) = \begin{cases} \big(t, \eta_{f,q(r(2\dot{q}(t)))}(v)\big) & \text{if} \quad 0 \le \dot{q}(t) \le \frac{1}{2} \\ \big(t, \eta_{g,q(r(2\dot{q}(t)-1))}(v)\big) & \text{if} \quad \frac{1}{2} \le \dot{q}(t) < 1 \end{cases}.$$

Then, $\varphi_{\ddot{f}\cdot\ddot{g}}$ is a bundle isomorphism and the diagram

$$\begin{tikzcd}
\{1\} \times S^1 \arrow[r, "\widetilde{i^{\ddot{f}\cdot\ddot{g}}}"] & S^1 \times S^1 & \\
E_0 \arrow[u, "(\varphi_{\ddot{f}\cdot\ddot{g}})\big|_{E_0}"] \arrow[r, "i^{\ddot{f}\cdot\ddot{g}}"'] & (\ddot{f}\cdot\ddot{g})^*(STM) \arrow[u, "\varphi_{\ddot{f}\cdot\ddot{g}}"'] \arrow[r, "\mu_{\ddot{f}\cdot\ddot{g}}"'] & STM
\end{tikzcd}$$

commutes.

Recall, $\mu_f^{-1} \circ Y_{X_f} = Y^*_{X_f}, \mu_f^{-1} \circ Y_{f'} = Y^*_{f'}$ are sections of $f^*(STM)$ of the form $t \mapsto (t, \alpha_f(t))$ and $t \mapsto (t, \beta_f(t))$ respectively. So $\varphi_f \circ Y^*_{Xf} = \widetilde{Y^*_{X_f}}, \varphi_f \circ Y^*_{f'} = \widetilde{Y^*_{f'}}$ are of the form $t \mapsto (t, \widetilde{\alpha_f}(t))$ and $t \mapsto (t, \widetilde{\beta_f}(t))$. Similarly, $\mu_g^{-1} \circ Y_{X_g} = Y^*_{X_g}, \mu_g^{-1} \circ Y_{g'} = Y^*_{g'}$ are defined by $t \mapsto (t, \alpha_g(t))$, $t \mapsto (t, \beta_g(t))$ and $\varphi_g \circ Y^*_{X_g} = \widetilde{Y^*_{X_g}}, \varphi_g \circ Y^*_{g'} = \widetilde{Y^*_{g'}}$ are of the form $t \mapsto (t, \widetilde{\alpha_g}(t))$ and $t \mapsto (t, \widetilde{\beta}_g(t))$. Note,

$\widetilde{\alpha_f}(t) = \eta_{f,t}\Big(Y^*_{X_f}\big(\dot{q}(t)\big)\Big)$, $\widetilde{\alpha_g}(t) = \eta_{g,t}\Big(Y^*_{X_g}\big(\dot{q}(t)\big)\Big)$, $\widetilde{\beta_f}(t) = \eta_{f,t}\Big(Y^*_{f'}\big(\dot{q}(t)\big)\Big)$, $\widetilde{\beta}_g(t) = \eta_{g,t}\Big(Y^*_{g'}\big(\dot{q}(t)\big)\Big)$ where $\dot{q}$ is again used because f and g are defined on I.

It is easy to check that $Y_{X_{\ddot{f}\cdot\ddot{g}}} = Y_{X_{\ddot{f}}} \cdot Y_{X_{\ddot{g}}}$, $Y_{(\ddot{f}\cdot\ddot{g})'} = Y_{\ddot{f}'} \cdot Y_{\ddot{g}'}$, $(Y_{X_{\ddot{f}}} \cdot Y_{X_{\ddot{g}}})^* = Y^*_{X_{\ddot{f}}} \cdot Y^*_{X_{\ddot{g}}}$, and $(Y_{\ddot{f}'} \cdot Y_{\ddot{g}'})^* = Y^*_{\ddot{f}'} \cdot Y^*_{\ddot{g}'}$. Then, it follows from the definition of $\varphi_{\ddot{f}\cdot\ddot{g}}$,

$$\Big(\varphi_{\ddot{f}\cdot\ddot{g}} \circ \big(Y^*_{X_{\ddot{f}}} \cdot Y^*_{X_{\ddot{g}}}\big)\Big)(t) = \begin{cases} \Big(t, \big(\eta_{f,q(r(2\dot{q}(t)))} \circ Y^*_{X_{\ddot{f}}}\big)\big(2\dot{q}(t)\big)\Big) & \text{if } \ 0 \leq \dot{q}(t) \leq \frac{1}{2} \\ \Big(t, \big(\eta_{g,q(r(2\dot{q}(t)-1))} \circ Y^*_{X_{\ddot{g}}}\big)\big(2\dot{q}(t)-1\big)\Big) & \text{if } \ \frac{1}{2} \leq \dot{q}(t) < 1 \end{cases}$$

$$= \begin{cases} \Big(t, \big(\eta_{f,q(r(2\dot{q}(t)))} \circ Y^*_{X_f}\big)\big(r\big(2\dot{q}(t)\big)\big)\Big) & \text{if } \ 0 \leq \dot{q}(t) \leq \frac{1}{2} \\ \Big(t, \big(\eta_{g,q(r(2\dot{q}(t)-1))} \circ Y^*_{X_g}\big)\big(r\big(2\dot{q}(t)-1\big)\big)\Big) & \text{if } \ \frac{1}{2} \leq \dot{q}(t) < 1 \end{cases}$$

$$= \begin{cases} \Big(t, \widetilde{\alpha_f}\big(q\big(r(2\dot{q}(t))\big)\big)\Big) & \text{if } \ 0 \leq \dot{q}(t) \leq \frac{1}{2} \\ \Big(t, \widetilde{\alpha_g}\big(q\big(r(2\dot{q}(t)-1)\big)\big)\Big) & \text{if } \ \frac{1}{2} \leq \dot{q}(t) < 1 \end{cases}.$$

Therefore, $\varphi_{\ddot{f}\cdot\ddot{g}} \circ \big(Y^*_{X_{\ddot{f}}} \cdot Y^*_{X_{\ddot{g}}}\big) = \big(\widetilde{Y^*_{X_{\ddot{f}}} \cdot Y^*_{X_{\ddot{g}}}}\big)$ is a reparameterization of the function where $t \mapsto \big(t, (\widetilde{\alpha_f} \cdot \widetilde{\alpha_g})(t)\big)$. Similarly, $\varphi_{\ddot{f}\cdot\ddot{g}} \circ \big(Y^*_{\ddot{f}'} \cdot Y^*_{\ddot{g}'}\big) = (\widetilde{Y^*_{\ddot{f}'} \cdot Y^*_{\ddot{g}'}})$ is reparameterization the function where $t \mapsto \big(t, (\widetilde{\beta_f} \cdot \widetilde{\beta}_g)(t)\big)$. Then, it follows from Theorem 3.2.8, $w(\ddot{f} \cdot \ddot{g}; X) = n$ where n is the integer representing the homotopy class of

$$\begin{aligned} j(t) &= \left(1, \left((\widetilde{\beta_f} \cdot \widetilde{\beta}_g) \times \left(\frac{1}{\widetilde{\alpha_f} \cdot \widetilde{\alpha_g}}\right)\right)(t)\right) \\ &= \left(1, \left(\left(\widetilde{\beta_f} \times \frac{1}{\widetilde{\alpha_f}}\right) \cdot \left(\widetilde{\beta}_g \times \frac{1}{\widetilde{\alpha_g}}\right)\right)(t)\right) \end{aligned}$$

in $\pi_1(\{1\} \times S^1, (1,1))$.

Since the first component of $\big(\widetilde{Y^*_{f'}} \cdot \overline{\overline{Y^*_{X_f}}}\big) \cdot \big(\widetilde{Y^*_{g'}} \cdot \overline{\overline{Y^*_{X_g}}}\big) : S^1 \to S^1 \times S^1$ is homotopic to $c_1 : S^1 \to S^1$ and the second component is homotopic to

$$\left(\widetilde{\beta_f} \times \frac{1}{\widetilde{\widetilde{\alpha_f}}}\right) \cdot \left(\widetilde{\beta}_g \times \frac{1}{\widetilde{\widetilde{\alpha_g}}}\right),$$

$$[j] = \left(\left[\widetilde{Y^*_{f'}}\right] \cdot \left[\widetilde{Y^*_{X_f}}\right]^{-1}\right) \cdot \left(\left[\widetilde{Y^*_{g'}}\right] \cdot \left[\widetilde{Y^*_{X_g}}\right]^{-1}\right).$$

Because $\pi_1(\{1\} \times S^1, (1,1))$ is being identified with $\mathbb{Z}$ under an isomorphism, the integer associated to $[j]$ is the sum of the integers associated to $\left[\widetilde{Y^*_{f'}}\right] \cdot \left[\widetilde{Y^*_{X_f}}\right]^{-1}$ and $\left[\widetilde{Y^*_{g'}}\right] \cdot \left[\widetilde{Y^*_{X_g}}\right]^{-1}$. Therefore, $w(\ddot{f} \cdot \ddot{g}; X) = w(f; X) + w(g; X)$. Equivalently, $w([f]_R \cdot [g]_R) = w([f]_R) + w([g]_R)$ so w is a homomorphism. $\square$

We finish this chapter with two more main results. With the following lemma, we prove for $[f]_R, [g]_R \in \pi_R^{or}(M, v_{x_0})$, $[f]_R = [g]_R$ if and only if $[f] = [g]$ and $w(f; X) = w(g; X)$. This result is then used along with a few lemmas to prove $\pi_R^{or}(M, v_{x_0})$ is isomorphic to $\pi_1^{or}(M, x_0) \times \mathbb{Z}$.

Lemma 3.3.7. *For closed curves f and g based at v_{x_0} such that $f \simeq g$ rel $\{x_0\}$, $Y_{X_f} \simeq Y_{X_g}$ rel $\{v_{x_0}\}$.*

Proof. Let $h_s : S^1 \to M$ be a homotopy between f and g based at x_0. Define $\dot{h}_s : S^1 \to STM$ by

$$\dot{h}_s(t) = \left(h_s(t), \frac{X(\widehat{h_s(t)})}{||X(\widehat{h_s(t)})||_{h_s(t)}}\right).$$

$\dot{h}_s$ is a homotopy between Y_{X_f} and Y_{X_g} based at v_{x_0}. Thus, $[Y_{X_f}] = [Y_{X_g}]$. $\square$

Theorem 3.3.8. *For orientation preserving, regular closed curves f and g, $f \simeq_R g$ rel $\{v_{x_0}\}$ if and only if $f \simeq g$ rel $\{x_0\}$ and $w(f; X) = w(g; X)$.*

Proof. Using Lemma 3.3.5, sufficiency is clear. To prove necessity, we begin by assuming $f \simeq g$ rel $\{x_0\}$ and $w(f;X) = w(g;X)$. For simplicity, we abuse notation letting $\overline{g}$ be a regular closed curve based at v_{x_0} such that $\overline{g} \in [g]_R^{-1}$. Recall, the homotopy sequence of the fibration $p^{\ddot{f}\cdot\ddot{\overline{g}}} : (\ddot{f}\cdot\ddot{\overline{g}})^*(STM) \to S^1$ based at e_0 is the exact sequence

$$\pi_2(S^1,1) \longrightarrow \pi_1\big((\ddot{f}\cdot\ddot{\overline{g}})^*(STM)_1, e_0\big) \xrightarrow{i_*^{\ddot{f}\cdot\ddot{\overline{g}}}} \pi_1\big((\ddot{f}\cdot\ddot{\overline{g}})^*(STM), e_0\big) \xrightarrow{p_*^{\ddot{f}\cdot\ddot{\overline{g}}}} \pi_1(S^1,1)$$

where $i^{\ddot{f}\cdot\ddot{\overline{g}}} : (\ddot{f}\cdot\ddot{\overline{g}})^*(STM)_1 \to (\ddot{f}\cdot\ddot{\overline{g}})^*(STM)$ is the inclusion map. Because $w : \pi_R^{or}(M, v_{x_0}) \to \mathbb{Z}$ is a homomorphism, $w([f]_R \cdot [g]_R^{-1}) = w([f]_R) - w([g]_R) = w(f;X) - w(g;X) = 0$. So $i_*^{\ddot{f}\cdot\ddot{\overline{g}}}(0) = \big[Y^*_{(\ddot{f}\cdot\ddot{\overline{g}})'}\big]\cdot\big[Y^*_{X_{\ddot{f}\cdot\ddot{\overline{g}}}}\big]^{-1}$ and $\big[Y^*_{(\ddot{f}\cdot\ddot{\overline{g}})'}\big]\cdot\big[Y^*_{X_{\ddot{f}\cdot\ddot{\overline{g}}}}\big]^{-1}$ is the identity of $\pi_1\big((\ddot{f}\cdot\ddot{\overline{g}})^*(STM), e_0\big)$.

The continuous function $\mu_{\ddot{f}\cdot\ddot{\overline{g}}} : (\ddot{f}\cdot\ddot{\overline{g}})^*(STM) \to STM$ induces the homomorphism $\mu_{\ddot{f}\cdot\ddot{\overline{g}}*} : \pi_1\big((\ddot{f}\cdot\ddot{\overline{g}})^*(STM), e_0\big) \to \pi_1(STM, v_{x_0})$. Then, $\mu_{\ddot{f}\cdot\ddot{\overline{g}}*}\Big(\big[Y^*_{(\ddot{f}\cdot\ddot{\overline{g}})'}\big]\cdot\big[Y^*_{X_{\ddot{f}\cdot\ddot{\overline{g}}}}\big]^{-1}\Big) = \big[Y_{(\ddot{f}\cdot\ddot{\overline{g}})'}\big]\cdot\big[Y_{X_{\ddot{f}\cdot\ddot{\overline{g}}}}\big]^{-1}$ is the identity of $\pi_1(STM, v_{x_0})$. Since $Y_{(\ddot{f}\cdot\ddot{\overline{g}})'} = Y_{\ddot{f}'}\cdot Y_{\ddot{\overline{g}}'}$ and $Y_{X_{\ddot{f}\cdot\ddot{\overline{g}}}} = Y_{X_{\ddot{f}}}\cdot Y_{X_{\ddot{\overline{g}}}}$,

$$\big[Y_{(\ddot{f}\cdot\ddot{\overline{g}})'}\big]\cdot\big[Y_{X_{\ddot{f}\cdot\ddot{\overline{g}}}}\big]^{-1} = \Big(\big[Y_{\ddot{f}'}\big]\cdot\big[Y_{\ddot{\overline{g}}'}\big]\Big)\cdot\Big(\big[Y_{X_{\ddot{f}}}\big]\cdot\big[Y_{X_{\ddot{\overline{g}}}}\big]\Big)^{-1}.$$

From Lemma 3.3.7, $Y_{X_{g\cdot\ddot{\overline{g}}}} \simeq Y_{X_{c_{x_0}}}$ rel $\{v_{x_0}\}$ where $c_{x_0} : S^1 \to M$ is the constant map at x_0. By definition, $Y_{X_{c_{x_0}}} = c_{v_{x_0}}$ where $c_{v_{x_0}} : S^1 \to STM$ is the constant map at v_{x_0}. Then, because $Y_{X_{g\cdot\ddot{\overline{g}}}} = Y_{X_g}\cdot Y_{X_{\ddot{\overline{g}}}}$ it follows that $[Y_{X_{\ddot{\overline{g}}}}] = [Y_{X_g}]^{-1}$. Hence,

$$\big[Y_{(\ddot{f}\cdot\ddot{\overline{g}})'}\big]\cdot\big[Y_{X_{\ddot{f}\cdot\ddot{\overline{g}}}}\big]^{-1} = \Big(\big[Y_{\ddot{f}'}\big]\cdot\big[Y_{\ddot{\overline{g}}'}\big]\Big)\cdot\Big(\big[Y_{X_{\ddot{f}}}\big]\cdot\big[Y_{X_g}\big]^{-1}\Big)^{-1}.$$

Since $\ddot{f} \simeq f$ rel $\{x_0\}$ and $f \simeq g$ rel $\{x_0\}$, Lemma 3.3.7 implies $\left[Y_{X_{\ddot{f}}}\right] \cdot \left[Y_{X_g}\right]^{-1} = \left[c_{v_{x_0}}\right]$. Since $\ddot{f}$ is a reparameterization of f, $Y_{\ddot{f}'}$ is a reparameterization of $Y_{f'}$ so $\left[Y_{\ddot{f}'}\right] = \left[Y_{f'}\right]$. Similarly, $\left[Y_{\ddot{\overline{g}}'}\right] = \left[Y_{\overline{g}'}\right]$. So

$$\left[Y_{(\ddot{f}\cdot\ddot{\overline{g}})'}\right] \cdot \left[Y_{X_{\ddot{f}\cdot\ddot{\overline{g}}}}\right]^{-1} = \left[Y_{f'}\right] \cdot \left[Y_{\overline{g}'}\right]$$

and $\left[Y_{f'}\right] \cdot \left[Y_{\overline{g}'}\right]$ is the identity of $\pi_1(STM, v_{x_0})$.

Recall $\phi_* : \pi_R(M, v_{x_0}) \to \pi_1(STM, v_{x_0})$ is the isomorphism defined by $[f]_R \mapsto [\phi(f)]$ where

$$\phi(f)(t) = \left(f(t), \frac{f'(t)}{||f'(t)||_{f(t)}}\right).$$

$\phi_*([f]_R) = \left[Y_{f'}\right]$ and $\phi_*([g]_R^{-1}) = \phi_*([\overline{g}]_R) = \left[Y_{\overline{g}'}\right]$ so $\phi_*([f]_R \cdot [g]_R^{-1}) = \left[Y_{f'}\right] \cdot \left[Y_{\overline{g}'}\right]$. Consequently, $[f]_R \cdot [g]_R^{-1}$ is the identity of $\pi_R(M, v_{x_0})$ and $[f]_R = [g]_R$. $\square$

Lemma 3.3.9. *Let $D \subset \mathbb{R}^2$ be an open disk and let $\check{X}$ be any nonzero, continuous vector field on D. For each $n \in \mathbb{Z}$, there exists a regular closed curve $a_n : S^1 \to D$ based at $v_{x_*} \in STD$ such that $w(a_n; \check{X}) = n$.*

Proof. On the disk, all nonzero, continuous vector fields are vector field homotopic to each other. Consequently, Lemma 3.3.4 implies we can specify $\check{X}$. Since $TD_x = \mathbb{R}^2$ for each $x \in D$, $STD_x = \{x\} \times S^1$. So $v_{x_*} = (x_*, \widehat{v_{x_*}}) \in \{x_*\} \times S^1$ and we can define $\check{X} : D \to TD$ such that $\check{X}(x) = (x, \widehat{v_{x_*}})$ for each $x \in D$.

We next orient $STD_{x_*} = \{x_*\} \times S^1$. Consider the isomorphism from S^1 to STD_{x_*} defined by $t \mapsto (x_*, t)$. Orienting S^1 in the counterclockwise direction, we orient STD_{x_*} by transporting the orientation of S^1 to STD_{x_*}.

Let $a_1 : S^1 \to D$ be the regular closed curve based at v_{x_*} that is depicted in Figure 3.1.

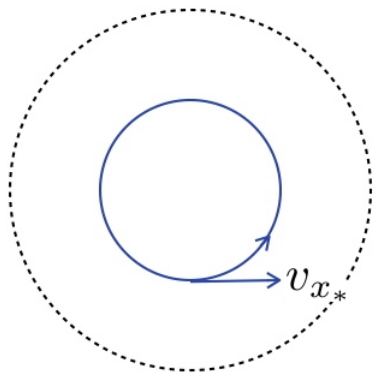

Figure 3.1: The regular closed curve a_1 based at v_{x_*} on D.

$Y_{a_1'} : S^1 \to STD$ is the vector field along a_1 where

$$t \mapsto \left(a_1(t), \frac{a_1'(t)}{||a_1'(t)||_{a_1(t)}} \right).$$

In Figure 3.2, $Y_{a_1'}(t)$ is pictured as blue vectors for four different values of t. $Y_{\check{X}_{a_1}} : S^1 \to STD$ is the vector field along a_1 such that $t \mapsto \big(a_1(t), \widehat{v_{x_*}}\big)$. Three of these vectors are pictured in Figure 3.2 in red. Since $Y_{a_1'}(1) = Y_{\check{X}_{a_1}}(1) = v_{x_*}$, we do not have a fourth red vector drawn at $a_1(1) = x_*$.

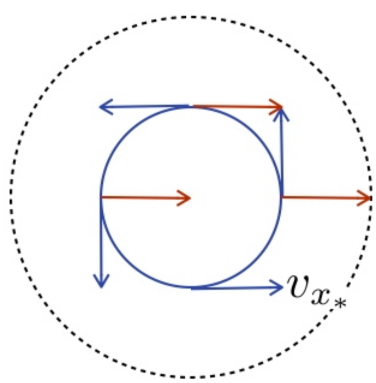

Figure 3.2: a_1 pictured with vectors $Y_{a_1'}(t)$ and $Y_{\check{X}_{a_1}}(t)$ for four values of t.

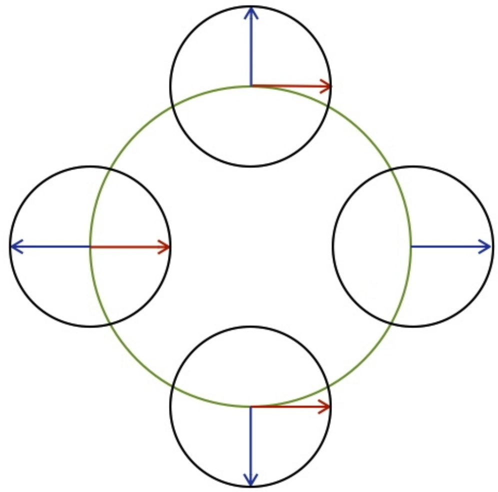

Figure 3.3: Modified version of $a_1^*(STD)$.

Since

$$a_1^*(STD) = \{(t, v) \in S^1 \times STD : a_1(t) = p(v)\},$$

$a_1^*(STD)_t \cong S^1$ for each $t \in S^1$. A modified version of $a_1^*(STD)$ is depicted in Figure 3.3. Here, S^1 is the large green circle and $a_1^*(STD)_t$ is drawn in black for four values of t. Within each $a_1^*(STD)_t$, $Y_{a_1'}^*(t)$ and $Y_{\check{X}_{a_1}}^*(t)$ are pictured.

Clearly, we can choose $\varphi : a_1^*(STD) \to S^1 \times S^1$ such that $\widetilde{Y_{\check{X}_{a_1}}^*} : S^1 \to S^1 \times S^1$ is defined by $t \mapsto (t, \widetilde{\alpha}(t)) = (t, 1)$ and $\widetilde{Y_{a_1'}^*} : S^1 \to S^1 \times S^1$ defined by $t \mapsto (t, \widetilde{\beta}(t))$ is homotopic to the function defined by $t \mapsto (t, t)$. So the integer associated to the homotopy class of the function

$$j(t) = \left(1, \left(\widetilde{\beta} \times \frac{1}{\widetilde{\alpha}}\right)(t)\right)$$

is 1. Therefore, $w(a_1; \check{X}) = 1$.

For each integer $n > 1$, let $a_n : S^1 \to D$ be a regular closed curve in the regular homotopy class given by taking $[a_1]_R$ under the group operation with itself n times. That is, $[a_1]_R \cdots [a_1]_R$ n times. $w(a_n; \check{X}) = w(a_1; \check{X}) + ... + w(a_1; \check{X}) = n$ by Theorem 3.3.6. For $n \leq -1$, let $a_n : S^1 \to D$ be a regular closed curve in the regular homotopy class $[a_{-n}]_R^{-1}$. Then, $w(a_n; \check{X}) = -w(a_{-n}; \check{X}) = n$. Lastly, for $n = 0$, let $a_0 = e$. Since $[e]_R$ is the identity of $\pi_R(D, v_{x_*})$ and $w : \pi_R(D, v_{x_*}) \to \mathbb{Z}$ is a homomorphism, $w(a_0; \check{X}) = w(e; \check{X}) = 0$. Thus, for each $n \in \mathbb{Z}$, there exists a regular closed curve $a_n : S^1 \to D$ such that $w(a_n; \check{X}) = n$. $\square$

Lemma 3.3.10. *For each $n \in \mathbb{Z}$, there exists a regular closed curve b_n on M based at v_{x_0} such that b_n is null-homotopic and $w(b_n; X) = n$.*

Proof. Let $\varepsilon : D \to M$ be a local diffeomorphism of $x_0 \in M$ such that $\varepsilon(x_*) = x_0$ and $d\varepsilon_{x_*}(\widehat{v_{x_*}}) = \widehat{v_{x_0}}$. Defining $\acute{\varepsilon} : STD \to STM$ by

$$v_x = (x, \widehat{v_x}) \mapsto \left(\varepsilon(x), \frac{d\varepsilon_x(\widehat{v_x})}{||d\varepsilon_x(\widehat{v_x})||_{\varepsilon(x)}} \right),$$

$\acute{\varepsilon}$ is a bundle isomorphism. To define winding numbers, an orientation of STM_{x_0} must be fixed. We orient STM_{x_0} by transporting the orientation of STD_{x_*} that was described in the previous lemma under the isomorphism $\acute{\varepsilon}\big|_{STD_{x_*}} : STD_{x_*} \to STM_{x_0}$.

Let $\check{X} : D \to TD$ be the continuous, nonzero vector field of D defined by

$$\check{X}(x) = \acute{\varepsilon}^{-1}\left(\frac{X\big(\varepsilon(x)\big)}{||X\big(\varepsilon(x)\big)||_{\varepsilon(x)}} \right).$$

From Lemma 3.3.9, $w(a_n; \check{X}) = n$. Let $b_n = \varepsilon \circ a_n$. Then,

$$\big(\acute{\varepsilon} \circ Y_{\check{X}_{a_n}}\big)(t) = \big(\acute{\varepsilon} \circ \check{X} \circ a_n\big)(t) = Y_{X_{b_n}}(t)$$

so $\acute{\varepsilon} \circ Y_{\check{X}_{a_n}} = Y_{X_{b_n}}$. By Theorem 2.1.6,

$$\big(\acute{\varepsilon} \circ Y_{a_n'}\big)(t) = \left(\varepsilon\big(a_n(t)\big), \frac{d\varepsilon_{a_n(t)}\big(a_n'(t)\big)}{||d\varepsilon_{a_n(t)}\big(a_n'(t)\big)||_{\varepsilon(a_n(t))}}\right) = Y_{b_n'}(t).$$

From Theorem 3.3.2, n is the unique element in $\pi_1(STD_{x_*}, v_{x_*})$ such that $i_{D*}(n) = [Y_{a_n'}] \cdot [Y_{\check{X}_{a_n}}]^{-1}$ where $i_D : STD_{x_*} \to STD$ is the inclusion map. Let $p_D : STD \to D$ be the projection of STD onto D. Because the diagram

$$\begin{tikzcd}
\pi_2(D, x_0) \arrow[r] & \pi_1(STD_{x_*}, v_{x_*}) \arrow[r, "i_{D*}"] \arrow[d, "(\acute{\varepsilon}|_{STD_{x_*}})_*"'] & \pi_1(STD, v_{x_*}) \arrow[r, "p_{D*}"] \arrow[d, "\acute{\varepsilon}_*"'] & \pi_1(D, x_*) \\
\pi_2(M, x_0) \arrow[r] & \pi_1(STM_{x_0}, v_{x_0}) \arrow[r, "i_*"] & \pi_1(STM, v_{x_0}) \arrow[r, "p_*"] & \pi_1(M, x_0)
\end{tikzcd}$$

is commutative and $\big(\acute{\varepsilon}_* \circ i_{D*}\big)(n) = [Y_{b_n'}] \cdot [Y_{X_{b_n}}]^{-1}$, $\Big(i_* \circ \big(\acute{\varepsilon}|_{STD_{x_*}}\big)_*\Big)(n) = [Y_{b_n'}] \cdot [Y_{X_{b_n}}]^{-1}$. Since STM_{x_0} was oriented by transporting the orientation of STD_{x_*} under the isomorphism $\acute{\varepsilon}\big|_{STD_{x_*}}$, $\big(\acute{\varepsilon}\big|_{STD_{x_*}}\big)_*(n) = n$. So $i_*(n) = [Y_{b_n'}] \cdot [Y_{X_{b_n}}]^{-1}$ and $w(b_n; X) = n$ by Theorem 3.3.2. Since $b_n(S^1) \subset \varepsilon(D)$ and $\varepsilon(D)$ is diffeomorphic to D, b_n is null-homotopic. $\square$

Lemma 3.3.11. *For each homotopy class $\alpha \in \pi_1^{or}(M, x_0)$, there exists a unique $[f]_R \in \pi_R^{or}(M, v_{x_0})$ such that $[f] = \alpha$ and $w([f]_R) = 0$.*

Proof. From the exact sequence given in the proof of Theorem 2.4.3, we know $p_* : \pi_1(STM, v_{x_0}) \to \pi_1(M, x_0)$ is surjective. Since $\pi_R(M, v_{x_0}) \cong \pi_1(STM, v_{x_0})$ by Corollary 2.3.4, it follows that there exists a regular closed curve $g \in \alpha$ that is based at v_{x_0} for each $\alpha \in \pi_1^{or}(M, x_0)$.

Let $n = w([g]_R)$ and let b_{-n} be the null-homotopic, regular closed curve described in Lemma 3.3.10. Since b_{-n} is null-homotopic, it is orientation preserving so Theorem 3.3.6 implies $w([g]_R \cdot [b_{-n}]_R) = 0$. Because $[g]_R \cdot [b_{-n}]_R = [(g \circ r) \cdot (b_{-n} \circ r)]_R$ and r is just a reparameterization of I, $[(g \circ r) \cdot (b_{-n} \circ r)] = [g \cdot b_{-n}] = [g] = \alpha$. Therefore, taking $f = (g \circ r) \cdot (b_{-n} \circ r)$, $[f]_R \in \pi_R^{or}(M, v_{x_0})$, $[f] = \alpha$, and $w([f]_R) = 0$. Uniqueness follows from Theorem 3.3.8. □

To prove the last main result, we utilize the following result of short exact sequences.

Theorem 3.3.12. (Conrad [2]). *Consider groups A, B, C with the short exact sequence*

$$1 \longrightarrow A \xrightarrow{k} B \xrightarrow{v} C \longrightarrow 1.$$

Let $s : B \to A$ be a homomorphism such that $s \circ k$ is the identity function on A. Then, $\lambda : B \to A \times C$ defined by $\lambda(b) = \big(s(b), v(b)\big)$ is an isomorphism.

Theorem 3.3.13. *$\pi_R^{or}(M, v_{x_0})$ is isomorphic to $\pi_1^{or}(M, x_0) \times \mathbb{Z}$ where the group structure of $\pi_R^{or}(M, v_{x_0})$ is the one described in Section 2.3.*

Proof. Define $k : \pi_1^{or}(M, x_0) \to \pi_R^{or}(M, v_{x_0})$ such that for each $\alpha \in \pi_1^{or}(M, x_0)$, $\alpha \mapsto [f]_R$ where $[f] = \alpha$ and $w([f]_R) = 0$. Then, k is injective and, as a result of Theorem 3.3.8, k is a homomorphism. From Lemma 3.3.10 and Theorem 3.3.6, $w : \pi_R^{or}(M, v_{x_0}) \to \mathbb{Z}$ is a surjective homomorphism. As a consequence of the uniqueness in Lemma 3.3.11, $\operatorname{im} k = \ker w$. Thus, we have the short exact sequence:

$$1 \longrightarrow \pi_1^{or}(M, x_0) \xrightarrow{k} \pi_R^{or}(M, v_{x_0}) \xrightarrow{w} \mathbb{Z} \longrightarrow 1$$

Next, we define a homomorphism $s : \pi_R^{or}(M, v_{x_0}) \to \pi_1^{or}(M, x_0)$. For each $[f]_R \in \pi_R^{or}(M, v_{x_0})$, let $s([f]_R) = [f]$. For $[f]_R, [g]_R \in \pi_R^{or}(M, v_{x_0})$, $s([f]_R \cdot [g]_R) = s([(f \circ r) \cdot (g \circ r)]_R) = [(f \circ r) \cdot (g \circ r)] = [f \cdot g] = [f] \cdot [g]$. Thus, s is a homomorphism. Next, we verify that $s \circ k$ is the identity on $\pi_1^{or}(M, x_0)$. Let $\alpha \in \pi_1^{or}(M, x_0)$. $k(\alpha) = [f]_R$ where $[f] = \alpha$ and $w([f]_R) = 0$. Then, $s(k(\alpha)) = s([f]_R) = [f] = \alpha$.

Let $\lambda : \pi_R^{or}(M, v_{x_0}) \to \pi_1(M, x_0) \times \mathbb{Z}$ be defined by

$$\lambda([f]_R) = \big(s([f]_R), w([f]_R)\big) = \big([f], w([f]_R)\big).$$

From Theorem 3.3.12, λ is an isomorphism. $\square$

CHAPTER 4

EXAMPLES

In this chapter, we look at several surfaces with specified continuous, nonzero vector fields. We can describe $\pi_R^{or}(M, v_{x_0})$ for each surface by listing the generators of the group as a result of $\pi_R^{or}(M, v_{x_0})$ being isomorphic to $\pi_1^{or}(M, x_0) \times \mathbb{Z}$.

4.1 Annulus

We examine the annulus A with respect to the continuous, nonzero vector field of Figure 4.1.

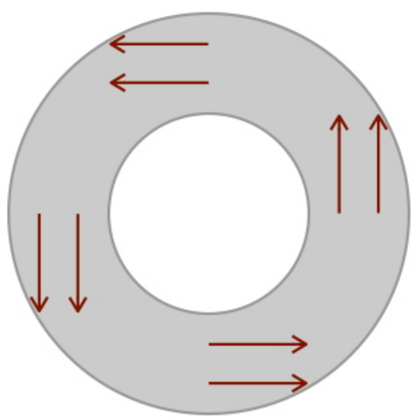

Figure 4.1: A continuous, nonzero vector field on the annulus.

Since the annulus is an orientable surface, $\pi_1^{or}(A, x_0) = \pi_1(A, x_0)$ and $\pi_R^{or}(A, v_{x_0}) = \pi_R(A, v_{x_0})$. $\pi_1(A, x_0) \cong \mathbb{Z}$ so $\pi_R(A, v_{x_0}) \cong \mathbb{Z} \times \mathbb{Z}$. The regular homotopy classes of the regular closed curves g_1 and g_2 of Figure 4.2 are the generators of $\pi_R(A, v_{x_0})$. This is because g_1 is null-homotopic and has winding number 1 while the homotopy class of g_2 generates $\pi_1(A, x_0)$ and g_2 has winding number 0. Thus, $[g_1]_R \mapsto (0, 1) \in \mathbb{Z} \times \mathbb{Z}$ and $[g_2]_R \mapsto (1, 0) \in \mathbb{Z} \times \mathbb{Z}$ under the isomorphism described in Theorem 3.3.13.

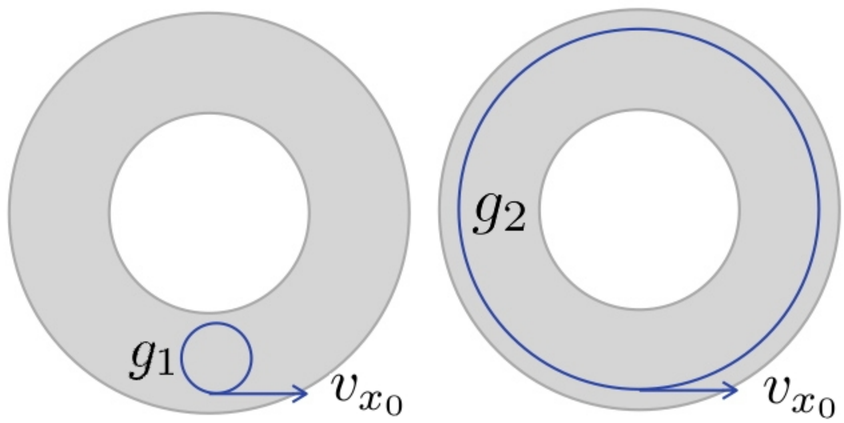

Figure 4.2: $[g_1]_R$ and $[g_2]_R$ generate $\pi_R(A, v_{x_0})$.

4.2 Möbius Band

Consider the Möbius band B with the continuous, nonzero vector field of Figure 4.3. $\pi_1(B, x_0) \cong \mathbb{Z}$ and is generated by an orientation reversing closed curve. Since the composition of two orientation reversing closed curves is orientation preserving (Lemma 3.1.6), it follows that the isomorphism from $\pi_1(B, x_0)$ to $\mathbb{Z}$ induces an isomorphism from $\pi_1^{or}(B, x_0)$ to $2\mathbb{Z}$ when it is restricted to $\pi_1^{or}(B, x_0)$. So $\pi_1^{or}(B, x_0) \cong \mathbb{Z}$ and $\pi_R^{or}(B, v_{x_0}) \cong \mathbb{Z} \times \mathbb{Z}$. The generators of $\pi_R^{or}(B, v_{x_0})$ are the regular homotopy classes of the regular closed curves pictured in Figure 4.4. $[g_1]_R \mapsto (0, 1) \in \mathbb{Z} \times \mathbb{Z}$ and $[g_2]_R \mapsto (1, 0) \in \mathbb{Z} \times \mathbb{Z}$.

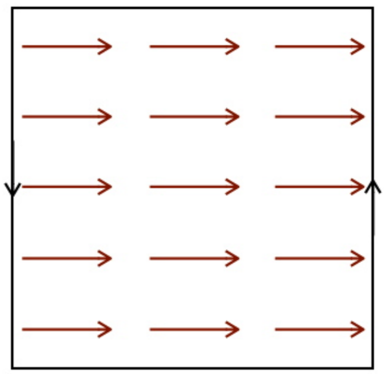

Figure 4.3: A continuous, nonzero vector field on the Möbius band.

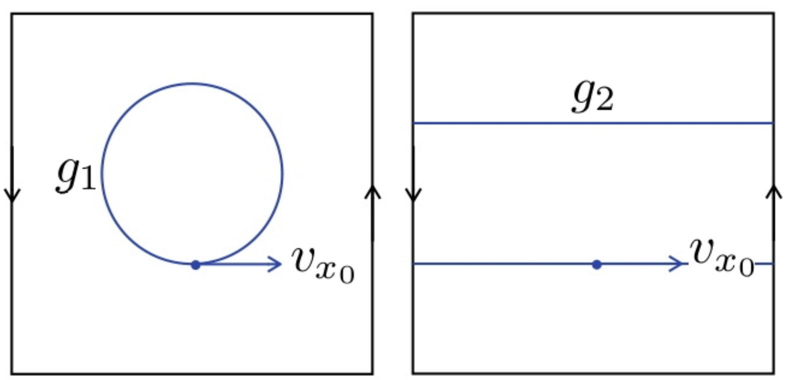

Figure 4.4: $[g_1]_R$ and $[g_2]_R$ generate $\pi_R^{or}(B, v_{x_0})$.

4.3 Torus

The torus T is viewed with respect to the continuous, nonzero vector field of Figure 4.5. $\pi_1(T, x_0) \cong \mathbb{Z} \times \mathbb{Z}$ so $\pi_R(T, v_{x_0}) \cong \mathbb{Z} \times \mathbb{Z} \times \mathbb{Z}$. $\pi_R(T, v_{x_0})$ is generated by the regular homotopy classes of g_1, g_2, and g_3 pictured in Figure 4.6. $[g_1]_R \mapsto (0, 0, 1)$ while $[g_2]_R \mapsto (1, 0, 0)$ and $[g_3]_R \mapsto (0, 1, 0)$ or vice versa depending on the isomorphism from $\pi_1(T, x_0)$ to $\mathbb{Z} \times \mathbb{Z}$ chosen.

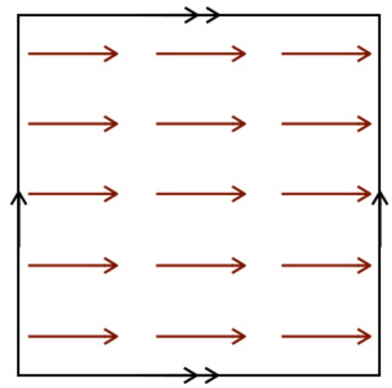

Figure 4.5: A continuous, nonzero vector field on the torus.

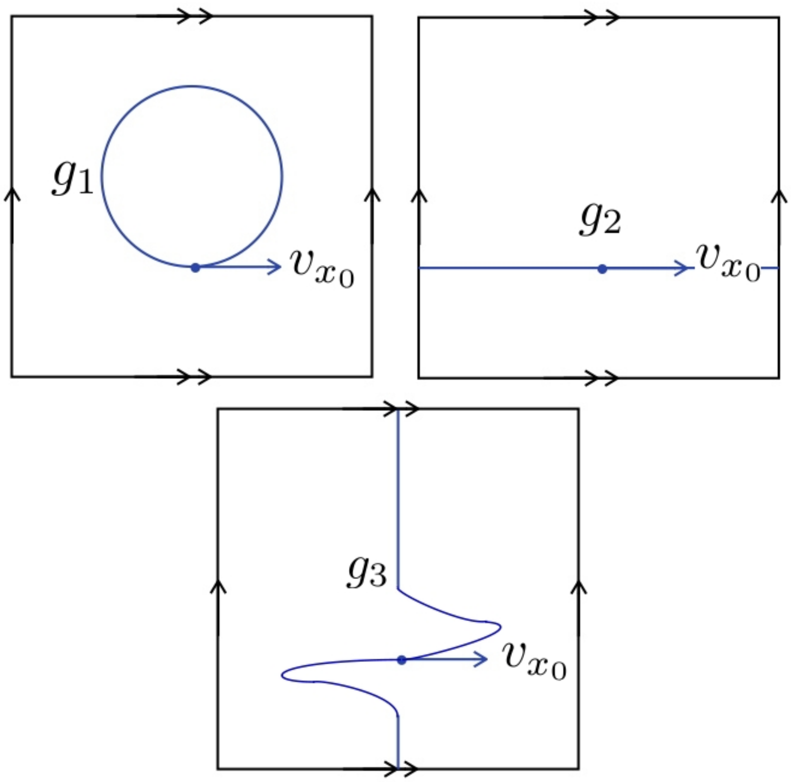

Figure 4.6: $[g_1]_R$, $[g_2]_R$, and $[g_3]_R$ generate $\pi_R(T, v_{x_0})$.

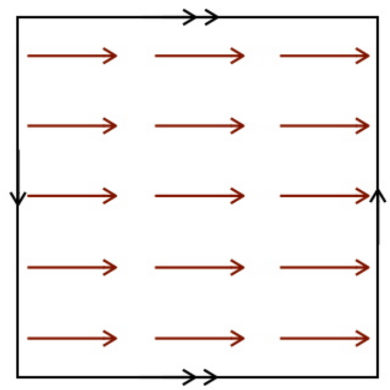

Figure 4.7: A continuous, nonzero vector field on the Klein bottle.

4.4 Klein Bottle

We view the Klein bottle K with respect to the vector field of Figure 4.7. $\pi_1(K, x_0) \cong \langle a, b | aba^{-1}b \rangle$ where a is identified with an orientation preserving closed curve while b is identified with an orientation reversing closed curve under the isomorphism. The relation $aba^{-1}b = 1$ can be used to uniquely write any word in this group in the form $a^m b^n$. Then, as a result of Lemma 3.1.6, any word with n odd is orientation reversing. Consequently, $\pi_1^{or}(K, x_0) \cong \langle a, b^2 | aba^{-1}b \rangle$.

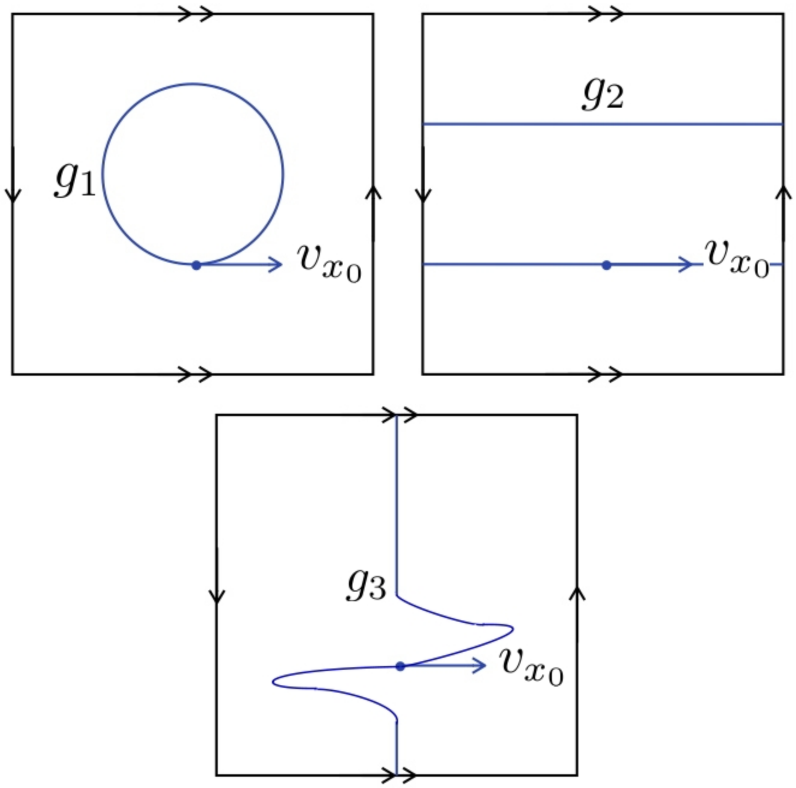

Figure 4.8: $[g_1]_R$, $[g_2]_R$, and $[g_3]_R$ generate $\pi_R^{or}(K, v_{x_0})$.

$[g_1]_R$, $[g_2]_R$, and $[g_3]_R$ generate $\pi_R^{or}(K, v_{x_0})$. g_1 is null-homotopic and has winding number 1. g_2 and g_3 have winding number 0 while $[g_2] \mapsto b^2 \in \langle a, b^2 | aba^{-1}b \rangle$ and $[g_3] \mapsto a \in \langle a, b^2 | aba^{-1}b \rangle$ under the isomorphism from $\pi_1^{or}(K, x_0)$ to $\langle a, b^2 | aba^{-1}b \rangle$.

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