# Complexity and Characterization of Set Splitting


Peter Bernstein, Tufts University

Cashous Bortner, University of Nebraska

Samuel Coskey, Boise State University

Shuni Li, Macalester College

Connor Simpson, Cornell University


## Discrepancy theory

Combinatorial discrepancy theory is a well-studied area of mathematics with applications to computational geometry, machine learning, probabilistic algorithm design, and other fields concerned with the regularity of distributions.

- Let $X$ be a finite set and let $\mathcal{B} = \{B_1, B_2, \ldots, B_n\} \subseteq 2^X$ be a collection of subsets. The **discrepancy of $\mathcal{B}$** is
$$\operatorname{disc}(\mathcal{B}) := \min_{S \subseteq X} \max_{B_i \in \mathcal{B}} \Big| |B_i \cap S| - |B_i \setminus S| \Big|.$$
- In a celebrated 1985 result, Joel Spencer gave a tight upper bound of
$$\operatorname{disc}(\mathcal{B}) \leq K\sqrt{n}$$
where $K$ is an absolute constant [5]. He later conjectured that no efficient algorithm exists to find a set $S$ witnessing that discrepancy is within his bound [1].
- In 2010, Bansal and others disproved the conjecture by giving an efficient algorithm to find such an $S$ [2].
- However, $\operatorname{disc}(\mathcal{B})$ can be much smaller than Spencer's bound, so Bansal's work prompts the following question:

  Is it efficient to determine whether $\operatorname{disc}(\mathcal{B}) \leq 1$ and to find a witness $S$ when this is the case?

  Our work provides an answer.

## Splittability

We use $[x]$ to denote the nearest integer to $x$, with free rounding if $x$ is an odd multiple of $\frac{1}{2}$. Let $\mathcal{B} = \{B_1, \ldots, B_n\} \subseteq 2^X$ be a collection of subsets of a set $X$, and fix $0 < p < 1$.

**$p$-Splittable**

$\mathcal{B}$ is ***p*-splittable** if there exists a $S \subseteq X$ such that for each $B_i \in \mathcal{B}$,
$$|B_i \cap S| = [p\,|B_i|].$$

Note that when $p = \frac{1}{2}$, being $p$-splittable is equivalent to having $\operatorname{disc}(\mathcal{B}) \leq 1$.

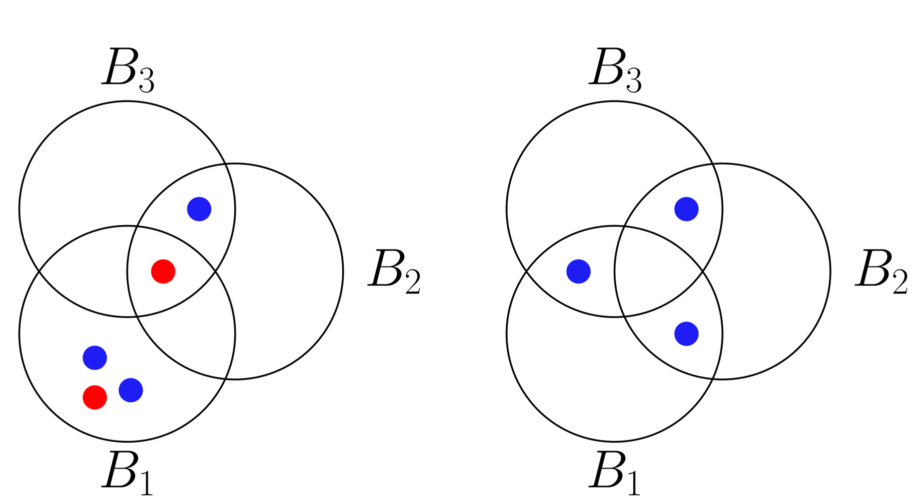



*Fig.* Splittable and unsplittable collections for $p = 1/2$

## Complexity of $p$-splitting

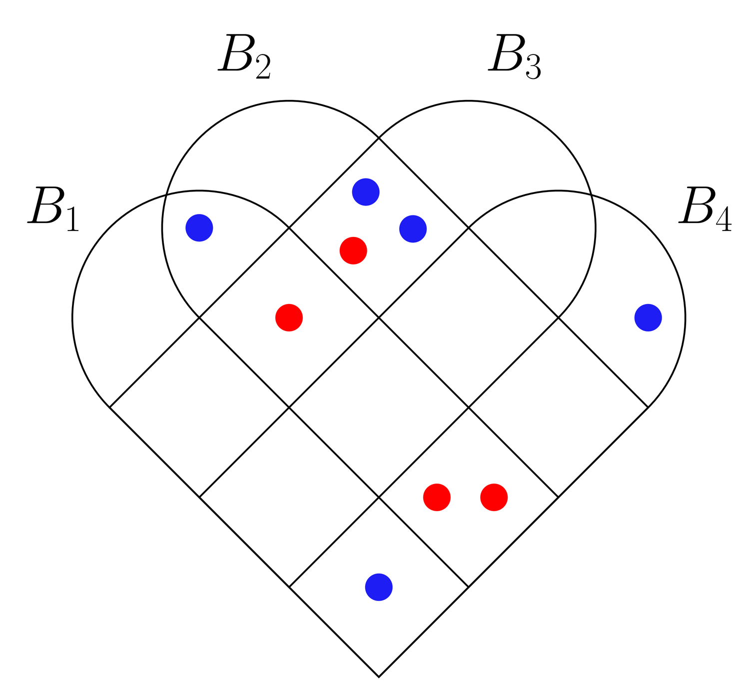
**Main Theorem**

Determining whether a collection is $p$-splittable is NP-Complete for any $0 < p < 1$.

*Selected proof techniques:*

Here, we outline some parts of our reduction from the NP-complete problem Zero-One Equations (ZOE) [4]. ZOE is stated as follows: Given a 0,1-matrix $A$, does there exist a 0,1-vector $\vec{y}$ such that $A\vec{y} = \vec{1}$, where $\vec{1}$ is the vector of ones?

- We can encode $\mathcal{B} = \{B_1, \ldots, B_n\} \subseteq 2^X$ in the form of a $0,1$-matrix $M$, which has a $1$ in its $(i,j)$ position precisely when element $j$ of the collection is in $B_i$.




$$\longrightarrow \begin{bmatrix} 1&1&0&0&0&0&1&1&1 \\ 1&1&1&1&1&0&1&1&0 \\ 0&1&1&1&1&0&0&0&0 \\ 0&0&0&0&0&1&1&1&1 \end{bmatrix} \begin{bmatrix} 0\\1\\1\\0\\0\\0\\1\\1\\0 \end{bmatrix} = \begin{bmatrix} 3\\4\\2\\2 \end{bmatrix}$$

*Fig.* A split collection and its corresponding matrix equation for $p = 1/2$

- To represent splitting $\mathcal{B}$, we apply $M$ to a 0,1-vector $\vec{x}$ encoding a potential solution $S \subseteq X$ to the splitting problem. The set $S$ corresponding to $\vec{x}$ is a valid solution if the $i$th entry of $M\vec{x}$ is equal to $[p|B_i|]$.
- We make use of this encoding by applying a polynomial-time construction that turns an arbitrary input to ZOE into a $p$-splitting problem in the form described above.

**Corollary**

Given a collection $\mathcal{B}$, determining whether there exists a set $S$ witnessing $\operatorname{disc}(\mathcal{B}) \leq 1$ is NP-complete.

## When is a collection $p$-splittable?

For general $p$, finding simple rules to tell when a collection of sets is $p$-splittable is very difficult, as one would expect given the theorem above. However, we do have criteria for some special collections.

**Lemma**

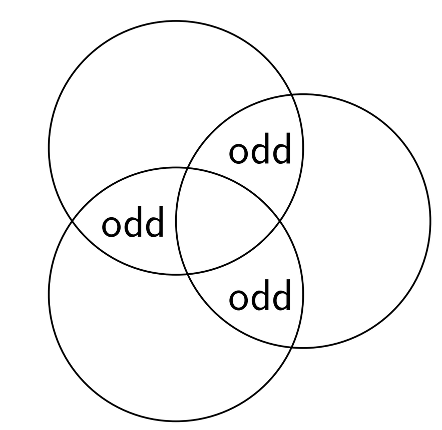
Let $\mathcal{B} = \{B_1, \ldots, B_n\}$, a collection of sets whose elements all lie in exactly $m$ sets.

- If $\mathcal{B}$ is $p$-splittable, then $\sum_{i=1}^{n} [p|B_i|]$ is divisible by $m$.
- If $m = 1$ or $m = n - 1$, the converse holds: if $\sum_{i=1}^{n} [p|B_i|]$ is divisible by $m$ then $\mathcal{B}$ is $p$-splittable.

## Criteria for $\frac{1}{2}$-splittability

While the main theorem implies it is hard to find splittability criteria in general, we have had some success with small $n$ and $p = \frac{1}{2}$. Some known results in this case are:

- Every collection of one or two sets is splittable.
- A collection of three sets is splittable if and only if it is not of the form [3]:




The situation becomes much more complex for four or more sets.

**4-Set Classification Theorem**

Every unsplittable collection of four sets falls into one of eleven simple patterns.

To prove this theorem we used a supercomputer to check all cases with a small number of elements, manually sorted the output, and generalized the conclusion using the lemma below.

**Lemma**

If $\mathcal{B}$ is splittable, then $\mathcal{B}$ remains splittable when an even number of elements are added to any of its Venn regions.

Computer experiments also lead us to the following:

**Conjecture**

Any collection of sets with no empty Venn regions is splittable.

## References & Acknowledgements


[1] N. Alon and J. Spencer. *The Probabilistic Method.* Wiley, 2000, p. 240.

[2] N. Bansal. "Constructive Algorithms for Discrepancy Minimization". In: *Proceedings of the 2010 IEEE 51st Annual Symposium on Foundations of Computer Science.* FOCS '10. IEEE Computer Society, 2010, pp. 3–10.

[3] D. Condon et al. "On Generalizations of Separating and Splitting Families". In: *submitted* (2015). arXiv: `1412.4683[math.CO]`.

[4] S. Dasgupta, C. Papadimitriou, and U. Vazirani. *Algorithms.* McGraw Hill, 2006, p. 254.

[5] J. Spencer. "Six Standard Deviations Suffice". In: *Transactions of the AMS* 289.2 (1985), pp. 679–706.



This work was supported by the NSF under grant No. DMS 1359425 and Boise State University.