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ARITHMAGONS AND GEOMETRICALLY INVARIANT MULTIPLICATIVE INTEGER PARTITIONS

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Abstract. In this article, we introduce a formal definition for integral arithmagons. Informally, an integral arithmagon is a polygonal figure with integer labeled vertices and edges in which, under a binary operation, adjacent vertices equal the included edge. By considering the group of automorphisms for the associated graph, we count the number of integral arithmagons whose exterior sum or product equals a fixed number.

1. Introduction

Arithmagons are numerical problems in which solvers look for ways to label a polygonal graph so that numbers on adjacent vertices combine under a binary operation to equal numbers on the edges as shown in Figure 1.

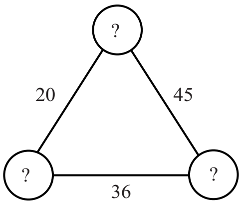

Figure 1. Example of a multiplicative arithmagon.

As a flexible context for generating problems of varying difficulty, arithmagons present nice opportunities for school-aged students to build algebraic reasoning skills while exploring properties of numbers and operations [**6, 7**]. The main question we explore in this article relates to how many such arithmagons can be constructed if the combined binary operation on the values on the edges must equal a fixed number N. We begin by considering multiplicative arithmagons. We then use this result to count additive arithmagons using a well-known relation between partitions of N and factorizations of q^N for any prime q. We obtain a

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well-defined count for both operations whenever all nonidentity vertex values are distinct. We also provide a Mathematica® function for obtaining these counts.

The paper is organized as follows. In Section 2, we treat arithmagons as a type of multigraph satisfying a system of equations. In Section 3, we use the automorphism groups of the associated multigraphs to determine the number of distinct arithmagons obtainable from a given set of vertex entries. This is used to address the following formal problem: Let $m(v_i, v_{i+1})$ denote the value on the edge that connects the vertices v_i and v_{i+1} of an arithmagon. Our goal is to find the number of distinct arithmagons with n vertices (up to rotations and reflections) that satisfy

$$m(v_1, v_2) \star m(v_2, v_3) \star \cdots \star m(v_{n-1}, v_n) \star m(v_n, v_1) = N \tag{1.1}$$

for a fixed $N \in \mathbb{N}$, where $\star$ is either addition or multiplication of integers.

2. FORMAL DEFINITION OF AN ARITHMAGON

In this section, we offer a formal definition of arithmagons. Loosely speaking, an arithmagon will be defined as the orbit of an undirected labeled multigraph.

To make this precise, let $\Gamma = (V, E)$ be an undirected, labeled, polygonal multigraph, with vertex set V and multiset of edges E. Since Γ is polygonal, we can order the vertices as $V = \{v_1, \ldots, v_n\}$ and record the multiplicity of the edges as $E = (A, m)$, where $A = \{(v_i, v_{i+1}) \mid 0 \le i \le n-1\} \cup \{(v_n, v_1)\}$ and $m\colon A \to \mathbb{Z}^{>0}$ is the multiplicity function.

It is well-known that if a permutation σ is an automorphism of the graph Γ, then $(\sigma(u), \sigma(v)) \in E$ if and only if $(u, v) \in E$ and that the set of all of these maps under composition forms the automorphism group $G = \operatorname{Aut}\Gamma$, [**1**]. We will be interested in the orbits of graphs, that is, sets of the form

$$\mathcal{O}(\Gamma) = \{\sigma(\Gamma) | \sigma \in G\},$$

and the stabilizers of a particular graph,

$$G_\Gamma = \{\sigma \in G \mid \sigma(\Gamma) = \Gamma\}.$$

Since we restrict our attention to polygonal graphs, we have $G = D_{2n}$, where n is the number of vertices in V.

Let $p\colon V \to \mathbb{Z}^{>0}$ be a map such that

$$p(i) \star p(i+1) = m(v_i, v_{i+1}) \tag{2.1}$$

$$p(n) \star p(1) = m(v_n, v_1), \tag{2.2}$$

where $\star$ is either addition or multiplication of integers. To simplify notation, we will define $m(v_n, v_{n+1}) := m(v_n, v_1)$.

An important remark is that these conditions are equivariant under the action of G. This compatibility allows us to define an arithmagon in the following way.

Definition 1. A *positive integral multiplicative (or additive) arithmagon* is the orbit of an undirected, labeled, polygonal multigraph $\Gamma = (V, A, m)$ under the action of $G = \operatorname{Aut}\Gamma$ and a map $p\colon V \to \mathbb{Z}^{>0}$ such that Equations (2.1) and (2.2) are satisfied by the multiplicative (or additive) operation $\star$.

Remark 1. For the purposes of the paper, we restrict ourselves to positive arithmagons. However, we could define more general integral arithmagraphs by considering directed multigraphs, where all edges between any pair of vertices have the same direction. We would say that the edge (v_i, v_j) is positively oriented if $i < j$ and negatively oriented if $i > j$. This would allow us to identify positive and negative edges on the arithmagraph. In this case, the map p would be allowed to take values in $\mathbb{Z}$.

3. Counting Problems

With the formal definition of arithmagons at hand, we can start counting the number of arithmagons for which the entries on the edges multiply or add to a fixed natural number. More precisely, we seek to count how many arithmagons satisfy

$$m(v_1, v_2) \star m(v_2, v_3) \star \cdots \star m(v_{n-1}, v_n) \star m(v_n, v_1) = N \tag{3.1}$$

for a fixed $N \in \mathbb{N}$.

By Equations (2.1) and (2.2), condition (3.1) becomes:

$$(p(1) \star p(1)) \star \cdots \star (p(n) \star p(n)) = N. \tag{3.2}$$

If $\star$ is addition, this implies $2\sum_i p(i) = N$. Thus, N is necessarily an even number. If $\star$ is multiplication, this implies $\prod_i p(i)^2 = N$ must be a perfect square. In the following, a set of numbers $\{p(i) \mid 1 \leq i \leq n\}$ satisfying (3.2) will be referred as a partition of $\sqrt{N}$ if $\star$ is multiplication or a partition of $N/2$ if $\star$ is addition. We will consider two partitions identical if they consist of the same elements, independently of permutations. This agrees with the traditional concept of additive and multiplicative partitions.

3.1. Symmetry Considerations

We will look at the counting problem for the additive and multiplicative cases separately. However, in both cases, we will take advantage of some symmetry considerations. To this end, we will consider the action of S_n on the arithmagons in more detail.

The standard action of S_n on a graph maps an edge (v_i, v_j) to $(\sigma(v_i), \sigma(v_j))$. If $\sigma \in \operatorname{Aut}\Gamma$, then $(\sigma(v_i), \sigma(v_j))$ is also an edge of Γ. This is not true for any other permutation in S_n. In particular, the standard action of $S_n \setminus G$ does not preserve the structure of an arithmagon. Therefore, we define the action of S_n as follows,

$$\sigma \cdot (V, A, m) = (\sigma \cdot V, A, \sigma \cdot m), \tag{3.3}$$

where $\sigma \cdot m \colon A \to \mathbb{Z}^{>0}$ is the unique map that satisfies conditions (2.1) and (2.2) with the new order of the vertices. We obtain the following lemma as an immediate consequence of this definition.

Lemma 1. *The action defined in* (3.3) *restricts to the standard action of* $G \subset S_n$.

While the action definition fixes problems with the standard action of S_n on arithmagons, it does not affect the action of the automorphism group G. Moreover, since S_n acts transitively on $V = \{v_1, \dots, v_n\}$ and V determines m uniquely, S_n acts transitively on the space of arithmagons with equal entries.

In order to count distinct arithmagons satisfying (3.2), we need to consider all the sets of entries for which the condition is satisfied, then count how many different arithmagons are yielded by each partition. All these numbers are added and this will yield the number of arithmagons with the desired property. Start by fixing a set of entries on the vertices $S = \{p(i) \mid 1 \le i \le n\}$ that satisfy the condition. Then we need to count how many configurations yield a different arithmagon. To do this, we will calculate the stabilizer of the arithmagon with entries in S. This calculation depends on the number of indices for which $p(i) = p(j)$. We consider the partition, I, on the set of indices induced by the equivalence relation $i \sim j$ if and only if $p(i) = p(j)$. Toward that end, form the list $I = \{[a_1], \dots, [a_l], [a_{l+1}], \dots, [a_k]\}$ where $|[a_i]| = 1$ for all $1 \le i \le l$ and $|[a_i]| > 1$ for all $l + 1 \le i \le k$.

Lemma 2. *Let $\Gamma = (V, A, m)$ be an arithmagon with vertex entries in $S = \{p(i) \mid 1 \le i \le n\}$. Define I and its classes as in the previous paragraph. Let G_Γ be the stabilizer of Γ. Then, the number of arithmagons with vertex entries in S is given by*

$$s(\Gamma) := |S_n \cdot \Gamma| = \frac{n!}{|G_\Gamma|}$$

and

$$|G_\Gamma| = \begin{cases} 2n \prod_{i=l+1}^{k} |[a_i]|! & \text{if } l \ge 3 \\ 2 \cdot (n-1)! & \text{if } l = 2 \text{ and } k \le 3 \\ n! & \text{if } l = 1 \text{ and } k \le 2 \end{cases}$$

when n is even, and

$$|G_\Gamma| = \begin{cases} 2n \prod_{i=l+1}^{k} |[a_i]|! & \text{if } l \ge 2 \\ n! & \text{if } l = 1 \text{ and } k \le 2 \end{cases}$$

when n is odd.

Proof. Since the action of S_n is transitive, the first statement follows from the Orbit-Stabilizer Theorem. Suppose that $l \ge 3$. Since the arithmagon is invariant under the permutation of the vertices that are labeled with a_i for $i \ge l+1$, a copy of $S_{|[a_i]|}$ can be embedded in the stabilizer of Γ for each $l + 1 \le i \le k$. Moreover, the actions of these symmetric groups commute with each other. Now, each element in D_{2n} leaves either 0 or 2 points fixed on the arithmagon if n is even and 0 or 1 if n is odd, but $l \ge 3$ guarantees that any permutation in any of the $S_{|[a_i]|}$ will fix at least 3 points, therefore these permutations do not generate any element of D_{2n} and vice-versa. This shows that when $l \ge 3$, $|G_\Gamma| = 2n \prod_{i=l+1}^{k} |[a_i]|!$. The previous argument still holds true when n is odd and $l = 2$ because in this case reflections have a single fixed point.

If n is even, $l = 2$, and $k = 3$, we have one number that appears $n-2$ times and two different numbers appear just once. Therefore, all the possible arithmagons consisting of two elements that appear just once are separated by $0, 1, \dots,$ or $n/2 - 1$ many copies of the same number. This shows that the stabilizer has order $2(n-1)!$.

Independently of the parity of n, if $l = 1$ and $k \leq 2$, then either, there is one number that appears $n-1$ times and another that appears just one time, or there is a number that appears n times. In both cases, the resulting arithmagon is stabilized by S_n. ☐

3.2. Counting Problem for Multiplicative Arithmagons

In Lemma 2, we determined the number of different arithmagons that can be obtained by permuting the vertices given a fixed set of vertex entries. So, if we want to find the number of arithmagons that satisfy condition (3.2), then we need to find the number of entries in the vertices that would satisfy this condition and multiply it by the number of arithmagons that can be obtained through permutations. In the case of multiplicative arithmagons, this would mean counting the number of factorizations of $\sqrt{N}$ into at most n factors. This is an open problem in number theory. Applications of recursive methods using Mathematica® can be found in [**5**], and a survey on recent advances can be found in [**4**].

We start with the simpler, but interesting case when $\sqrt{N} = q_1 \cdots q_m$ for m distinct primes.

Proposition 1. *If n is odd, then the number of integral, positive multiplicative n-arithmagons with edge values multiplying to $N = q_1^2 \cdots q_m^2$ (i.e., that satisfy (3.2)) is given by*

$$1 + \frac{1}{2} \sum_{l=2}^{n} \frac{(n-1)!}{(n-l)!} S(m, l),$$

where $S(m, l)$ represents the Stirling number of the second kind. If n is even, the number of such arithmagons is equal to

$$1 + \frac{n}{2} S(m, 2) + \frac{1}{2} \sum_{l=3}^{n} \frac{(n-1)!}{(n-l)!} S(m, l).$$

Proof. It is well-known that the number of factorizations of a square free number $\sqrt{N} = q_1 \cdots q_m$ in l factors is given by the Stirling number of the second kind, $S(m, l)$ (see Chapter 6 of [**2**]). Since $\sqrt{N} = q_1 \cdots q_m$, either $p(i) \neq p(j)$ or $p(i) = p(j) = 1$ when $i \neq j$. Then l is the number of entries that are not equal to 1 and $n - l$ is equal to the number of vertices that must get a 1 assigned to them. By Lemma 2, the stabilizer of the arithmagon is of order either $n!$, $2(n-1)!$, or $2n(n-l)!$. Now, the proposition follows from multiplying each $S(m, l)$ by its corresponding order and adding each of these terms. ☐

The assumption that $\sqrt{N}$ is a square-free number is rather stringent. We can solve a slightly different counting problem by using the recursive algorithm presented in [**5**], which is based on Hughes-Shallit's reasoning in [**3**]. In particular, we can count the number of multiplicative arithmagons with n vertices whose vertex entries multiply to (any perfect square) N and all non-identity vertex labels are different.

Theorem 1. *Let* $P_d(l, N)$ *denote the number of multiplicative partitions of* N *in* l *distinct factors. The number of multiplicative arithmagons with* n *vertices satisfying* (3.2) *for any perfect square* N *and such that the only entry that can appear more than once is* 1 *is equal to*

$$\sum_{l=1}^{n} \left\lceil \frac{(n-1)!}{2(n-l)!} \right\rceil P_d(l, \sqrt{N}).$$

Proof. The proof mimics the proof of Proposition 1 and thus, is omitted. □

Remark 2. Notice that Proposition 1 is a special case of Theorem 1 since $P_d(l, \sqrt{N}) = S(m, l)$ when $\sqrt{N} = q_1 \cdots q_m$, and because the latter condition forces the non-unit entries on the vertices to be distinct.

An interesting result occurs when $N = q^{2r}$ for some prime q. It is well-known that the number of factorizations of $\sqrt{N}$ in l factors is the same as the number of partitions of r in exactly l parts (c.f. [**4**]). This is because a partition $\{r_1, \ldots, r_l\}$ of r gives an unordered factorization $q^{r_1} \cdots q^{r_2}$ of $\sqrt{N}$ and vice-versa. This induces a natural equivalence between multiplicative arithmagons whose exterior product is of the form q^{2r} and the additive arithmagons whose exterior sum equals $2r$. Two equivalent multiplicative and additive pentagonal arithmagons are shown in Figure 3.2. We record this in the following proposition.

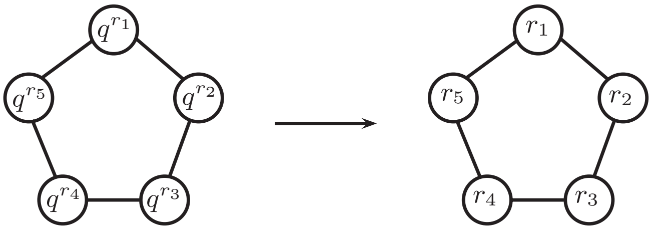

Figure 2. Equivalence between a multiplicative and an additive pentagonal arithmagon.

Proposition 2. *The number of multiplicative arithmagons with exterior multiplication equal to* $N = q^{2r}$ *is the number of additive arithmagons with exterior sum equal to* $2r$.

3.3. Counting Problem for Additive Arithmagons

As a result of Proposition 2 and Theorem 1, we can count the number of additive arithmagons for which all non-zero entries are distinct.

Corollary 1. *The number of additive arithmagons that satisfy* (3.2) *for* $N \in 2\mathbb{N}$ *and whose all non-zero entries are distinct is equal to*

$$\sum_{l=1}^{n} \left\lceil \frac{(n-1)!}{2(n-l)!} \right\rceil P_d(l, q^{N/2})$$

for any prime q.

4. Example and Conclusions

To illustrate the results, consider the problem of counting the number of integral, positive multiplicative pentagonal arithmagons ($n = 5$) with an exterior product $N = 900 = 2^2 3^2 5^2$. Since $\sqrt{N} = 30$ is a square-free number, we apply Proposition 1 to obtain that the number of such arithmagons is

$$1 + \sum_{l=2}^{5} \frac{4!}{2(5-l)!} S(3, l) = 13.$$

These 13 pentagonal arithmagons are shown in Figure 3. To support the computations, we provide a Mathematica$^{®}$ function we found useful for calculating the number of multiplicative arithmagons with distinct non-unit entries that have an exterior product equal to N. Applying Corollary 1, the function can be used to calculate the number of additive arithmagons whose exterior sum equals N by evaluating $s(q^N, n)$ for some prime q.

The code from [**5**] that we will use for our calculation is the following:

```
DistinctUnorderedFactorizations[m_,1]={{}};
DistinctUnorderedFactorizations[1,n_]={};
DistinctUnorderedFactorizations[0,n_]={};
DistinctUnorderedFactorizations[m_,n_/;PrimeQ[n]]:= If[m<n,{},{{n}}]

DistinctUnorderedFactorizations[m_,n_]:=DistinctUnorderedFactoriza-
tions[m,n]=Flatten[Function[d,Prepend[#,d]&/@DistinctUnordered-
Factorizations[d-1,n/d]] /@Rest[Select[Divisors[n],#≤m&]],1]

DistinctUnorderedFactorizations[n_]:=DistinctUnorderedFactoriza-tions[n,n]
```

Using this code we can now write the code for the function that calculates the number of arithmagons:

```
s[N_,n_]:=Plus @@ (#*Ceiling[(n-1)!/(2(n-Range[Length[#]])!)])&
[Length /@ Split[Sort[DistinctUnorderedFactorizations[Sqrt[N]],Length
[#1]<Length[#2]&],Length[#1]==Length[#2]&]]
```

4.1. Conclusions

We started this article with a formal definition of an arithmagon that is compatible with the intuitive definition and that offers the additional structure of a graph. This introduced consideration of symmetries and the group of automorphisms of

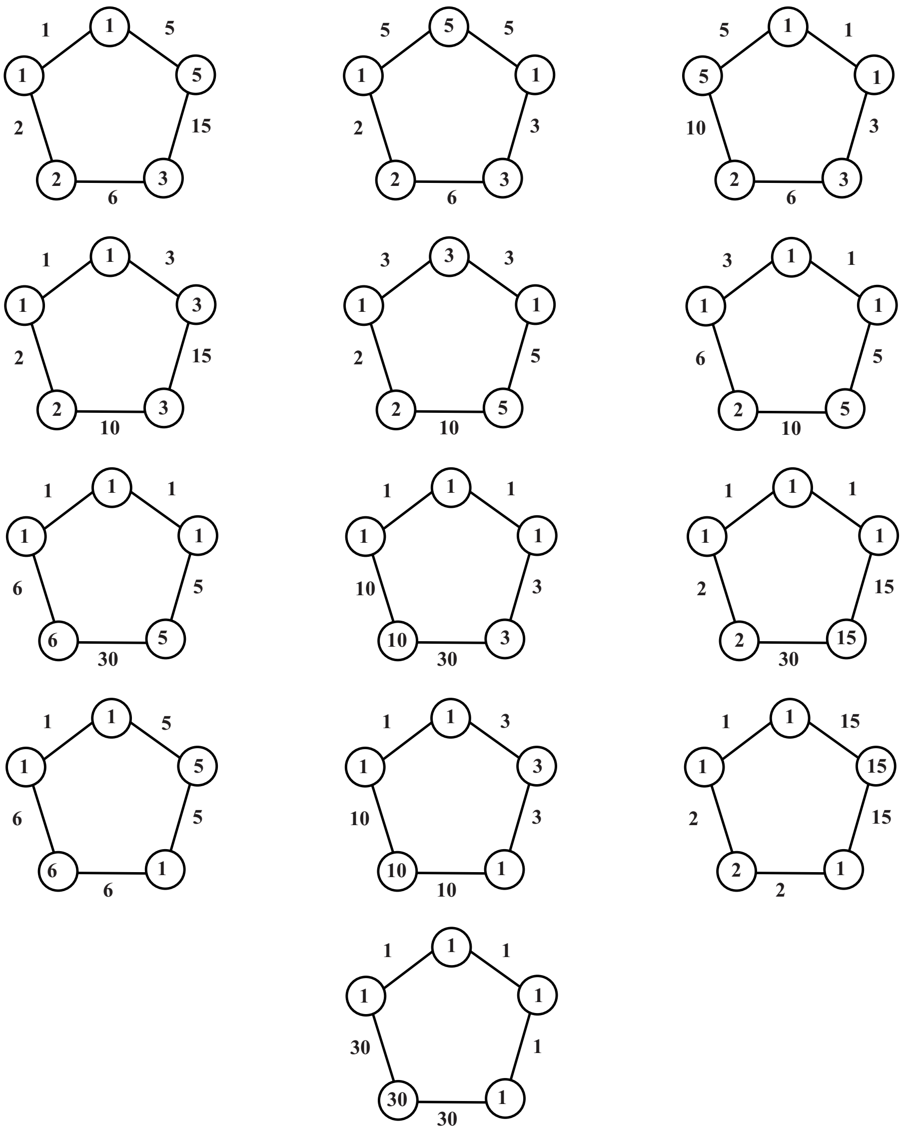

Figure 3. Positive multiplicative pentagonal arithmagons with an exterior product $N = 900$.

the graph. Using these symmetries and recursive methods, we solved counting problems pertaining to the number of multiplicative and additive arithmagons with no repeated non-unit entries. In the case of multiplicative arithmagons, this included all possible arithmagons when N is the perfect square of a square-free number. We illustrated one of the main results with the help of a Mathematica® function based on [**5**].

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