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A Refinement of Michener's Example Classification

Laurie O. Cavey
Boise State University

Margaret T. Kinzel
Boise State University

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Laurie O. Cavey
Boise State University
lauriecavey@boisestate.edu

Margaret T. Kinzel
Boise State University
mkinzel@boisestate.edu

In this paper we propose a refinement of Michener's (1978) well-known example classification based on data from university mathematicians. The refinement takes into account the mathematician's perspective on the role of examples in doing mathematics. More specifically, our work provides insight into the ways in which mathematicians talk about using examples in their scholarly work and their work with students. The proposed classification has the potential to inform our work as teachers as we strive to create opportunities to engage students in authentic mathematical work.

Keyword: Advanced Mathematical Thinking

Examples play a significant role in teaching and learning mathematics. Often it is with a carefully formulated example that subtleties in a definition or an algorithmic process can be detected. Examples make it possible to consider generalities, but can also limit one's mathematical perspective. Ideally, mathematical learning experiences provide opportunities to develop a rich array of examples that contribute to students' problem-solving skills and understanding.

Interestingly, what is known about the role of examples in doing mathematics is not adequately informed by the mathematician's perspective. Few studies on examples or related topics have mathematician participants (c.f. Lockwood, Ellis, & Knuth, 2013). On the other hand, there is evidence that mathematicians frequently generate examples in the process of validating a proof given by a peer (Weber, 2008) and that mathematicians use different types of examples in conjecture-related work (Lockwood, Ellis, & Knuth, 2013). In addition, given the number of proofs without words publications, it is reasonable to assume that mathematicians value opportunities to "see the general in the specific" (Mason & Pimm, 1984). That is, mathematicians sometimes use an example to make a proof. However, we have little evidence that speaks to the mathematicians' perspective on the use of examples in teaching or how examples support their own mathematical work.

Michener's seminal work (1978) presented an epistemology of mathematics from her perspective as a mathematician that classified example types for the development of mathematical thinking: start-up examples, reference examples, model examples, and counterexamples. Classifying an example is necessarily tied to the purpose the example serves in supporting mathematical learning. Studies on the use of examples with students continue to reference Michener's classification (e.g., Alcock & Inglis, 2008; Watson & Mason, 2002; Zaslavsky & Shir, 2005). Because of this, we wondered whether Michener's classification is useful when examining the ways mathematicians use examples in their work. In particular, we asked: do mathematicians' articulations of the uses of examples align with these classes? Do the classes capture the range of the mathematicians' articulations related to the uses of examples?

We describe our initial refinement of Michener's epistemology based on data from an earlier study where we examined how mathematicians make sense of definitions (Kinzel, Cavey, Walen, & Rohrig, 2011). We then use preliminary analysis of data from a current study focused on mathematicians' use of examples more generally to propose further refinements.

Background

Michener's (1978) epistemology of mathematical knowledge presents three interrelated categories of items: results (theorems), concepts (definitions, heuristics, advice), and examples (illustrative material). The interrelatedness of these categories is described in terms of a predecessor/successor relationship: Examples may lead to the construction of definitions and/or theorems or alternatively may serve to illustrate a definition or theorem. These relationships are connected through the notion of "dual relations" and the identification of "dual items." The dual items of a particular example are the ingredient concepts and results needed to construct the example and the results motivated by the example. Similar duals can be defined for results and concepts. Thus, examples are integrally entwined with the other categories of items.

Michener further states that because not all examples are "created equal," those more noteworthy deserve further attention and can be grouped into epistemological classes. These classes are not intended to be disjoint: A single example can play more than one role, perhaps even within a particular context or situation. It is these classes of examples in which we are interested. Michener's four classes of examples are paraphrased below (1978, pp. 366–368).

- *Start-up examples* are used to introduce a new subject by motivating basic concepts and useful intuitions, such as using a specific picture or visual representation to highlight key features of graphs of monotonically increasing sequences.
- *Reference examples* are widely applicable and provide a common point of contact, and so tend to be referred to repeatedly. This class can also include standard cases used to verify one's understanding of a particular concept or result as when a particular function is used to illustrate what it means for a graph to be concave down.
- *Model examples* suggest and summarize the expectations and default assumptions about results and concepts. The absolute value function could be a model example for the idea that a continuous function need not be differentiable on its domain.
- *Counterexamples* sharpen the distinctions between concepts and are used to show a statement is not true. The function $f(x) = x^3, x \in \Re$ is a counterexample to the statement that all cubic polynomial functions have two local extreme values.

Goldenberg and Mason (2008) elaborated on Michener's classes, pointing specifically to the role of "pertinent nonexamples" (p. 184) in clarifying the sorting of items into things *that are* versus things *that are not*. Nonexamples are not equivalent to counterexamples in that there may not be a statement whose truth is in question; however, the identification of nonexamples can also serve to sharpen distinctions or interpretations of results and concepts. Nonexamples may also be used to develop one's intuition (start-up example) or to verify one's established understanding of a concept (reference). The primary use of nonexamples is to highlight contrast, which can overlap with the purposes of other classes. In this paper, we include nonexamples as a fifth class in our framework, paying particular attention to instances when nonexamples can be classified otherwise. As we describe in the next section, data from our earlier research supported the inclusion of this fifth class.

Methodology

Interview data from our earlier research, which focused on how mathematicians make sense of definitions, revealed themes related to the role of examples within that work (Kinzel, Cavey, Walen, & Rohrig, 2011). Nine mathematicians participated in interviews in which they were asked to first describe how they make sense of a new definition for themselves and then how they support students in making sense of new definitions. Examples, and the purposes for using examples, were prominent in the mathematicians' descriptions. Making sense of definitions is a key component of mathematical work but does not capture the full range of that work. We used data from the earlier study to begin a refinement of Michener's framework, then returned to the same group of mathematicians to explore the role of examples beyond the context of definitions. In the follow-up study we asked the mathematicians to review the classes of examples presented in the framework and consider (1) how well the framework reflects the ways in which they use examples, (2) if there are uses of examples not represented in the framework, and (3) if the framework is a useful means for thinking about mathematical work, either for themselves or for students. As we are still collecting data from the follow-up study, what follows are the methods used to establish the initial refinement.

Interview data were transcribed and pseudonyms were assigned to the participants. Individual research team members reviewed transcripts to identify instances where the mathematicians made a reference to the use of an example (*example-instances*); coded transcripts were then shared and example-instances verified. To be identified as an example-instance, an articulation by the mathematician either explicitly included the word "example" or referred to illustrating a concept or result in some way; for instance, a description of how the mathematician uses examples to clarify which items fit a definition and which do not counts as an example-instance. The instances we identified did not always include the articulation of a specific concept or mathematical object. Where possible, the specific concept or result within the instance was identified (e.g., articulating the usefulness of providing visual examples to illustrate the concept of collinear points).

The second stage of analysis focused on coding the example-instances based on Michener's (1978) classes of examples. We focused the data analysis through two questions: (a) What types of examples did the mathematicians describe? and (b) How did they describe the purpose of each example? We began this part of the analysis with a discussion of Michener's classes and a group analysis of one transcript to clarify the shared understanding of the classes. This involved using transcript data to clarify distinctions between the classes. These distinctions often focused on the perspective of the learner as well as the intended use of the example by the mathematician. After these criteria were established, the remaining transcripts were analyzed by individual team members; two other team members then verified the coding. Within this analysis, we encountered several articulations related to the construction or analysis of *things that are not*; a function that would *not* meet the criteria of a particular definition, for instance. We introduced *nonexamples* as a fifth class in the framework to capture these articulations.

To illustrate the coding process, we present several example-instances and the resulting classification. Consider the articulation from Adam in response to what helps him to make sense of a new definition: "I start off with things that are familiar to me . . . I would start going through the list of standard examples that I have in my head for these." This instance was coded as *reference* because we inferred from this statement that Adam's "standard examples" were widely applicable. On the other hand, Sam responded to the same question as follows: "the simplest thing to try first is just to look at the specific concrete examples … you kind of get a feeling of how this specific definition is working." This instance was coded as a *start-up,* since we inferred from "get a feeling of how this specific definition is working" that Sam's purpose was to develop his intuition. About three-fourths of the way into the interview with Greg he began explaining the importance of using examples with students where the "main feature" of the concept is "worked out that we actually want to transport by those examples." By this, we inferred that he was articulating the importance of using an example that illustrates critical features, and thus coded it as *model*.

After we coded all the example-instances, we examined the instances within each class to identify themes in how mathematicians talked about using examples to support either their own or their students' learning. Broad epistemological themes emerged within each class of examples. Recall that mathematicians were asked to reflect on their own processes/experiences as well as on those they intend for students. Because of this, it was necessary to identify the intended learner within example-instances. Each instance could potentially refer to the mathematician, to their students, or to a hypothetical learner. When we refer to the intended use of an example, we always mean in reference to the learner, whether it be the mathematician or a potentially hypothetical student. However, the epistemological themes that emerged address both the instructor's and the learner's perspective. It was the identification of these themes that led to the refinement of Michener's classification of examples.

Initial Refinement of Michener's Example Classification

In this section we provide a description of our initial refinement of Michener's example classification. Data analysis led to clarification on the classes already noted in the literature (Michener, 1978; Watson & Mason, 2005). Informal conversations with mathematicians about our proposed refinement indicate that these categories are useful but may not be exhaustive, especially in relation to work with results or theorems. For this reason, we anticipated data from the follow-up study to lead to further expansion and clarification of our initial refinement.

See Table 1 for a list of the classes in our initial refinement and a summary of the purposes associated with each class.

Table 1: Initial Refinement Of Michener's Example Classification

Class	Purpose
Start-up	develop intuitive notions consider what is and what is not to isolate concept check initial understanding through generating examples
Reference	widely applicable or standard case isolate new subclasses of mathematical objects
Model	demonstrate salient features consider interplay between features of definition and example
Counterexample	refute the truth of a statement in question
Nonexample	demonstrate what *is not* show "control" of a definition consider particular features of a definition

Start-Up Examples

In presenting Michener's classification, Watson and Mason (2005) describe start-up examples as those from which "basic problems, definitions, and results can be conjectured at the beginning of learning some theory and can be 'lifted' to the general case" (p. 64). The articulations from the mathematicians provide further detail into particular ways in which this conjecturing might be supported. For example, analyzing a collection of examples can help to isolate essential features of a new concept. The activity of generating examples of a new idea contributes to clarifying one's emerging understanding of the idea. Creating variations of known examples can serve to further demonstrate or clarify one's understanding. From the learner's perspective, examples in this class should be familiar objects that also illustrate key features of the new concept.

Mathematicians described using examples to develop intuition about the features of the concept that distinguish it from other related ideas. Greg articulated that students may not know to what to pay attention at first: "And then I might try to work out with them a little bit, you know, what could be mathematically interesting there? What are the features there? . . . Then state a formal definition and then go and do plenty of examples to kind of work on that." Ned articulated this use of examples in his own work: "But if it was foreign or I read it and realized that I didn't truly understand it, then I probably would try to come up with an example. Which is a hard thing because you need your example easy enough to understand, yet hard enough so that it eliminates what needs to be eliminated." Wes gave a similar response: "And how is it new? And can I think of something that does this as well as something that doesn't do this?" Across these instances, the focus was on using examples of *things that are* as well as examples of *things that are not* to develop intuition for the essential features of a newly encountered definition.

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Reference Examples

Reference examples are intended to be widely applicable and available for consideration as a standard case. A learner may return to one or more reference examples while in the process of developing understanding of a new idea or subclass of mathematical objects; perhaps using a familiar example of a group to explore a new property, for instance. A learner may have a known and familiar set of standard examples that are consciously used to verify or extend understanding. As with model examples, the level of awareness of the learner with respect to the essential features of these examples is critical in their appropriate use. Overuse of a specific case could lead to confusion between aspects of the particular with aspects of the more general concept.

Wes and Adam specifically discussed using familiar objects to *understand an unfamiliar subclass of objects*. Wes shared an experience of reading a student thesis in which a particular property was introduced. In considering whether he understood the property as it related to the thesis, he asked himself: “It says it does this, or these things do that, so why is this (stably free module) different from this thing that we have (free module).” To make sense of the notion, he noted that he began thinking of examples of free modules with which he was familiar and tried to identify which would have the new property and which would not. Adam expressed this same idea in his own work; a general practice for him when encountering a new idea is to check his “list of standard examples” to see which of those objects illustrate the new idea. Adam also explained how he uses this process with students. The instance he shared related to introducing the concept of algebraic groups; he drew students’ attention to familiar sets with operations (such as integers under addition) and then engaged them in determining which of these familiar things met the criteria of a group. In this way, Adam seemed to intend that these familiar instances could become reference examples for the students for the concept of group.

Model Examples

Model examples are intended as paradigmatic and generic, and can be used to convey salient features of an idea. As noted by Sam, a model example may emerge from the analysis of a collection of examples; one exemplar from the collection may serve as an illustration of the desired concept. The presentation of a model example can serve to highlight the interplay between the use of the example and one’s understanding of the idea; that is, the activity of determining *why* the presented exemplar qualifies as a model example potentially interacts with one’s understanding. Poor choice of a model example can cloud one’s understanding of an idea, in that this awareness of essential features may be (perhaps implicitly) compromised.

In choosing examples to present to students as potential model examples, Greg emphasized the need for the example to illustrate the key features, and that it be “not too trivial,” yet also “not too complicated” as that could interfere with “seeing” the features of the concept. Marc also spoke of choosing pictures to convey relevant features to students. Ned stated that he uses the interplay between example and concept as a means for checking students’ understanding. After presenting a (model) example, he makes small variations, such as changing a positive slope to negative. In his experience, students who were able to see the key features were less likely to be distracted by the variation. That is, we would argue that Ned was determining whether students saw the initial presentation as a model example for the presented idea; if so, students were more likely to be able to identify key components and perhaps not be distracted by variations that did not alter the underlying concept.

Nonexamples

Nonexamples clarify distinctions between *what is* and *what is not*, and thus are used to demonstrate the importance of key features of a concept. The purpose of a nonexample can overlap the purpose of start-up, model, or reference examples. A collection of examples can be used to draw attention to common features; contrasting such a collection with nonexamples serves to sharpen distinctions. The successful generation of relevant nonexamples can be taken as an indication of the "control" one has with regard to a concept or result. A learner's explicit attention to aspects of nonexamples is an indication of depth of understanding.

Mathematicians discussed using nonexamples to refine one's understanding of a definition or concept. In particular, the generation of nonexamples can focus attention on salient features and the purpose of those features. Often, this involved dropping or violating one or more criteria within the definition and asking, in Ned's words, "how does that change the outcome of what is permissible?" Sadie stated that nonexamples give a "different perspective" through analyzing what "it can't be." Sam acknowledged that "find[ing] an example in which this fails" can be challenging for students, but can be a critical step in developing understanding of the definition by forcing one to "look deeper at what things are."

Counterexamples

Counterexamples are used to refute a statement. Some become well-known and used often. While not an explicit focus of our analysis, we see some common aspects. In particular, the learner's awareness of why an instance qualifies as a counterexample for a given statement is key, and could serve as an indication of the learner's understanding of the underlying ideas.

Current Work

We are currently in the midst of a follow-up study focused on mathematicians' use of examples more broadly. We asked the mathematicians who participated in the definitions research study to read the results of that work, including our initial refinement of Michener's example classification prior to a face-to-face interview. We also asked each mathematician to respond to the following questions in writing prior to the interview:

- How well does the proposed example-use framework reflect the way you think about the purposes of different types of examples in your mathematical work? Please address this question as it relates to your scholarly work (writing, research, etc.) and teaching.
- Is there is a type of example that you use that is not articulated in the framework? Is there an example type that you rarely use? Please explain.
- Is the framework useful as a means of thinking about the different purposes of examples in doing mathematics? If so, in what ways?
- Is the framework useful as a means of thinking about teaching mathematics? If so, in what ways?

The mathematicians we have interviewed articulated benefits associated with thinking carefully about the purposes of different types of examples in their teaching but not in relation to their own mathematical work. They describe using examples for the purposes described in the initial refinement along with other purposes more closely related to their scholarly work. Thus far, it appears that there may be one or two other example classes that warrant defining. In particular, two mathematicians noted how examples can be preceded by definitions and results, providing the impetus for *existence examples*. The purpose of this class is to demonstrate the existence of a mathematical object and is thus distinct from the other classes. Greg noted, "You may actually indirectly prove that a thing exists without actually constructing it" while Evan noted odd perfect numbers as an instance where the definition has preceded examples. Another possible category is that of *boundary examples*—those that support understanding of "where the boundaries are" (Evan) with a concept or result. It not yet clear from the data whether this category stands on its own. The data suggest that there is overlap with the purposes of reference and nonexamples in that boundary examples can be used to identify subclasses of objects or to demonstrate control of a definition. Further analysis is needed to clarify these distinctions. Moreover, careful consideration of the epistemological value of potential new classes is needed. As noted by Michener (1978), the classes are not meant to be exhaustive but rather particularly informative in relation to thinking about how to fully support students in learning mathematics.

Discussion

Research indicates that students benefit from generating their own examples rather than passively accepting examples given by the teacher (Dahlberg & Housman, 1997; Watson & Mason, 2002; Sowder, 1980; Weber, Porter, & Housman, 2008). In our work, we observed mathematicians attending to perceived benefits of example generation both for themselves and for their students. Generating examples was seen to help build intuition about a new concept (start-up), to sharpen distinctions (non-example), and to reveal or verify understanding (start-up and/or model). In general, the ability to construct an appropriate example was taken as evidence of some level of understanding. Being able to then modify or create variations of the example could be further evidence of understanding. Constructing nonexamples was seen as more challenging, but was also seen as evidence of working knowledge of a concept.

Watson and Mason (2005) describe *example spaces* as a metaphor for the psychological structure of the ideas and examples associated with a particular concept. An interesting feature within the mathematicians' articulations was the interplay between examples and related ideas. For instance, Adam talked of drawing on his set of "standard cases" of objects to make sense of a new concept. He may think of standard examples of groups to make sense of a newly encountered type of group. Using Watson and Mason's metaphor, he pulls reference examples from one example space to be used as start-up examples in a related space. In discussing their work with students, the mathematicians talked of beginning with objects that should be familiar or known to students, using these as start-up examples for a new concept, or establishing them as reference examples for a concept. These descriptions align with Watson and Mason's characterization of mathematical activity as the reorganization of example spaces.

Our work aligns with work by Lockwood and colleagues (2012, 2013) in which they analyzed mathematicians' articulations of the role of examples within the context of exploring conjectures. We particularly agree with Lockwood, et al. (2012) as to the importance of "intentional example exploration" (p. 157). Our application of Michener's classification relies on identifying the intended role of the example; for instance, a model example is most powerful when the learner recognizes the features that exemplify the concept.

Implications

In general, refinement of the example classification provides the mathematics education community with a common language about the role of examples in teaching and learning mathematics. Having a common language is important for future advances in the area. Of course, this work also raises important questions regarding the role of exemplification in the teaching and learning of mathematics. In particular, how well does the classification capture the role of examples within a broader context of mathematical activity? How might the classification be used to guide instructional design? Could deliberate attention to the purposes support the selection of, presentation of, and plans for student engagement with examples?

A common language for example types can support more intentional selection of examples within instruction. From an instructional design perspective, one might attend to the intended purpose of an example to determine its place within the unit of study. Following Adam's suggestion, for example, a textbook author might include one or more start-up examples of a concept prior to introducing a definition. Likewise, reference and model examples may be better placed after a definition has been presented. Further, a classroom teacher might be more explicit about the role of examples, providing support to students' interpretations and use of examples.

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