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***Abstract*—This paper examines a shortcoming of the classical phasor diagram of a salient-pole synchronous machine based on the well-established two-reaction theory. Unlike in the phasor diagram of a smooth-air-gap machine, it is not possible to readily identify the internally-developed electromagnetic power of a salient-pole synchronous machine from this phasor diagram. By defining new machine reactances, a single equivalent circuit of a salient-pole synchronous machine is proposed together with a phasor diagram where the internally-developed electromagnetic power is made apparent. The revised two-reaction theory is illustrated using the mathematical model of a two-phase salient-pole synchronous machine whose equations are manipulated using complex space vectors instead of traditional matrix transformations.**

I. Introduction

At the 2000 North American Power Symposium, a paper by R. H. Park on the dq0 transformation ranked second in a survey of papers having had the most impact in power engineering over the last century [1]. The dq0 transformation, also known as the Park transformation, is still today the basis for the investigation of synchronous machine dynamics in power systems.

Historically, the development of a versatile mathematical model of a synchronous machine began with the two-reaction theory of Blondel, Doherty and Nickle, and others [2-3], and was later generalized by Park in his papers defining an ideal synchronous machine and outlining the theory of the dq0 transformation [4-5].

Subsequent refinements of Park's theory have been made in the development of equivalent damper windings in the direct and quadrature axes, the quantification of magnetic saturation, and the determination of machine parameters during subtransient, transient and steady-state analyses [6-9].

In this paper, we examine two shortcomings of the two-reaction theory. The first is a lack of a single equivalent circuit for a salient-pole synchronous machine in steady state. The second is the inability of the classical phasor diagram to determine the internally-developed electromagnetic power of the machine using relevant complex vectors.

We propose to remedy these shortcomings by analyzing a two-phase salient-pole synchronous machine for simplicity. Damper windings are not considered in this analysis as the aim is to produce a new steady-state circuit and its phasor diagram.

II. Theoretical Review

Let us consider a two-phase salient-pole synchronous machine with two stator windings (a,b) and one field winding (f). The mathematical model of this machine comprises three voltage equations and a second-order mechanical equation of motion. Using Kimbark's notation [7], the voltage equations are expressed in motor notation as the following set of three first-order differential equations,

$$v_a = R_s i_a + \frac{d\lambda_a}{dt} \tag{1}$$

$$v_b = R_s i_b + \frac{d\lambda_b}{dt} \tag{2}$$

$$v_f = R_f i_f + \frac{d\lambda_f}{dt} \tag{3}$$

together with the flux-current relationships

$$\begin{bmatrix} \lambda_a \\ \lambda_b \\ \lambda_f \end{bmatrix} = \begin{bmatrix} L_{aa} & L_{ab} & L_{af} \\ L_{ba} & L_{bb} & L_{bf} \\ L_{fa} & L_{fb} & L_{ff} \end{bmatrix} \begin{bmatrix} i_a \\ i_b \\ i_f \end{bmatrix} \tag{4}$$

where

$$L_{aa} = L_s + L_m \cos 2\theta \tag{5}$$

$$L_{bb} = L_s - L_m \cos 2\theta \tag{6}$$

$$L_{ab} = L_{ba} = L_m \sin 2\theta \tag{7}$$

$$L_{af} = L_{fa} = M_f \cos\theta \tag{8}$$

$$L_{bf} = L_{fb} = M_f \sin\theta \tag{9}$$

$$L_{ff} = L_f \tag{10}$$

and where $\theta(t) = \omega t + \theta_o$ is the electrical angle of rotation of the rotor shaft in steady state. By defining the following complex space vectors,

$$\overrightarrow{v_s} = v_a + jv_b \tag{11}$$

$$\overrightarrow{\lambda_s} = \lambda_a + j\lambda_b \tag{12}$$

$$\overrightarrow{i_s} = i_a + ji_b \tag{13}$$

the voltage equations can be concisely formulated as

$$\overrightarrow{v_s} = R_s \overrightarrow{i_s} + \frac{d\overrightarrow{\lambda_s}}{dt} \tag{14}$$

$$v_f = R_f i_f + \frac{d\lambda_f}{dt} \tag{15}$$

together with the flux-current vector relations

$$\overrightarrow{\lambda_s} = L_s\overrightarrow{i_s} + L_m e^{j2\theta}\overrightarrow{i_s^*} + M_f e^{j\theta} i_f \quad (16)$$

$$\lambda_f = \frac{1}{2}M_f e^{-j\theta}\overrightarrow{i_s} + \frac{1}{2}M_f e^{j\theta}\overrightarrow{i_s^*} + L_f i_f \quad (17)$$

Assuming a magnetically-linear coupling field, the magnetic coenergy of this machine is given by

$$W_m' = \frac{1}{2}\lambda_a i_a + \frac{1}{2}\lambda_b i_b + \frac{1}{2}\lambda_f i_f \quad (18)$$

It is straightforward to show that this coenergy can also be expressed as

$$W_m' = \Re e\left\{\frac{1}{2}\overrightarrow{\lambda_s}\overrightarrow{i_s^*}\right\} + \frac{1}{2}\lambda_f i_f \quad (19)$$

Expanding this last expression yields

$$\begin{aligned} W_m' &= \Re e\left\{\frac{1}{2}L_s|\overrightarrow{i_s}|^2 + \frac{1}{2}L_m e^{j2\theta}(\overrightarrow{i_s^*})^2 + \frac{1}{2}M_f e^{j\theta}\overrightarrow{i_s^*} i_f\right\} \\ &\quad + \frac{1}{4}M_f e^{-j\theta}\overrightarrow{i_s} i_f + \frac{1}{4}M_f e^{j\theta}\overrightarrow{i_s^*} i_f + \frac{1}{2}L_f i_f^2 \\ &= \Re e\left\{\frac{1}{2}L_m e^{j2\theta}(\overrightarrow{i_s}^*)^2 + M_f e^{j\theta}\overrightarrow{i_s^*} i_f\right\} \\ &\quad + \frac{1}{2}L_s|\overrightarrow{i_s}|^2 + \frac{1}{2}L_f i_f^2 \end{aligned} \quad (20)$$

By taking the partial derivative of the coenergy with respect to the mechanical rotor shaft angle $\theta_m = (2/p)\theta$, where p is the number of poles per phase, we obtain the following expression for the developed electromagnetic torque

$$\begin{aligned} T_e &= \frac{\partial W_m'}{\partial \theta_m} = \left(\frac{p}{2}\right)\frac{\partial W_m'}{\partial \theta} \\ &= \left(\frac{p}{2}\right)\Re e\left\{jL_m e^{j2\theta}(\overrightarrow{i_s^*})^2 + jM_f e^{j\theta}\overrightarrow{i_s^*} i_f\right\} \end{aligned} \quad (21)$$

III. Classical Two-Reaction Theory

If we define fictitious dq space vectors rotating with the rotor reference frame,

$$\overrightarrow{v_{dq}} = \overrightarrow{v_s} e^{-j\theta} \quad (22)$$

$$\overrightarrow{\lambda_{dq}} = \overrightarrow{\lambda_s} e^{-j\theta} \quad (23)$$

$$\overrightarrow{i_{dq}} = \overrightarrow{i_s} e^{-j\theta} \quad (24)$$

the stator voltage equation (14) can be transformed into the following vector equation

$$\overrightarrow{v_{dq}} = R_s\overrightarrow{i_{dq}} + \frac{d\overrightarrow{\lambda_{dq}}}{dt} + j\omega\overrightarrow{\lambda_{dq}} \quad (25)$$

After substituting all complex space vectors with their rectangular variables, that is, $\overrightarrow{v_{dq}} = v_d + jv_q$, $\overrightarrow{\lambda_{dq}} = \lambda_d + j\lambda_q$ and $\overrightarrow{i_{dq}} = i_d + ji_q$, the following two real equations are obtained,

$$v_d = R_s i_d + \frac{d\lambda_d}{dt} - \omega\lambda_q \quad (26)$$

$$v_q = R_s i_q + \frac{d\lambda_q}{dt} + \omega\lambda_d \quad (27)$$

Expressed in dq variables, the original flux-current relations,

$$\overrightarrow{\lambda_s} = L_s\overrightarrow{i_s} + L_m e^{j2\theta}\overrightarrow{i_s^*} + M_f e^{j\theta} i_f \quad (28)$$

$$\lambda_f = \frac{1}{2}M_f e^{-j\theta}\overrightarrow{i_s} + \frac{1}{2}M_f e^{j\theta}\overrightarrow{i_s^*} + L_f i_f \quad (29)$$

become

$$\overrightarrow{\lambda_{dq}} = L_s\overrightarrow{i_{dq}} + L_m\overrightarrow{i_{dq}^*} + M_f i_f \quad (30)$$

$$\lambda_f = \frac{1}{2}M_f\overrightarrow{i_{dq}} + \frac{1}{2}M_f\overrightarrow{i_{dq}^*} + L_f i_f \quad (31)$$

Expanding these equations in rectangular form,

$$\lambda_d = L_d i_d + M_f i_f \quad (32)$$

$$\lambda_q = L_q i_q \quad (33)$$

$$\lambda_f = M_f i_d + L_f i_f \quad (34)$$

where the d-axis and q-axis inductances, L_d and L_q, are respectively defined as

$$L_d = L_s + L_m \quad (35)$$

$$L_q = L_s - L_m \quad (36)$$

The developed electromagnetic torque can be expressed in dq variables as

$$\begin{aligned} T_e &= \left(\frac{p}{2}\right)\Re e\left\{jL_m(e^{j\theta}\overrightarrow{i_s^*})^2 + jM_f(e^{j\theta}\overrightarrow{i_s^*})i_f\right\} \\ &= \left(\frac{p}{2}\right)\Re e\left\{jL_m(\overrightarrow{i_{dq}^*})^2 + jM_f\overrightarrow{i_{dq}^*} i_f\right\} \end{aligned} \quad (37)$$

By replacing $\overrightarrow{i_{dq}}$ with its rectangular coordinates,

$$\overrightarrow{i_{dq}} = i_d + ji_q \quad (38)$$

the standard form of the developed electromagnetic torque of a salient-pole synchronous machine is obtained as

$$T_e = \left(\frac{p}{2}\right)\Re e\left\{jL_m(i_d - ji_q)^2 + jM_f i_f(i_d - ji_q)\right\}$$

$$= \left(\frac{p}{2}\right)(M_f i_q i_f + 2L_m i_d i_q) \quad (39)$$

$$= \left(\frac{p}{2}\right)(\lambda_d i_q - \lambda_q i_d) \quad (40)$$

IV. Classical Phasor Diagram

The steady-state machine voltage equations are obtained by setting the time derivatives of the d-axis and q-axis flux linkages to zero in Eqs. (26)-(27). Using capital letters to denote steady-state quantities, these equations become

$$V_d = R_s I_d - \omega\Lambda_q \quad (41)$$

$$V_q = R_s I_q + \omega\Lambda_d \quad (42)$$

These two equations can each be represented by a circuit in the d-axis or q-axis as shown in Figure 1. The two circuits are coupled via the steady-state speed voltages induced in opposite axes, $\omega\Lambda_d$ and $\omega\Lambda_q$.

The internally-developed electromagnetic power is obtained by adding the power contributions of the two speed voltages in the two subcircuits yielding

$$P_e = \omega(\Lambda_d I_q - \Lambda_q I_d) = \omega_m T_e \quad (43)$$

where $\omega_m = (2/p)\omega$ is the mechanical rotor shaft speed in steady state. This power expression can also be expressed as

$$\begin{aligned} P_e &= \omega\Lambda_d I_q - \omega\Lambda_q I_d \\ &= \omega(L_d I_d + M_f I_f)I_q - \omega L_q I_q I_d \\ &= (X_d - X_q)I_d I_q + E_{fd} I_q \end{aligned} \quad (44)$$

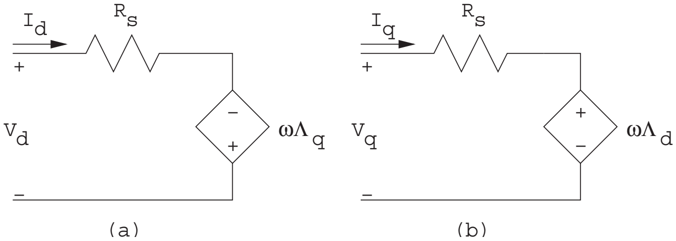

Fig. 1. d-axis and q-axis Circuits of a Synchronous Machine

where $E_{fd} = \omega M_f I_f$, $X_d = \omega L_d$ and $X_q = \omega L_q$. This last expression makes apparent the power developed due to each of the reluctance term and the synchronous power of the second term found in smooth-air-gap machines.

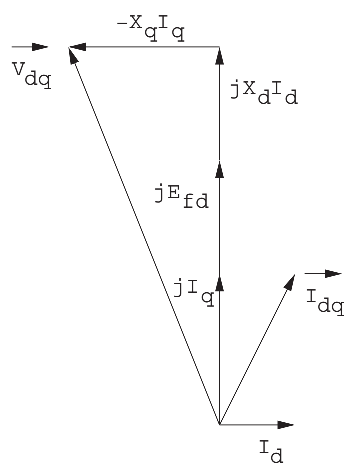

Fig. 2. Classical Phasor Diagram of a Salient-Pole Synchronous Machine

The classical phasor diagram is obtained by combining the steady-state equations in both axes as

$$\overrightarrow{V_{dq}} = R_s\overrightarrow{I_{dq}} + jX_d I_d + jX_q(jI_q) + jE_{fd} \qquad (45)$$

Figure 2 is a pictorial representation of Eq. (45) with the stator resistance neglected. The phasor diagram in this picture resembles Kimbark's phasor diagram [7, p. 70] and differs slightly from the classical one using space vectors instead of complex quantities such as (jE_{fd}) instead of $\overrightarrow{E_{fd}}$.

The input real power into the machine is equal to

$$P_{in,2\phi} = 2 \times \frac{1}{2}\Re e\{\overrightarrow{V_{dq}}\overrightarrow{I_{dq}^*}\} = V_d I_d + V_q I_q \quad (46)$$

This input power is equal to the internally-developed electromagnetic P_e since stator resistance and magnetic core losses are being neglected in the equivalent circuit representation. The synchronous power term $(E_{fd}I_q)$ is recognizable from the complex product of (jE_{fd}) and $(jI_q)^*$. However, the reluctance power term is not readily identifiable in this classical phasor diagram. In the next section, we propose a new single-circuit representation of a salient-pole synchronous machine along with a modified phasor diagram where the two terms of the internally-developed electromagnetic power are readily identifiable.

V. Revised Phasor Diagram

By defining the following reactances,

$$X^+ = \frac{X_d + X_q}{2} \qquad (47)$$

$$X^- = \frac{X_d - X_q}{2} \qquad (48)$$

the steady-state voltage equations (41)-(42) can be manipulated to yield the following phasor equation which is represented by the single equivalent circuit shown in Figure 3:

$$\overrightarrow{V_{dq}} = (R_s + jX^+)\overrightarrow{I_{dq}} + jX^-\overrightarrow{I_{dq}^*} + jE_{fd} \qquad (49)$$

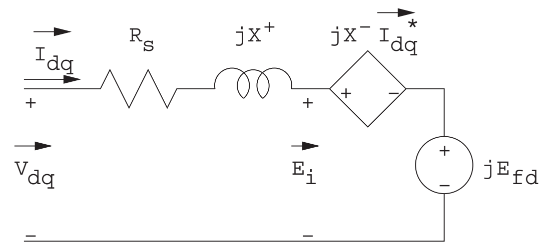

Fig. 3. Proposed Single Equivalent Circuit of a Salient-Pole Synchronous Machine

If we define an internal voltage $\overrightarrow{E_i}$ as the sum of the dependent voltage source $(jX^-\overrightarrow{I_{dq}^*})$ and the independent voltage source (jE_{fd}),

$$\begin{aligned}\overrightarrow{E_i} &= jE_{fd} + jX^-\overrightarrow{I_{dq}^*} \\ &= jE_{fd} + jX^-(I_d - jI_q) \\ &= jE_{fd} + jX^-I_d + X^-I_q \qquad (50)\end{aligned}$$

then we can easily verify that the internally-developed electromagnetic power P_e is equal to the real power absorbed by this internal voltage by expanding the following equation

$$\begin{aligned}P_e &= \Re e\{\overrightarrow{E}_i\overrightarrow{I_{dq}^*}\} \\ &= \Re e\{\left(jE_{fd} + jX^-I_d + X^-I_q\right)(I_d - jI_q)\} \\ &= 2X^-I_dI_q + E_{fd}I_q \qquad (51) \\ &= (X_d - X_q)I_dI_q + E_{fd}I_q \qquad (52)\end{aligned}$$

A phasor diagram for the single equivalent circuit can be drawn as shown in Figure 4 where the stator resistance has been omitted for simplicity. The internally-developed power P_e can now be identified as the real part of the complex product of this internal voltage voltage $\overrightarrow{E_i}$ and the conjugate of the current vector $\overrightarrow{I_{dq}} = (I_d + jI_q)$. By decomposing the current vector into its real and imaginary components, $\overrightarrow{I_{dq}} = I_d + jI_q$, the internally-developed power P_e becomes readily available as the sum of three terms:

- a power term $(E_{fd}I_q)$ resulting from the complex product of (jE_{fd}) and $(jI_q)^*$;
- a power term $(X^-I_dI_q)$ resulting from the complex product of (jX^-I_d) and $(jI_q)^*$; and
- a power term $(X^-I_dI_q)$ resulting from the complex product of (X^-I_q) and $(I_d)^*$.

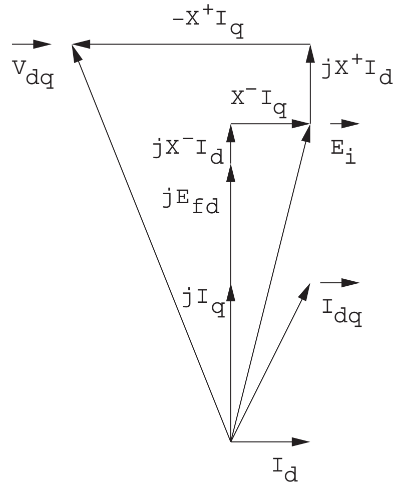

Fig. 4. Proposed Phasor Diagram of a Salient-Pole Synchronous Machine

VI. Torque Production in a Salient-Pole Synchronous Machine

In steady state, Equations (30)-(31) describing the flux-current relationships can be expressed as

$$\overrightarrow{\Lambda_{dq}} = L_s\overrightarrow{I_{dq}} + L_m\overrightarrow{I_{dq}^*} + M_f\overrightarrow{I_f} \quad (53)$$

$$\overrightarrow{\Lambda_f} = \frac{1}{2}M_f\overrightarrow{I_{dq}} + \frac{1}{2}M_f\overrightarrow{I_{dq}^*} + L_f\overrightarrow{I_f} \quad (54)$$

where $\overrightarrow{I_f} = I_f e^{j0^o}$ and $\overrightarrow{\Lambda_f} = \Lambda_f e^{j0^o}$.

In this form, it is apparent that there are three complex current vectors in the air gap of a salient-pole synchronous machine, namely, $\overrightarrow{I_{dq}}$, $\overrightarrow{I_{dq}^*}$ and $\overrightarrow{I_f}$. These three current vectors are depicted in Figure 5 and are relative to a dq reference frame attached to the rotor. The field current vector $\overrightarrow{I_f}$ is oriented along the field axis which is assumed to be colinear with the direct axis. The stator current vectors $\overrightarrow{I_{dq}}$ and $\overrightarrow{I_{dq}^*}$ are complex conjugate vectors with respect to the direct axis.

In steady state, a previous expression of the electromagnetic torque, Equation (37), can be successively rewritten as

$$T_e = \left(\frac{p}{2}\right)\Re e\left\{jM_f\overrightarrow{I_f}\overrightarrow{I_{dq}^*} + jL_m(\overrightarrow{I_{dq}^*})^2\right\} \quad (55)$$

$$= \left(\frac{p}{2}\right)\left(M_f\overrightarrow{I_f}\times\overrightarrow{I_{dq}} + L_m\overrightarrow{I_{dq}^*}\times\overrightarrow{I_{dq}}\right) \quad (56)$$

where the cross product expressions have been intuitively defined as

$$\overrightarrow{I_f}\times\overrightarrow{I_{dq}} = \Re e\left\{j\overrightarrow{I_f}\overrightarrow{I_{dq}^*}\right\} = \Im m\left\{\overrightarrow{I_f^*}\overrightarrow{I_{dq}}\right\} \quad (57)$$

$$\overrightarrow{I_{dq}^*}\times\overrightarrow{I_{dq}} = \Re e\left\{j(\overrightarrow{I_{dq}^*})^2\right\} = \Im m\left\{\overrightarrow{I_{dq}}\overrightarrow{I_{dq}}\right\} \quad (58)$$

Noting that

$$\overrightarrow{I_f}\times\overrightarrow{I_{dq}} = \frac{1}{2}\overrightarrow{I_f}\times\overrightarrow{I_{dq}} + \frac{1}{2}\overrightarrow{I_{dq}^*}\times\overrightarrow{I_f} \quad (59)$$

the last torque expression, Equation (56), can be expanded as

$$T_e = \left(\frac{p}{2}\right)\left(\frac{1}{2}M_f\overrightarrow{I_f}\times\overrightarrow{I_{dq}} + \frac{1}{2}M_f\overrightarrow{I_{dq}^*}\times\overrightarrow{I_f} + L_m\overrightarrow{I_{dq}^*}\times\overrightarrow{I_{dq}}\right) \quad (60)$$

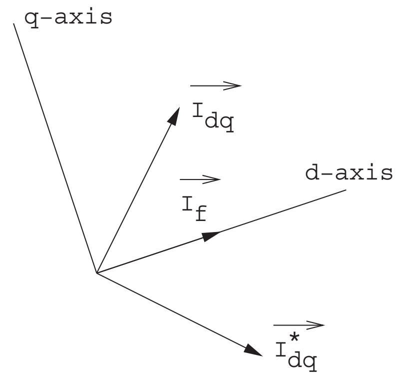

Fig. 5. Torque Production in a Salient-Pole Synchronous Machine Interpreted as the Pairwise Interaction of Three Complex Current Vectors

Torque production in a salient-pole synchronous machine can now be interpreted as the pairwise magnetic interaction of three complex current vectors in the air gap of the machine. The interaction of the conjugate current vectors $\overrightarrow{I_{dq}}$ and $\overrightarrow{I_{dq}^*}$ is responsible for the reluctance component of the torque which is absent in round-rotor synchronous machines.

VII. Conclusion

In this paper, we have extended the two-reaction theory of a two-phase salient-pole synchronous machine by proposing a single equivalent circuit in dq coordinates. A new phasor diagram for this equivalent circuit identifies an internal voltage responsible for the internally-developed electromagnetic power in a straightforward manner. The torque production in a salient-pole synchronous machine has also been explained using the interaction of three complex current vectors in the air gap of the machine.

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