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A BOUND FOR THE WARING RANK OF THE DETERMINANT VIA SYZYGIES

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ABSTRACT. We show that the Waring rank of the 3×3 determinant, previously known to be between 14 and 18, is at least 15. We use syzygies of the apolar ideal, which have not been used in this way before. Additionally, we show that the symmetric cactus rank of the 3×3 permanent is at least 14.

1. INTRODUCTION

Let $F \in \mathbb{k}[x_1, \dots, x_n]$ be a homogeneous polynomial of degree d . The **Waring rank** of F , denoted $\text{rk}(F)$, is the least number of terms r in an expression for F as a combination of d th powers of linear forms, $F = c_1 \ell_1^d + \dots + c_r \ell_r^d$, for some $c_i \in \mathbb{k}$ and linear forms ℓ_i , or ∞ if no such expression exists. For example, $xy = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2$ (as long as \mathbb{k} has characteristic not equal to 2), so $\text{rk}(xy) \leq 2$, and in fact $\text{rk}(xy) = 2$: if $\text{rk}(xy) < 2$, then $xy = c\ell^2$ for some ℓ , contradicting uniqueness of factorization. Similarly,

$$xyz = \frac{1}{24} \left\{ (x+y+z)^3 - (x+y-z)^3 - (x-y+z)^3 - (-x+y+z)^3 \right\},$$

showing $\text{rk}(xyz) \leq 4$ (as long as \mathbb{k} has characteristic not equal to 2 or 3); and one can show that $\text{rk}(xyz) = 4$. As these examples suggest, upper bounds for Waring rank often come from explicit expressions, while lower bounds must come from other considerations. The Waring rank is finite for all forms of degree d strictly less than the characteristic of \mathbb{k} , or all d if \mathbb{k} has characteristic zero.

Recent introductions to Waring rank include [17], [37], [49], and the comprehensive [34]. Interest in Waring rank arises from the intimate connection with the geometry of secant varieties (see, for example, [4]) and from connections with geometric complexity theory [38]. See also very recently observed connections with parametrized algorithms [43].

Waring rank is notoriously difficult to compute. In fact, computing Waring rank is NP-hard [31, Conjecture 13.2], [47, Theorem 6]. Polynomials whose Waring rank is known include quadratic forms ($d = 2$) and binary forms ($n = 2$), general polynomials [1, 42], monomials [16, 14], certain highly symmetric products of linear forms [50], reducible cubic forms [18], and elementary symmetric polynomials of odd degree [40]. Algorithms to compute Waring rank have been studied (see, for example, [8]) but are not practical in most cases.

For $d \geq 1$ let $\det_d \in \mathbb{k}[x_{1,1}, \dots, x_{d,d}]$ be the generic determinant, $\det_d = \det(x_{i,j})_{1 \leq i,j \leq d}$. It is well known that $\text{rk}(\det_2) = 4$, but no other values are known (apart from the trivial case $d = 1$). It is known that $14 \leq \text{rk}(\det_3) \leq 18$; in fact there have been several proofs of $\text{rk}(\det_3) \geq 14$, which we review below.

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Our main result is that $\text{rk}(\det_3) \geq 15$, which we show using syzygies of the apolar ideal. We also show that the symmetric cactus rank of the 3×3 permanent is greater than or equal to 14 (it was previously known to be greater than or equal to 10). Along the way, we also give a new proof of the known results that the Waring rank and symmetric cactus rank of \det_3 , and the Waring rank of the 3×3 permanent, are greater than or equal to 14.

2. BACKGROUND

In this section we review background material, including the important technical tool of apolarity; cactus rank and border rank; conciseness and indecomposability; and a review of previous results on Waring rank, cactus rank, and border rank of the determinant and permanent.

2.1. Apolarity. The primary tool in most studies of Waring rank, including this one, is apolarity. We review the needed material here. For a more thorough treatment see, for example, [34].

Let $S = \mathbb{k}[x_1, \dots, x_n]$ be a polynomial ring. Denote by $V = S_1 = \text{span}\{x_1, \dots, x_n\}$ the vector space of linear forms in S ; we identify S with the symmetric algebra $S(V)$. We denote by y_1, \dots, y_n the dual basis of the dual vector space V^* , and set $T = S(V^*)$, which we identify with $\mathbb{k}[y_1, \dots, y_n]$. We let elements of T act on S , with each y_i acting as the operator $\frac{\partial}{\partial x_i}$. We denote this action by \circ , as in $y_i \circ x_i^4 = 4x_i^3$; explicitly,

$$y_1^{a_1} \cdots y_n^{a_n} \circ x_1^{d_1} \cdots x_n^{d_n} = \begin{cases} \prod_{i=1}^n \frac{d_i!}{(d_i - a_i)!} x_i^{d_i - a_i}, & \text{if } a_i \leq d_i \text{ for all } i, \\ 0 & \text{otherwise,} \end{cases}$$

extended bilinearly over sums. This makes S into a T -algebra, and in fact a graded T -algebra, although the grading as a T -algebra is reversed from the standard grading as a polynomial ring: the action takes $T_a \times S_d \rightarrow S_{d-a}$, where $S_k = 0$ if $k < 0$.

When we discuss matrix polynomials such as the determinant, which depend on the entries of a matrix, we may use doubly-indexed variables such as $x_{1,1}, \dots, x_{d,d}$ and corresponding dual variables $y_{1,1}, \dots, y_{d,d}$.

Let $F \in S$ be a homogeneous form of degree d . We denote by $\text{Derivs}(F)$ the vector space spanned by F and all of its derivatives of all orders, equivalently the T -submodule of S generated by F . The **apolar ideal** of F , or **annihilating ideal**, denoted F^\perp , is the ideal $\{\theta \in T : \theta \circ F = 0\}$. It is a homogeneous ideal. It is the kernel of the quotient map $T \rightarrow \text{Derivs}(F) \cong T/F^\perp$, $\theta \mapsto \theta \circ F$.

An ideal I is called **apolar to F** if $I \subseteq F^\perp$. A scheme Z is called **apolar to F** if its defining ideal is apolar to F . When Z is a projective scheme, by the defining ideal we mean the saturated ideal of Z .

The relevance of apolarity for Waring rank is largely (perhaps entirely) due to the following well-known statement.

Lemma 1 (Apolarity Lemma). *Let $S = \mathbb{k}[x_1, \dots, x_n]$ and $V = S_1 = \text{span}\{x_1, \dots, x_n\}$. Let $F \in S$ be a homogeneous polynomial of degree d . Assume \mathbb{k} has characteristic 0 or $> d$. Let $Z \subset \mathbb{P}V$ be a closed subscheme. Let $\nu_d : \mathbb{P}V \rightarrow \mathbb{P}S_d$ be the d th Veronese map, and let $\nu_d(Z)$ be the image of Z under this map. Then $[F] \in \mathbb{P}S_d$ lies in the linear span of $\nu_d(Z)$ if and only if Z is apolar to F .*

Here the linear span of a closed subscheme of projective space is the smallest reduced linear subspace containing the scheme, equivalently the linear subspace defined by the linear forms

in the saturated ideal of the scheme. For a proof of the Apolarity Lemma see, for example, [5, §4] or [49, §4.1].

For Waring rank, the most relevant case is when Z consists of a finite, reduced set of points. Suppose $Z = \{[\ell_1], \dots, [\ell_r]\} \subset \mathbb{P}S_1$; here each ℓ_i is a linear form. Then $\nu_d(Z) = \{[\ell_1^d], \dots, [\ell_r^d]\}$, and to say that $[F]$ lies in the linear span of $\nu_d(Z)$ is precisely to say that $F = c_1 \ell_1^d + \dots + c_r \ell_r^d$ for some scalars c_i . Thus the Waring rank of F is equal to the minimum degree of a zero-dimensional reduced projective scheme apolar to F , equivalently, the minimum degree of a one-dimensional radical homogeneous ideal I contained in F^\perp .

This (reduced, zero-dimensional) version of the Apolarity Lemma may be found in many references, for example [34, Lemma 1.15, Theorem 5.3], [44, §1.3], [23, Propositions 1.3.9-10]. It goes back considerably further, to the 1851 work of Sylvester [48], in the case of binary forms ($n = 2$).

2.2. Cactus rank and border rank. When homogeneous polynomials are identified with symmetric tensors, Waring rank of a homogeneous polynomial is identified with the **symmetric rank** of a symmetric tensor.

The **symmetric cactus rank** of a homogeneous form $F \in S_d$, denoted $\text{crk}(F)$, is the minimum degree of a zero-dimensional, not necessarily reduced, closed subscheme $Z \subset \mathbb{P}(V)$ apolar to F (so that $[F]$ lies in the span of $\nu_d(Z)$). Equivalently, it is the minimum degree of a one-dimensional saturated, but not necessarily reduced, homogeneous ideal $I \subset F^\perp$. Evidently $\text{crk}(F) \leq \text{rk}(F)$.

The **symmetric border rank** of a homogeneous form $F \in S_d$, denoted $\text{brk}(F)$, is the minimum r such that F lies in the Zariski closure of the locus of forms of rank less than or equal to r . This locus, or its projectivization, is the r th secant variety of the Veronese variety: the border rank of F is less than or equal to r if and only if $[F]$ lies in the secant variety of \mathbb{P}^{r-1} s spanned by points on the Veronese variety. Evidently $\text{brk}(F) \leq \text{rk}(F)$.

The **symmetric border cactus rank** of a homogeneous form $F \in S_d$, denoted $\text{bcrk}(F)$, is the minimum r such that F lies in the Zariski closure of the locus of forms of cactus rank less than or equal to r . Evidently $\text{bcrk}(F) \leq \text{brk}(F)$ and $\text{bcrk}(F) \leq \text{crk}(F)$.

Remark 2. One may similarly define the rank, cactus rank, border rank, and border cactus rank of general (non-symmetric) tensors. In that context it is important to distinguish the rank from the symmetric rank, cactus rank from the symmetric cactus rank, etc.

Here, however, we deal exclusively with homogeneous polynomials and with the symmetric versions of these ranks. Therefore we use the shorter terms *rank* (or Waring rank), *border rank*, *cactus rank*, and *border cactus rank*, but we always mean the symmetric versions of these quantities.

An analogue of the Apolarity Lemma for border rank has been introduced very recently, see [12]. For several more variant notions of rank, see [3]. For examples with $\text{crk}(F_1) > \text{brk}(F_1)$ and $\text{crk}(F_2) < \text{brk}(F_2)$, see [11].

2.3. Conciseness and indecomposability. For a homogeneous form $F \in S_d = \text{Sym}^d(V)$, the following are equivalent:

- (1) F cannot be written as a polynomial in fewer variables, even after a linear change of coordinates. More precisely, if $F \in \text{Sym}^d(W)$ for some $W \subseteq V$, then $W = V$.
- (2) The projective hypersurface defined by $F = 0$ is not a cone.
- (3) $\text{Derivs}(F)_1 = S_1 = V$.

(4) F^\perp does not contain any linear forms.

A form meeting these conditions is called **concise** with respect to V [15].

In general, as long as \mathbb{k} has characteristic 0 or $> d$, an arbitrary degree d form F lies in $\text{Sym}^d(\text{Derivs}(F)_1)$, and is always concise with respect to $\text{Derivs}(F)_1$. The **essential variables** of F are the elements of the subspace $\text{Derivs}(F)_1$, or just the elements of a basis for it. For example, the essential variables of $xy + xz$ are x and $y + z$.

A **direct sum decomposition** of a homogeneous form $F \in S_d$ is an expression

$$F = G(\ell_1, \dots, \ell_k) + H(\ell_{k+1}, \dots, \ell_n)$$

where G, H are nonzero forms of degree d and

$$S_1 = \text{span}\{\ell_1, \dots, \ell_k\} \oplus \text{span}\{\ell_{k+1}, \dots, \ell_n\},$$

equivalently $\ell_1, \dots, \ell_n \in S_1$ are linearly independent. A form is **indecomposable** (as a direct sum) if it admits no direct sum decomposition. For example, $\det_2 = ad - bc$ is decomposable, with the decomposition $G = ad, H = -bc$; xy cannot be decomposed as $xy = G(x) + H(y)$, but $xy = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2$, so xy is decomposable with $\ell_1 = x+y, \ell_2 = x-y$. See [13] for more on direct sum decompositions and indecomposable forms.

2.4. Previous results on the determinant and permanent. By Laplace expansion, \det_3 and per_3 are each the sum of 6 monomials of the form xyz . Expanding each monomial as a sum of powers, then, $\text{rk}(\det_3)$ and $\text{rk}(\text{per}_3)$ are each at most $6 \text{rk}(xyz) = 24$. For larger d , we have

$$(1) \quad \text{rk}(\det_d), \text{rk}(\text{per}_d) \leq d! \cdot \text{rk}(x_1 \cdots x_d) = d!2^{d-1}.$$

Better upper bounds follow from identities found by Derksen [21], Krishna–Makam [36], Glynn [28], and Conner–Gesmundo–Landsberg–Ventura [20]. The identity of Krishna–Makam is

$$\begin{aligned} \det_3 &= x_{1,1}(x_{2,2} + x_{2,3})(x_{3,1} + x_{3,3}) \\ &\quad + (x_{1,2} + x_{1,3})x_{2,1}x_{3,2} \\ &\quad - (x_{1,1} + x_{1,3})x_{2,2}x_{3,1} \\ &\quad - x_{1,2}(x_{2,1} + x_{2,3})(x_{3,2} + x_{3,3}) \\ &\quad + (x_{1,2} - x_{1,1})x_{2,3}(x_{3,1} + x_{3,2} + x_{3,3}). \end{aligned}$$

(Derksen had earlier found a similar identity.) The identity implies $\text{rk}(\det_3) \leq 5 \text{rk}(xyz) = 20$. As observed by Derksen, Laplace expansion shows inductively that

$$(2) \quad \text{rk}(\det_d) \leq \left(\frac{5}{6}\right)^{\lfloor d/3 \rfloor} d!2^{d-1}.$$

Glynn’s identity is

$$\text{per}_d = \frac{1}{2^{d-1}} \sum_{\substack{\epsilon \in \{\pm 1\}^d \\ \epsilon_1 = 1}} \prod_{i=1}^d \sum_{j=1}^d \epsilon_i \epsilon_j x_{i,j}.$$

For example,

$$\begin{aligned} 4 \operatorname{per}_3 &= (x_{1,1} + x_{1,2} + x_{1,3})(x_{2,1} + x_{2,2} + x_{2,3})(x_{3,1} + x_{3,2} + x_{3,3}) \\ &\quad - (x_{1,1} + x_{1,2} - x_{1,3})(x_{2,1} + x_{2,2} - x_{2,3})(x_{3,1} + x_{3,2} - x_{3,3}) \\ &\quad - (x_{1,1} - x_{1,2} + x_{1,3})(x_{2,1} - x_{2,2} + x_{2,3})(x_{3,1} - x_{3,2} + x_{3,3}) \\ &\quad + (x_{1,1} - x_{1,2} - x_{1,3})(x_{2,1} - x_{2,2} - x_{2,3})(x_{3,1} - x_{3,2} - x_{3,3}). \end{aligned}$$

This expresses the $d \times d$ permanent as a sum of 2^{d-1} terms, each of rank 2^{d-1} . Therefore

$$(3) \quad \operatorname{rk}(\operatorname{per}_d) \leq 2^{2^{d-2}}.$$

In particular, $\operatorname{rk}(\operatorname{per}_3) \leq 16$.

In [35] it was shown that \det_3 cannot be written as a sum of products of linear forms with fewer than 5 summands, nor can per_3 be written as a sum of products of linear forms with fewer than 4 summands. So there is no possibility of getting better upper bounds for Waring rank of \det_3 or per_3 by improving the above identities. Nevertheless, Conner–Gesmundo–Landsberg–Ventura see [20] have very recently found an explicit expression with 18 terms to show $\operatorname{rk}(\det_3) \leq 18$. For brevity, we write a linear form on a space of matrices by simply writing a matrix of its coefficients, so

$$\begin{pmatrix} a_{1,1} & \cdots \\ \vdots & \end{pmatrix} \quad \text{represents the linear form} \quad a_{1,1}x_{1,1} + \cdots.$$

Let $\theta = \exp(2\pi i/6)$. Then:

$$\begin{aligned} 18 \det_3 &= \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}^3 + \begin{pmatrix} -\theta & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \theta^{-1} \end{pmatrix}^3 + \begin{pmatrix} -\theta^{-1} & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & \theta \end{pmatrix}^3 \\ &\quad + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}^3 + \begin{pmatrix} -1 & 0 & 0 \\ 0 & 0 & -\theta^{-1} \\ 0 & -\theta & 0 \end{pmatrix}^3 + \begin{pmatrix} \theta^{-1} & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -\theta & 0 \end{pmatrix}^3 \\ &\quad + \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^3 + \begin{pmatrix} 0 & \theta & 0 \\ -\theta^{-1} & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^3 + \begin{pmatrix} 0 & \theta^{-1} & 0 \\ -\theta & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}^3 \\ &\quad + \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{pmatrix}^3 + \begin{pmatrix} 0 & -\theta & 0 \\ 0 & 0 & \theta^{-1} \\ -1 & 0 & 0 \end{pmatrix}^3 + \begin{pmatrix} 0 & -\theta^{-1} & 0 \\ 0 & 0 & \theta \\ -1 & 0 & 0 \end{pmatrix}^3 \\ &\quad + \begin{pmatrix} 0 & 0 & 1 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}^3 + \begin{pmatrix} 0 & 0 & 1 \\ \theta^{-1} & 0 & 0 \\ 0 & \theta & 0 \end{pmatrix}^3 + \begin{pmatrix} 0 & 0 & 1 \\ \theta & 0 & 0 \\ 0 & \theta^{-1} & 0 \end{pmatrix}^3 \\ &\quad + \begin{pmatrix} 0 & 0 & -1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}^3 + \begin{pmatrix} 0 & 0 & \theta^{-1} \\ 0 & -\theta & 0 \\ 1 & 0 & 0 \end{pmatrix}^3 + \begin{pmatrix} 0 & 0 & \theta \\ 0 & -\theta^{-1} & 0 \\ 1 & 0 & 0 \end{pmatrix}^3, \end{aligned}$$

see [20, Theorem 2.11].

Conner–Gesmundo–Landsberg–Ventura also showed that $\text{brk}(\det_3) \leq 17$, see [20, Theorem 2.12]. At this time we do not know of upper bounds for the border rank of the permanent beyond that simply the border rank is less than or equal to the Waring rank.

The rank and border rank of a general cubic form in 9 variables are 19 [1]. The cactus rank of a general cubic form in 9 variables is at most 18 [6].

As for lower bounds, the catalecticant, or flattening, lower bound (see [39]) shows

$$\text{rk}(\det_3) \geq \text{brk}(\det_3) \geq 9$$

and

$$\text{rk}(\text{per}_3) \geq \text{brk}(\text{per}_3) \geq 9$$

and for all d , for $F_d \in \{\det_d, \text{per}_d\}$,

$$(4) \quad \text{rk}(F_d) \geq \text{brk}(F_d) \geq \binom{d}{\lfloor d/2 \rfloor}^2 \approx \frac{4^d}{\pi d/2}.$$

The catalecticant lower bound is also a lower bound for the cactus rank and border cactus rank: $\text{crk}(\det_3) \geq \text{bcrk}(\det_3) \geq 9$, and so on, see [34, Theorem 5.3D].

In [39] this was improved to $\text{rk}(\det_3) \geq 14$, $\text{rk}(\text{per}_3) \geq 12$ using a lower bound for $\text{rk}(F)$ in terms of the singularities of the hypersurface defined by F . This also gave improved lower bounds for $\text{rk}(\det_d)$ and $\text{rk}(\text{per}_d)$, but the improvement was just quadratic in d . Compare the exponential lower bound in (4), the exponential upper bound in (3), and the factorial upper bound in (2).

Shafiei [46] used a bound of Ranestad-Schreyer [45] to show

$$\text{rk}(\det_3) \geq \text{crk}(\det_3) \geq 10, \quad \text{rk}(\text{per}_3) \geq \text{crk}(\text{per}_3) \geq 10,$$

and for all d , for $F_d \in \{\det_d, \text{per}_d\}$,

$$\text{rk}(F_d) \geq \text{crk}(F_d) \geq \frac{1}{2} \binom{2d}{d} \approx \frac{4^d}{2\sqrt{\pi d}}.$$

Although this gives a worse bound for $d = 3$, it is asymptotically better than the bound arising from singularities.

Using a bound for the rank of forms invariant under a group action, [22] showed $\text{rk}(\det_3) \geq \text{crk}(\det_3) \geq 14$, and

$$\text{rk}(\det_d) \geq \text{crk}(\det_d) \geq \binom{2d}{d} - \binom{2d-2}{d-1} \approx \frac{3}{4} \binom{2d}{d} \approx \frac{3 \cdot 4^d}{4\sqrt{\pi d}}.$$

(The bound for invariant forms does not seem to give any interesting results for the permanent, because its stabilizer is too small.)

Farnsworth [26] used Koszul-Young flattenings to show

$$\text{rk}(\det_3) \geq \text{brk}(\det_3) \geq 14, \quad \text{rk}(\text{per}_3) \geq \text{brk}(\text{per}_3) \geq 14,$$

and $\text{brk}(\det_4) \geq 38$, along with modestly improved lower bounds for $\text{brk}(\det_d)$ and $\text{brk}(\text{per}_d)$ when $d \geq 5$.

Gałaszka [27] showed that certain lower bounds for border rank are in fact also lower bounds for cactus rank and border cactus rank. Specifically, Gałaszka shows that lower bounds for border rank arising from Young flattenings are in fact lower bounds for cactus rank and border cactus rank, if the Young flattenings arise from vector bundles. For the $d \geq 5$ cases, Farnsworth uses Young flattenings that indeed arise from vector bundles. It is not immediately clear whether the Young flattenings used by Farnsworth for the $d = 3, 4$ cases

also arise from vector bundles. If so, then combining Farnsworth’s and Gałaszka’s results gives $\text{crk}(\text{per}_3) \geq 14$.

In summary, three (at least) proofs of $\text{rk}(\det_3) \geq 14$ have been given [39, 22, 26]. The chronologically first proof was strictly for Waring rank; the second proof gave a bound for cactus rank, in addition to Waring rank; and the third proof gave a bound for border rank, in addition to Waring rank. We are aware of only one proof of $\text{rk}(\text{per}_3) \geq 14$, in [26]. That proof gave a bound for border rank, in addition to Waring rank. It seems that up to now, the best bound for the cactus rank of the 3×3 permanent is $\text{crk}(\text{per}_3) \geq 10$ [46], unless Farnsworth’s and Gałaszka’s results may be combined to yield $\text{crk}(\text{per}_3) \geq 14$.

3. BOUND FOR WARING RANK VIA SYZYGIES

In order to illustrate the method we will use for determinant and permanent, we illustrate briefly with a proof of $\text{rk}(xyz) \geq 4$ (which is well-known by other means). The explicit expression given in the introduction shows $\text{rk}(xyz) \leq 4$. Suppose that $xyz = \ell_1^3 + \dots + \ell_r^3$. Note that xyz is concise, since each variable can be obtained as a derivative of xyz : $x = \frac{\partial^2}{\partial y \partial z}(xyz)$, and so on. This means that the ℓ_i must span $V = \text{span}\{x, y, z\}$, so already we have $r \geq 3$. If $r < 4$, then the ℓ_i must give a basis for V . Up to linear change of coordinates, $I = I(\{[\ell_1], [\ell_2], [\ell_3]\})$ is given by $I = (tu, tv, uv)$, where t, u, v are suitable coordinates. The minimal graded free resolution of T/I has graded Betti numbers

$$\begin{array}{rcccc} & & 0 & 1 & 2 \\ \text{total:} & 1 & 3 & 2 & \\ 0: & 1 & . & . & \\ 1: & . & 3 & 2 & \end{array}$$

Now we compare the apolar ideal. Here it is more convenient to replace xyz with $x_1x_2x_3$, with dual variables y_1, y_2, y_3 . We have $(x_1x_2x_3)^\perp = (y_1^2, y_2^2, y_3^2)$, whose resolution is Koszul, with graded Betti numbers

$$\begin{array}{rcccc} & & 0 & 1 & 2 & 3 \\ \text{total:} & 1 & 3 & 3 & 1 & \\ 0: & 1 & . & . & . & \\ 1: & . & 3 & . & . & \\ 2: & . & . & 3 & . & \\ 3: & . & . & . & 1 & \end{array}$$

In particular, $\beta_{2,3}(T/I) = 2$, while $\beta_{2,3}(T/(xyz)^\perp) = 0$.

In general, an inclusion $I \subseteq J$ does not imply $\beta_{i,j}(T/I) \leq \beta_{i,j}(T/J)$. However, we do get an inequality in the lowest-degree strand of the resolution of the larger ideal, that is, along the first non-trivial row of the Betti table. For our purposes the following special case is sufficient:

Proposition 3. *Suppose that $I \subseteq J$ are homogeneous ideals in the polynomial ring T with standard grading, and that J contains no linear forms. Then $\beta_{i,i+1}(T/I) \leq \beta_{i,i+1}(T/J)$ for all i .*

See for example [24, Proposition 8.11] for a more general statement.

Now the inclusion $I \subset (xyz)^\perp$, and the fact that $(xyz)^\perp$ contains no linear form, means that $\beta_{2,3}(T/I) \leq \beta_{2,3}(T/(xyz)^\perp)$. This contradiction shows $\text{rk}(xyz) > 3$.

The graded Betti numbers of T/per_3^\perp are as follows:

	0	1	2	3	4	5	6	7	8	9
total:	1	36	160	315	404	404	315	160	36	1
0:	1
1:	.	36	160	315	288	116
2:	116	288	315	160	36	.
3:	1

These are easy to compute in Macaulay2 [29]. For our purposes, the important point is that $\beta_{5,6}(T/\text{det}_3^\perp) = 100$ and $\beta_{5,6}(T/\text{per}_3^\perp) = 116$.

Remark 5. The syzygies of det_d^\perp and per_d^\perp have been studied by Alper and Rowlands [2], who give combinatorial interpretations of some syzygies.

3.2. Ideals of points.

Proposition 6. *Let $I \subset S = \mathbb{k}[y_1, \dots, y_9]$ be a one-dimensional saturated homogeneous ideal of degree 13 containing no linear form. Then $\beta_{5,6}(T/I) \geq 140$.*

Proof. Peeva [41] showed that the Betti numbers of T/I are obtained by consecutive cancellations from those of T/L where L is the lex-segment ideal with the same h -vector as I . So $\beta_{5,6}(T/I) \geq \beta_{5,6}(T/L) - \beta_{4,6}(T/L)$. (A priori subtracting $\beta_{6,6}(T/L)$ would also be possible, but the condition that I contains no linear form means that neither does L , so $\beta_{6,6}(T/L) = 0$.)

The h -vector of T/I must start with $(1, 8, \dots)$ since T/I contains no linear form. The entries of the h -vector sum to 13, the degree of T/I . After the 8, the entries of the h -vector are nonincreasing, by the Macaulay bounds, see [9, Theorem 4.2.10]. Explicitly, if the h -vector's d th entry, counted from 0, is less than or equal to d , then the Macaulay bounds imply that from that point on the h -vector is nonincreasing. So if the h -vector of T/I starts with $(1, 8, 2, \dots)$ or $(1, 8, 1, \dots)$ then the remainder is nonincreasing. If it starts with $(1, 8, 3, \dots)$ then because the entries add up to 13 the vector must be $(1, 8, 3, 1)$. Hence the h -vector of T/I must be one of $(1, 8, 4)$, $(1, 8, 3, 1)$, $(1, 8, 2, 2)$, $(1, 8, 2, 1, 1)$ or $(1, 8, 1, 1, 1, 1)$.

The Betti numbers for the lex-segment ideals with these h -vectors are given as follows. For h -vector $(1, 8, 4)$:

	0	1	2	3	4	5	6	7	8
	1	37	175	399	539	455	237	70	9
0:	1
1:	.	32	141	300	379	300	147	41	5
2:	.	5	34	99	160	155	90	29	4

For h -vector $(1, 8, 3, 1)$:

	0	1	2	3	4	5	6	7	8
	1	37	175	399	539	455	237	70	9
0:	1
1:	.	33	148	321	414	335	168	48	6
2:	.	3	20	57	90	85	48	15	2
3:	.	1	7	21	35	35	21	7	1

For h -vector $(1, 8, 2, 2)$:

	0	1	2	3	4	5	6	7	8
	1	36	168	378	504	420	216	63	8
0:	1
1:	.	34	154	336	434	350	174	49	6
2:
3:	.	2	14	42	70	70	42	14	2

For h -vector $(1, 8, 2, 1, 1)$:

	0	1	2	3	4	5	6	7	8
	1	36	168	378	504	420	216	63	8
0:	1
1:	.	34	154	336	434	350	174	49	6
2:	.	1	7	21	35	35	21	7	1
3:
4:	.	1	7	21	35	35	21	7	1

For h -vector $(1, 8, 1, 1, 1, 1)$:

	0	1	2	3	4	5	6	7	8
	1	36	168	378	504	420	216	63	8
0:	1
1:	.	35	161	357	469	385	195	56	7
2:
3:
4:
5:	.	1	7	21	35	35	21	7	1

These can be produced in Macaulay2 with code such as the following.

```
lexSeg = (R,d,i) -> ideal ((basis(d,R))_{0..(numcols basis(d,R)-i-1)})
lexIdeal = L -> (
  R := QQ[q_1..q_(L_1)]; trim sum for i to #L-1 list lexSeg(R,i,L_i));
```

```
betti res lexIdeal(1,8,4,0)
betti res lexIdeal(1,8,3,1,0)
```

and so on. (To be precise, this code produces the Artinian reductions of the desired one-dimensional ideals. But they have the same graded Betti numbers.)

This shows that in all cases, after taking into account the possible cancellations, the Betti number $\beta_{5,6}(T/I) \geq 140$. □

In the following theorem, the only new result is that $\text{crk}(\text{per}_3) \geq 14$, improving the previously known $\text{crk}(\text{per}_3) \geq 10$ [46].

Theorem 7. $\text{rk}(\det_3) \geq \text{crk}(\det_3) \geq 14$ and $\text{rk}(\text{per}_3) \geq \text{crk}(\text{per}_3) \geq 14$.

Proof. Let I be any one-dimensional saturated homogeneous ideal which is apolar to \det_3 or per_3 . Since \det_3 and per_3 are concise, \det_3^\perp and per_3^\perp contain no linear forms, hence neither does I . If I has degree 13 then $\beta_{5,6}(T/I) \geq 140$ by Proposition 6. However, by Proposition 3, $\beta_{5,6}(T/I) \leq \beta_{5,6}(T/\det_3^\perp) = 100$ or $\beta_{5,6}(T/I) \leq \beta_{5,6}(T/\text{per}_3^\perp) = 116$, a contradiction. □

Remark 8. The Betti table of the minimal graded free resolution of the coordinate ring of a general set of 13 points in \mathbb{P}^8 is as follows:

	0	1	2	3	4	5	6	7	8
	1	36	168	378	504	420	216	63	8
0:	1
1:	.	32	136	266	280	140	2	.	.
2:	10	49	24	4

See [25] or [7, §5]. This implies, by semicontinuity of graded Betti numbers, that the graded Betti number $\beta_{5,6}$ is at least 140 for an arbitrary set of 13 points in \mathbb{P}^8 , not lying on a hyperplane. So, why not just give this argument; why the longer argument given above?

In fact, this shorter argument implies the inequality for any closed, nondegenerate zero-dimensional subscheme of \mathbb{P}^8 which can be deformed into a set of 13 distinct points—a *smoothable* scheme. This would be sufficient if we only cared about ideals of reduced schemes, corresponding to Waring rank, or smoothable schemes, corresponding to so-called smoothable rank.

But there exist non-smoothable schemes, see for example [33, 32]. (These give non-smoothable schemes of degree less than 13, in spaces of lower dimension than \mathbb{P}^8 . But the degree can be increased by adding disjoint points, and the embedding dimension can be increased by re-embedding.) So this shorter argument is not sufficient to deal directly with arbitrary schemes.

On the other hand, it is known that every scheme of minimal degree that spans a given form is Gorenstein (if R is a zero-dimensional scheme such that $[F]$ lies in the linear span of R and R is not Gorenstein, then there is a subscheme $R' \subset R$ of strictly lower degree and whose linear span still includes $[F]$, see [10, Lemma 2.3]). Thus we can restrict attention to Gorenstein schemes. And it is known that every Gorenstein scheme of degree at most 13 in \mathbb{P}^8 is smoothable [19] (under some mild assumptions on the field). Combining these results gives an alternative proof of Theorem 7.

Remark 9. A similar computation shows that a one-dimensional ideal I of degree 14 in \mathbb{P}^8 containing no linear form has $\beta_{5,6}(T/I) \geq 70$, and 70 does occur (in particular for the ideal of a general set of 14 reduced points). This lower bound is far too weak to use in our argument to prove $\text{rk}(T/\det_3) > 14$. Instead we will use the bounds for ideals of degree 13.

3.3. Conciseness after subtracting.

Lemma 10. *Let F be any concise, indecomposable form and let ℓ be any linear form. Then $F - \ell^d$ is concise.*

Proof. If $F - \ell^d$ is not concise, then either ℓ is an essential variable of $F - \ell^d$, in which case $F = \ell^d + (F - \ell^d)$ is not concise, or else ℓ is independent of the essential variables of $F - \ell^d$, in which case $F = \ell^d + (F - \ell^d)$ is a direct sum decomposition of F . \square

It is shown in [13] that \det_d and per_d are indecomposable for $d \geq 3$. And it is easy to check that \det_d and per_d are concise.

Corollary 11. *For any linear form ℓ , $\det_d - \ell^d$ and $\text{per}_d - \ell^d$ are concise.*

3.4. Apolarity after subtracting. Now we consider the apolarity of $\det_3 - \ell^3$ for a linear form ℓ . Our goal is to show $\beta_{5,6}(T/(\det_3 - \ell^3)^\perp) < 140$ for every linear form ℓ .

Proposition 12. *Fix $d > 0$. Assume that \mathbb{k} is closed under taking d th roots. Let X be the generic $d \times d$ matrix $X = (x_{i,j})$, $\det_d = \det X$, $S = \mathbb{k}[X] = \mathbb{k}[x_{1,1}, \dots, x_{d,d}]$. Let $\ell \in S$ be an arbitrary nonzero linear form. There is a linear change of coordinates leaving \det_d invariant and taking ℓ^d to $\lambda(x_{1,1} + \dots + x_{k,k})^d$ for some $1 \leq k \leq d$ and some scalar $\lambda \neq 0$. In addition, if $1 \leq k \leq d - 1$, then we can take $\lambda = 1$.*

Proof. Let $s_1, s_2 \in \mathrm{SL}_d$. Our linear change of coordinates in S will be the substitution of X with the matrix product $s_1 X s_2$. We have $\det_d = \det(X) = \det(s_1 X s_2)$, which is to say that for any s_1 and s_2 , this coordinate change leaves \det_d invariant.

Write $\ell = \sum_{i,j} a_{i,j} x_{i,j}$. Let $A = (a_{i,j})$ be the $d \times d$ matrix of coefficients of ℓ . Then $\ell = \mathrm{tr}(AX^t)$. The coordinate change takes ℓ to

$$\mathrm{tr}(A(s_1 X s_2)^t) = \mathrm{tr}(A s_2^t X^t s_1^t) = \mathrm{tr}(s_1^t A s_2^t X^t) = \mathrm{tr}(A' X^t),$$

where $A' = s_1^t A s_2^t$. We can choose row operations and column operations to make A diagonal, and swap rows and columns so that the first $k = \mathrm{rk}(A)$ diagonal entries of A are nonzero and the rest are zero. Multiply the i th row of A by $a_{i,i}^{-1}$ and the d th row of A by $a_{i,i}$, for i from 1 to k or $d - 1$, whichever is less. If $k < d$, then we have reached A' with diagonal entries consisting of k ones followed by $d - k$ zeros, as desired.

If $k = d$, then at this point our matrix has diagonal entries consisting of $d - 1$ entries of 1 followed by $\det(A)$, which is nonzero. Multiply each of the first $d - 1$ rows by a d th root of $\det(A)$ and the last row by the reciprocal to ensure that all the diagonal entries of A' are equal, i.e., A' is a scalar multiple of the identity, namely $A' = cI$, $c = \det(A)^{1/d}$. Then $\ell^d = (cx_{1,1} + \dots + cx_{d,d})^d = \lambda(x_{1,1} + \dots + x_{d,d})^d$, $\lambda = c^d = \det(A)$. \square

This means that when we consider $\det_3 - \ell^d$, we can reduce to the three following cases:

- (1) $\det_3 - x_{1,1}^3$,
- (2) $\det_3 - (x_{1,1} + x_{2,2})^3$,
- (3) $\det_3 - \lambda(x_{1,1} + x_{2,2} + x_{3,3})^3$ for a nonzero scalar λ .

Proposition 13. $\beta_{5,6}(T/(\det_3 - x_{1,1}^3)^\perp) = \beta_{5,6}(T/(\det_3 - (x_{1,1} + x_{2,2})^3)^\perp) = 100$.

Proof. A direct computation in Macaulay2. \square

We will next show that $\beta_{5,6}(T/(\det_3 - \lambda(x_{1,1} + x_{2,2} + x_{3,3})^3)^\perp) < 140$ for all $\lambda \in \mathbb{k}$.

It is convenient to simplify notation by replacing $x_{1,1}, \dots, x_{3,3}$ with x_1, \dots, x_9 , with dual variables y_1, \dots, y_9 , where

$$X = \begin{pmatrix} x_{1,1} & x_{1,2} & x_{1,3} \\ x_{2,1} & x_{2,2} & x_{2,3} \\ x_{3,1} & x_{3,2} & x_{3,3} \end{pmatrix} = \begin{pmatrix} x_1 & x_2 & x_3 \\ x_4 & x_5 & x_6 \\ x_7 & x_8 & x_9 \end{pmatrix}, \quad Y = \begin{pmatrix} y_{1,1} & y_{1,2} & y_{1,3} \\ y_{2,1} & y_{2,2} & y_{2,3} \\ y_{3,1} & y_{3,2} & y_{3,3} \end{pmatrix} = \begin{pmatrix} y_1 & y_2 & y_3 \\ y_4 & y_5 & y_6 \\ y_7 & y_8 & y_9 \end{pmatrix}.$$

It is also convenient for computations to introduce μ and homogenize as in the following statement.

Proposition 14. *Let $\mu, \lambda \in \mathbb{k}$, $\mu \neq 0$, and let $F = F(\mu, \lambda) = \mu \det_3 - \lambda(x_1 + x_5 + x_9)^3$. Then F^\perp is generated by the following 36 linearly independent quadrics.*

$$\begin{aligned}
 F^\perp = & (y_2^2, y_3^2, y_4^2, y_6^2, y_7^2, y_8^2, y_1^2 - y_9^2, y_5^2 - y_9^2, \\
 & y_1y_2, y_1y_3, y_1y_4, y_1y_7, y_2y_3, y_2y_5, y_2y_8, y_3y_6, y_3y_9, \\
 & y_4y_5, y_4y_6, y_4y_7, y_5y_6, y_5y_8, y_6y_9, y_7y_8, y_7y_9, y_8y_9, \\
 & y_1y_6 + y_3y_4, y_1y_8 + y_2y_7, y_2y_9 + y_3y_8, \\
 & y_2y_6 + y_3y_5, y_4y_8 + y_5y_7, y_4y_9 + y_6y_7, \\
 & y_1y_5 + y_2y_4 - y_9^2, y_1y_9 + y_3y_7 - y_9^2, y_5y_9 + y_6y_8 - y_9^2, \\
 & \mu y_1^2 - 6\lambda(y_6y_8 + y_3y_7 + y_2y_4)).
 \end{aligned}$$

The above generators were found with the assistance of Macaulay2.

Proof. It is easy to check that all of the listed generators annihilate F . Let H be the ideal generated by the 36 listed generators on the right hand side, so $H \subseteq F^\perp$.

By Corollary 11, F is concise, so F^\perp contains no linear form. Therefore the Hilbert function of F^\perp begins with $1, 9, \dots$. By Gorenstein symmetry, the full Hilbert function is $1, 9, 9, 1$. Therefore $(F^\perp)_2$ has codimension 9 in the space of quadrics in $T = \mathbb{k}[y_1, \dots, y_9]$. That space has dimension $\binom{8+2}{2} = 45$. So F^\perp contains 36 linearly independent quadrics.

The 36 quadrics generating H are linearly independent because they have distinct leading monomials. So H agrees with F^\perp in degree 2 (as well as degrees less than 2).

Since F has degree 3, F^\perp contains all forms of degree 4, so F^\perp certainly does not have any generator of degree greater than 4.

And F^\perp does not have a generator of degree equal to 4. Indeed, [13, Proposition 1.6] asserts that if G is a form of degree d such that G^\perp has a minimal generator of degree $d + 1$, then G must be a power of a linear form. But F is concise in 9 variables by Corollary 11, so F is not a power of a linear form (such powers are concise with respect to 1 variable).

Finally we eliminate the possibility that F^\perp has any generators in degree 3. One computes directly that the first 35 generators of H generate a codimension 2 space of cubics, while $(F^\perp)_3$ has codimension 1. The element $\mu y_1^3 - 6\lambda y_1 y_6 y_8$ is in H (using the 36th generator), but not in the subideal generated by the first 35 quadrics, because each of those first 35 quadrics, considered as a polynomial on matrices, vanishes on the identity matrix, while $\mu y_1^3 - 6\lambda y_1 y_6 y_8$ does not. This shows that H_3 has codimension 1, so H coincides with F^\perp in degree 3. Therefore the cubics in F^\perp are generated by the quadrics, and thus the generators of F^\perp are the quadrics. \square

Proposition 15. *With notation as above, $\beta_{5,6}(T/F^\perp) < 140$.*

Proof. We may assume the field \mathbb{k} is algebraically closed, since the graded Betti numbers do not change under field extension. Let $\tilde{T} = T[\mu, \lambda] = \mathbb{k}[y_1, \dots, y_9, \mu, \lambda]$, where y_1, \dots, y_9 have degree 1 and μ, λ have degree 0. Let $\tilde{H} \subset \tilde{T}$ be the ideal generated by the quadrics listed above, where μ, λ are read as variables instead of scalars. (This is not the apolar ideal of “ \tilde{F} ”; that apolar ideal would include $\mu^2, \mu\lambda, \lambda^2$.) One computes in Macaulay2 a graded free

resolution \tilde{C} of \tilde{T}/\tilde{H} with graded Betti numbers as follows.

	0	1	2	3	4	5	6	7	8	9
total:	1	36	160	316	424	479	399	196	45	1
0:	1
1:	.	36	160	315	289	135	56	28	8	1
2:	.	.	.	1	135	344	343	168	37	.
3:	1

In particular, $\beta_{5,6}(\tilde{T}/\tilde{H}) = 135 < 140$. Substituting any values of $\mu, \lambda \in \mathbb{k}$, $\mu \neq 0$, takes \tilde{H} to $H = F^\perp$ and takes \tilde{C} to a complex C that resolves T/H .

We claim that this complex C is exact. For all values $\mu, \lambda \in \mathbb{k}$, $\mu \neq 0$, $F = F(\mu, \lambda)$ is a concise cubic form in 9 variables. So the Hilbert function of T/F^\perp starts with $(1, 9, \dots)$. By Gorenstein symmetry it is $(1, 9, 9, 1)$. This is constant with respect to μ, λ , so the family of algebras $T/F(\mu, \lambda)^\perp$ has the same Hilbert polynomial (the constant 20) at every closed point of $\text{Spec } \mathbb{k}[\mu^{\pm 1}, \lambda]$. We can choose a constant monomial cobasis for the family of ideals $F(\mu, \lambda)^\perp$ (i.e., a set of monomials whose images give a basis for the algebras $T/F(\mu, \lambda)^\perp$), so that the family $T/F(\mu, \lambda)^\perp$ is locally free over $\mathbb{k}[\mu^{\pm 1}, \lambda]$. Therefore the Hilbert polynomial of the family $T/F(\mu, \lambda)^\perp$ is constant (at all points, not just closed points), so the family is flat over $\mathbb{k}[\mu^{\pm 1}, \lambda]$ [30, Theorem III.9.9]. By induction on homological degree, the kernels of maps in \tilde{C} are flat. Substituting μ_0, λ_0 for μ, λ is tensoring with $\mathbb{k}[\mu^{\pm 1}, \lambda]/(\mu - \mu_0, \lambda - \lambda_0)$; by flatness, exactness is preserved.

Now the minimal graded free resolution of T/F^\perp is a quotient of C . So $\beta_{5,6}(T/F^\perp) \leq \beta_{5,6}(C) = 135$. □

Remark 16. C is not necessarily minimal. It seems possible that in fact C is never minimal, and $\beta_{5,6}(T/F^\perp) = 100$ for all μ, λ with $\mu \neq 0$.

Theorem 17. $\text{rk}(\det_3) \geq 15$.

Proof. If \mathbb{k} has characteristic 2 or 3 then \det_3 cannot be expressed as a sum of cubes, so $\text{rk}(\det_3) = \infty$. Indeed, in any cube ℓ^3 , the coefficient of any term of the form $x_i x_j x_k$ with i, j, k distinct is divisible by 6.

So we can assume \mathbb{k} has characteristic 0 or strictly greater than 3. Then all the above discussion about apolarity applies (since it assumed that the field had characteristic 0 or strictly greater than the degree of the polynomial under consideration). If \mathbb{k} is not closed under taking cube roots, then we pass to an extension of \mathbb{k} . This can only decrease the Waring rank, so it is sufficient to get a lower bound over the extension field. We may thus assume that \mathbb{k} is closed under cube roots.

We have $\beta_{5,6}(T/(\det_3 - \ell^3)^\perp) < 140$ for all linear forms ℓ . Therefore $\text{rk}(\det_3 - \ell^3) \geq 14$ for all ℓ . Hence $\text{rk}(\det_3) \geq 15$. □

4. REMARKS

One might try to prove $\text{rk}(\det_3) \geq 16$ by considering $\det_3 - \ell^3 - m^3$. After a change of linear coordinates puts ℓ in normal form, a further linear change of coordinates to normalize m must now fix both \det_3 and ℓ . To simplify, we can assume $\ell = x_1 + x_5 + x_9$, with coefficient matrix given by the identity matrix. Indeed, since $\det(Y) \circ \det_d = 6 \neq 0$, there is no power sum decomposition of \det_d consisting entirely of terms whose coefficient matrices are singular. There is always at least one term with coefficient matrix of full rank, which we choose to be ℓ .

Now the joint stabilizer of \det_3 and the identity matrix is GL_3 acting by conjugation. Under this action we can put the coefficient matrix of m in Jordan canonical form. There are three cases: the coefficient matrix of m may have Jordan type $(1, 1, 1)$, $(2, 1)$, or (3) . One may show that $\det_3 - \ell^3$ is indecomposable and concise, hence $\det_3 - \ell^3 - m^3$ is concise. So as before, it is sufficient to show $\beta_{5,6} < 140$ for each $\mu \det_3 - \lambda \ell^3 - \kappa m^3$, where m involves unknown (variable) eigenvalues. Unfortunately at this point computer computations are prohibitive; we were not able to compute the needed free resolutions and graded Betti numbers.

The currently best known bounds in the case $d = 4$ are $50 \leq \mathrm{rk}(\det_4) \leq 160$. The syzygies of \det_4^\perp and of 49 or 50 points in \mathbb{P}^{15} do not seem to be suitable for this approach.

To try to improve the bound for $\mathrm{rk}(\mathrm{per}_3)$ by similarly considering $\mathrm{per}_3 - \ell^3$ involves a great number of cases. The group of linear coordinate changes that leaves invariant the permanent is precisely the product of the torus acting by scaling rows and columns (with total scaling 1), the permutation group acting on rows and columns, and the transposition. This group has many orbits, including the orbit of a matrix with all entries nonzero and independent. Then at most we can ensure that the coefficient matrix of ℓ has 5 entries of 1 (the first entries in each row and column), the other 4 entries being independent variables. We were not able to compute the apolar ideal, resolution, and graded Betti numbers in this case.

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REFERENCES

1. J. Alexander and A. Hirschowitz, *Polynomial interpolation in several variables*, J. Algebraic Geom. **4** (1995), no. 2, 201–222. 1, 6
2. Jarod Alper and Rowan Rowlands, *Syzygies of the apolar ideals of the determinant and permanent*, to appear in J. Alg. Combin. (2019) 9
3. Alessandra Bernardi, Jérôme Brachat, and Bernard Mourrain, *A comparison of different notions of ranks of symmetric tensors*, Linear Algebra Appl. **460** (2014), 205–230. 3
4. Alessandra Bernardi, Enrico Carlini, Maria Virginia Catalisano, Alessandro Gimigliano, and Alessandro Oneto, *The hitchhiker guide to: Secant varieties and tensor decomposition*, arXiv:1812.10267 [math.AG], Dec 2018. 1
5. Alessandra Bernardi, Joachim Jelisiejew, Pedro Macias Marques, and Kristian Ranestad, *On polynomials with given Hilbert function and applications*, Collect. Math. **69** (2018), no. 1, 39–64. 3

6. Alessandra Bernardi and Kristian Ranestad, *On the cactus rank of cubic forms*, J. Symbolic Comput. **50** (2013), 291–297. 6
7. Mats Boij, *Artin level modules*, J. Algebra **226** (2000), no. 1, 361–374. 11
8. Jerome Brachat, Pierre Comon, Bernard Mourrain, and Elias Tsigaridas, *Symmetric tensor decomposition*, Linear Algebra Appl. **433** (2010), no. 11–12, 1851–1872. 1
9. Winfried Bruns and Jürgen Herzog, *Cohen-Macaulay rings*, Cambridge Studies in Advanced Mathematics, vol. 39, Cambridge University Press, Cambridge, 1993. 9
10. Weronika Buczyńska and Jarosław Buczyński, *Secant varieties to high degree Veronese reembeddings, catalecticant matrices and smoothable Gorenstein schemes*, J. Algebraic Geom. **23** (2014), no. 1, 63–90. 11
11. Weronika Buczyńska and Jarosław Buczyński, *On differences between the border rank and the smoothable rank of a polynomial*, Glasg. Math. J. **57** (2015), no. 2, 401–413. 3
12. Weronika Buczyńska and Jarosław Buczyński, *Apolarity, border rank and multigraded hilbert scheme*, arXiv:1910.01944 [math.AG], 2019. 3
13. Weronika Buczyńska, Jarosław Buczyński, Johannes Kleppe, and Zach Teitler, *Apolarity and direct sum decomposability of polynomials*, Michigan Math. J. **64** (2015), no. 4, 675–719. 4, 11, 13
14. Weronika Buczyńska, Jarosław Buczyński, and Zach Teitler, *Waring decompositions of monomials*, J. Algebra **378** (2013), 45–57. 1
15. Enrico Carlini, *Reducing the number of variables of a polynomial*, Algebraic geometry and geometric modeling, Math. Vis., Springer, Berlin, 2006, pp. 237–247. 4
16. Enrico Carlini, Maria Virginia Catalisano, and Anthony V. Geramita, *The solution to the Waring problem for monomials and the sum of coprime monomials*, J. Algebra **370** (2012), 5–14. 1
17. Enrico Carlini, Nathan Grieve, and Luke Oeding, *Four lectures on secant varieties*, Connections between algebra, combinatorics, and geometry, Springer Proc. Math. Stat., vol. 76, Springer, New York, 2014, pp. 101–146. 1
18. Enrico Carlini, Emanuele Ventura, and Cheng Guo, *Real and complex Waring rank of reducible cubic forms*, J. Pure Appl. Algebra **220** (2016), no. 11, 3692–3701. 1
19. Gianfranco Casnati, Joachim Jelisiejew, and Roberto Notari, *Irreducibility of the Gorenstein loci of Hilbert schemes via ray families*, Algebra Number Theory **9** (2015), no. 7, 1525–1570. 11
20. Austin Conner, Fulvio Gesmundo, Joseph M. Landsberg, and Emanuele Ventura, *Kronecker powers of tensors and strassen’s laser method*, arXiv:1909.04785 [cs.CC], 2019. 4, 5, 6
21. Harm Derksen, *On the nuclear norm and the singular value decomposition of tensors*, Found. Comput. Math. **16** (2016), no. 3, 779–811. 4
22. Harm Derksen and Zach Teitler, *Lower bound for ranks of invariant forms*, J. Pure Appl. Algebra **219** (2015), no. 12, 5429–5441. 6, 7
23. Igor V. Dolgachev, *Classical algebraic geometry: A modern view*, Cambridge University Press, Cambridge, 2012. 3
24. David Eisenbud, *The geometry of syzygies*, Graduate Texts in Mathematics, vol. 229, Springer-Verlag, New York, 2005, A second course in commutative algebra and algebraic geometry. 7
25. David Eisenbud and Sorin Popescu, *Gale duality and free resolutions of ideals of points*, Invent. Math. **136** (1999), no. 2, 419–449. 11
26. Cameron Farnsworth, *Koszul-Young flattenings and symmetric border rank of the determinant*, J. Algebra **447** (2016), 664–676. 6, 7
27. Maciej Gałaszka, *Vector bundles give equations of cactus varieties*, Linear Algebra Appl. **521** (2017), 254–262. 6
28. David G. Glynn, *The permanent of a square matrix*, European J. Combin. **31** (2010), no. 7, 1887–1891. 4
29. Daniel R. Grayson and Michael E. Stillman, *Macaulay2, a software system for research in algebraic geometry*, Available at <http://www.math.uiuc.edu/Macaulay2/>. 9
30. Robin Hartshorne, *Algebraic geometry*, Springer-Verlag, New York, 1977, Graduate Texts in Mathematics, No. 52. 14
31. Christopher J. Hillar and Lek-Heng Lim, *Most tensor problems are NP-hard*, J. ACM **60** (2013), no. 6, Art. 45, 39. 1

32. Mark E. Huibregtse, *Some elementary components of the Hilbert scheme of points*, Rocky Mountain J. Math. **47** (2017), no. 4, 1169–1225. 11
33. A. Iarrobino and J. Emsalem, *Some zero-dimensional generic singularities; finite algebras having small tangent space*, Compositio Math. **36** (1978), no. 2, 145–188. 11
34. Anthony Iarrobino and Vassil Kanev, *Power sums, Gorenstein algebras, and determinantal loci*, Lecture Notes in Mathematics, vol. 1721, Springer-Verlag, Berlin, 1999, Appendix C by Iarrobino and Steven L. Kleiman. 1, 2, 3, 6
35. Nathan Ilten and Zach Teitler, *Product ranks of the 3×3 determinant and permanent*, Canad. Math. Bull. **59** (2016), no. 2, 311–319. 5
36. Siddharth Krishna and Visu Makam, *On the tensor rank of 3×3 permanent and determinant*, arXiv:1801.00496 [math.CO], Jan 2018. 4
37. J. M. Landsberg, *Tensors: Geometry and Applications*, Graduate Studies in Mathematics, vol. 128, American Mathematical Society, Providence, RI, 2012. 1
38. ———, *Geometric complexity theory: an introduction for geometers*, Ann. Univ. Ferrara Sez. VII Sci. Mat. **61** (2015), no. 1, 65–117. 1
39. J. M. Landsberg and Zach Teitler, *On the ranks and border ranks of symmetric tensors*, Found. Comput. Math. **10** (2010), no. 3, 339–366. 6, 7
40. Hwangrae Lee, *Power sum decompositions of elementary symmetric polynomials*, Linear Algebra Appl. **492** (2016), 89–97. 1
41. Irena Peeva, *Consecutive cancellations in Betti numbers*, Proc. Amer. Math. Soc. **132** (2004), no. 12, 3503–3507. 9
42. Elisa Postingshel, *A new proof of the Alexander-Hirschowitz interpolation theorem*, Ann. Mat. Pura Appl. (4) **191** (2012), no. 1, 77–94. 1
43. Kevin Pratt, *Waring rank, parameterized and exact algorithms*, Jul 2018, arXiv:1807.06194 [cs.DS]. 1
44. Kristian Ranestad and Frank-Olaf Schreyer, *Varieties of sums of powers*, J. Reine Angew. Math. **525** (2000), 147–181. 3
45. ———, *On the rank of a symmetric form*, J. Algebra **346** (2011), 340–342. 6
46. Sepideh Masoumeh Shafei, *Apolarity for determinants and permanents of generic matrices*, J. Commut. Algebra **7** (2015), no. 1, 89–123. 6, 7, 8, 10
47. Yaroslav Shitov, *How hard is the tensor rank?*, arXiv:1611.01559 [math.CO], 2016. 1
48. J.J. Sylvester, *On a remarkable discovery in the theory of canonical forms and of hyperdeterminants*, originally published in Philosophical Magazine, vol. 2, 1851, pp. 391–410. Paper 41 in *Mathematical Papers*, Vol. 1, Chelsea, New York, 1973, originally published by Cambridge University Press in 1904., 1851. 3
49. Zach Teitler, *Geometric lower bounds for generalized ranks*, arXiv:1406.5145 [math.AG], Jun 2014. 1, 3
50. Zach Teitler and Alexander Woo, *Power sum decompositions of defining equations of reflection arrangements*, J. Algebraic Combin. **41** (2015), no. 2, 365–383. 1

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