MODULES OVER LOCALIZED GROUP RINGS FOR GROUPS MAPPING ONTO FREE GROUPS

by

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ABSTRACT

In 1964, Paul Cohn showed that if F is a finitely-generated free group, and Q a field, then all ideals in the group ring $Q[F]$ are free as $Q[F]$-modules. In particular, all finitely-generated submodules of free $Q[F]$-modules are free. In 1990, Cynthia Hog-Angeloni reproved this theorem using techniques from geometric group theory. Leaning on Hog-Angeloni's methods, we prove an analogous statement for crossed products $D * F$, with D a division ring.

With this result in hand, we prove that if $G = H \rtimes F$, the semi-direct product of H with F, so that the group ring D[G] may be localized at the sub-group ring $k[H] - \{0\}$, then the resulting localized group ring also has the property that finitely-generated submodules of free modules are free.

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CHAPTER 1

INTRODUCTION

In 1964, Paul Cohn proved in [1] that, if Q is a field and F a finitely-generated free group, then all ideals in the group ring $Q[F]$ are free as $Q[F]$-modules, with unique rank. In particular, this shows that all finitely-generated submodules of free $Q[F]$-modules are free. This theorem was used by algebraic topologists to obtain a complete homotopy classification for compact, connected 2-complexes with free fundamental group (see [13]). Cohn's argument is technically difficult, and in 1990, Cynthia Hog-Angeloni ([3]) offered an alternate proof of this fact, relying on geometric arguments related to the free action of F on a tree.

In this thesis, we lean on Hog-Angeloni's methods to generalize this result. Let k be a division ring. If G the semidirect product $H \rtimes F$, where $k[H]$ is a domain satisfying the Ore condition, then the group ring $k[G]$ can be localized at $k[H] - \{0\}$, and the resulting localized group ring, call it R, has the property that all finitely-generated submodules of free R-modules are free.

Chapter 2 takes an informal tone to remind readers of the basic ideas of group actions and trees. Content in Chapter 2 potentially new to the reader includes the crossed product structure, which may be viewed as a generalized group ring.

Chapter 3 adapts Hog-Angeloni's arguments to the crossed product $D * F$, where D is a division ring and F is a finitely-generated free group. Using the arguments, we conclude that finitely-generated submodules of free $D * F$-modules are free.

Chapter 4 provides the basic theory of the localization of noncommutative rings, focusing specifically upon the Ore localization of domains.

Chapter 5 applies the Ore localization theory to group rings. Given $H \leq G$, the chapter gives sufficient conditions for the localization of $k[G]$ at $k[H] - \{0\}$ to be possible, and explores the consequences of this localization, as discussed above.

Here we make a short comment about terms that will be used throughout the text. All groups are written multiplicatively, with identity e. The term "ring" refers to a ring with identity. A "domain" is a ring which is free from zero divisors, with no assumption made about commutativity. The term "division ring" is used to describe a ring where all non-zero elements are invertible, while the term "field" is reserved for commutative division rings. "Modules" are assumed to be left modules.

CHAPTER 2

BASIC NOTIONS

This chapter gives basic definitions and constructions that will be important in later chapters. Most of the information presented here should be familiar to the reader, including the notions of group actions and trees. The crossed product structure, which can be viewed as a generalized group ring, is potentially new to the reader. Readers interested in further reading on crossed products are directed to [9].

2.1 Group Actions

The concept of group action is fundamental in the study of group theory. Here is a definition, which may be found in [10]:

Definition 2.1. Let X be a set, and G a group. A **group action** of G on X is a function $\sigma : G \times X \to X$, usually denoted by juxtaposition, where we write gx in place of $\sigma(g, x)$. The group action must satisfy:

1. $ex = x$ for all $x \in X$.

2. $(gh)x = g(hx)$ for all $g, h \in G$ and $x \in X$.

Example 2.2. Take $X = \{\mathbf{1}, \mathbf{2}, \mathbf{3}, \mathbf{4}, \mathbf{5}\}$ and $G = S_5$, the symmetric group on 5 elements. The action of G on X is easily defined by letting the elements of G act in their natural way. For example, the permutation $(12)(34) \in S_5$ would act on $\mathbf{1} \in X$ by $(12)(34)\,\mathbf{1} = \mathbf{2} \in X$.

Given an element $x \in X$, we define the **orbit** of x (under the action of G) as $\mathcal{O}_G(x) = \{gx : g \in G\}$. It is easily verified that the collection $\{\mathcal{O}_G(x) : x \in X\}$ forms a partition of X. Define the **stabilizer** of x in G, denoted by G_x, as the set of elements of G that fix x. That is, $G_x = \{g \in G : gx = x\}$. It is apparent that G_x is a subgroup of G. The following terms are important in describing how G acts on X.

- The action of G on X is **transitive** if every element of X is contained in the same orbit under the action of G. This is equivalent to the statement that for every $x, y \in X$, there exists a $g \in G$ such that $gx = y$.

- The action of G on X is **trivial** if for every $g \in G$ and $x \in X$, we have that $gx = x$. Given some $x_0 \in X$, we may say that G acts trivially on x_0 if $gx_0 = x_0$ for every $g \in G$.

- The action of G on X is **faithful** if the only element of G that fixes every $x \in X$ is the identity.

- Suppose that whenever there exists $g \in G$ and $x \in X$ such that $gx = x$, this implies that $g = e$. Thus, the identity e is the only element of G that fixes *any* element of X. Such an action is called **free.**

It is common for the set X to possess some structure (say, as a ring or a graph). In these situations, we want the group action to respect that structure. As a result, the action of G on X will be subject to other requirements. For example, when working with crossed products (which will be introduced later in this chapter), it will be important to consider actions of a group G on a ring D, where the action respects the ring structure of D. In particular, we will be interested in actions $\sigma : G \times D \to D$ where for each fixed $g \in G$, the map $\sigma(g, \cdot) : D \to D$ is an automorphism. Thus, the action σ assigns to every $g \in G$ an automorphism of D, and we can view such an action as a map $\alpha : G \to \mathrm{Aut}(D)$ given by $\alpha : g \mapsto \sigma(g, \cdot)$. In this case, we might say that G acts on D via the map α. If σ is truly a group action, then the second condition of Definition 2.1 shows that the map α must be a homomorphism. However, we will consider situations where α is not a homomorphism, so the map $G \times X \to X$ does not strictly adhere to the definition of a group action.

2.2 Trees

In this paper, we make use of group actions upon trees to prove results about group rings and crossed products. Here, we briefly introduce graphs and trees, and provide a basic result about groups acting on trees. For the reader unfamiliar with these topics, [7] provides an excellent introduction to the topic of geometric group theory, which includes these ideas. We begin with a definition:

Definition 2.3. A **graph** Γ consists of a set $V(\Gamma)$ of **vertices**, and another set $E(\Gamma)$ of **edges**. To each edge $e \in E(\Gamma)$, we associate two (not necessarily distinct) vertices, say $u, v \in V(\Gamma)$, called the **ends** of e. Write $\mathrm{Ends}(e) = \{u, v\}$.

Graphs are usually visualized as a collection of points (the vertices of the graph), with lines (edges) connecting the points. An line is drawn between two points u and v if the graph contains an edges f with $\mathrm{Ends}(f) = \{u, v\}$. If $u, w \in V(\Gamma)$, an **edge path** (or just **path**) from u to w is a finite sequence that alternates between vertices and edges of Γ, such that each edge is preceded and followed by its ends. For example, the sequence $(u = v_0, f_1, v_1, f_2, ..., v_{n-1}, f_n, v_n = w)$ forms an edge path,where $v_0, ..., v_n \in V(\Gamma)$, $f_1, ..., f_n \in E(\Gamma)$, and $\mathrm{Ends}(f_i) = \{v_i, v_{i-1}\}$.

An edge path can be visualized as a walk taken from one vertex to another, when one is only allowed to walk along edges joining vertices. A path is **reduced** if it contains no subsequence of the form $(..., u, f, v, f, u, ...)$, where the edge f was taken from the vertex u to v, and then immediately taken back to u. We can reduce an edge path by removing all backtracks. An edge path is **trivial** if it contains exactly one vertex and no edges, so that the walk taken along the edge path does not move from the initial vertex. A non-trivial, reduced path that starts and ends at the same vertex is called a **cycle**. Given any two vertices of Γ, there is no guarantee that a path between them exists. We say that Γ is **connected** if an edge path exists between any pair of vertices in Γ.

A **tree** is a connected graph that contains no cycles. This is equivalent to the assertion that, given any vertices u, v in a tree Γ, there is a unique, reduced edge path joining u to v. This property will be exploited in the next chapter, where much of the work is done by considering the geometry of trees.

Given a graph Γ, a **symmetry** of Γ is a bijection ρ, which takes $V(\Gamma)$ into $V(\Gamma)$, and $E(\Gamma)$ into $E(\Gamma)$. This function must respect the graph structure of Γ. That is, if $f \in E(\Gamma)$ with $\mathrm{Ends}(f) = \{u, v\}$, then we must have $\mathrm{Ends}(\rho(f)) = \{\rho(u), \rho(v)\}$. The set of all symmetries of Γ is written $\mathrm{Sym}(\Gamma)$, which is easily seen to be a group under composition. If G is a group acting on a tree Γ, then there is a map $\sigma : G \to \mathrm{Sym}(\Gamma)$. Typically, we will write the function $\sigma(g)$ as σ_g. Moreover, the action of G on Γ is usually expressed by juxtaposition, so that, for $g \in G$, $f \in E(\Gamma)$, and $v \in V(\Gamma)$, we write gf in place of $\sigma_g(f)$ and gv in the place of $\sigma_g(v)$.

Given a group G and a generating set $\{a_i\}$, we may form the **Cayley graph** of

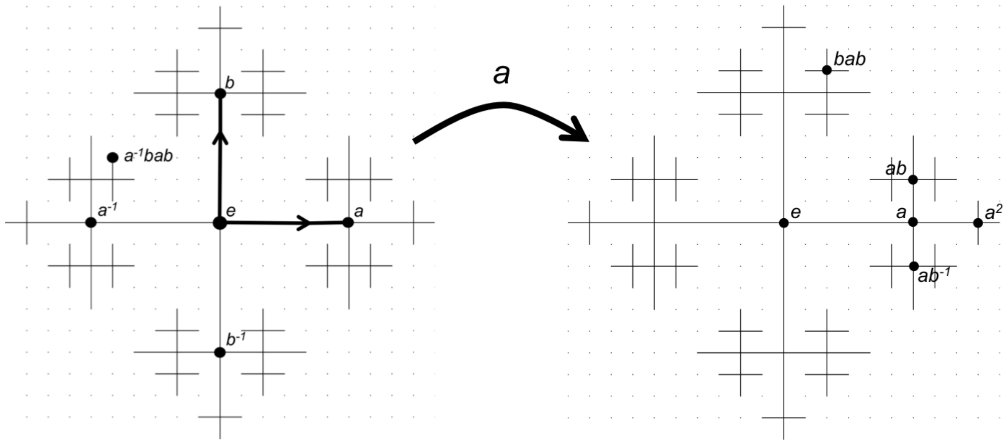

Figure 2.1: The figure on the left shows the Cayley graph of F_2, which F_2 acts freely upon. Several vertices labeled. The figure on the right shows the image of these vertices under the action of a on the tree.

G. The group G always acts freely upon its Cayley graph, and the vertices of Γ are in one-to-one correpsondence with the elements of G. If G is a free group, and the generating set is a basis, then the Cayley Graph of G is a tree. In fact, we have the following well-known result, which will be important in the next chapter. The theorem may be found in [7], Theorem 3.20.

Theorem 2.4. *If G is a group, then G is free if and only if G acts freely on a tree.*

Example 2.5. Let $F_2 = \langle a, b \rangle$ be the free group on generators a and b. Then, F_2 acts its Cayley graph Γ, which is an infinite tree. Part of Γ is pictured in Figure 2.1. From any vertex $v \in V(\Gamma)$, we see four edges leaving v. The upward vertical edge is identified with the b, the generator of F_2, and the downward edge with b^{-1}. Similarly, we identify a with the horizontal edges leaving v in the rightward direction, and identify a^{-1} with the horizontal edge leaving the vertex in the leftward direction.

In fact, in this tree we can identify every vertex of Γ with some element of F_2. Starting at the vertex corresponding to the identity e, if we pass along the edge identified with a, we obtain the vertex $ea = a$. Similarly, the vertex a^{-1} is obtained from moving from e to the left by one edge. If we travel upward from a^{-1}, along a b edge, we obtain the vertex $a^{-1}b$. Then, moving rightward along a, we obtain the vertex $a^{-1}ba$, and moving upward again transports us to the vertex $a^{-1}bab$, as labeled in Figure 2.1.

How does F_2 act on Γ? Given some $v \in V(\Gamma)$, remember that we can identify v with an element of F_2, call it h. Then, given some $a \in F_2$, we define the action of a on v by $av = ah$, where ah is the vertex of Γ associated to the element $ah \in F_2$. As an example, given the vertex $a^{-1}bab$, the action of a on this vertex results in the vertex $a(a^{-1}bab) = bab$. Geometrically, we may view the action as shifting the vertices of Γ to the right, as in Figure 2.1.

Once we define the action of F_2 on the vertices of Γ, the action of F_2 on the edges of Γ is completely determined. The fact that F_2 acts freely on Γ follows easily, when we keep in mind that the vertices of Γ are in one-to-one correspondence with the elements of F_2.

2.3 Crossed Products

If G is a group and D is a ring with identity, then the (ordinary) **group ring** $D[G]$ is formed by taking sums of the form $\sum_{g \in G} q_g g$, where only finitely many of the $q_g \in D$ are non-zero. Addition in the group ring is given in the obvious way, and multiplication is given by

$$\left(\sum_{g \in G} q_g g\right)\left(\sum_{h \in G} r_h h\right) = \sum_{g \in G}\left(\sum_{h \in G} q_g r_h gh\right),$$

where gh denotes the product of g with h in G. It is easy see that the $D[G]$ is a ring with identity $1e$, where 1 is the identity of D, and e is the identity of G. Typically, the multiplicative identity of $D[G]$ is simply written as 1.

In studying group rings, other algebraic objects arise by altering the multiplication in the group ring. An algebraic structure arising in this manner is the crossed product, which is defined as follows. Suppose G is a group acting on a ring D so that there is a map $\sigma : G \to \mathrm{Aut}(D)$, which is not necessarily a homomorphism. Denote the automorphism $\sigma(g)$ by σ_g. Let $\bar{G} = \{\bar{g} : g \in G\}$ be a copy of G, which will be used to define multiplication in what follows. Then, we can form the **crossed product** $D * G$ by taking sums of the form $\sum_{g \in G} r_g \bar{g}$, where only finitely many of the r_g are non-zero. Addition is taken in the obvious way, and we can view $D * G$ as a free, left D-module with basis $\bar{G}$. In order to make $D * G$ a ring, multiplication in $D * G$ is defined using two unexpected rules:

1. (Skewing) For $g \in G$ and $r \in D$, we have $\bar{g}r = \sigma_g(r)\bar{g}$.

2. (Twisting) If $g, h \in G$, then $\bar{g}\bar{h} = \tau(g, h)\overline{gh}$, where $\bar{g}\bar{h}$ denotes their product in $D * G$, gh is the product in G, and $\tau(g, h)$ is some unit in D. Thus, when multiplying $\bar{g}\bar{h}$, the result differs from $\overline{gh}$ by some unit.

Using this definition, it is not apparent that multiplication in $D * G$ is associative. In fact, additional assumptions must be imposed in order to guarantee the associativity of $D * G$, assumptions on the twisting function $\tau : G \times G \to U(D)$, the units of D. For our purposes, however, $D * G$ will always constructed beginning with a group ring $R[G]$, so that associativity will not be a concern.

The notation $D * G$ is ambiguous, because the crossed product is not uniquely determined by D and G. That is, given a ring D and a group G, it is certainly possible to define multiple crossed product structures $D*G$, with each crossed product essentially different. Therefore, to be precise, we would need to specify the skewing function σ and twisting τ whenever introducing a crossed product. However, this ambiguity rarely leads to confusion, and we often say "a crossed product $D*G$" rather than "the crossed product $D*G$" in order to emphasize this fact.

It is clear that every unit in D is a unit in $D*G$. Does the same follow for the elements of G? That is, given some $g \in G$, is the $\bar{g} \in D*G$ invertible? The answer is in the affirmative. Before showing this, we need to find the multiplicative identity of $D*G$. Our first thought is that $1\bar{e}$ should be the identity. In fact, we have that for any $x = \sum_{g \in G} q_g g \in D*G$,

$$\begin{aligned}
1\bar{e}\left(\sum_{g\in G} q_g \bar{g}\right) &= \sum_{g\in G} 1\bar{e} q_g \bar{g} \\
&= \sum_{g\in G} 1\sigma_e\left(q_g\right)\bar{e}\bar{g} \\
&= \sum_{g\in G} q_g \tau\left(e,g\right)\bar{g}.
\end{aligned}$$

Combining this will a similar equality for right-multiplication by $1\bar{e}$, we see that $1\bar{e}$ is the identity of $D*G$ if and only if $\tau(e,g) = \tau(g,e) = 1$ for every $g \in G$. In fact, we lose no generality in making such an assumption (see [9], p. 3), so that we will always assume in our crossed products that $1\bar{e}$ is the identity of $D*G$. We will typically abbreviate this element as 1 or $\bar{1}$. It is interesting to note that using a slightly different twisting function would yield a different multiplicative identity in $D*G$. After setting our multiplicative identity, we may embed D into $D*G$ via the map $d \mapsto d\bar{1}$. In general, the twisted multiplication prevents G from embedding into $D*G$. Now that we have a multiplicative identity in $D*G$, we may prove the following.

Proposition 2.6. *If $g \in G$, then $\bar{g} \in D * G$ is invertible, with $\bar{g}^{-1} = u\overline{g^{-1}}$ for some unit $u \in D$.*

Proof. One can verify that the $\bar{g}\left(\sigma_g^{-1}\left(\tau\left(g, g^{-1}\right)\right)\overline{g^{-1}}\right) = 1\bar{e}$, and $\left(\tau\left(g^{-1}, g\right)^{-1}\overline{g^{-1}}\right)\bar{g} = 1\bar{e}$, so $\bar{g}$ is invertible on both sides. The associativity of $D * G$ asserts that the left and right inverse must be equal, so that $\sigma_g^{-1}\left(\tau\left(g, g^{-1}\right)\right) = \tau\left(g^{-1}, g\right)^{-1}$. Note that their common value is a unit in D. □

If the skewing in $D * G$ is trivial, so that $\sigma_g \in \mathrm{Aut}\,(D)$ is the identity function for each $g \in G$, $D * G$ is known as a **twisted group ring**, denoted by $D^t[G]$. When the twisting is trivial, so that $\tau(g, h) = 1$ for each $g, h \in G$, the crossed product is a **skew group ring,** denoted by DG. In the case of skew group rings, G embeds into DG, so that we may remove the overbars and write elements of DG as $\sum_{g \in G} q_g g$. If both the skewing and twisting are trivial, then $D * G$ is just $D[G]$, the ordinary group ring.[1]

Where would these objects arise in the study of group rings? The following construction can be found in [9], p. 2, and will be important in Chapter 5. Let R be a ring and G be a group, with N a subgroup. We wish to relate the group ring $R[G]$ in some way to N and the collection of cosets G/N. Set $H = G/N$. Because H is a collection of cosets, to each coset $x \in H$ fix some coset representative $\bar{x} \in G$, so that $x = N\bar{x}$. Then, let $\bar{H} = \{\bar{x} : x \in H\} \subset G$ be the collection of all coset representatives. It follows that G is the disjoint union $\bigcup_{x \in H} N\bar{x}$. Therefore, $R[G]$ is the direct sum $\sum_{x \in H} R[N]\,\bar{x}$, and every element of $R[G]$ can be expressed uniquely in the form $\sum_{x \in H} f_x \bar{x}$ where $f_x \in R[N]$. This means that $R[G]$ can be viewed as

[1] While there does not seem to be a consensus, the notation used here ($D^t[G]$, DG, and $D[G]$) is common in the literature.

a free $R[N]$-module with basis $\bar{H}$. If $N \lhd G$, then $H = G/N$ is a group. Setting $D = R[N]$, the structure relating $R[G]$ with N and H is a crossed product $D * H$.

To verify that multiplication in this structure is skewed and twisted, we begin by investigating the twisted multiplication of elements in $\bar{H}$. Given $\bar{x}$ and $\bar{y} \in \bar{H}$, we have $N\bar{x}N\bar{y} = N\overline{xy}$. This means that $\bar{x}\bar{y} \in N\overline{xy}$, and therefore $\bar{x}\bar{y} = n\overline{xy}$ for some $n \in N$, which is a unit in D. Beginning with elements of H and passing to their corresponding preimages in $\bar{H}$, we then have a function $\tau : H \times H \to N \subset U(D)$, where $U(D)$ denotes the set of units in D. It follows that $\bar{x}\bar{y} = \tau(x, y)\overline{xy}$, which gives the twisting.

To study the skewing, note that because $N \lhd G$, we have $gDg^{-1} = D$ for each $g \in G$. Thus, every element of G induces an automorphism on D via conjugation. In particular, the elements of $\bar{H}$ induce an automorphism on D. This gives a map $\sigma : H \to \mathrm{Aut}(D)$ defined by $\sigma : x \mapsto \sigma_x$, where σ_x is the automorphism that conjugates by $\bar{x}$. It follows that

$$\begin{aligned} \bar{x}f &= \left(\overline{x}f\overline{x}^{-1}\right)\overline{x} \\ &= \sigma_x(f)\,\bar{x}, \end{aligned}$$

which is the crossed product's skewing. Note that, because of the twisting in the crossed product, the map σ is not necessarily a group homomorphism. Also, the crossed product $D * H$ is actually a twisted group ring if the elements of N commute with the elements of G, and the crossed product is a skew group ring if the coset representatives $\bar{x}$ can be chosen so that the product of any of our chose coset representatives is another chose coset representative. This happens when the exact sequence $e \to N \to G \to H \to e$ splits, i.e., when G is the semidirect product $N \rtimes H$.

CHAPTER 3

$D * F$-MODULES

3.1 Introduction

Suppose Q is a field and F a finitely-generated free group. In 1964, Paul Cohn proved that all finitely-generated ideals in the group ring $Q[F]$ are free as $Q[F]$-modules. Using this result, we may also deduce that finitely-generated submodules of free $Q[F]$-modules are free. Cohn's proof was technical and difficult, using what he termed a "weak reduction algorithm." In 1990, Hog-Angeloni proved the same result in [3], using the geometry of $Q[F]$ to greatly simplify the arguments. This chapter generalizes these ideas to crossed products $D * F$, with D a division ring and F a finitely-generated free group.

By borrowing ideas from Hog-Angeloni, we offer a geometric interpretation for the elements of $D * F$. Using these geometric notions, we implement a method to reduce the collective "diameter" of collections of elements from $D * F$. By doing this, we are able to prove that finitely-generated submodules of free $D * F$-modules are free.

Section 3.2 introduces the geometric notions that will be important in the rest of this chapter. Then, Section 3.3 builds on these ideas to also give a geometric interpretation for linear combinations of elements from $D * F$, which we view as a $D * F$-module. Then, Section 3.4 introduces a way to replace linearly dependent generating sets with linearly independent generating sets.

It is worth noting that we are primarily interested in $D*F$-modules in this chapter. For this reason, if M is a $D*F$-module, then we use bold faced font for the elements $\mathbf{x} \in M$ to distinguish them from elements of $D*F$ (the "scalars") and elements of F. This has the potential to be confusing when $M = D*F$. In this case, we use bold font for the elements of $D*F$ that we consider to be in the module, and we do not bold the elements of $D*F$ that are "scalars."

3.2 The Geometry of $D*F$

In what follows, let D be a division ring, and F a finitely-generated free group with identity e. If T is the Cayley graph of F, then F acts freely on T, and T is a tree. Further, in T, there is a bijective correspondence between F and $V(T)$, the vertices of T. Thus, we may view the elements of F as vertices in T, and vice versa.

In any tree Γ, any two $u, v \in V(\Gamma)$ have a unique reduced path defined between them. For what follows, we will use the term **geodesic** to describe that path. This path includes a number of edges of Γ, so we can define the length of a path to be the number of edges in that path. We can measure the distance between two vertices $u, v \in V(\Gamma)$ by calculating the length of the geodesic between the vertices. In addition, if w is the midpoint of an edge in T, then we can measure the distance betwen w and a vertex u by drawing a geodesic from w to u, and counting the half-edge traversed by taking w to its neighboring vertex as $1/2$ of an edge.

The following situation will arise frequently in the geometric arguments later this chapter. For this reason, we deal with it once in order to avoid reproving the statement several times.

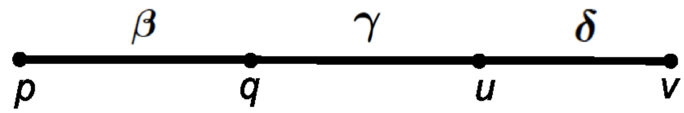

Figure 3.1: An illustration of Proposition 3.1.

Proposition 3.1. *Suppose p, q, u, v are four distinct vertices or midpoints of edges in a tree Γ. Let β denote the geodesic between p and q, γ the geodesic betwen q and u, and δ the geodesic between u and v. Given that β is disjoint from γ, and δ is disjoint from γ, consider the path from p to v obtained by following β from p to q, then γ from q to u, and finally δ from u to v. This path is reduced, so that it is the unique geodesic from p to v.*

Proof. Figure 3.1 gives a graphical intepretation for the hypotheses of the proposition. Suppose on the contrary that the path is not reduced. Because γ is disjoint from β and δ, and all three of these paths are reduced. Because the path from p to v is not reduced, it must be due to cancelling edges coming from β and δ. It follows that β and γ share an edge, and in addition, the paths share a vertex, call it z.

Beginning at the point q, we may follow β until we come to the point z. Then, we may take the path δ from z to u, and finally take γ from u back to q. This produces a loop in the tree Γ. Further, the loop may not fully reduce to a trivial path, because the geodesic γ is disjoint from β and δ. Thus, we have a contradiction to the fact that Γ is a tree, and the edge path from p to v, described in the statement of the proposition, must be reduced. $\square$

Choose some element $\mathbf{x} \in D * F$. Then, $\mathbf{x} = \sum_{g \in F} q_g \overline{g}$, where only finitely many of the q_g are non-zero. Geometrically, we can view $\mathbf{x}$ as a finite collection of vertices in T, with each vertex assigned a non-zero coefficient. Keeping in mind that the

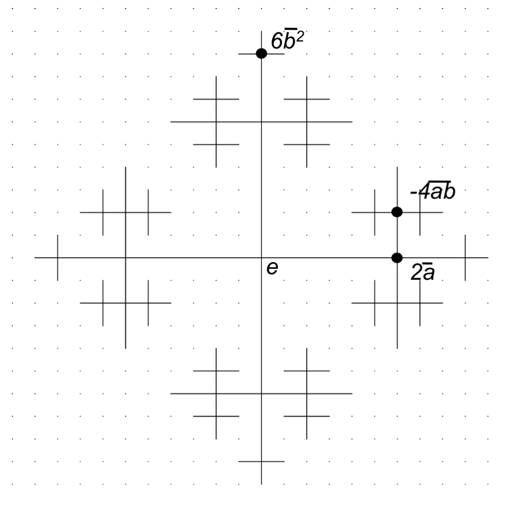

Figure 3.2: A geometric representation for $\mathbf{x} = 2\bar{a} - 4\overline{ab} + 6\overline{b^2} \in D * F$.

vertices of T are identified with the elements of F, the vertices included in $\mathbf{x}$ are those vertices g with $q_g \neq 0$. The coefficient assigned to each vertex is q_g. The summands $q_g\overline{g}$ with $q_g \neq 0$ will be called **points** of $\mathbf{x}$. Alternatively, if $q_g \neq 0$, we might also refer to the vertex g as a point in $\mathbf{x}$. Using this interpretation for elements of $D * F$, we may view $D * F$ as the set of all finite collections of vertices in T, each with an assigned non-zero coefficient in D.

Example 3.2. Taking D as the rationals, and F as the free group on two generators a and b, suppose $\mathbf{x} = 2\bar{a} - 4\overline{ab} + 6\overline{b^2}$. Figure 3.2 offers the geometric interpretation for $\mathbf{x}$.

We define the **distance** of $\mathbf{x}$, written $\operatorname{dist}(\mathbf{x})$, as the length of the longest geodesic from a point of $\mathbf{x}$ to the identity vertex e. Intuitively, this is a measure of how far $\mathbf{x}$ lies from the vertex e. With this definition, for any $q \in D$, we have $\operatorname{dist}(q\bar{e}) = 0$. We define $\operatorname{dist}(0) = -1$. The collection of points of $\mathbf{x}$ must contain some point that attain this maximal distance to the vertex e. Such points will be called **extreme points** of $\mathbf{x}$.

The **diameter** of $\mathbf{x}$, written $\operatorname{diam}(\mathbf{x})$, is the length of the longest geodesic from one point of $\mathbf{x}$ to another, where it is understood that the coefficients assigned to the endpoints are non-zero. Under this definition, for any $g \in G$ we have $\operatorname{diam}(q\bar{g}) = 0$, so the elements of $D * F$ containing only one point have diameter 0. Define $\operatorname{diam}(0)$ to be -1. We might also use the term diameter to refer to a reduced geodesic between points in $\mathbf{x}$ that attains the length $\operatorname{diam}(\mathbf{x})$. The definition of diameter also provides a notion of a radius for $\mathbf{x}$, by defining the radius of $\mathbf{x}$ to be $\operatorname{diam}(\mathbf{x})/2$.

Remark 3.3. Let us phrase these terms in a slightly more familiar context. If we let $D = \mathbb{Q}$, and $F = \mathbb{Z}$, then we may take $D * F$ to be the ordinary group ring $\mathbb{Q}[\mathbb{Z}]$. Given some $\mathbf{x} \in \mathbb{Q}[\mathbb{Z}]$, we have $\mathbf{x} = \sum_{i=-\infty}^{\infty} q_i t^i$, where only finitely many q_i are non-zero. In fact, the ring $\mathbb{Q}[\mathbb{Z}]$ is a Laurent polynomial ring. Then, points of $\mathbf{x}$ would be those terms $q_i t^i$ with $q_i \neq 0$. The term $\operatorname{dist}(\mathbf{x})$ would be the maximum of the set $\{|i| : q_i \neq 0\}$. Further, $\operatorname{diam}(\mathbf{x})$ would be $\max\{|i-j| : q_i, q_j \neq 0\}$, the difference between the highest and lowest degree terms of $\mathbf{x}$.

Example 3.4. Let $\mathbf{x}$ be as in Example 3.2. Then, $\operatorname{dist}(\mathbf{x}) = 2$, because the maximal number of edge paths between a vertex in $\mathbf{x}$ and the vertex e is 2. In fact, 2 is the distance between ab and e, and it is also the distance between b^2 and e. Therefore, the points $-4\overline{ab}$ and $6\overline{b^2}$ are extreme points of $\mathbf{x}$. We also have that $\operatorname{diam}(\mathbf{x}) = 4$, which is the distance between the vertices b^2 and ab. The reduced edge path between b^2 and ab is a diameter of $\mathbf{x}$.

Now that diameter has been defined, we may also obtain a notion of the center of an element of $D * F$. Given $\mathbf{x} \in D * F$, define the **barycenter** of $\mathbf{x}$, denoted by

$\hat{\mathbf{x}}$, as the midpoint of the geodesic between two points in $\mathbf{x}$ whose length attains the diameter of $\mathbf{x}$. If $\operatorname{diam}(\mathbf{x})$ is even, then $\hat{\mathbf{x}}$ is a vertex in T (whose assigned coefficient might be zero). If $\operatorname{diam}(\mathbf{x})$ is odd, then $\hat{\mathbf{x}}$ is a midpoint of an edge in T. If u is the endpoint of a diameter with $\hat{\mathbf{x}}$ as a midpoint, we call the geodesic joining $\hat{\mathbf{x}}$ to u a radius of $\mathbf{x}$. Note that the radius has length $\operatorname{diam}(\mathbf{x})/2$.

Example 3.5. This continues Examples 3.2 and 3.4. Because the geodesic between b^2 and ab has the vertex e as a midpoint, it follows that e is the barycenter of $\mathbf{x}$. That is, $\hat{\mathbf{x}} = e$. Note that e is the barycenter of $\hat{\mathbf{x}}$, even though the coefficient assigned to e by $\mathbf{x} \in D * F$ is zero. Then, the geodesics joining e to b^2 and e to ab are radii of $\mathbf{x}$, and they have length $2 = \operatorname{diam}(\mathbf{x})/2$.

Given the definition of barycenter, it is not immediately clear that the barycenter of $\mathbf{x}$ is well-defined. Is it possible that $\mathbf{x}$ would contain two diameters with distinct midpoints? We prove this cannot happen by using the following propositions:

Proposition 3.6. *Let y be the midpoint of a diameter of $\mathbf{x}$. Suppose q is any vertex in the tree T, and let β be the geodesic from q to y. If we set $r = \operatorname{diam}(\mathbf{x})/2$, then there is some other point of $\mathbf{x}$, call it p, so that: (1) The distance from p to y is r, and (2) The geodesic from p to y is disjoint from β.*

Proof. As C is the midpoint of a diameter in $\mathbf{x}$, there exists points u and v of $\mathbf{x}$ so that the geodesic between u and v is a diameter, and C is a midpoint of that geodesic. Then, the distance between C and u is r, as is the distance between C and v. Let γ denote the radius between C and u, and δ the radius between y and v. Note that γ must be disjoint from δ, because they are each half of the geodesic between u and v. In addition, the radii γ and δ both have length r.

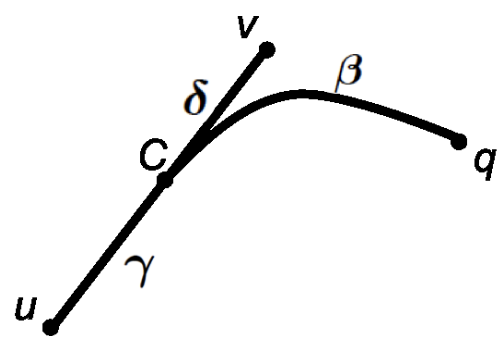

Figure 3.3: The relationship between the points and paths decribed in the proof of Proposition 3.6

Given some point q of $\mathbf{x}$, consider the geodesic β running between q and C. As β leaves C, it cannot use the same edge as both γ and δ, because γ and δ are disjoint Without loss of generality, suppose that β does not leave C using the same edge as γ. Then, β must be totally disjoint from γ, because if β intersected γ at some point, the edge path beginning at C formed by following β to the intersection with γ, and γ back to C would be a cycle in the tree T. Thus, β and γ must be totally disjoint. By setting $p = u$, we are done. □

An immediate consequence of this proposition that if q is a point of $\mathbf{x}$, and C a midpoint of a diameter, then the distance from q to y must be no larger than $\operatorname{diam}(\mathbf{x})/2$. Otherwise, we could find some point p of $\mathbf{x}$ so that the geodesic from p to q is longer than $\operatorname{diam}(\mathbf{x})$, which contradicts the maximality of $\operatorname{diam}(\mathbf{x})$. We exploit this in the following proposition.

Proposition 3.7. *If $\mathbf{x} \in D * F$, then the barycenter of $\mathbf{x}$ is well-defined.*

Proof. Suppose $\mathbf{x}$ contains two diameters, with respective midpoints w_1 and w_2. Then, w_1 and w_2 are two possible candidates for the barycenter of $\mathbf{x}$, and we want to show that $w_1 = w_2$. Set $r = \operatorname{diam}(\mathbf{x})/2$, so that r is the radius of $\mathbf{x}$.

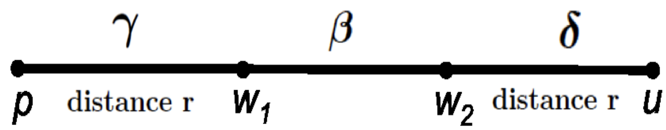

Figure 3.4: The relationship between the points described in the proof of Proposition 3.7.

Let β denote the edge path between w_1 and w_2, and suppose the length of β is $l > 0$. We want to arrive at a contradiction, and it will follow that $l = 0$, so $w_1 = w_2$. Using the Proposition 3.6 with $C = w_1$ and $q = w_2$, we can find some point p so that the distance from w_1 to p is r, and the geodesic from w_1 to p, call it γ, is disjoint from β. Applying the proposition again, this time with $C = w_2$ and $q = w_1$, we can find some other point, call it u, so that the distance from w_2 to u is r, and the geodesic from w_2 to u, call it δ is disjoint from β. See Figure 3.4 for an illustration.

The edge path from p to w_1 is reduced and disjoint from β. Further, the edge path between w_2 and u is reduced and disjoint from β. It follows from Proposition 3.1 that the edge path from p to u obtained by traveling from p to w_1 (distance r) and then from w_1 to w_2 via β (distance l) and finally from w_2 to u (distance r) gives a reduced edge path with length $2r + l > \operatorname{diam}(\mathbf{x})$. This is a contradiction, and it follows that $l = 0$. Thus $w_1 = w_2$, and the barycenter is well-defined. □

Given that $\hat{\mathbf{x}}$ is well-defined, we prove the following result, which will be useful in the proof of several lemmas in Section 3.4.

Proposition 3.8. *If p is an extreme point of $\mathbf{x}$, then the geodesic from p to the vertex e passes through $\hat{\mathbf{x}}$, and p is the endpoint of a diameter of $\mathbf{x}$.*

Proof. Suppose that the distance from an extreme point to the vertex e is s. Set $r = \operatorname{diam}(\mathbf{x})/2$. Let β denote the geodesic between p and the vertex e. If we

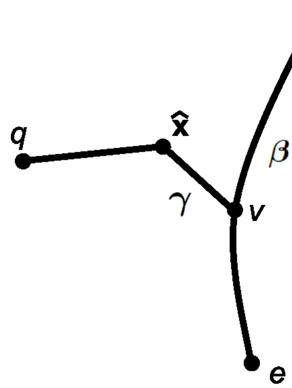

Figure 3.5: An illustration of the proof of Proposition 3.8.

consider the geodesic from $\hat{\mathbf{x}}$ to e, it must intersect β at some point (possibly only at e). Then, let v denote the point where these two geodesics intersect, and γ the edge path from $\hat{\mathbf{x}}$ to v. If the length of γ is 0, then $\hat{\mathbf{x}}$ lies on the edge path β. Thus, we need to show that the length of γ is 0.

Note that because of our choice of v, the path obtained by following γ from $\hat{\mathbf{x}}$ to v, and then β from v to p is reduced, and therefore, the geodesic from $\hat{\mathbf{x}}$ to p. Because r is the length of any radius of $\mathbf{x}$, the distance betwen $\hat{\mathbf{x}}$ and p must be at most r. Because v lies on this geodesic, its distance from p must be at most r. Further, because the distance from p to e is s, it follows that the distance from v to e must be at least $s - r$. By Proposition 3.6, we can find a point q of $\mathbf{x}$ so that the geodesic between q and $\hat{\mathbf{x}}$ has length r, and is disjoint from the geodesic between p and $\hat{\mathbf{x}}$. Therefore, the path between q and $\hat{\mathbf{x}}$ is disjoint from γ. See Figure 3.5.

Suppose that $\hat{\mathbf{x}} \neq v$, so the length of γ is larger than 0. We may form a geodesic from q to e by passing from q to $\hat{\mathbf{x}}$ (length r), and then from $\hat{\mathbf{x}}$ to v via γ (length > 0) and finally from v to e (length $\geq s - r$) along the path β. This edge path is reduced by Proposition 3.1. Thus, the distance from q to e is larger than s, which contradicts the maximality of s. We conclude that the length of γ must be 0, so $\hat{\mathbf{x}} = v$, and $\hat{\mathbf{x}}$ lies on the geodesic β, as desired.

What is the distance from $\hat{\mathbf{x}}$ to e? If this distance were larger than $s - r$, then using the same logic above, the distance from the point q to e would be larger than s. Thus, the distance from $\hat{\mathbf{x}}$ to e is $s - r$. Because the distance from p to e is s, and $\hat{\mathbf{x}}$ lies on the geodesic between p and e, it follows that the distance from p to $\hat{\mathbf{x}}$ is exactly r. Then, the distance from p to q is $2r = \operatorname{diam}(\mathbf{x})$, p is the endpoint of a diameter of $\mathbf{x}$. $\square$

The group F acts on T, and in particular, F acts on the vertices of T. Let $\mathbf{x} \in D * F$. Because $\mathbf{x}$ is a can be viewed as a collection of vertices in T, so we obtain an action of F on $\mathbf{x}$ by letting F act on the vertices of $\mathbf{x}$. Algebraically, for $\bar{h} \in \bar{F}$ and $\mathbf{x} = \sum_{g \in F} q_g \bar{g}$, we have

$$\bar{h}\mathbf{x} = \sum_{g \in F} \sigma_h (q_g) \, \tau(h, g) \, \overline{hg}$$

where σ_h is some automorphism of D, and $\tau(h, g)$ is some invertible element of D. Thus, we notice that the coefficient associated to $\overline{hg}$ by $\bar{h}\mathbf{x}$ is non-zero if and only if the coefficient assigned to $\bar{g}$ by $\mathbf{x}$ is non-zero.

The vertex hg is simply the vertex g under the action of h. Therefore, given some $\bar{h} \in \bar{F}$, a geometric interpretation for $\bar{h}\mathbf{x}$ can be obtained by letting h act on the vertices of $\mathbf{x}$, and slightly altering the coefficients at each resulting vertex. In fact, the vertices used in $\bar{h}\mathbf{x}$ are precisely the collection of vertices obtained by letting h act on the vertices of $\mathbf{x}$.

For each $h \in F$, the action of h on T is a symmetry of T. It follows that the distances between the vertices of $\mathbf{x}$ is preserved in the action by $\bar{h}$ on $\mathbf{x}$, and diam $\left(\bar{h}\mathbf{x}\right) = \text{diam}\,(\mathbf{x})$. In addition, it is also clear that the barycenters of $\mathbf{x}$ and $\bar{h}\mathbf{x}$ satisfy the relationship $\widehat{\bar{h}\mathbf{x}} = h\hat{\mathbf{x}}$, where $h\hat{\mathbf{x}}$ is the action of $h \in F$ on the vertex or edge midpoint $\hat{\mathbf{x}}$.

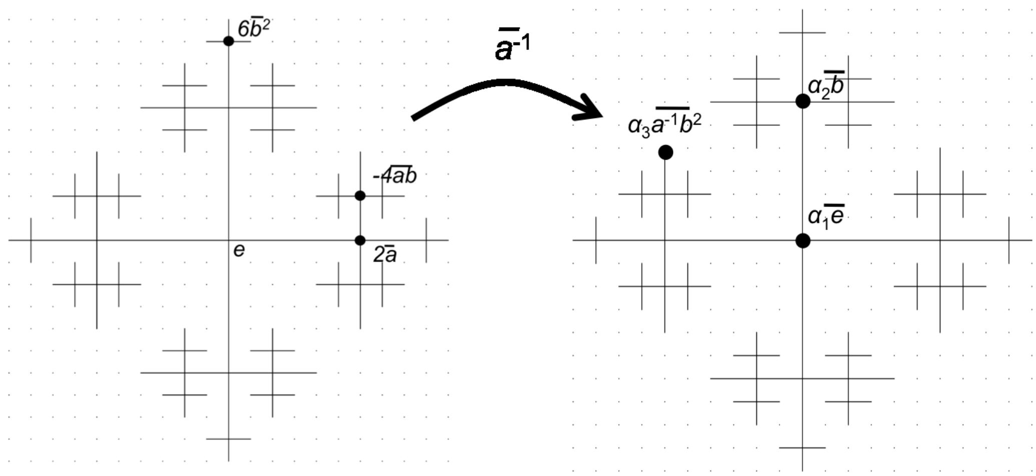

Figure 3.6: A geometric interpretation for the action of $\overline{a^{-1}}$ on $\mathbf{x}$.

Example 3.9. This continues the previous examples. The action of $\overline{a^{-1}}$ on $\mathbf{x}$ gives

$$\overline{a^{-1}}\mathbf{x} = \sigma_{a^{-1}}(2)\,\tau\left(a^{-1},a\right)\bar{e} - \sigma_{a^{-1}}(4)\,\tau\left(a^{-1},ab\right)\bar{b} + \sigma_{a^{-1}}(6)\,\tau\left(a^{-1},b^2\right)\overline{ab^2}.$$

For simplicity, write $\overline{a^{-1}}\mathbf{x}$ as $\alpha_1\bar{e} + \alpha_2\bar{b} + \alpha_3\overline{a^{-1}b^2}$. Then, by looking at Figure 3.6, we see that the action of $\overline{a^{-1}}$ on $\mathbf{x}$ shifts vertices of $\mathbf{x}$ to the left, and changes the coefficients. Further, we observe that diam $\left(\overline{a^{-1}}\mathbf{x}\right) = \text{diam}\,(\mathbf{x}) = 4$, and that the barycenter $\widehat{\overline{a^{-1}}\mathbf{x}}$ is a^{-1}. Notice that this is the same as $a^{-1}\hat{\mathbf{x}}$, because $\hat{\mathbf{x}} = e$.

3.3 Linear Combinations in $D * F$

Using the geometric language discussed in the previous section, we may study the geometry of linear combinations of elements in $D * F$. Let $\mathbf{x}_1, ..., \mathbf{x}_n \in D * F$ (the module) and $\alpha_1, ..., \alpha_n \in D * F$ (the ring). Then, consider the linear combination $\sum_{i=1}^{n} \alpha_i \mathbf{x}_i$. By writing $\alpha_i = \sum_{g \in F} q_{i,g}\bar{g}$, we may rewrite the linear combination as $\sum_{i=1}^{n} \left(\sum_{g \in F} q_{i,g}\bar{g}\mathbf{x}_i \right)$. Consider each summand $q_{i,g}\bar{g}\mathbf{x}_i$ with $q_{i,g} \neq 0$, and find the summand(s) such that $\operatorname{dist}(q_{i,g}\bar{g}\mathbf{x}_i)$ is maximized. These summands will be called **extreme summands**, because they are the summands with the maximal distance from the vertex e. Each extreme summand each contain at least one point $q_g\bar{g}$ that realizes this maximum distance. Call such points **extreme points** of the linear combination.

Remark 3.10. Let us try to phrase these terms in a more familiar context: the ring $\mathbb{Q}[x]$ of polynomials in one variable over the rationals, which we view as a $\mathbb{Q}$-module. Given a linear combination $q_1 p_1(x) + \cdots + q_n p_n(x)$, with $q_i \in \mathbb{Q}$ and $p_i \in \mathbb{Q}[x]$, the analogue for extreme summands of the linear combination are those $q_i p_i(x)$ with maximal degree. Then, the extreme points are the terms of each $q_i p_i(x)$ that realize this maximal degree.

3.4 $D * F$-modules

Consider the left $D * F$-module J generated by the finite collection $\mathbf{x}_1, ...\mathbf{x}_n \in J$. We write $M = \langle \mathbf{x}_1, ..., \mathbf{x}_n \rangle$. We would like to replace this set with a linearly independent set $\mathbf{y}_1, ..., \mathbf{y}_m$, with $m \leq n$, so that $\langle \mathbf{y}_1, ..., \mathbf{y}_m \rangle = M$. In this section, we present an algorithm that allows us to do this.

First, we present an algorithm for finitely-generated submodules of $D * F$, which are simply ideals in $D * F$. Given a collection $\mathbf{x}_1, ...\mathbf{x}_n \in D * F$, the algorithm repeatedly decreases $\sum \operatorname{diam}(\mathbf{x}_i)$ until the non-zero elements of the collection are linearly independent. At that point, we may remove all zero elements to obtain a basis for finitely-generated ideals in $D * F$.

Our eventual goal is to prove that all finitely-generated submodules of free $D * F$-modules are free. By using the result in $D * F$ (which is free with rank 1), we may translate the result to modules with higher rank. Because we will eventually work in free modules with rank larger than 1, it will be useful to introduce matrices into our discussion.

Given an $m \times n$ matrix M, with entries in $D * F$, each row of M corresponds to an elements of the module $(D * F)^n$. Then, we define the **row space** of M to be the submodule of $(D * F)^n$ generated by the rows of M. We also have the following definition.

Definition 3.11. A $n \times n$ matrix A, with entries in $D * F$, is called an **elementary matrix** if A is either the $n \times n$ identity matrix I_n, or A may be obtained from I_n in one of the following ways:

1. A is obtained by interchanging two rows of I_n.
2. A is obtained by multiplying one row of I_n by a unit.
3. A is obtained by changing one off-diagonal entry of I_n from a 0 to some non-zero element of $D * F$.

Under these requirements, it is easy to see that the elementary matrices are invertible matrices, and generate a group under multiplication. Let $E_n(D * F)$ denote the group of $n \times n$ matrices generated by the elementary matrices.

Left-multiplication by elements of $E_n(D * F)$ preserves the row space of a given matrix, as we prove in the following proposition.

Proposition 3.12. *Let $A \in E_m(D * F)$ and M be an $m \times n$ matrix with entries in $D * F$. If J is the row space of M, then the row space of of AM is also J.*

Proof. Let $\mathbf{x}_1, ..., \mathbf{x}_m \in (D * F)^n$ denote the rows of M. Then, $J = \langle \mathbf{x}_1, ..., \mathbf{x}_m \rangle$. It suffices to show that if A is an elementary $m \times m$ matrix, then the rows of AM generate J, because every element of $E_m(D * F)$ is a finite product of elementary matrices.

If A is an elementary matrix, then A has one of the three forms given in Definition 3.11. For the first two forms, it is obvious that the rows of AM also generate the module J, so suppose that A has the third form. Then, A is identical to the identity matrix, except for having some $r \in D * F$ in some non-diagonal entry, say the k-th row and the l-th column. If $\mathbf{y}_1, ..., \mathbf{y}_m$ denote the rows of AM, then we have the relationship $\mathbf{y}_i = \mathbf{x}_i$ for $i \neq k$, and $\mathbf{y}_k = \mathbf{x}_k + r\mathbf{x}_l$. In this case, though, it is clear that $\langle \mathbf{y}_1, ..., \mathbf{y}_m \rangle = \langle \mathbf{x}_1, ..., \mathbf{x}_m \rangle$, and the result follows. $\square$

With the definition of elementary matrices and this proposition in hand, we are prepared to prove the following result.

Theorem 3.13. *Given a linearly dependent collection $\mathbf{x}_1, ..., \mathbf{x}_n \in D * F$, with each $\mathbf{x}_i \neq 0$, there exists $\mathbf{y}_1, ..., \mathbf{y}_n \in D * F$ and a matrix $A \in E_n(D * F)$ so that*

$$A \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix} = \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix},$$

and $\sum_{i=1}^n \operatorname{diam}(\mathbf{y}_i) < \sum_{i=1}^n \operatorname{diam}(\mathbf{x}_i)$.

Note that, by Proposition 3.12, we also have the equality $\langle \mathbf{x}_1, ..., \mathbf{x}_n \rangle = \langle \mathbf{y}_1, ..., \mathbf{y}_n \rangle$.

The proof of the theorem is long and involved, so it will be broken into smaller pieces. First, we set up the proof: because $\mathbf{x}_1, ..., \mathbf{x}_n$ are linearly dependent, there exists $\alpha_1, ..., \alpha_n \in D * F$, not all zero, such that $\sum_{i=1}^{n} \alpha_i \mathbf{x}_i = 0$. Writing $\alpha_i = \sum_{g \in F} q_{i,g} \bar{g}$, we may rewrite the linear combination as $\sum_{i=1}^{n} \left(\sum_{g \in F} q_{i,g} \bar{g} \mathbf{x}_i \right)$. For what follows, we only consider summands with $q_{i,g} \neq 0$. Moreover, because we care about the geometry of this situation, and not the coefficients $q_{i,g}$ (except that they are non-zero), we will omit $q_{i,g}$ from our discussion. Consider the set of extreme points of the linear combination (those points of maximal distance from the identity vertex e) and extreme summands (those $\bar{g}\mathbf{x}_i$ with $q_{i,g} \neq 0$ containing an extreme point of the linear combination). Set s as the distance from extreme points of the linear combination to the vertex e. Of the extreme summands, choose one with maximal diameter. Without loss of generality, we will call this $\bar{h}\mathbf{x}_k$, for some $h \in F$ and a fixed $k \in \{1, ..., n\}$. Set $r = \operatorname{diam}\left(\bar{h}\mathbf{x}_k\right)/2$. This means that the greatest distance any point of $\bar{h}\mathbf{x}_k$ may have from the barycenter $\widehat{\bar{h}\mathbf{x}_k}$ is r. Among the extreme summands, let $\mathcal{R}$ be the set of those summands that contain an extreme point p, such that the geodesic from p to the vertex e passes through $\widehat{\bar{h}\mathbf{x}_k}$, the barycenter of $\bar{h}\mathbf{x}_k$. Such extreme points with this property will be called **special extreme points**, and $\mathcal{R}$ is the collection of summands containing special extreme points.

The proof of the theorem proceeds with the following lemmas.

Lemma 3.14. *The extreme summand $\bar{h}\mathbf{x}_k$ is in the collection $\mathcal{R}$, and the distance from $\widehat{\bar{h}\mathbf{x}_k}$ to the vertex e is $s - r$. Further, every special extreme point is at distance r from $\widehat{\bar{h}\mathbf{x}_k}$.*

Proof. In order to show that $\bar{h}\mathbf{x}_k$ is in $\mathcal{R}$, we show that any extreme point of $\bar{h}\mathbf{x}_k$ is a special extreme point. Let p be an extreme point of $\bar{h}\mathbf{x}_k$. Then, Proposition 3.8

shows that the geodesic from p to e passes through $\widehat{\bar{h}\mathbf{x}_k}$, and p is a special extreme point. Because $\bar{h}\mathbf{x}_k$ contains a special extreme point, it is a member of the collection $\mathcal{R}$.

Furthermore, in the proof of Proposition 3.8, we showed that p must be the endpoint of a diameter of $\bar{h}\mathbf{x}_k$, so the distance between p and $\widehat{\bar{h}\mathbf{x}_k}$ is r. Because the geodesic between p and e passes through $\widehat{\bar{h}\mathbf{x}_k}$, it follows that the distance from $\widehat{\bar{h}\mathbf{x}_k}$ to e is $s-r$. From this, we may conclude that the distance from *any* special extreme point to $\widehat{\bar{h}\mathbf{x}_k}$ is r. $\square$

Lemma 3.15. *If p is any extreme point of some $\bar{g}\mathbf{x}_i \in \mathcal{R}$, then p is special. Moreover, if $\bar{g}\mathbf{x}_i \in \mathcal{R}$, and $r' = \operatorname{diam}(\bar{g}\mathbf{x}_i)/2$, then the distance from $\widehat{\bar{g}\mathbf{x}_i}$ to $\widehat{\bar{h}\mathbf{x}_k}$ is $r-r'$.*

Proof. Notice that, due to our choice of $r = \operatorname{diam}\left(\bar{h}\mathbf{x}_k\right)/2$, we must have $r \geq r'$. Because $\bar{g}\mathbf{x}_i \in \mathcal{R}$, it follows that $\bar{g}\mathbf{x}_i$ has a special extreme point, call it q. Then, Proposition 3.8 shows that the geodesic from q to e must pass through $\widehat{\bar{g}\mathbf{x}_i}$. Moreover, q is the endpoint of a diameter in $\bar{g}\mathbf{x}_i$, its distance from $\bar{g}\mathbf{x}_i$ is r'. From these two facts, it follows that the distance from $\widehat{\bar{g}\mathbf{x}_i}$ is $s-r'$.

Because q is special, the geodesic from q to e must also pass through $\widehat{\bar{h}\mathbf{x}_k}$. Note that the distance from q to $\widehat{\bar{g}\mathbf{x}_i}$ is r', and the distance from q to $\widehat{\bar{h}\mathbf{x}_k}$ is r (by Lemma 3.14). Thus, starting at q and moving to the vertex e, we first pass through $\widehat{\bar{g}\mathbf{x}_i}$, and then through $\widehat{\bar{h}\mathbf{x}_k}$. This shows that $\widehat{\bar{h}\mathbf{x}_k}$ lies on the geodesic between $\widehat{\bar{g}\mathbf{x}_i}$ and e, and using the fact that the distance from $\widehat{\bar{h}\mathbf{x}_k}$ to e is $s-r$, it follows that the distance between $\widehat{\bar{h}\mathbf{x}_k}$ and $\widehat{\bar{g}\mathbf{x}_i}$ is $r-r'$.

Let p be any extreme point of $\widehat{\bar{g}\mathbf{x}_i}$. Then, the geodesic from p to e passes through $\widehat{\bar{g}\mathbf{x}_i}$. After passing through $\widehat{\bar{g}\mathbf{x}_i}$, our work above shows that the geodesic must also pass through $\widehat{\bar{h}\mathbf{x}_k}$. This shows that p is a special extreme point. $\square$

Lemma 3.16. *If q is any point of some $\bar{g}\mathbf{x}_i \in \mathcal{R}$, then the distance from q to $\widehat{\bar{h}\mathbf{x}_k}$ is at most r.*

Proof. Let $r' = \operatorname{diam}(\bar{g}\mathbf{x}_i)/2$. Lemma 3.15 shows that the distance from $\widehat{\bar{g}\mathbf{x}_i}$ to $\widehat{\bar{h}\mathbf{x}_k}$ is $r - r'$. If q is any point of some $\bar{g}\mathbf{x}_i$, its distance to $\widehat{\bar{g}\mathbf{x}_k}$ must be at most r'. Because the distance from $\widehat{\bar{g}\mathbf{x}_i}$ to $\widehat{\bar{h}\mathbf{x}_k}$ is $r - r'$, it follows that the distance from q to $\widehat{\bar{h}\mathbf{x}_k}$ is at most r. □

Lemma 3.17. *The extreme summand $\bar{h}\mathbf{x}_k$ is the only copy of $\mathbf{x}_k$ appearing in $\mathcal{R}$. That is, if $\bar{g}\mathbf{x}_k \in \mathcal{R}$, then $g = h$.*

Consider the subtree of T formed by taking all vertices with distance at most r away from $\widehat{\bar{h}\mathbf{x}_k}$. Note that any diameter of this subtree must have midpoint $\widehat{\bar{h}\mathbf{x}_k}$, using the same arguments to show that the barycenter is well-defined (Proposition 3.7). By Lemma 3.16, every $\bar{g}\mathbf{x}_i \in \mathcal{R}$ must be contained in this subtree. Moreover, if any $\bar{g}\mathbf{x}_i \in \mathcal{R}$ has diameter $2r$, then it must be true that $\widehat{\bar{g}\mathbf{x}_i} = \widehat{\bar{h}\mathbf{x}_k}$. Suppose that there is some $g \in F$ so that $\bar{g}\mathbf{x}_k \in \mathcal{R}$. Then, because the action of $\bar{g}$ on $\mathbf{x}_k$ preserves the diameter of $\mathbf{x}_k$, we have

$$\operatorname{diam}(\bar{g}\mathbf{x}_k) = \operatorname{diam}(\mathbf{x}_k) = \operatorname{diam}\left(\bar{h}\mathbf{x}_k\right) = 2r.$$

Thus, $\widehat{\bar{g}\mathbf{x}_k} = \widehat{\bar{h}\mathbf{x}_k}$. This implies that $g\hat{\mathbf{x}}_k = h\hat{\mathbf{x}}_k$ (see the remarks on p. 23), and $h^{-1}g\hat{\mathbf{x}}_k = \hat{\mathbf{x}}_k$. Thus, the action of $h^{-1}g$ on $\hat{\mathbf{x}}_k$ is trivial. However, because the action of F on T is free, we have $h^{-1}g = e$, and $g = h$. An important consequence of this is that

$$\sum_{\bar{g}\mathbf{x}_i \in \mathcal{R}} q_{i,g}\bar{g}\mathbf{x}_i = q_{k,h}\bar{h}\mathbf{x}_k + \sum_{\bar{g}\mathbf{x}_i \in \mathcal{R},\ i \neq k} q_{i,g}\bar{g}\mathbf{x}_i.$$

Thus, if we gather all $\mathbf{x}_k$ terms in the sum, the coefficient on $\mathbf{x}_k$ is a unit in $D * F$.

Lemma 3.18. *The summands in $\mathcal{R}$ satisfy* $\operatorname{diam}\left(\sum_{\bar{g}\mathbf{x}_i \in \mathcal{R}} q_{i,g}\bar{g}\mathbf{x}_i\right) < \operatorname{diam}(\mathbf{x}_k)$.

Proof. The sum $\sum_{\bar{g}\mathbf{x}_i \in \mathcal{R}} q_{i,g}\bar{g}\mathbf{x}_i$ is formed by taking a collection of summands from the linear combination $\sum_{i=1}^{n} \alpha_i \mathbf{x}_i = 0$. Furthermore, the definition of $\mathcal{R}$ is the collection of all summands containing special extreme points. Because $\sum_{i=1}^{n} \alpha_i \mathbf{x}_i = 0$ and $\mathcal{R}$ contains all special extreme points, the coefficients of all special extreme points must vanish in the sum $\sum_{\bar{g}\mathbf{x}_i \in \mathcal{R}} q_{i,g}\bar{g}\mathbf{x}_i$. For simplicity, set $\mathbf{w} = \sum_{\bar{g}\mathbf{x}_i \in \mathcal{R}} q_{i,g}\bar{g}\mathbf{x}_i$.

Once again, consider the subtree formed in the proof of Lemma 3.17. The element $\mathbf{w}$ is contained in the subtree, because every summand in the definition of $\mathbf{w}$ is contained in the subtree. We want to show that $\operatorname{diam}(\mathbf{w}) < \operatorname{diam}(\mathbf{x}_k) = 2r$. Suppose on the contrary that this were not the case. Then, we would have $\operatorname{diam}(\mathbf{w}) = 2r$, and the barycenter $\hat{\mathbf{w}}$ would be $\widehat{\bar{h}\mathbf{x}_k}$ (by arguments in the proof of Lemma 3.17). Consider a geodesic β between two points of $\mathbf{w}$ with the length of β as $2r$. Note that, by the definition of diameter, if we view the endpoints of β as points of $\mathbf{w}$, the coeffiecients on the endpoints must be non-zero. We will show that one of the endpoints must have distance s from the vertex e, so the endpoint is an extreme point. By 3.15, this point must be a special extreme point, which contradicts the fact that the coefficient assigned to special extreme points must vanish in $\mathbf{w}$.

Because $\widehat{\bar{h}\mathbf{x}_k}$ is the barycenter of $\mathbf{w}$, it must be the midpoint of β, so that the distance from $\widehat{\bar{h}\mathbf{x}_k}$ to the endpoints is r. Futhermore, the radii extending from $\widehat{\bar{h}\mathbf{x}_k}$ to the endpoints of β are disjoint, so at least one of these radii must be disjoint from the edge path from $\widehat{\bar{h}\mathbf{x}_k}$ to e. Call the end of this radius u. Then, traveling from u to $\widehat{\bar{h}\mathbf{x}_k}$ (distance r), and then traveling to e (distance $s - r$) show that the distance from u to e is s. Thus, u is a special extreme point, which is a contradiction. It follows that $\operatorname{diam}(\mathbf{w}) < \operatorname{diam}(\mathbf{x}_k)$, as desired. $\square$

We are now ready to prove Theorem 3.13.

Proof. For $1 \leq k \leq n$, define $\mathbf{y}_k = \sum_{\bar{g}\mathbf{x}_i \in \mathcal{R}} q_{i,g}\bar{g}\mathbf{x}_i$, and $\mathbf{y}_i = \mathbf{x}_i$ for $i \neq k$. Note that it is possible to have $\mathbf{y}_k = 0$. Clearly, $\sum \operatorname{diam}(\mathbf{y}_i) < \sum \operatorname{diam}(\mathbf{x}_i)$, because $\operatorname{diam}(\mathbf{y}_k) < \operatorname{diam}(\mathbf{x}_k)$, and $\operatorname{diam}(\mathbf{y}_i) = \operatorname{diam}(\mathbf{x}_i)$ for $i \neq k$.

In the sum defining $\mathbf{y}_k$, we may pull the $\mathbf{x}_k$ term out, as in the proof of Lemma 3.17. Then, collecting all terms in the sum using the same $\mathbf{x}_i$, we can write

$$\mathbf{y}_k = q_{k,h}\bar{h}\mathbf{x}_k + \sum_{\substack{i \in \{1,\dots,n\} \\ i \neq k}} f_i \mathbf{x}_i$$

for some $f_i \in D * F$. Then, form the $n \times n$ matrix A in the following way. For every $1 \leq i \leq n$, $i \neq k$, let the i-th row of A have a 1 on the diagonal, a zero in every other entry. For the k-th row, put f_i in the i-th column, where $i \neq k$, and $q_{k,h}\bar{h}$ in the k-column. It follows that $A \begin{bmatrix} \mathbf{x}_1 & \cdots & \mathbf{x}_n \end{bmatrix}^T = \begin{bmatrix} \mathbf{y}_1 & \cdots & \mathbf{y}_n \end{bmatrix}^T$. It is not difficult to see that the matrix A is generated by elementary matrices, so $A \in E_n(D * F)$. This completes the proof. $\square$

Repeated applications of Theorem 3.13 can take a linearly dependent generating set, and transform it into a generating set whose non-zero elements are linearly independent. We prove this in the following theorem.

Theorem 3.19. *If $\mathbf{x}_1, \dots, \mathbf{x}_n \in D * F$, then there exists $\mathbf{y}_1, \dots, \mathbf{y}_n \in D * F$ and a matrix $B \in E_n(D * F)$ so that*

$$B \begin{bmatrix} \mathbf{x}_1 \\ \vdots \\ \mathbf{x}_n \end{bmatrix} = \begin{bmatrix} \mathbf{y}_1 \\ \vdots \\ \mathbf{y}_n \end{bmatrix},$$

and the collection of non-zero $\mathbf{y}_i$ *are linearly independent.*

Again, by Proposition 3.12, the statement also implies that $\langle \mathbf{x}_1, ..., \mathbf{x}_n \rangle = \langle \mathbf{y}_1, ..., \mathbf{y}_n \rangle$.

Proof. We prove the result via the following inductively-defined algorithm, which begins with a collection $\mathbf{x}_1, ..., \mathbf{x}_n$. In order to track of the iterations of our algorithm, we will relabel each $\mathbf{x}_i$ with the index $\mathbf{x}_{0,i}$. Each step of the algorithm will increase the first index value by 1.

Step 1. Given the collection $\mathbf{x}_{0,1}, \mathbf{x}_{0,2} ..., \mathbf{x}_{0,n}$, if the non-zero elements of this collection are linearly independent over $D * F$, then we set $\mathbf{y}_i = \mathbf{x}_{0,i}$ for each $1 \leq i \leq n$, and take B to be the identity $n \times n$ matrix, and we are done. In particular, if $\mathbf{x}_{0,1} = \mathbf{x}_{0,2} = \cdots = \mathbf{x}_{0,n} = 0$, then we are done. Otherwise, let $\mathbf{x}_{0,1}, ..., \mathbf{x}_{0,k}$ be the non-zero elements of the collection (if need be, relabel), with $k \leq n$. By Theorem 3.13, because the non-zero $\mathbf{x}_{0,1}, ..., \mathbf{x}_{0,k}$ are linearly dependent, there exists $\mathbf{x}_{1,1}, ..., \mathbf{x}_{1,k} \in D * F$ and a matrix $A \in E_k (D * F)$ such that

$$A \begin{bmatrix} \mathbf{x}_{0,1} & \cdots & \mathbf{x}_{0,k} \end{bmatrix}^T = \begin{bmatrix} \mathbf{x}_{1,1} & \cdots & \mathbf{x}_{1,k} \end{bmatrix}^T$$

and $\sum_{i=1}^{k} \operatorname{diam}(\mathbf{x}_{1,i}) < \sum_{i=1}^{k} \operatorname{diam}(\mathbf{x}_{0,i})$. We can expand A to an $n \times n$ matrix $A_1 \in E_n (D * F)$, by adding rows and columns so that a 1 occurs in the diagonal of the new rows and columns, and zeros everywhere else. For $k < i \leq n$, we set $x_{1,i} = 0$ $(= x_{0,i}$, by assumption) and we have the equation

$$A_1 \begin{bmatrix} x_{0,1} & \cdots & x_{0,n} \end{bmatrix}^T = \begin{bmatrix} x_{1,1} & \cdots & x_{1,n} \end{bmatrix}^T$$

Further,

$$\sum_{i=1}^{n} \operatorname{diam}(\mathbf{x}_{1,i}) < \sum_{i=1}^{n} \operatorname{diam}(\mathbf{x}_{0,i}).$$

The next step is defined inductively. After completing the first j steps, we have a matrix $A_j \in E_n(D * F)$, so that

$$A_j \begin{bmatrix} \mathbf{x}_{0,1} & \cdots & \mathbf{x}_{0,n} \end{bmatrix}^T = \begin{bmatrix} \mathbf{x}_{j,1} & \cdots & \mathbf{x}_{j.n} \end{bmatrix}^T, \tag{3.1}$$

where $\mathbf{x}_{j,1}, ..., \mathbf{x}_{j,n} \in D*F$ are such that $\sum_{i=1}^{n} \operatorname{diam}(\mathbf{x}_{j,i}) < \sum_{i=1}^{n} \operatorname{diam}(\mathbf{x}_{j-1,i})$. Then, the $j+1$-th step is:

Step $j+1$. Given the collection $\mathbf{x}_{j,1}, ..., \mathbf{x}_{j,n}$, we may permute the $\mathbf{x}_{j,i}$ so that the non-zero elements of the collection occur at the beginning of the list. This permuation of the $\mathbf{x}_{j,i}$ must also be accompanied by permuting that n rows of A_j in the same manner, so we maintain the relationship given in Equation (3.1) Then, after permuting, suppose that $\mathbf{x}_{j,1}, ..., \mathbf{x}_{j,r}$ are the non-zero elements of the collection, where $r \leq n$. If $\mathbf{x}_{j,1}, ..., \mathbf{x}_{j,r}$ are linearly independent, we can set $\mathbf{y}_i = \mathbf{x}_{j,i}$ for each i, and and $B = A_j$, and halt the algorithm. Otherwise, by Theorem 3.13, there exists $\mathbf{x}_{j+1,1}, ..., \mathbf{x}_{j+1,r} \in D * F$ and a matrix $A \in E_r(D * F)$ so that

$$A \begin{bmatrix} \mathbf{x}_{j,1} & \cdots & \mathbf{x}_{0,r} \end{bmatrix}^T = \begin{bmatrix} \mathbf{x}_{j+1,1} & \cdots & \mathbf{x}_{j+1,r} \end{bmatrix}^T$$

and $\sum_{i=1}^{r} \operatorname{diam}(\mathbf{x}_{j+1,i}) < \sum_{i=1}^{r} \operatorname{diam}(\mathbf{x}_{j,i})$. Then, we can expand A to an $n \times n$ matrix $A' \in E_n(D * F)$, by adding rows and columns that are 1 only on the diagonal of A', and zero everywhere else. For $r < i \leq n$, set $\mathbf{x}_{j+1,i} = \mathbf{x}_{j,i}$ $(= 0$, by assumption). We have:

$$A' \begin{bmatrix} x_{j,1} & \cdots & x_{j,n} \end{bmatrix}^T = \begin{bmatrix} x_{j+1,1} & \cdots & x_{j+1,n} \end{bmatrix}^T,$$

where $\sum_{i=1}^{n} \operatorname{diam}(\mathbf{x}_{j+1,i}) < \sum_{i=1}^{n} \operatorname{diam}(\mathbf{x}_{j+1,i})$. Define the matrix $A_{j+1} \in E_n(D * F)$ as $A' A_j$. Then,

$$A_{j+1} \begin{bmatrix} \mathbf{x}_{0,1} & \cdots & \mathbf{x}_{0.n} \end{bmatrix}^T = \begin{bmatrix} \mathbf{x}_{j+1,1} & \cdots & \mathbf{x}_{j+1,n} \end{bmatrix}^T.$$

We proceed to the next step of the algorithm.

The claim is that this algorithm must eventually halt. To justify the claim, notice that for any step j, $\sum_{i=1}^{n} \operatorname{diam}(\mathbf{x}_{j,i})$ is bounded below by $-n$, in which case $\operatorname{diam}(\mathbf{x}_{j,i}) = -1$ for each i, and $\mathbf{x}_{j,i} = 0$. Because the sum of the diameters of the generators is finite, strictly decreasing after each step, the algorithm must halt. $\square$

We are now ready to present the main result from the chapter:

Theorem 3.20. *Suppose that D is division ring with F a finitely-generated free group. If N is a finitely-generated submodule of a free $D * F$-module J, then N is also a free module.*

Proof. Every generator of N may only use a finite number of basis elements from J. Because N is finitely-generated, say with m generators, without loss of generality, we may assume that J is free with finite rank, say with rank n Writing each generator of N as a row vector, we may form an $m \times n$ matrix M, where each row of M is the row vector for a generator for of J. We may assume that every column of M contains at least one non-zero entry, otherwise we may disregard that column. The argument continues by applying Theorem 3.19 to each column of M.

If the non-zero elements of the first column of M are not linearly independent, then 3.19 says that we can find some $A_1 \in E_m(D * F)$ so that the non-zero entries of the first column of $A_1 M$ is linearly independent. If the non-zero entries of the second column of $A_1 M$ are not linearly independent, we can find a matrix A_2 so that the non-zero entries of the second column of $A_2 A_1 M$ are linearly independent.

Our concern at this point is that, while the non-zero entries of the first column of $A_1 M$ are linearly independent, is this also true for the non-zero entries of the first row of $A_2 A_1 M$? In fact, this is the case. By multiplying on the left by $A_2 \in E_m(D * F)$, we are essentially performing row operations on the matrix $A_1 M$. Thus, the entries of the first column of $A_2 A_1 M$ are linear combinations of the entries of the first column of $A_1 M$. Because the non-zero entries of the first column of $A_1 M$ are linearly independent, this must also be true of $A_2 A_1 M$.

We continue this process of multiplying by matrices from $E_m(D * F)$ to ensure that the non-zero entries of each column column are linearly independent. If we do this for each of the n columns, we obtain a matrix $M' = A_n A_{n-1} \cdots A_1 M$, so that for each fixed column of M', the non-zero entries of that column are linearly independent. Because M' is obtained from M by elementary matrices, it follows from Proposition 3.12 that the rows of M' generate the same $D * F$-module as the rows of M, which is the module J.

It is possible that M' contains rows of zeros. By discarding zero rows, and reordering the remaining rows of M, we can bring M' to the following form:

$$M' = \begin{bmatrix} * & & & & & & \\ * & & & & & & \\ \vdots & & & & & & \\ & * & & & & & \\ & & * & & & & \end{bmatrix}$$

where $*$ represents the first non-zero entry of a given row. Because the non-zero entries of each column are linearly independent, it follows that the rows of M' are linearly independent, and form a basis for J. Thus, J is a free module. $\square$

The following is an easy corollary. Recall that a right zero divisor is an element $\mathbf{x} \in D * F$, $\mathbf{x} \neq 0$, so that there is some non-zero $a \in D * F$ with $a\mathbf{x} = 0$.

Corollary 3.21. *$D * F$ is a domain.*

Proof. It suffices to show that $D * F$ contains no right zero divisors. Let $\mathbf{x} \in D * F$ be non-zero. We show that $\mathbf{x} \in D * F$ is not a right zero divisor. The left ideal $J = \langle \mathbf{x} \rangle$ can be viewed as a left $D * F$-module, which is a submodule of $D * F$. Thus, by the theorem, we can find some $\mathbf{y} \in D * F$ such that $\mathbf{y}$ serves as a basis for J. Notice that $\mathbf{y}$ cannot be a right zero divisor, otherwise the set $\{\mathbf{y}\}$ would not be linearly independent. Because $\mathbf{x} \in J$, there are some $a, b \in D * F$ with $a\mathbf{x} = \mathbf{y}$ and $b\mathbf{y} = \mathbf{x}$. By substituting, we notice that $ab\mathbf{y} = \mathbf{y}$, and because y is not a right zero divisor, a and b are units in $D * F$. Thus, $\mathbf{x}$ and $\mathbf{y}$ differ by a unit, and because $\mathbf{y}$ is not a right zero divisor, it easily follows that $\mathbf{x}$ is also not a right zero divisor. $\square$

CHAPTER 4

ORE LOCALIZATION

4.1 Introduction

Suppose k is a division ring, and G a group. If we could embed the ordinary group ring $k[G]$ into a division ring Q, then all modules over Q would be free. In particular, submodules of free Q-modules would be free. Sometimes it is not necessary to invert every element of $k[G]$ in order to ensure that finitely-generated submodules of free modules are free. For example, in the group ring $\mathbb{Q}[F]$, with F a finitely free group, submodules of free $\mathbb{Q}[F]$-modules are free, even though $\mathbb{Q}[F]$ clearly has elements that are not units.

If G is a group of the form $H \rtimes F$, then our work on p. 11 shows that the group ring $k[G]$ can be viewed as a crossed product $k[H] * F$. If we can embed $k[G]$ into a larger ring Q so that the elements of $k[H]$ are invertible, then it seems reasonable that this resulting structure would have the form $D * F$, where D is a ring containing the multiplicative inverses of $k[H]$. In the previous chapter, we proved that $D * F$ satisfies the property that all finitely-generated submodules of free $D * F$-modules are free. Thus, if M is a finitely-generated submodule of a free $k[G]$-module, then M has a basis when viewed as module over $D * F$.

In this chapter, we consider the following problem: Given a ring R and $S \subset R$, what conditions guarantee the existence of a ring Q so that R embeds into Q and the elements of S are invertible when mapped into Q? A special case for this problem comes from taking $S = R - \{0\}$, in which case Q would be a division ring (and because $R \to Q$ is an embedding, R must be a domain). The process of creating inverses for the elements of a ring R is called **localization.** If $S \subset R$, and we wish to only construct inverses for the elements of S, then this process is referred to as the localization of R at S. In this chapter, we present the theory of Ore localization, which offers the most intuitive way to localize R at S.

Ring theorists often study localization in a more general setting. By not requiring that the map $R \to Q$ to be an embedding, a ring theorist also does not need the assumption that the elements of S do not divide zero in R. However, because we are primarily interested in embeddings, we do assume that S contains no zero divisors.

The theory contained in this chapter can be found in many texts on localization and noncommutative rings. References include [12] and [4].

4.2 Conditions to Localize R at S

Given a ring R with some subset S, we wish to construct a ring Q and a map $\varphi : R \to Q$ so that for each $s \in S$, the element $\varphi(s) \in Q$ is invertible. Further, wish the map φ to be an embedding, so that we can view $R \subset Q$. The most obvious way to construct Q is to mimic the construction of the rationals from the integers. Thus, we will construct Q from "fractions" with elements of R in the numerator and elements of S in the denominator.

What should the fractions in Q look like? Mimicking the construction of $\mathbb{Q}$ from $\mathbb{Z}$, we might first attempt to define Q as the set of fractions of the form $\frac{a}{b}$ with $a \in R$ and $B \in S$, where the fractions are equipped with some equivalence relation. However, this notation is ambiguous because $\frac{a}{b}$ could mean $b^{-1}a$ or $a^{-1}b$. In the commutative setting, this makes little difference, but because R might not be commutative (and therefore Q might not be commutative), we must specify which "side" of the fraction should contain the denominator. Because it seems more intuitive to write the denominator on the right, as in $a/b = ab^{-1}$, we will write our fractions in this manner, although we could do the following construction with denominators on the left in a similar manner. We now turn our attention to finding conditions on R and S that guarantee that such a ring Q exists.

What must be true of S if such a ring Q exists? An element $r \in R$ is said to be **regular** if it is not a zero divisor. If there were some $s \in S$ which divides zero in R, then it is clear that map $R \to Q$ cannot be an embedding. Thus, every element of S must be regular. Further, we must guarantee that $0 \notin S$ to avoid trying to invert 0. In addition, we need $1 \in S$, so that every element of R can embed into Q via the map $r \mapsto r1^{-1}$. Lastly, the product of invertible elements in Q needs to be invertible, so S should be multiplicatively closed. A set S that satisfies these conditions is called a **multiplicative**[1] subset of R. We now give a definition for our ring Q.

Definition 4.1. Let R be a ring with S a multiplicative subset of R. Suppose that a ring Q exists, along with an embedding $\varphi : R \to Q$, so that every element of Q can be written in the form $\varphi(a)\varphi(b)^{-1}$, with $a \in R$ and $b \in S$. Then, Q is called the **right ring of fractions** for R with respect to S. The ring Q is usually written as RS^{-1}.

Given this definition, it is not immediately clear that the right ring of fractions is unique in anyway. Is it possible for there to be two right rings of fractions for R with

[1] In the more general theory, the definition of multiplicative sets does not include the assumption that the elements of S do not divide zero in R.

respect to S, say Q and Q', which are not isomorphic? The following propositions, the first of which may be found in [12], can be used to show that this may not occur.

Proposition 4.2. *Let R be a ring with S a multiplicative subset, so that a right ring of fractions exists with respect to S, call the ring of fractions Q. Further, suppose that $\varphi : R \to Q$ is an embedding. Then, whenever there is a ring homomorphism $\psi : R \to Q'$ such that $\psi(s)$ is invertible in Q for every $s \in S$, there exists a unique homomorphism $\sigma : RS^{-1} \to Q$ such that $\sigma \circ \varphi = \psi$.*

Proof. Given $a \in R$ and $b \in S$, define $\sigma : Q \to Q'$ by $\sigma\left(\varphi(a)\varphi(b)^{-1}\right) = \psi(a)\psi(b)^{-1}$. We need to show that σ is well-defined on the elements of Q. To show this, suppose that $\varphi(a)\varphi(b)^{-1} = \varphi(r)\varphi(s)^{-1}$. Then, we have the following equation in R:

$$\begin{aligned} \varphi(a)\varphi(s) &= \varphi(r)\varphi(b) \\ \varphi(as) &= \varphi(rb). \end{aligned}$$

Beause φ is an embedding, it follows that

$$as = rb.$$

From this, we conclude that

$$\begin{aligned} \psi(as) &= \psi(rb) \\ \psi(a)\psi(s) &= \psi(r)\psi(b) \\ \psi(a)\psi(b)^{-1} &= \psi(r)\psi(s)^{-1} \\ \sigma\left(ab^{-1}\right) &= \sigma\left(rs^{-1}\right). \end{aligned}$$

Thus, σ is well-defined. It is also not difficult to show that σ is a homomorphism. Further, the fact that $\sigma \circ \varphi = \psi$ and that σ also follows quickly. $\square$

Proposition 4.3. *Suppose Q and Q' are both right rings of fractions for R with respect to S, with embeddings $\varphi : R \to Q$ and $\psi : R \to Q'$. Then, there is an isomorphism between Q and Q'.*

Proof. The previous proposition gives us a unique map $\sigma : Q \to Q'$, which is a homomorphism, so that $\sigma \circ \varphi = \psi$. Reversing the roles of Q and Q', we also obtain a unique map $\sigma' : Q' \to Q$, where $\sigma' \circ \psi = \varphi$. Then, we have the equations

$$\begin{aligned} \sigma' \circ \sigma \circ \varphi &= \varphi \\ \sigma \circ \sigma' \circ \psi &= \psi. \end{aligned}$$

Thus, $\sigma' \circ \sigma : Q \to Q$. By the previous proposition (taking $Q' = Q$), $\sigma' \circ \sigma$ is the unique homomorphism so that $(\sigma' \circ \sigma) \circ \varphi = \varphi$. Because id_Q is another such function, we have that $\mathrm{id}_Q = \sigma' \circ \sigma$. Similarly, $\mathrm{id}_{Q'} = \sigma \circ \sigma'$, and it follows that σ and σ' are the desired isomorphisms. $\square$

In light of this proposition, every right ring of fractions for R with respect to S is essentially the same. For this reason, we can say "the right ring of fractions" rather than "a right ring of fractions," and use the notation RS^{-1} to denote this ring with no ambiguity.

Given S a multiplicative subset of R, we wish to construct the ring RS^{-1}. We know that when RS^{-1} exists, it is unique. However, at this point, it is not clear when RS^{-1} exists. If S is a multiplicative subset of R, we say that S is **right permutable** if for every $s \in S$ and $r \in R$ we have $rS \cap sR \neq \emptyset$. Equivalently, S

is right permutable if and only if for every $s \in S$ and $r \in R$, there exists $t \in S$ and $x \in R$ such that $sx = rt$. Thus, right permutability can be viewed as a weak version of commutativity, because it allows us to take an element of S and an element of R, and find a common multiple. Notice that right permutability is trivially satisfied if R is a commutative.

A similar notion can be defined for left permutability[2], although it is not true that left and right permutability are equivalent in general settings. We will see in the next chapter that left and right permutability are equivalent in group rings, though.

If S is a right permutable, multiplicative subset of R, then S is sometimes called a **right denominator set** in R. The following well-known thereom justifies this term, and gives necessary and sufficient conditions for the existence of RS^{-1}. The proof of this theorem (without making the assumption that the elements of S are regular in R) may be found in [12].

Theorem 4.4. *Suppose that R is a ring with S a multiplicative subset of R Then, the right ring of fractions RS^{-1} exists if and only if S is right permutable.*

Proof. As this is an "if and only if" statement, we need to prove two directions.

(RS^{-1} exists implies right permutable) To prove this direction, if RS^{-1} exists, pick any $s \in S$ and $r \in R$. Then, by the definition of RS^{-1}, the product $s^{-1}r$ must have the form ab^{-1} for some $a \in R$ and $b \in S$. By clearing denominators, we obtain

$$\begin{aligned} s^{-1}r &= ab^{-1} \\ rb &= sa. \end{aligned}$$

This shows that S is right permutable.

[2]The left permutability replaces $rS \cap sR \neq \emptyset$ with $Sr \cap Rs \neq \emptyset$.

(Right permutable implies RS^{-1} exists) We prove this direction by defining an equivalence relation $\sim$ on $R \times S$, and then defining operations on $R \times S/\sim$ to make this set form a ring.

When considering elements of $R \times S$, we view the first coordinate as the numerators and the second as the denominators of the fractions we wish to form. Then, define a relation $\sim$ on $R \times S$ by $(r, s) \sim (p, q)$ if there exists $x, y \in R$ such that $sx, qy \in S$ and $(rx, sx) = (py, qy)$, where equality means that the first coordinates are equal, and the second coordinates are equal. Essentially, the definition of $\sim$ means that we can regard two "fractions" as the same under $\sim$ if they can be brought to the same denominator, and after bringing them to the same denominator, the numerators are also equal.

We can show that $\sim$ is an equivalence relation. Reflexivity and symmetry of $\sim$ are trivial, so we will only prove transitivity. Suppose that $(r, s) \sim (p, q)$ and $(p, q) \sim (a, b)$. Then, there exists $x_1, y_1, x_2, y_2 \in R$ such that $sx_1, qy_1, qx_2, by_2 \in S$ and $(rx_1, sx_1) = (py_1, qy_1)$, $(px_2, qx_2) = (ay_2, by_2)$. Because S is right permutable, there exists $c \in R$ and $d \in S$ such that $(qy_1)\, c = (qx_2)\, d \in S$. Because $q \in S$, and elements of q are not zero divisors, we cancel the q's to obtain $y_1 c = x_2 d$. Then,

$$\begin{aligned}
(rx_1 c, sx_1 c) &= (py_1 c, qy_1 c) \\
&= (px_2 d, qx_2 d) \\
&= (ay_2 d, by_2 d)\,.
\end{aligned}$$

This shows that $(r, s) \sim (a, b)$, and $\sim$ is an equivalence relation. Then, consider the collection of equivalence classes $R \times S/\sim$. Looking toward our end goal, we denote $R \times S/\sim$ by RS^{-1}, and write the equivalence class associated to $(a, b) \in R \times S$ as

$a/b \in RS^{-1}$. An important observation is that, by the definition of $\sim$, if $a/b \in RS^{-1}$ and $c \in R$ is such that $bc \in S$, then $a/b = (ac)\,/\,(bc)$.

We define addition on RS^{-1} as follows:

$$r/s + p/q = (rc + pd)\,/u$$

where $u = sc = qd \in S$, for some $c \in R$ and $d \in S$. We can find such a u by the right permutability of S. Essentially, to add two elements of RS^{-1}, we bring them to a common denominator and then add the numerators. Multiplication is defined in RS^{-1} by:

$$r/s \cdot p/q = (rd)\,/\,(qc)$$

where $c \in S$ and $d \in R$ satisfy $sd = pc$, and the existence of such c and d is guaranteed by the right permutability of S. The motivation for this definition makes more sense if we proceed through some intermediate steps. Suppose that $sd = pc$. Momentarily ignoring the fact that sd, pc might not be in S, we have

$$\begin{aligned} r/s \cdot p/q &= (rd)\,/\,(sd) \cdot (pc)\,/\,(qc) \\ &= (rd)\,/\,(pc) \cdot (pc)\,/\,(qc) \end{aligned}$$

Then, because pc occurs in the denominator of the first fraction and the numerator of the second, we can “cancel” it to obtain

$$r/s \cdot p/q = (rd)\,/\,(qc)$$

Here, we skip over some details. One can verify that these operations are well-defined on the equivalence classes of RS^{-1}. Further, equipped with these operations, RS^{-1} forms a ring with additive identity $0/1$, and multiplicative identity $1/1$.

If we define a map $\varphi : R \to RS^{-1}$ by $\varphi : r \mapsto r/1$, then φ is easily verified to be a ring homomorphism. Furthermore, if $r/1 = 0/1$, then there is some $x, y \in R$ so that $1x = 1y \in S$ (so $x \in S$), and $(rx, x) = (0, y)$, where equality is taken in $R \times S$. Thus, $rx = 0 \in R$, and because $x \in S$, which does not contain zero divisors, it follows that $r = 0$. This shows that the map φ is an embedding, and we can view R as a subring of RS^{-1} by identifying elements of R with their image under φ.

If we view $S \subset RS^{-1}$, then every $s \in S$ can be written $s = s/1$. Using the definition of multiplication on $S \times R/\sim$, we have $s/1 \cdot 1/s = 1/1 = 1/s \cdot s/1$, so every element of S is invertible in RS^{-1}. Additionally, every element RS^{-1} has the form ab^{-1} for $a \in R$ and $b \in S$.

We have shown that the ring RS^{-1} is a right ring of fractions for the ring R with respect to S, and it follows that the right ring of fractions exists. Further, every right ring of fractions is isomorphic to RS^{-1} as above. $\square$

The process of creating a right ring of fractions for a ring R is called **Ore localization**, after Oystein Ore. Concentrating the case where R is a domain, and $S = R - \{0\}$, Ore introduced the right permutability condition to solve the problem of embedding a domain R into a division ring Q. For this reason, if R is a domain so that for every $r \in R$ and $s \in R - \{0\}$, we have $sR \cap r\,(R - \{0\}) \neq \emptyset$, then R is said to satisfy the right **Ore condition.** Equivalently, a domain R satisfies the right Ore condition if for every non-zero $s, r \in R$, we have $sR \cap rR \neq \{0\}$. Any domain that satisfies the right Ore condition is called a right **Ore domain.** Essentially, the Ore condition means that any finite collection of non-zero elements from R can be brought to a common, non-zero multiple by multiplying on the right, as shown in the following proposition.

Proposition 4.5. *If R is a right Ore domain, given a finite collection of non-zero $r_1, ..., r_n \in R$, there exists non-zero $s_1, ..., s_n \in R$, such that $r_1 s_1 = r_2 s_2 = \cdots = r_n s_n$.*

Proof. The proof will be by induction on n. For $n = 2$, this is just the right Ore condition on R. Then, we assume that any collection of n non-zero elements of R can be brought to a common multiple, and consider the collection of non-zero $r_1, ..., r_n, r_{n+1} \in R$. By assumption, there exists non-zero $t_1, ..., t_n \in R$ so that $r_1 t_1 = \cdots = r_n t_n$. Denote their common value by u. Then, $u \neq 0$. Because R satisfies the Ore condition, there exists non-zero s and $s' \in R$ so that $us = r_{n+1}s'$. Then, $r_1 t_1 s = r_2 t_2 s = \cdots = r_n t_n s = r_{n+1} s'$. By writing $s_i = t_i s$ for $1 \leq i \leq n$, and $s_{n+1} = s'$, the proof is complete. □

Besides Ore domains, another special case of Theorem 4.4 is the case where $S = R^{\times}$, the regular elements of R. In this case, the ring RS^{-1} is called the **right classical ring of quotients** for R. This ring has the interesting property that every element is either a zero divisor or a unit.

4.3 Some Examples and Properties of Ore Localization

At this point, while we have necessary and sufficient conditions for the existence of the ring RS^{-1}, we do not have any examples of rings R with multiplicative subsets S satisfying the hypotheses of Theorem 4.4. Let us find some examples. It is trivial to prove that taking R to be any commutative domain and S any multiplicative subset of R, then S is right permutable, so that RS^{-1} exists. For example, if I is a prime ideal in R, then we could take $S = R - I$, and we could form RS^{-1}.

As another example, if we take R to be a division ring and S a multiplicative subset, S is easily verified to be right permutable, so RS^{-1} exists. We can also see that R is a right ring of fractions for itself, so Proposition 4.3 shows RS^{-1} is isomorphic to R.

The following proposition shows that if R contains ideals that are free with rank larger than 1, then R is not an Ore domain.

Proposition 4.6. *Let R be a right Ore domain. Then, a right ideal $I \neq \{0\}$ in R is a free right R-module if and only if it is principal.*

Proof. Suppose that I is a free, right R-module. If the rank of I were larger than one, pick two distinct basis elements from I, call them x and y. Note that x and y must both be non-zero. Because R is a right Ore domain, there exists non-zero $s, t \in R$ so that $xs = yt$. Then, $xs - yt = 0$, and x and y are not linearly independent. This is a contradiction, and it follows that the rank of I cannot be larger than one. Thus, I is principal.

If I is non-zero, right, principal ideal in the domain R, then it is clear that the generator for I serves as a basis for I as a right R-module. Thus, I is free. □

The following proposition shows that Ore domains are common in the study of ring theory.

Proposition 4.7. *If R is a right Noetherian domain, then R is a right Ore domain.*

Proof. We will prove this by contrapositive. Assume that R is a domain that is not a right Ore domain, so R does not satisfy the right Ore condition. Thus, there exists $x \in R$ and $y \in S$ so that $xR \cap yR = \emptyset$. We will produce a strictly increasing chain of right ideals, and deduce that R is not right Noetherian. For $n \geq 0$, define the right ideal $I_n = xR + yxR + y^2xR + \cdots + y^nxR$. Then, we have the ascending chain

$$I_0 \subset I_1 \subset \cdots .$$

The claim is that $I_n \subsetneq I_{n+1}$ for every n, so the chain does not stabilize. If this were not case, let n be the smallest natural number for which $I_n = I_{n+1}$. It follows that $y^{n+1}x \in I_n$, and we can write

$$y^{n+1}x = \sum_{i=0}^{n} y^i x a_i$$

where not every $a_i = 0$. Let r be the smallest natural number so that $a_r \neq 0$, so that $a_r \in S$. Note that $r < n+1$. Then, because we are in a domain, we can cancel a y^r from both sides to obtain:

$$\begin{aligned}
y^{n+1-r}x &= \sum_{i=r}^{n} y^{i-r} x a_i \\
y^{n+1-r}x &= xa_r + \sum_{i=r+1}^{n} y^{i-r} x a_i \\
y^{n+1-r}x - \sum_{i=r+1}^{n} y^{i-r} x a_i &= xa_r \\
y\left(y^{n-r}x - \sum_{i=r+1}^{n} y^{i-r-1} x a_i\right) &= xa_r.
\end{aligned}$$

This contradicts the assumption that x and y are such that $xR \cap yR = \{0\}$, and it follows that $I_n \subsetneq I_{n+1}$ for every n. Thus, R is not Noetherian. $\square$

Because right Artinian rings are also right Noetherian (see [2], Corollary 2.28) this implies the following:

Proposition 4.8. *If R is a right Artinian domain, then R is a right Ore domain.*

Everything we have done above has been for right rings of fractions, and the right Ore condition. One might ask whether there are rings that have a right ring of fractions, but not a left ring of fractions, or equivalently, whether there are rings where right permutability does not imply left permutability. In fact, such rings do exist. An example can be constructed from a skew polynomial ring. See [2], Theorem 5.4, for the construction.

CHAPTER 5

LOCALIZED GROUP RINGS

In this chapter, we apply the theory of Ore localization to group rings, and then deduce an interesting result about modules over localized group rings, for groups of the form $H \rtimes F$, where F is a finitely-generated free group. Because we would like to avoid zero divisors, we also make the standing assumption that all groups we consider are such that $k[G]$ is a domain, unless noted otherwise.

5.1 The Ore Condition and Group Rings

Given a group G, when can we embed $k[G]$ into a division ring? While other forms of localization might be possible, the Ore condition says that $k[G]$ has a right ring of fractions (taking $S = k[G] - \{0\}$) if and only if for every $x \in S$ and $y \in k[G]$, we have $x(k[G]) \cap yS \neq \{0\}$. In a domain, this is equivalent to the statement that for every $x, y \neq 0$, we have $x(k[G]) \cap y(k[G]) \neq \{0\}$. In the previous chapter, we noted that the existence of a right ring of fractions does not guarantee the existence of a left ring of fractions. However, in the case of group rings, it is true that $k[G]$ has a right ring of fractions if and only if it also has a left ring of fractions. To prove this fact, we introduce the function $\lambda : k[G] \to k[G]$ given by $\lambda\left(\sum_{g \in G} a_g g\right) = \sum_{g \in G, a_g \neq 0} a_g^{-1} g^{-1}$. We notice that λ is self-inverting and if $x \in k[G]$ is non-zero, then $\lambda(x)$ is also non-zero.

Proposition 5.1. *The function λ reverses the multiplication of $k[G]$. That is, if $x, y \in k[G]$, then $\lambda(xy) = \lambda(y)\lambda(x)$.*

Proof. Write $x = \sum_{g\in G} a_g g$ and $y = \sum_{g \in G} b_g g$. Then,

$$\begin{aligned} xy &= \left(\sum_{g\in G, a_g \neq 0} a_g g\right)\left(\sum_{h\in G} b_h h\right) \\ &= \sum_{g\in G}\left[\sum_{h\in G} a_g b_h gh\right]. \end{aligned}$$

Therefore,

$$\begin{aligned} \lambda(xy) &= \lambda\left(\sum_{g\in G}\left[\sum_{h\in G} a_g b_h gh\right]\right) \\ &= \sum_{h\in G, b_h\neq 0}\left(\sum_{g\in G, a_g\neq 0} (a_g b_h)^{-1}(gh)^{-1}\right) \\ &= \sum_{h\in G, b_h\neq 0}\left(\sum_{g\in G, a_g\neq 0} b_h^{-1} h^{-1} a_g^{-1} g^{-1}\right) \\ &= \left(\sum_{h\in G, b_h\neq 0} b_h^{-1} h^{-1}\right)\left(\sum_{g\in G, a_g\neq 0} a_g^{-1} g^{-1}\right) \\ &= \lambda(y)\lambda(x). \end{aligned}$$

□

If H is a subgroup of G, then λ maps $k[H]$ into $k[H]$. The following proposition shows that whenever λ maps $S \subset k[G]$ into itself, then right and left permutability are equivalent for S. In particular, they are equivalent for $k[H]$.

Proposition 5.2. *Let k be a division ring and G a group so that $k[G]$ is a domain. Suppose that S is a multiplicative subset of $k[G]$ so that $\lambda(S) \subset S$. Then, the ring*

$(k[G])S^{-1}$ exists if and only if $S^{-1}(k[G])$ also exists. Moreover, these two rings of fractions are isomorphic.

Proof. For the first part of the statement, it suffices to show that S is right permutable if and only if it is left permutable.

Assume S is right permutable. Then, given $x \in S$ and $y \in k[G]$, we need to show that there exists $s \in S$ and $r \in k[G]$ such that $rx = sy$. By assumption, S is invariant under λ, so that $\lambda(x) \in S$ and $\lambda(y) \in k[G]$. Because S is right invertible, there exists $s_0 \in S$ and $r_0 \in k[G]$ such that $\lambda(x) r_0 = \lambda(y) s_0$. Then,

$$\begin{aligned} \lambda(\lambda(x) r_0) &= \lambda(\lambda(y) s_0) \\ \lambda(r_0)\lambda(\lambda(x)) &= \lambda(s_0)\lambda(\lambda(y)) \\ \lambda(r_0)x &= \lambda(s_0)y. \end{aligned}$$

This shows that $k[G]$ also satisfies the left Ore condition. The proof that left permutability implies right permutability is nearly identical, and it follows that the ring $(k[G])S^{-1}$ exists if and only if $S^{-1}(k[G])$ also exists. These rings are naturally isomorphic. □

In particular, the above proposition implies that $k[G]$ satisfies the left Ore condition (that is, $k[G] - \{0\}$ is left permutable) if and only if $k[G]$ satisfies the right Ore condition. It follows that $k[G]$ is a left Ore domain if and only if $k[G]$ is a right Ore domain. For this reason, we often omit "right" or "left" in our discussion, because the two are equivalent.

It is known that for some groups G, the group ring $k[G]$ does not satisfy the Ore condition. For example, if $F_2 = \langle x, y\rangle$, the free group on two generators, then it is well-know that $k[F_2]$ does not satisfy the Ore condition. A full proof of this fact may be found in [5]. However, this statement is obvious to those readers familiar with the augmentation ideal of the group ring $k[F_2]$, which can be viewed as a free right $k[F_2]$-module with basis $x-1, y-1 \in k[F_2]$. Then, by Proposition 4.6, $k[F_2]$ is not an Ore domain. In fact, if G is any group with F_2 as a subgroup, then $k[G]$ does not satisfy the Ore condition.

It is not difficult to deduce that when A is an abelian group, $k[A]$ satisfies the Ore Condition. Besides the abelian groups, are there any classes of groups for $k[G]$ that satisfy the Ore condition, for every G in the class? To answer this question, we first need to explain some terminology. A **directed set** I is a set equipped with a preorder $\preceq$ such that for every $a, b \in I$ there exists a $c \in I$ such that $a \preceq c$ and $b \preceq c$. Let $\mathcal{A}$ be a collection of groups indexed by a directed set I, so that if $G_i, G_j \in \mathcal{A}$ with $i \preceq j$, then $G_i \subset G_j$. Then, the **directed union** of $\mathcal{A}$ is $\bigcup_{i \in I} G_i$. Thus, directed unions can be viewed as a generalization of countable, increasing unions. Let $\mathcal{C}$ denote the class of elementary amenable groups, which is the the class of groups so that:

1. $\mathcal{C}$ contains every abelian and finite group.

2. $\mathcal{C}$ is closed under extensions.

3. $\mathcal{C}$ is closed under directed unions.

For example, the fundamental group of the Klein bottle is an elementary amenable group, being the extension of $\mathbb{Z}$ by $\mathbb{Z}$. We have the following results, which demonstrate the nice properties of group rings of elementary amenable groups. The first result comes from [6] and the second from [5].

Theorem 5.3. *If G is a torsion-free elementary amenable group, and k is a division ring, with $k * G$ a crossed product, then $k * G$ is a domain. In particular, $k[G]$ is a domain.*

Theorem 5.4. *If G is a torsion-free elementary amenable group, then $k[G]$ is a right Ore domain.*

Example 5.5. If G is the Klein bottle group $G = \langle x, y \mid yx = xy^{-1} \rangle$, the group ring $\mathbb{Q}[G]$ is an Ore domain, as G is an extension of $\mathbb{Z}$ by $\mathbb{Z}$.

When working with a group ring $k[G]$, we often do not want to localize the entire ring, but rather just some subset. We are primarily interested in examples where $G = H \rtimes F$, so that $k[H]$ is an Ore domain, and F is a finitely-generated free group. Then, the group ring $k[G]$ has the form $k[H] * F$. From Corollary 3.21, we know that $D * F$ is a domain if D is a division ring. At this point, however, we can make no statement about the group ring $k[G]$. For this reason, in order to localize $k[G]$ at $S = k[H] - \{0\}$, we need to first show that the elements of S do not divide zero in $k[G]$. Then, the fact that S is multiplicative easily follows. In addition, we also need to show that S is permutable subset of $k[G]$. We do this in the following propositions, whose proofs are adapted from arguments used by Rosset in [11].

Proposition 5.6. *If $H \lhd G$, so that $k[H]$ is a domain, then the elements of $k[H]$ do not divide zero in $k[G]$.*

Proof. Pick some non-zero $x \in k[G]$ and non-zero $y \in k[H]$. We will show $yx \neq 0$, and $xy \neq 0$. To this end, write $k[G]$ in the form $k[H] * G/H$, so that x can be written uniquely in the form $\sum_{\bar{g} \in G/H} f_g \bar{g}$.

We have the following equation:

$$xy = \left(\sum_{g \in G/H} f_g \bar{g} \right) y = \sum_{g \in G/H} f_g \sigma_g (y) \bar{g}.$$

We have that $\sigma_g (y) \in k [H]$ for each g. Because x is non-zero, at least one of the f_g is non-zero. Because $k [H]$ is a domain, it follows that $f_g \sigma_g (y) \neq 0$ for some $g \in G$, and therefore $xy \neq 0$. Similarly,

$$yx = y \left(\sum_{g \in G/H} f_g \bar{g} \right) = \sum_{g \in G/H} y f_g \bar{g}.$$

Because $k [H]$ is a domain, and $y, f_g \in k [H]$, we may use similar reasoning to deduce that $yx \neq 0$. $\square$

Proposition 5.7. *Let k be a division ring and G a group, where $k [G]$ is not necessarily a domain. If $H \triangleleft G$ and $k [H]$ is an Ore domain, then the set $k [H] - \{0\}$ is permutable as a subset of $k [G]$.*

This proof is also adapted from a proof by Rosset in [11].

Proof. Write $S = k [H] - \{0\}$ and $R = k [G]$. It suffices to show that S is right permutable, because right and left permutability are equivalent for S. Write $y = \sum_{g \in G} q_g g$. Because $q_g \neq 0$ for only finitely many g, we can write $y = \sum_{i=1}^{n} q_i g_i$, where $q_i \neq 0$ for each i.

For $1 \leq i \leq n$, define $z_i = (q_i g_i)^{-1} x (q_i g_i)$. We know that each $z_i \in S = k [H] - \{0\}$ because $x \in k [H] - \{0\}$ and $H \triangleleft G$. By assumption, $k [H]$ satisfies the Ore condition, and we can use Proposition 4.5 to find some $s_1, ..., s_n$ such that

$z_1 s_1 = z_2 s_2 = \cdots = z_n s_n$. Denote their common value by u, and note that $u \in S$. Then,

$$\begin{aligned} yu &= \sum_{i=1}^{n} q_i g_i u \\ &= \sum_{i=1}^{n} q_i g_i z_i s_i \\ &= \sum_{i=1i}^{n} q_i g_i \left[(q_i g_i)^{-1} x q_i g_i \right] s_i \\ &= \sum_{i=1}^{n} x q_i g_i s_i \\ &= x \left(\sum_{i=1}^{n} q_i g_i s_i \right). \end{aligned}$$

Thus, $yS \cap xR \neq \emptyset$, and $S = k[H] - \{0\}$ is a right permutable subset of $k[G]$. $\square$

5.2 Localized Group Rings

From Proposition 5.7, we have the following. Suppose G is a group so that $k[G]$ is a domain, and $H \triangleleft G$. Then, if $k[H]$ is an Ore domain, we can form the right ring of fractions $(k[G]) S^{-1}$, where $S = k[H] - \{0\}$. As an example, we can take H to be a torsion-free elementary amenable group, and G to be any group. Then, even though $k[H \rtimes G]$ is not necessarily a domain, we may localize $k[H \rtimes G]$ at $k[H] - \{0\}$. Then, we may localize this group ring at $k[H] - \{0\}$. We show that the localized group ring constructed in this way can be viewed as a crossed product $D * (G/H)$, where $D = (k[H]) S^{-1}$ is a division ring.

Theorem 5.8. *Suppose k is a division ring and G a group. If $H \triangleleft G$ so that $k[H]$ is an Ore domain, then we may form the crossed product $D * G/H$, where D is the* *division ring* $k[H](k[H]-\{0\})^{-1}$.

Proof. Set $S = k[H]-\{0\}$. By Theorem 4.4, we can construct the right ring of fractions $(k[G])S^{-1}$. Set $Z = G/H$. Using the discussion on p. 11, we can write $k[G]$ as a crossed product $k[H]*Z$, with skewing function $\sigma : Z \to \operatorname{Aut}(k[H])$, and twisting $\tau : Z \times Z \to U(k[H])$, the units of $k[H]$. Every element of $(k[G])S^{-1} = (k[H]*Z)S^{-1}$ has the form $\left(\sum_{z\in Z} f_z\bar{z}\right)s^{-1}$, for $f_z \in k[H]$ and some $s \in S$. We would like to move the s^{-1} to the other "side" of the linear combination, with the goal to eventually view $(k[H]*Z)S^{-1}$ as a crossed product $D*Z$, where $D = k[H]S^{-1}$ is a division ring. For any $z \in Z$, we will show that the map $\sigma_z \in \operatorname{Aut}(k[H])$ extends to a map $\bar{\sigma}_z \in \operatorname{Aut}(D)$ so that $\bar{z}d = \bar{\sigma}_z(d)\bar{z}$ for each $d \in D$. From this, we will be able to deduce that $(k[G])S^{-1}$ is a crossed product $D*Z$.

Once we know that σ_z extends to $\bar{\sigma}_z$, how will we use this fact? First, given element of $(k[G])S^{-1}$ written in the form $\left(\sum_{z\in Z} f_z\bar{z}\right)s^{-1}$, we can rewrite this as $\sum_{z\in Z} f_z\bar{\sigma}_z(s)^{-1}\bar{z}$. Because $f_z\bar{\sigma}_z(s)^{-1} \in D$, we again rewrite this as $\sum_{z\in Z} d_z\bar{z}$, where $d_z \in D$, and the elements of $(k[G])S^{-1}$ can be written in the form of elements from a crossed product $D*Z$. Moreover, we can show that $(k[G])S^{-1}$ exhibits the crossed product structure. For a twisting function, we may use the same twisting τ from $k[H]*Z$, because the multiplication of elements of $\bar{Z}$ should be unchanged in $D*Z$. Then, for a skewing function, we may use $\bar{\sigma} : Z \to \operatorname{Aut}(Z)$ given by $\bar{\sigma}(z) = \bar{\sigma}_z$.

Using the arguments above, we need only prove that the function $\sigma_z \in \operatorname{Aut}(k[H])$ extends to $\bar{\sigma}_z \in \operatorname{Aut}(D)$, and that $\bar{\sigma}_z$ satisfies the skewing property $\bar{z}d = \bar{\sigma}_z(d)\bar{z}$. Recall from the discussion on p. 11 that the $\bar{z} \in k[H]*Z = k[G]$ are actually elements of G, which have been picked as preimages of $z \in G/H$ under the map

$G \to G/H$. Then, we defined our skewing function σ_z to be the automorphism of $k[H]$ that arises from conjugating by $\bar{z}$, so that for $r \in k[H]$, we have

$$\begin{aligned} \bar{z}r &= \left(\bar{z}r\bar{z}^{-1}\right)\bar{z} \\ &= \sigma_z(r)\,\bar{z}. \end{aligned}$$

We would like to extend σ_z to $\bar{\sigma}_z \in \operatorname{Aut}(D)$ by thinking of $\bar{\sigma}_z$ as a function that conjugates elements of D by $\bar{z}$. We do this by

$$\begin{aligned} \bar{\sigma}_z(z) &= \bar{z}\left(rs^{-1}\right)\bar{z}^{-1} \\ &= \left(\bar{z}r\bar{z}^{-1}\right)\bar{z}\left(s^{-1}\right)\bar{z}^{-1} \\ &= \sigma_z(r)\,\bar{z}s^{-1}\bar{z}^{-1} \\ &= \sigma_z(r)\left[\bar{z}s\bar{z}^{-1}\right]^{-1} \\ &= \sigma_z(r)\,\sigma_z(s)^{-1} \end{aligned}$$

We have that $\bar{z}d = \left(\bar{z}d\bar{z}^{-1}\right)\bar{z} = \bar{\sigma}_z(d)\,\bar{z}$, as desired. Moreover, as $\bar{\sigma}_z$ is just a map that conjugates by $\bar{z}$, it is evident that $\bar{\sigma}_z$ is an automorphism of D. Further, $\bar{\sigma}_z$ extends σ_z in the following way. Given some element $r \in k[H]$, we can view r as an element of D as $r \cdot 1^{-1}$. Then, $\bar{\sigma}_z\left(r \cdot 1^{-1}\right) = \sigma_z(r)\,\sigma_z(1)^{-1} = \sigma_z(r) \cdot 1^{-1}$, which is the embedding of the element $\sigma_z(r)$ into D. Thus, $\bar{\sigma}_z$ extends σ_z. We have shown that for each $z \in Z$, the automorphism $\sigma_z \in \operatorname{Aut}(k[H])$ extends to an automorphism $\bar{\sigma}_z \in \operatorname{Aut}(D)$, and that $\bar{\sigma}_z$ is a skewing of D. Therefore, using the remarks above, we have proven the theorem. $\square$

Theorem 5.9. *If k is a division ring, F is a finitely-generated free group, and H is any group so that $k[H]$ is an Ore domain. Set $S = k[H] - \{0\}$. Then, the* *localized group ring $k[H \rtimes F]S^{-1}$ can be viewed as a crossed product $D * F$, where $D = k[H]S^{-1}$. Moreover, if M is a finitely-generated submodule of a free module over the localized group ring, then M is free.*

Proof. This follows from Theorem 5.8 and Theorem 3.20. $\square$

Now, suppose that G is a group that maps onto a finitely-generated free group F. Then, if H is the kernel of this map, we have the short exact sequence

$$e \to H \to G \to F \to e.$$

Because F is free, this short exact sequence splits, and it follows that $G = H \rtimes F$. Thus, Theorem 5.9 applies to any group G mapping onto a free group, as long as the kernel of this map satisfies the Ore condition.

Here are two corollaries.

Corollary 5.10. *If H is a group so that $k[H]$ is an Ore domain, then $k[H \rtimes F]$ is a domain.*

Proof. This follows from the fact that $k[H \rtimes F]$ embeds into a crossed product $D*F$, which we showed in Corollary 3.21 to be a domain. $\square$

Corollary 5.11. *Given the hypotheses of Theorem 5.9, finitely-generated projective modules over the localized group ring R are free.*

Proof. If P is a finitely-generated projective R-module, then P is a direct summand of a free R-module. Thus, P is a finitely-generated submodule of a free R-module, and P is free. $\square$

Example 5.12. In 2010, Misseldine constructed finitely-generated, non-free, stably free modules over the Klein bottle group $\langle a, b \mid aba^{-1}b \rangle = \mathbb{Z} \rtimes \mathbb{Z}$ in [8]. If we take $G = \mathbb{Z} \rtimes \mathbb{Z}$, and $H = \langle a \rangle \cong \mathbb{Z}$, then H is torsion-free elementary amenable, normal subgroup of G, and G/H is free. Therefore, we may form the localized group ring $(k[G])(k[H] - \{0\})^{-1}$. The localized group ring can be expressed in the form $D * \mathbb{Z}$, where D is a division ring, and it follows that all finitely-generated projective modules over the localized group ring are free. In particular, the modules that Misseldine constructed are free when viewed as modules over the localized group ring.

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