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Prospective Teachers' Learning in Geometry: Changes in Discourse and Thinking

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Abstract

This study investigates changes in prospective teachers' levels of geometric thinking and the development of their geometric discourses in the classification of quadrilaterals. To examine prospective teachers' thinking about geometry, this study connects Sfard's discursive framework to another, namely the van Hiele theory. Findings of the study reveal discursive similarities and differences in participants' geometric discourses within the same van Hiele level, as well as changes in geometric discourse as a result of changes in levels of geometric thinking. The study also investigates the usefulness of a discursive framework in providing rich descriptions of prospective teachers' thinking processes.

Keywords: discourse, geometry, K-8 prospective teachers, van Hiele levels

Introduction

In the mathematics education research community, investigations of how students learn mathematics have defined mathematical learning as actively building new knowledge from experience and prior knowledge, moving to a higher level of thinking, or as changes in discourse. Other researchers have developed methods to measure learning quantitatively. The question that served as the impetus for this study was: "What do prospective teachers learn in geometry from their preparation for the work of teaching geometry?" It can be argued that this study does little to answer the question because of the complexity of participants' learning, and of the context in which these students were observed. However, my effort is to conceptualize these participants' mathematical thinking through their mathematical discourses as evidence of their learning, thereby adding some information to the two perspectives of learning as moving to a higher level of geometric thinking and as changes in discourse.

The term "level of geometric thinking" came from the van Hieles (1959/1985), and they used the term to describe a process of learning a new language because "each level has its own linguistic symbols" (p.4). The van Hiele levels of thinking reveal the importance of language use, and emphasize that language is a critical factor in movement through the levels; however, the word "language" is not clearly defined. Moschkovich (2010) argued that the language of mathematics does not mean a list of vocabulary words or grammar rules, but rather the communicative competence necessary and sufficient for competent participation in mathematical discourse. Sfard (2008) used a discursive approach inspired by Vygotsky to make a distinction between language and discourse – language is a tool, whereas discourse is an activity in which the tool is used or mediates. This perspective provides an understanding of mathematics as a social and discursive accomplishment in which talk, diagrams, representations, and mathematical objects play an important role. Consequently, mathematics learning requires several modes of communication (Sfard, 2002).

Many researchers have attempted to develop frameworks to examine discourse in learning mathematics. As an example, Sfard's (2008) communicational approach to

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mathematical learning provides a notion of mathematical discourse that distinguishes her framework from others in several ways. In particular, Sfard (2002) argues that the knowing of mathematics is synonymous with the ability to participate in mathematics discourse. From this perspective, conceptualizing mathematical learning as the development of a discourse and investigating learning means getting to know the ways in which children modify their discursive actions in these three respects: "its vocabulary, the visual means with which the communication is mediated, and the meta discursive rules that navigate the flow of communication and tacitly tell the participants what kind of discursive moves would count as suitable for this particular discourse, and which would be deemed inappropriate." (p.4) Therefore, Sfard's discursive framework is grounded in the assumption that thinking is a form of communication and that learning mathematics is learning to modify and extend one's discourse.

Theoretical Framework

In Sfard's (2008) *Thinking as Communicating: Human Development, the Growth of Discourses, and Mathematizing,* she introduces her discursive framework, a systematic approach to analyzing the discursive features of mathematical thinking. To examine the development of geometric discourse, this study connects Sfard's discursive framework to the van Hiele theory (see van Hiele, 1959/1985; 1986). The van Hiele theory describes the development of students' levels of thinking in geometry. The levels, numbered 1 to 5, are described as visual, descriptive, theoretical, formal logic and rigor. Connecting Sfard's work with the van Hiele theory provides a new perspective with which to revisit the van Hiele levels as the discursive development of geometric discourses. According to Sfard, geometric discourse features mathematical vocabulary specifically relating to geometric shapes, definitions, axioms and proofs, and so on. Mathematical discourses are distinguishable by the following four features:

- Word use: Mathematical words that signify mathematical objects or processes
- Visual mediator: Symbolic artifacts, related especially for particular communication
- Narratives: Any text, spoken or written, which is framed as description of objects, of relations between processes with or by objects, and which is subject to endorsement or rejection, that is to be labeled as true or false.
- Routines: Repetitive patterns characteristic of the given mathematical discourse.

These features interact with one another in a variety of ways. For example, endorsed narratives contain mathematical words and provide the context in which mathematical words are used; mathematical routines are apparent in the use of visual mediators and produce narratives; visual mediators are used in the construction of endorsed narratives, and so on. With this discursive lens, this study investigates students' geometric thinking through the analyses of their geometric discourses. In this paper, I will share findings regarding two questions. First, what are the changes in participants' van Hiele levels and their geometric discourses? And, what additional information does Sfard's (2008) discursive framework provide with regard to the levels of geometric thinking?

Method

Participants

All participants in the study were pre-service teachers enrolled in a Midwestern university teacher education program. They were required to complete a sequence of two mathematics content courses designed for teachers. The first of these courses dealt with numbers and operations and the second with measurement and geometry. The participants of the study were pre-service teachers enrolled in the measurement and geometry course; most of them were juniors and sophomores, and a few were seniors. All sixty-three students enrolled in the course in the fall of 2010 participated in the pre and posttest, both tests being given as class assignments. Twenty participants voluntarily participated in the interview part of the study soon after they took the pre and posttest.

All participants enrolled in the geometry and measurement course for teachers used Parker and Baldridge's (2007) textbook, *Elementary Geometry for Teachers*. Ten chapters are included in this textbook, all discussing mathematical topics related to geometry and measurement for prospective elementary school teachers. Most participants had studied geometry in K-12, therefore the contents of this study related to triangles, quadrilaterals and proof introduced in Chapter 2 (*Geometric Figures*) and Chapter 4 (*Deductive Geometry*) were partly review to them. For example, in Chapter 2 students were introduced to triangles and parallelograms. The discussion included the introduction of angles, perpendicular and parallel lines, as well as the classification of quadrilaterals. In the classification of quadrilaterals, students studied parallelograms, rectangles, rhombuses, squares, trapezoids, and kites. In Chapter 4 students learned how to derive new geometric facts from previously known facts using logical arguments. For instance, in the beginning of Chapter 4, where a problem of finding an unknown angle in a quadrilateral leads to an unknown angle proof, students learned from a natural computation to deduce a general fact about a quadrilateral. Later in the chapter, students learned to construct proofs for congruent triangles, and to use congruent triangles to verify properties of quadrilaterals. Thus, these participants were introduced to the topics in this study by the textbook for the course.

van Hiele Geometry Test

Many mathematics educators have used van Hiele theory to determine students' levels of mathematical thinking. In order to identify suitable survey instruments for the study, literature on the van Hiele levels was reviewed. The van Hiele Geometry Test used in the Cognitive Development and Achievement in Secondary School Geometry (CDASSG) project, was chosen because it was carefully designed and tested by the researchers of the project (see Usiskin, 1982). This test was used as the pretest and posttest instrument to determine the van Hiele level of the students. The van Hiele Geometry Test contains 25 multiple-choice items, distributed into five van Hiele levels: Items 1-5 (Level 1), Items 6-10 (Level 2), Items 11-15 (Level 3), Items 16-20 (Level 4) and Items 21-25 (Level 5). These items are designed to identify students' geometric thinking at the five van Hiele levels. For example, Items 1 to 5 of are designed to identify students' thinking related to van Hiele Level 1, at which figures are judged according to their appearance. Items 5 to10 identify students' behaviors related to van Hiele Level 2, at which figures are described according to their properties. The van Hiele Geometry Test was used to provide information on students' levels of geometric thinking at the two end points of the study: the beginning (pretest) and the end of the semester (posttest). The analyses of the pretest and posttest helped to determine students' changes in geometric thinking resulting from participating the measurement and geometry class.

Interview Tasks

The goals of the interviews were (1) to gather information about participants' knowledge of triangles and quadrilaterals, as well as the parts of the triangles and quadrilaterals (e.g., angles and sides), (2) to examine participants' abilities to verify their claims and derive mathematical proofs, and (3) to probe further into participants' geometric discourses as revealed through these mathematical activities. Three tasks and corresponding interview protocols were designed for the interviews. I will share two tasks to narrow the scope of this paper.

Task One: sorting geometric figures. Task One presented eighteen geometric shapes labeled with capital letters (see Figure 1). Among these eighteen polygons, thirteen were quadrilaterals, four were triangles, and one was a hexagon. Sixteen of these polygons were chosen from the van Hiele Geometry Test Items 1-5. Two more polygons were added: Q, a quadrilateral, and S, a triangle with no pair of sides equal. These shapes have also commonly appeared in elementary and middle school textbooks used by many researchers over the past two decades to categorize students' geometric thinking with respect to the van Hiele levels (e.g., Mayberry, 1983; Burger & Shaughessy, 1986; Gutierrez, Jaime, & Fortuny, 1991).

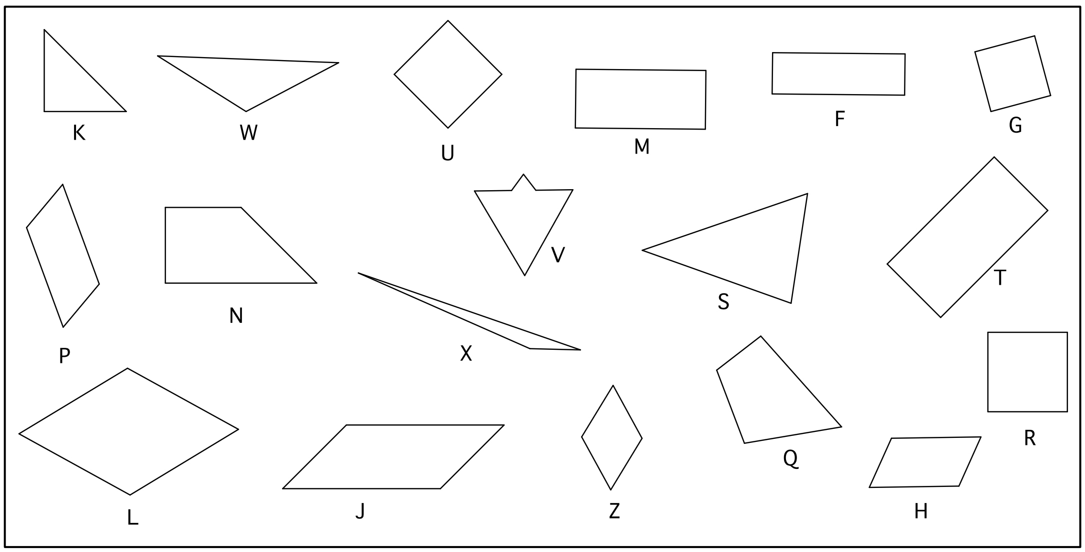


Figure 1. Task One: Sorting Geometric Figures

Task One was presented to participants at the beginning of the interviews, and each participant was asked to sort the eighteen polygons into groups. After the first round of sorting, each participant was asked to regroup the polygons. For example, some participants sorted the polygons into a group of rectangles (U, M, F, T, R, G), and a group of triangles (X, K, W, S). The questions "Can you describe each group to me?" and "Can you find another way to sort these shapes into groups?" allowed participants to produce narratives about triangles and quadrilaterals based on their knowledge of polygons. Analysis of the act of grouping gave information on how participants classified triangles and quadrilaterals.

At the end of Task One, each participant was asked to write the definitions of rectangle, square, parallelogram, rhombus, trapezoid and isosceles triangle, and their written narratives were collected. This information revealed how participants defined these mathematical terms, and how they made connections between a name and a recognized parallelogram, as well as how the quadrilaterals were related to one another.

Task Two: investigating properties of parallelograms. Task Two of the interview had two components. The first component, divided into Part A and Part B, was designed to collect participants' drawings of parallelograms (see Table 1), and to gather more information on their knowledge of parallelograms.

Table 1
Investigating the Properties of Parallelograms

Draw a parallelogram in the space below.
○ What can you say about the angles of this parallelogram? ○ What can you say about the sides of this parallelogram? ○ What can you say about the diagonals of this parallelogram?
Draw a new parallelogram that is different from the one you drew previously.
○ What can you say about the angles of this parallelogram? ○ What can you say about the sides of this parallelogram? ○ What can you say about the diagonals of this parallelogram?

Part A began with *"Draw a parallelogram in the space below,"* and asked participants to describe the angles, sides and diagonals of the parallelogram. The follow-up questions probed participants' familiarity with different aspects of parallelograms. For instance, the question, "What can you say about the angles of this parallelogram?" was to find out about participants' familiarity with the angles of parallelograms. Part B started with *"In the space below, draw a new parallelogram that is different from the one you drew previously,"* and asked participants to describe the angles, sides and diagonals of the new parallelogram. This part of the task investigated how participants defined parallelograms and their thinking about *different* parallelograms. After participants completed Parts A and B, they were presented with pictures of parallelograms not included in their drawings. Four pictures of parallelograms were prepared for the interviews, consisting of a parallelogram, a rhombus, a rectangle and a square. Figure 2 shows the four parallelograms.

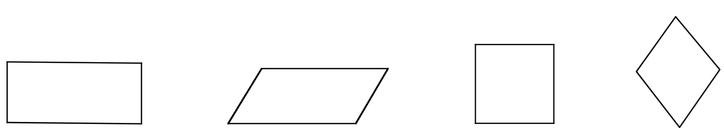


Figure 2. Pictures of four parallelograms

The purpose of these pictures was to encourage discussion of different parallelograms and their parts. The task was designed to explore why a participant included some parallelograms but excluded others. For example, after a participant drew a picture of a parallelogram in Part A and drew a rectangle as a different parallelogram in Part B of Task Two, I presented a picture of a square, and asked whether it was also a parallelogram and why. Thereby, I gathered more information about participants' understandings of parallelograms, and was able to gain insights missed in Task One regarding participants' ways of identifying and defining parallelograms. A set of interview scripts was designed to further aid in analyzing participants' understanding of parallelograms. These scripts were written to help participants make claims about the angles, sides and diagonals of parallelograms. Task Two shed additional light on participants' knowledge of parallelograms and familiarity with their angles, sides and diagonals.

Data Collection

The study used three primary data sources, (1) written responses to the van Hiele Geometry Test (see Usiskin, 1982) (from the pretest and posttest), (2) transcripts from two in-depth interviews, the first conducted right after pretest, and the second right after posttest, and (3) other written artifacts (students' written statements, answer sheets to the tasks during the interviews).

Data collection took place in four phases: (1) 63 students took the pretest, a 35-minute van Hiele Geometry Test administered during the first class of the fall semester of 2010; (2) 20 students volunteered to participate in a 90-minute interview one week after they participated in the pretest; (3) the same 63 students participated in the posttest at the end of the semester; and (4) 90-minute interviews with the same 20 students who participated in the interviews at the beginning of the semester. All tests were collected and analyzed. All interviews were video and audio recorded. All interview data were transcribed and analyzed.

Findings

Summary of Changes in van Hiele Geometry Test Performance

Whole group (*n*=63) results from the van Hiele Geometry Test showed that most students in the study had moved one or two van Hiele levels, and the majority of the students' levels of geometry thinking were at van Hiele levels 2 or 3 after their participation in the college geometry course. There were improvements in answering questions related to van Hiele levels 1 to 3 at the posttest (see Figure 3). In particular, most participants did better in the following ways:

- More than 95% of the participants correctly named triangles, squares, rectangles, and parallelograms at the posttest.
- More than 95% of the participants at the posttest correctly identified the properties of isosceles triangles, squares, rectangles, and rhombi related to their sides, angles and diagonals.
- About 90% of the posttest participants correctly used logical statements regarding triangles, squares, rectangles, and parallelograms.

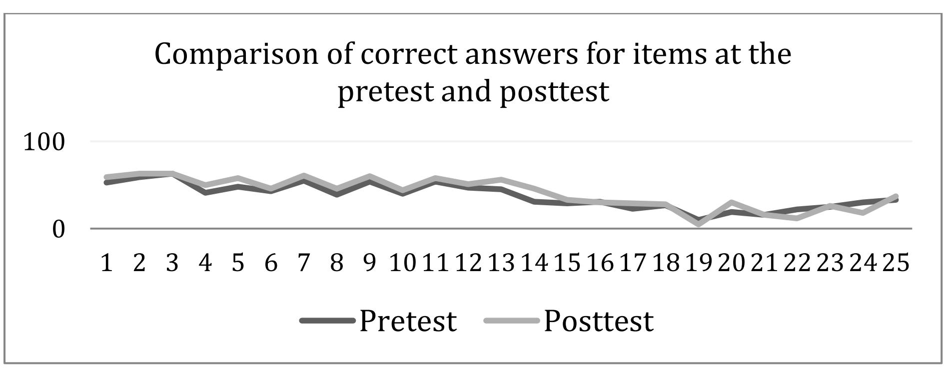


Figure 3. Comparison of correct answers for items at pre-and post-test

These changes showed that participants gained familiarity with figures like triangles, squares, rectangles, rhombi and parallelograms, and with their properties (Items 1-15), which predicted their levels of thinking operating at up to van Hiele Level 3. The comparisons of van Hiele pretest and posttest levels revealed students' weaknesses in using deductive reasoning to construct proofs and abstract thinking (Items 16-25), which indicated that their levels of thinking were not yet operating at level 4 or level 5. Given these test results, one conclusion is that the geometry course helped students to move from a lower van Hiele level to Level 3. However, the van Hiele test also showed that a student entering the class at Level 3 likely would stay at Level 3. However, that results was consistent with the fact that the course was designed for future elementary and middle school teachers, and that the course materials emphasized activities mostly at Levels 1 to 3, with only a brief introduction to constructing proofs. This study did not look at how teaching or the use of the textbook affected these students' learning, but certainly the textbook and course instruction contributed in some degree to these prospective teachers' learning about geometric figures and their properties. The van Hiele Geometry pre- and post-test served as a frame to gather general information about students' competencies and their thinking as a whole, but it did not provide details on changes in students' thinking at an individual level. For in-depth analyses of participants' thinking, twenty participants were interviewed soon after the pretest and posttest.

Summary of Changes in Geometric Discourse

To analyze interviewees' geometric discourses in the context of quadrilaterals and triangles, I devoted my attention to interviewees' familiarity with polygons in regard to their *word use*, including use of the names of polygons (parallelogram, rectangle, etc.), and the names of the parts of polygons (angle, side, etc.). Also, I analyzed interviewees' various routines while engaging in solving geometric tasks during the interviews; these routines included *routines of sorting, identifying, defining, conjecturing* and *substantiating.*

In this study different routines were involved given the nature of the tasks: the *routine of sorting* is a set of routine procedures that describes repetitive actions in classifying polygons (e.g, by their family appearances or by visual properties); the *routine of identifying* is a set of routine procedures that describes repetitive actions in identifying polygons (e.g., by visual recognition or by partial properties check); and the *routine of defining* is a set of repetitive actions related to how polygons are described or defined (e.g, by visual properties or by mathematical definition). In endorsed narratives such as mathematical definitions or axioms, the *routine of recalling,* a subcategory of the *routine of defining,* is a set of repetitive actions using previously endorsed narratives (e.g, I remember this definition because I learned it), and "it can indicate a lot not just about how the narratives were memorized, but also about how they were constructed and substantiated originally" (Sfard, 2008, p.236).

With regard to performing mathematical tasks, "guessing and checking" are seen as common activities. The *routine of conjecturing* is a set of repetitive actions that describe a process of how a conjecture is formed; and the *routine of substantiating* is a set of patterns describing a process of using endorsed narratives to produce new narratives that are true (e.g., an informal or formal proof using a triangle congruence criterion).

To better understand how learning takes place, and how mathematical concepts are developed, it is helpful to conceptualize mathematical learning as the development of a discourse, or a change in discourse. Among the twenty interviewees, some showed a change in their van Hiele levels from lower to higher according to the van Hiele Geometric Test conducted at the beginning of the semester (the pretest) and at the end of semester (posttest), whereas others showed no changes. The following sections detail analyses comparing van Hiele test results and geometric discourses for two participants, namely Amy and Sam.

Case 1: Changes in Amy's Geometric Discourse

The van Hiele Geometry Test showed that Amy was at Level 1 at the pretest, and ten weeks later she had moved two van Hiele levels to Level 3 at the posttest. Amy was interviewed after both tests. Amy's changes in geometric discourse can be summarized as follows:

- Amy's routines changed from grouping polygons according to their family appearances at the pre-interview, to classifying polygons according to their visual properties and definitions.
- Amy's use of the names of parallelograms changed from describing the parallelograms as collections of unstructured quadrilaterals that share some physical appearances at the pre-interview, to using the names as collections of quadrilaterals that share common descriptive narratives at the post-interview.
- Amy did not prove or disprove congruent parts of the polygons at an object level; nor did she use informal or formal mathematical proofs at the meta level.

Task One, *Sorting Geometric Shapes,* was used to analyze interviewees' *routines of sorting*, *identifying* and *defining* polygons. Interviewees were asked to classify these polygons into groups, without being given measurement information. The analyses showed that there was a change in Amy's routines of grouping, from using visual recognition to group quadrilaterals according to their family appearances at the pre-interview, to classifying quadrilaterals according to their common descriptive narratives (i.e., definitions and properties). For example, at the Pre-Interview, Amy stated, "I group them solely on their amount of sides," and sorted the polygons into three groups on her first attempt: 1) 3 sides, consisting of K, W, X, and S; 2) 4 sides, consisting of U, M, F, G, P, T, L, J, H, R and Z; and 3) trapezoids, including V and Q. Figure 4 shows Amy's grouping of quadrilaterals.

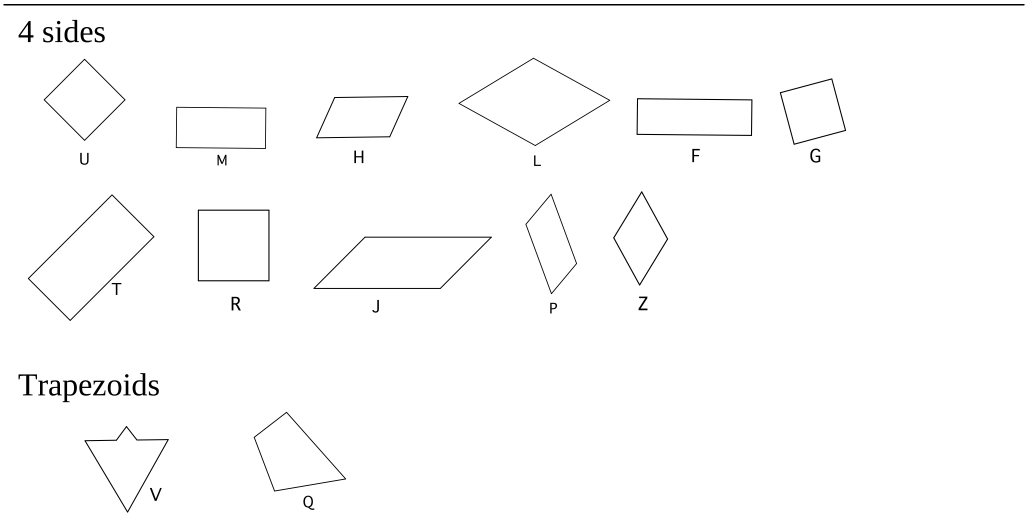


Figure 4. Amy's grouping of quadrilaterals in the pre-interview.

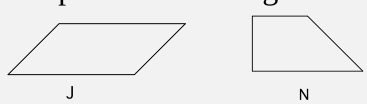
Amy then regrouped the 4-sided polygons according to their family appearances, with the names of *square*, *rectangle*, *parallelogram*, and *rhombus*. For example, Figure 5 shows two of the groups: the *rectangles* and the *squares.*

4a. Squares 4b. Rectangles

Figure 5. Amy's regrouping of the quadrilaterals at the pre-interview.

When I asked Amy why she regrouped the quadrilaterals in this way, she responded, "I know this figure (U, a square) and this figure (M, a rectangle) are different, but they both belong to the same quadrilateral group." The right trapezoid (N) was not included in any of the groups again at this second attempt. I asked Amy if I could put J (a parallelogram) and N together, and the following conversation took place:

Interviewer: Can we put these two together?



Amy: I wouldn't believe so… Just because this [pointing at Fig. N] shows the angle… it doesn't have the properties of a square or a rectangle, the sides…measurement… it does have four sides, but no…. congruent parts.

When it came to classifying quadrilaterals, Amy's routine procedures focused on the appearances of the polygons and how their appearances related to their family names. It was evident that Amy identified polygons with visual recognition.

At the post-interview the same task was performed. In contrast to her earlier performance, Amy grouped the polygons by "looking at the numbers of sides solely," and she sorted the 18 polygons into three groups: 1) *Triangles*, including all the 3-sided polygons in the task; 2) *5-sided*, consisting of V; and 3) *Quadrilaterals*, including all the 4-sided polygons in the task. Figure 6 compares Amy's first groupings from both interviews with some examples of each group.

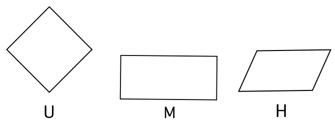
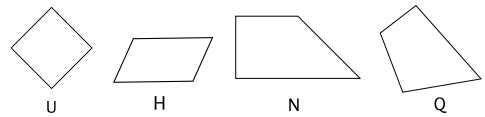
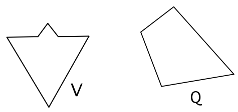
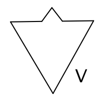
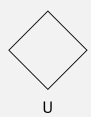
Before	Ten Weeks Later
4 sides *(Fig. N & Q are missing)* 	Quadrilaterals *(All quadrilaterals are included)* 
Trapezoid 	5-sided 

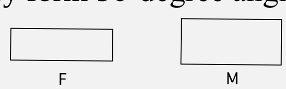
Figure 6. A comparison of Amy's grouping of polygons at both interviews.

At the post-interview, Amy included both N (a right trapezoid) and Q (a 4-sided figure) into the quadrilateral group with the help of her defining routine. At the pre-interview Amy did not include N in any groups because "it does have four sides, but… not congruent parts," but at the Post-Interview she included N in the quadrilateral group because it was "a four sided figure with one distinct pair of parallel sides." I asked Amy to regroup the quadrilaterals, and her response is shown below:

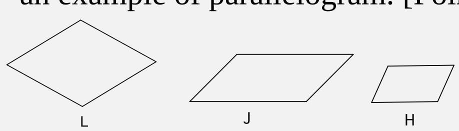
18a Amy: Quadrilaterals, you know that you have your square because … each forms 90-degree and all the side lengths are equal. [Pointing at U]



18c. Amy: These are rectangles because two sides and those two sides are the same. But again they form 90-degree angles… [Pointing at F and M]

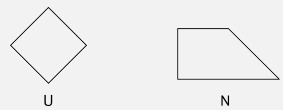


18e.Amy: Opposite angles are equal and opposite sides are equal, so these three would be an example of parallelogram. [Pointing at L, J and H]



At the post-interview, Amy was able to use her definitions of *square*, *rectangle*, *parallelogram* and *rhombus* to identify and to regroup the quadrilateral group. She regrouped quadrilaterals into: 1) *squares*, including U, G, and R; 2) *rectangles*, including M, F, T; 3) *rhombus*, consisting of Z; and 4) *parallelograms*, including L, J, H. When I asked Amy if I could put U (a square) and N (a right trapezoid) together, Amy responded:

Interviewer: Can U and N group together?

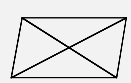


Amy: They can group together as both being same amount of sides…but in terms like property…no … they both have two parallel sides, but a trapezoid cannot be branched off with parallelograms into rectangles and squares….

This dialogue indicates Amy's ability to compare U and N, not only focusing on the "same amount of sides," but also on visual properties, like "they both have parallel sides."

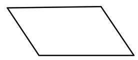
Amy's responses to the questions in Task Two also revealed changes in her geometric discourses. This led to my conclusion that Amy's identifying routines changed from self-evident visual recognition at the pre-interview to identifying visual properties and using definitions of parallelograms to draw conclusions about the angles and the sides of parallelograms at the post-interview. For example, during the pre-interview, Amy declared that the two parallelograms she produced in Task Two were different because "I would change the sizes of it [side]." Amy described the second drawing as, "it's a rectangle… but it's not the typical looking parallelogram." In response to the questions about the angles of the parallelograms, Amy expressed her frustrations on the angles, "I am still stuck on the question on what it means by the angles, … Usually when I'm talking about angles, we have measurements…[pausing] I feel like the angles would be the same … just based on how it looks."

Moreover, during the pre-interview, Amy also made intuitive claims about the angles of a parallelogram and a rectangle using direct recognition. For example, Amy assumed that the angles were "the same for the opposites" in a parallelogram using direct recognition. In this case, the question "How do you know [they are the same]?" did not lead to any substantiations of the claim, nor lead her to endorse any narratives using definitions; instead Amy's final conclusion was reached by direct visual recognition which was self-evident. This routine pattern also appeared when Amy was discussing the diagonals of a parallelogram:

17. Interviewer	What can you say about the diagonals of this parallelogram? 
18. Amy	The diagonals would be equal…
19. Interviewer	How do you know the diagonals are equal?
20. Amy	You have to measure and make sure these were, all their sides were the same…right here [pointing at the sides], would all equal… on each side all equaling the same parts.

Amy declared a narrative about the diagonals of the parallelogram stating that, "the diagonals would be equal." The diagonals of this parallelogram are not equal, as can be detected with a ruler. However, Amy did not check because her direct recognition was intuitive and also self-evident. Ten weeks later, the same task was performed again. The change in Amy's identifying routines was evident. Table 2 summarizes Amy's course of actions in response to the question, "What can you say about the angles of the parallelogram?" at the post-interview.

Table 2
Amy's Routines of Verifying for the Angles of Parallelograms at the Post-Interview

Q: "What can you say about the angles of this parallelogram?"		
	 Parallelogram	 Rectangle
Declared Narratives	"Opposite angles equal and … they don't form 90-degreee angle"	"…you could say that the opposite angles are equal, and in this one all angles are equal"
Q: "How do you know?"		
Routines	a. Visually identify partial properties of a parallelogram by checking the condition of opposite angles *(Identifying routine)* b. Describe a parallelogram with no right angles *(Defining routine-recalling)*	a. Visually identify partial properties of a rectangle by checking the condition of opposite angles *(Identifying routine)* b. Describe a rectangle with right angles *(Defining routine – recalling)*
Declared Narratives	"I would just say the property of parallelogram"	"It has properties of parallelogram. It's a rectangle"

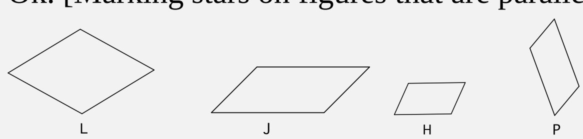
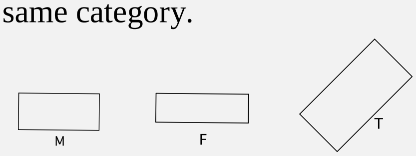
Recall that, at the Pre-Interview, Amy did not know how to draw a conclusion about the angles in a parallelogram without measurements. At the Post-Interview, Amy was able to discuss the angles of parallelograms using the properties of a parallelogram (*defining routine*). For example, when Amy declared the narrative "opposite angles are equal and they don't form a 90-degree angle," she identified that this 4-sided polygon was a parallelogram (*identifying routine*) and described the parallelogram, as it had no right angles using defining routines. Similarly, Amy was able to identify the differences of the angles between two parallelograms: a *parallelogram*, "opposite angles are equal and … they don't form a 90-degreee angle" and a *rectangle*, "the opposite angles are equal, and in this one [rectangle] all angles are equal."

In this scenario, we begin to see the change in Amy's routines of identifying, from visual recognition, to identifying visual properties of the angles in a parallelogram. Amy's routines of defining also showed a use of definitions of parallelograms to justify her claims at the Post-Interview. However it is important to note that Amy's routine of defining was more of a *recalling*, as it appeared to be memorization of facts.

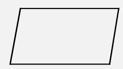
Analyzing Amy's use of words regarding these quadrilaterals helped me to understand her thinking about parallelograms, and about the relations among the angles, sides and diagonals of parallelograms, as well as how the concepts of these geometric figures were developed. The following findings provide more information regarding Amy's meanings for the words *parallelogram*, *rectangle*, *square* and *rhombus*. I begin my analyses of Amy's word use with this conversation from the Pre-Interview:

15. Interviewer	What is a parallelogram?
16. Amy	A parallelogram is when two sides of each side… all four are parallel to the opposite one…
17. Interviewer	What is a rectangle?
18. Amy	A rectangle is the two longer sides… the shorter ones … but in more technical terms, I am sure that they have congruency on both of those sides too
19. Interviewer	What is a square?
20. Amy	The square is all four of the sides are completely the same
21. Interviewer	What is a rhombus?
22. Amy	A rhombus… is a square… is just tilted [giggling]

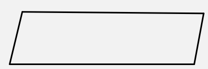
Amy's narratives concerning *parallelogram*, *rectangle*, *square*, and *rhombus* are descriptive and visual at the Pre-Interview. Amy gave a descriptive narrative about *rectangles* based on physical appearance, "a rectangle is the two longer sides [and two] shorter ones… have congruency on both…sides." Amy made connections between *squares* and *rhombi* according to visual appearances, and declared narratives, "a rhombus is a square," because "they both have four equal sides," and "[it] is just a tilted [square]." Amy's ways of defining parallelograms triggered the way she classified them. For example, when Amy was asked to identify all the parallelograms from a set of given figures, her response was as follows (see earlier analyses about the *routine of sorting*):

53. Interviewer	Can you identify all the parallelograms on this sheet? [Pointing to task One]
56a. Amy	Ok. [Marking stars on figures that are parallelograms]
	
56b. Amy	…Now for these ones, these could be actually… be considered parallelograms. Based on the side measures … even though they are rectangles… they could be in the same category.
	

Amy's classification was no coincidence. In Amy's written narratives about the *rectangles*, she wrote, "rectangle is when 2 sides are differing from the other 2 sides, however, opposite sides are equal in length," and for the *parallelograms*, she wrote, "parallelogram is when 2 parallel sides are congruent in length." To Amy, the word *parallelogram* was a family name of figures having opposite sides that were parallel, and having two long sides and two short sides. For example, when I asked Amy to draw a parallelogram, she provided the following:

1. Interviewer	Draw a parallelogram.
2. Amy	[Amy drew a figure that looked like this]: 
3. Interviewer	How do you know this is a parallelogram?
4. Amy	The opposite are equal in length… with the different sides parallel, they are the same length.

Next, I asked Amy to draw a new parallelogram different from the one she drew:

23. Interviewer	Draw a new parallelogram that is different from the one you drew.
24a. Amy	[Amy drew a figure that looked like this]: 
24b. Amy	All I know is to change the size of it, but that's more of a rectangle…. But it's not a typical looking parallelogram…
30a. Amy	I feel it's a rectangle, but rectangles can still have the properties of a parallelogram… just a broad term for it.
33. Interviewer	Can you say a little more about why this parallelogram [rectangle] is different from this one [the parallelogram Amy drew earlier]
34. Amy	They aren't. Technically, they're probably not different, that one just looks more like a stereotypical parallelogram [pointing at the parallelogram]. In terms of properties, there is nothing different.

Amy drew two parallelograms: "a stereotypical parallelogram" and a "not typical looking parallelogram." After Amy drew these parallelograms, I presented a picture of a square and a picture of a rhombus. Amy did not think a square and a rhombus were parallelograms because "to be a parallelogram, you have to have two long sides and two short ones, here all sides are equal and it is square." In the case of a rhombus, Amy responded, "this is similar to the square that you just showed me, … is a rhombus or just a square." From these conversations, it is evident that to Amy the word *parallelogram* signified two types of polygons, as summarized in Figure 7.

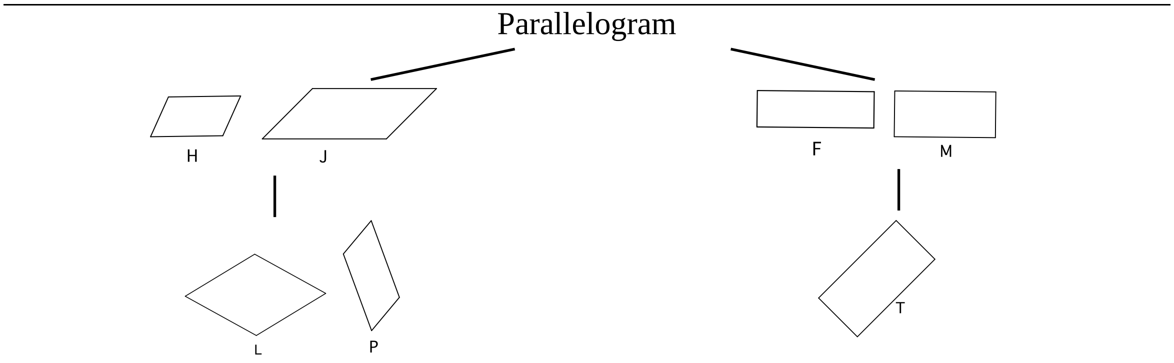
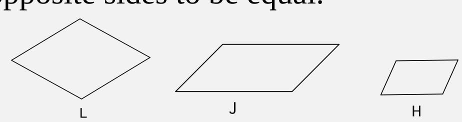


Figure 7. Amy's use of the word parallelogram at the pre-interview.

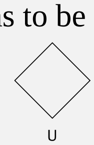
Amy's use of the word *parallelogram* signified a collection of unstructured polygons by their family appearances. This family appearance included figures appearing to have opposite sides equal and parallel, and in particular, two opposite sides longer than the other two opposite sides. However there was no explicit mention of the necessary condition that these figures be 4-sided, nor of any condition on the angles in rectangles. At the post-interview, when I asked Amy to identify all the parallelograms from eighteen polygons, her response was as follows:

19. Interviewer	What are the parallelograms here? [Pointing to Task One]
20. Amy	L and J and H will be just parallelograms, but all of these figures [pointing to figures that are squares, rectangles and rhombus] will be parallelograms, because…they all fit into the greater property of opposite angles and opposite sides to be equal.

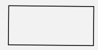
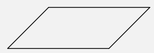
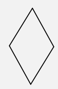


Amy identified two groups of parallelograms: one group contained figures that were "just parallelograms," and the other group contained figures that "fit into the greater property of opposite angles and opposite sides to be equal." As our conversation continued during the post-interview, Amy provided the following narratives about the parallelograms:

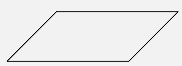
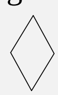
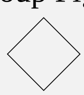
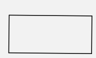
51. Interviewer	What is a square?
52. Amy	A square is when all the angles form right angles and they are all the same they are all 90 degrees…and each side length also has to be the same. [Pointing at U]



53. Interviewer	What is a rectangle?

54. Amy	A rectangle, each angle is 90 degrees but these sides are the same and parallel, and this one is the same and parallel, but not all 4 of them are the same, necessarily [Pointing at M]
	 M
55. Interviewer	What is a parallelogram?
56. Amy	Um… a parallelogram is when opposite sides are equal and opposite angles are both equal… [Pointing at J]
	 J
63. Interviewer	What is a rhombus?
64. Amy	Sides are all the same. Does not form 90-degree angle as rhombus alone. [Pointing at Z]
	 Z

To better understand her word meaning in the context of parallelograms, I asked Amy if I could group J and Z together, and group U and M together. Her response was "yeah." The following conversation gives Amy's responses to these questions ten weeks later:

37. Interviewer	Can I group Fig. J and Fig. Z together?
	 J  Z
38. Amy	Mm Hmm.
39. Interviewer	Why is that?
40. Amy	Mm… because they both have opposite sides parallel and both opposite angle measures are equal.
45. Interviewer	Can I group Fig. U and Fig. M together?
	 U  M
46. Amy	Yeah, you can because U has the same property as M. The only differences is that M does not have all the same sides length, so M would not have all the properties as U, but U has all the properties of M…

In these conversations during the post-interview, more dimensions were added to Amy's use of the word *parallelogram*. At the pre-interview, the word *parallelogram* only signified polygons that fit into the physical appearances of parallelograms and rectangles, whereas at the post-interview, the word *parallelogram* signified a family of polygons that share common descriptive narratives.

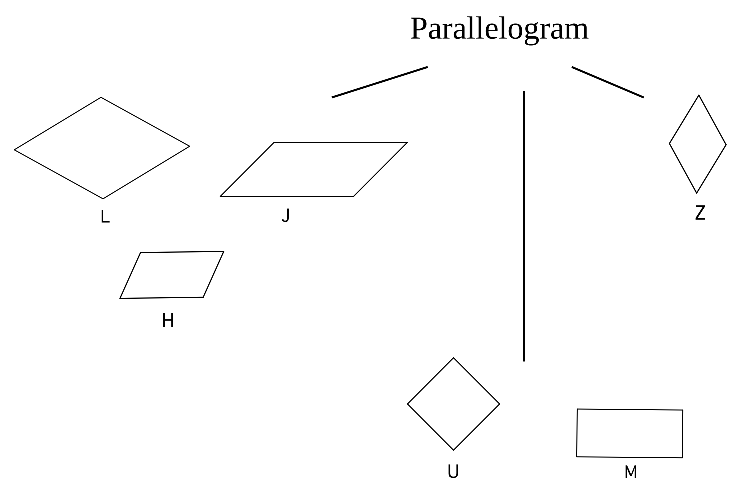


Figure 8. Amy's use of the word parallelogram at the post-interview.

As shown in Figure 8, the word *parallelogram* signified to Amy a common family name for all figures that "have opposite sides parallel and opposite angles equal." This diagram illustrates how parallelograms were inter-connected. For example, Amy identified that "as a rhombus alone" [it] does not form a 90-degree angle, and "sides are all the same." A rhombus was different from a square with regard to the angles: "all the angles form right angles…and each side length also has to be the same." However Amy did not mention how rectangles were different from parallelograms.

In Amy's case, we learned that the test results were limited to the general categorization of the van Hiele levels at the time, which changed from Level 1 to Level 3. However, investigating Amy's geometric discourse further regarding the use of mathematical words, as well as her routine procedures when engaged in mathematical activities, provided rich descriptions of her geometric thinking and how that thinking changed over time from Level 1 to Level 3.

Case 2: Changes in Sam's Geometric Discourse.

Sam was a college sophomore at the time of the interviews. The van Hiele Geometry Test showed that she was at Level 2 for the pretest, and stayed at Level 2 according to the posttest ten weeks later. Sam was interviewed after both tests, and her interview data was analyzed. A summary of findings about Sam's geometric discourse follows:

- Sam's routines of sorting polygons remained the same.
- Sam's routine of substantiation changed from descriptions about the processes of activities using transformations such as reflection, translation, and rotation at the pre-interview, to constructions of newly endorsed narratives using propositions and definitions at the post-interview.
- When verifying congruent figures, Sam chose Side-Side-Angle as the conditions of verification at the pre-interview, which was incorrect, whereas at the post-interview Sam

chose Angle-Side-Angle and Side-Angle-Side, valid congruence criteria for the verification of congruent triangles.

- There were changes in Sam's use of mathematical terminology such as the names of quadrilaterals and their parts.

Sam's routine procedures for sorting polygons were observed and analyzed in Task One. During the pre-interview, when Sam was asked to sort polygons into groups, her first question was, "Am I doing it on the assumption that those are right angles [pointing at the angles of a square], and by itself can I assume anyway?" Sam's first attempt at sorting geometric shapes "in terms of the numbers of sides they had" resulted in the following: 1) 3-sided figures, consisting of all triangles; 2) 4-sided figures, consisting of all quadrilaterals; and 3) 6-sided figures, which included only V (a hexagon). When I asked Sam to subdivide the 4-sided group, her first reaction was, "If I can assume that the sides appear to be parallel to each other," while pointing to the opposite sides of a parallelogram. Sam then rearranged her 4-sided group into three subgroups, and subsequently, she rearranged the 3-sided group into three subgroups as well. See Figure 9 for details of Sam's subgrouping of the quadrilaterals on the first attempt at the pre-interview.

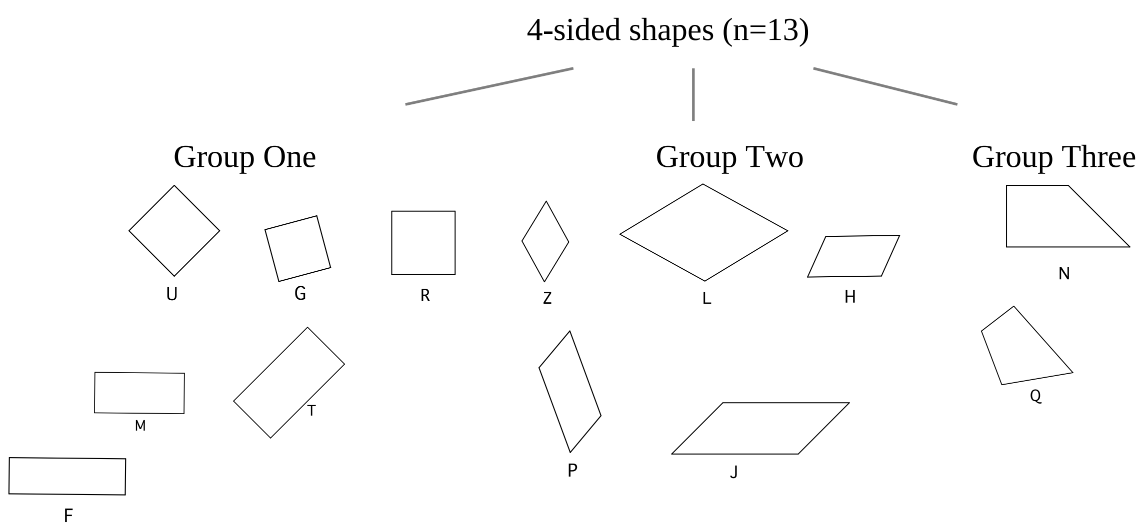
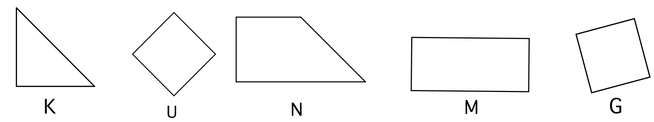


Figure 9. Sam's subdivision of quadrilaterals.

Figure 9 presents three subgroups for quadrilaterals: squares/rectangles, parallelograms and a group of 4-sided figures that do not fit into the descriptions of the two previous groups. Sam made the characteristics of each group very clear. For example, Sam talked about the parallelograms group consisting only of the parallelograms that "don't have right angles," and the squares/rectangles group consisting of figures that "have four sides, all right angles, pairs of sides are parallel and have the same length." On the first attempt, Sam's courses of action in response to the questions about sorting geometric figures focused on characteristics of angles (e.g., right angles) and sides (e.g., parallel sides or equal sides). During the interview, Sam did not use measurement tools such as rulers or protractors to check the angles and sides of the figures, but instead chose geometric figures under the assumptions that "the sides appeared parallel" and "angles are right angles."

When I asked Sam to regroup the figures differently, her first response was, “I want to separate them into shapes containing right angles and shapes that do not contain right angles.” Among the eighteen geometric figures in Task One, Sam included figures with at least one right angle in Group One, and included the figures with no right angles in Group Two. See Figure 10 for some examples.

Group One: shapes contain right angles (n=8)



Group Two: shapes do not contain right angles (n=10)

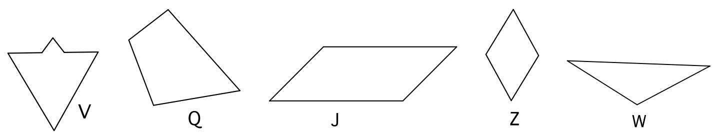
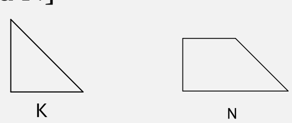
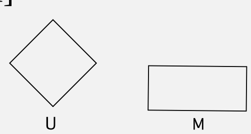


Figure 10. Examples of Sam’s regrouping at the pre-interview

Figure 10 presents a variety of figures in each group, where Sam simply divided figures that have right angles from those figures that do not have right angles. I then asked Sam to subdivide Group One, and she provided the following response:

Interviewer	Can you subgroup Group One?
Sam	I guess for Group One [subgroup 1]…I could take squares and non-squares …I could take shapes that have acute angles…like triangle…like K and N [pointing at K and N] 
Sam	I am sure that by defining the second group [subgroup 2] …I don’t know they kind seem exclusive.... will follow figures that don’t contain acute angles [pointing at U and M] 

Sam continued to talk about her strategies for subdividing Group Two. She divided Group Two into two subgroups that do not contain right angles: one with figures that have at least one set of parallel sides, and the other with figures that have no parallel sides. Figure 11 illustrates the two subgroups in Group Two.

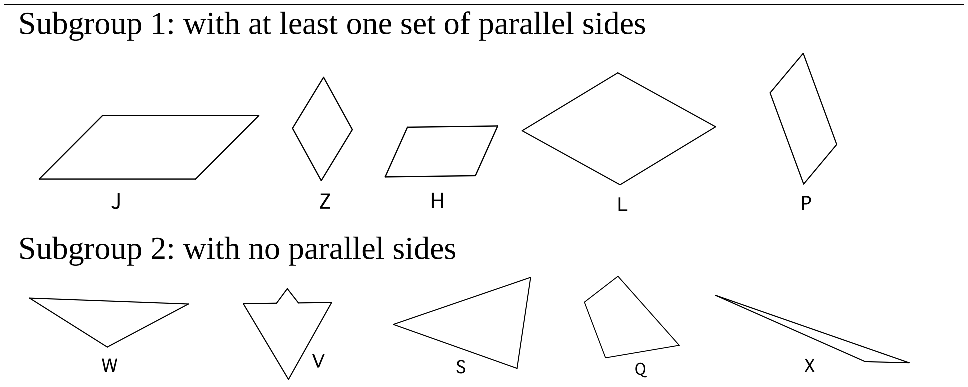
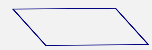
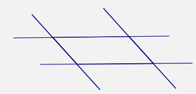
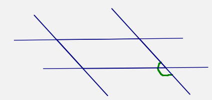
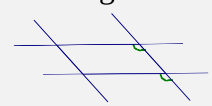
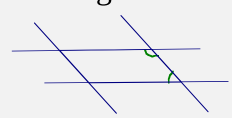


Figure 11. The two subgroups of Group Two at the Pre-Interview.

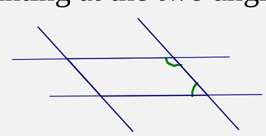
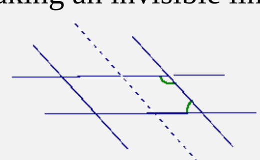
During the regrouping, Sam's courses of actions for sorting geometric shapes focused mostly on the characteristics of the angles of figures, not the sides of figures. That is, Sam first divided the entire group of figures into two groups, depending on whether the figures had a right angle or not; and then divided Group One into two subgroups based on whether the figures had an acute angle or not. Sam divided Group Two according to whether the figures had parallel sides or not; half of the figures in Group Two were parallelograms.

Ten weeks later I interviewed Sam again, and found no change in her routine procedures for sorting geometric shapes when compared to those of the pre-interview. For example, at the post-interview, when I asked Sam to group the figures she said, "the first thing I want to do is separate them by numbers of sides, like I did last time [at the pre-interview]." When I asked Sam to regroup the figures, she replied, "This [group] just assumes that all figures appeared to have right angles … Group Two could just be all the figures that don't contain right angles." Therefore Sam's routine procedures for grouping polygons at the post-interview were similar to what she did at the pre-interview. Although I did not find any changes in Sam's routine procedures in classifying geometric figures between the time of the pre-interview and the post-interview, I noted changes in her routine procedures of substantiation of narratives.

Recall that a *routine of substantiation* is a set of patterns describing a process of using endorsed narratives to produce new narratives that are true. For instance, in the context of this study, a routine of substantiation describes what an interviewee did, step-by-step, to substantiate her/his declared statements that opposite sides are equal in a parallelogram. One important finding in Sam's geometric discourse was the changes in her routines of substantiation as observed and analyzed in Task Two. Task Two asked interviewees to draw two parallelograms that are different from each other, and then to discuss the angles, sides and diagonals of those two parallelograms. At my request, Sam drew a parallelogram and declared, "in this parallelogram all angles should add up equal to 360°." After my prompt for substantiation, "How do you know that all angles add up to 360°?" Sam produced the following:

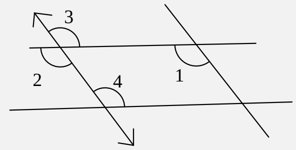
14a. Sam	Well, when you have parallel sides, you can extend all the sides…	Sam's drawing:  Sam extended the sides of parallelogram: 
14b. Sam	…it's 180 degrees and they're complementary angles …	Pointing at the two angles that form straight angles: 
14c. Sam	…but you can see that this angle really just matches this angle	Pointing at the two angles: 
14d. Sam	…so you know that these two angles together are gonna equal 180 degrees	Pointing at the two angles: 

In the preceding substantiation, Sam first drew extended lines on the sides of the parallelogram, in saying "you have parallel sides… you can extend…" [14a], and identified a vertex angle and its corresponding exterior angle forming a "complementary angle" [14b]; and she then identified an adjacent vertex angle transversal to the same exterior angle and made an intuitive claim about the two angles, "you …see this angle…matches this angle" [14c]. Sam concluded that the two adjacent vertices of a parallelogram added up to 180 degrees [14d]. Using this endorsed narrative, "two angles equal 180 degrees," Sam continued her substantiation to the final step:

24a. Sam	From this diagram and the parallel sides, these two angles add up to 180 degrees…	Pointing at the two angles: 
24b. Sam	… the fact that it's just like a mirror image, the two 180 sets of angles are just gonna add up to 360 degrees.	Making an invisible line: 

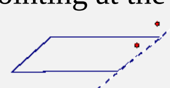
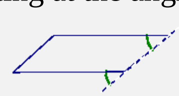
Sam used her previously endorsed narrative, "two angles add up to 180," and then endorsed a new narrative, "two sets of 180 degrees angles add up to 360 degrees" because these are a "mirror image" (i.e., a reflection) of each other, and drew a reflection line (i.e., the dashed line in 24b). Sam's substantiation of the narrative, "all angles add up to 360 degrees" was intuitive and self-evident, although the reflection line that Sam drew was not a line of reflection of the parallelogram. Mathematically, this parallelogram only has point symmetry, symmetry with respect to the center of the parallelogram (i.e., where the diagonals intersect), and not line symmetry. In this example, Sam used a "mirror image" (i.e., reflection) to draw a conclusion that all the angles add up to 360 degrees.

During the pre-interview, Sam frequently used reflections, rotations and translations in her substantiations of narratives. For example, when I asked Sam to verify her claim that "two opposite angles (i.e., ∠ 1 and ∠ 4) are equal," she provided the following response:

37.Interviewer	How do you know this angle is equal to this? [Pointing at ∠1 and ∠4] 
38a. Sam	…this angle [pointing at ∠1] can just be slid over to this position and create this angle [pointing at ∠2] …
38b. Sam	…this line [drawing arrowhead on the line] can be rotated so that this angle [pointing at ∠2] now becomes this angle [pointing at ∠3].
38c. Sam	…this angle [pointing at ∠3] at this intersection, can just be slid down and then be in this angle's position [pointing at ∠4].
38d. Sam	So these two angles are equal [Pointing at ∠1 and ∠4]

In this case, Sam used words such as "slid over," "rotated" and "slid down" to indicate a sequence of movements preformed to substantiate the claim that "two opposite angles are equivalent." Lines and angles are static mathematical objects, but Sam used these sequences of imaginary movements to complete her substantiation; and through Sam's description, these imaginary movements became visible to me. It can be argued that Sam's substantiation relied on the processing of the activities of mathematical objects (object level), rather than on discussions about these mathematical objects (meta level) at the time of the pre-interview.

Ten weeks later, Sam used mathematical axioms and propositions to verify her claims, a meta level activity for substantiations, in addition to describing what happened to these geometric figures using transformations. The following brief substantiation was typical for Sam at the post-interview:

16a. Sam	… angles on a straight line add up to 180 degrees…	Extending one side of the parallelogram with a dashed line, and pointing at the two angles: 
16b. Sam	…this angle here is the same as this angle… Because parallel lines meet a third line at the same angle.	Pointing at the angles: 

16c. Sam	By the same reason [referring to 16a], this angle added to this angle equals 180 degrees…	Pointing at the two angles: 
16d. Sam	…these two also add up to 180 degrees	Pointing at the two angles: 
16e. Sam	…for a similar reason, these two angles add up to 180 degrees...	Pointing at the two angles: 
16f. Sam	Together they equal 360 degree…	

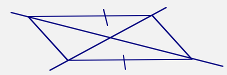
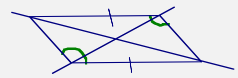
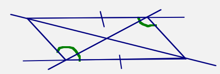
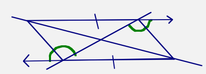
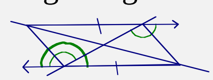
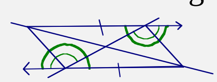
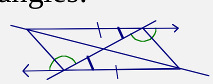
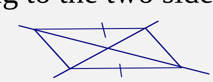
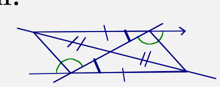
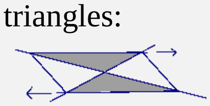
Sam made the same statement as previously about the angles of a parallelogram, "all added together they equal 360°." In contrast to Sam's routines of substantiation at the pre-interview, this example shows two changes that are evident: The first is that in each step of substantiation, Sam provided endorsed narratives (e.g., mathematical axioms and propositions, etc.) as evidence instead of reasoning intuitively. For example, Sam explained how two corresponding angles are equivalent, not because you "can see it" as in the pre-interview, but because "parallel lines meet a third line at the same angle." The second change occurred in the post-interview when Sam's concluded "all angles [in a parallelogram] add up to 360 degree." At the pre-interview, she argued this based on the assumption of the "mirror image," whereas at the post-interview Sam reached her conclusion by repeating a similar proof that "two angles add up to 180 degrees" for two adjacent angles in a parallelogram [16e-f]. Thus, one change in Sam's *routine of substantiation* was the shift from descriptions of processes and actions at the object level towards the meta-level. The maturity of the meta-level of substantiation is also revealed in Sam's substantiation of congruent triangles.

In the following example, I will describe the changes in Sam's *routines of substantiation* of two congruent triangles that I observed between the pre-interview and the post-interview. To describe these changes I looked at two aspects: 1) change from the use of transformations in the process of substantiation at the object level, to the use of mathematical axioms at the meta level; and 2) the change in the choices of elements needed for verification of congruent triangles.

During the interviews, participants were asked to substantiate their declared narratives about the angles, sides and diagonals of a parallelogram. For example, when asked for substantiation of the narratives, "opposite sides are equal," "opposite angles are equal" and /or "diagonals bisect each other," some interviewees would support their narratives by using rulers and protractors to measure the sides and angles, whereas other interviewees would try to use mathematical proofs to verify their statements. Using triangle congruency to substantiate that opposite sides and angles were congruent in a parallelogram was a common method students utilized.

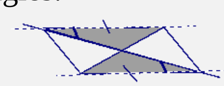
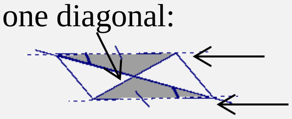
During my interviews with Sam, when asked for substantiation of declared narratives about the sides and angles of a parallelogram, Sam's first response was, "other than just measuring them?" Sam expected to substantiate her declared narratives without using the measurement tools at both the pre-Interview and the post-interview. As an example, the following are Sam's routine procedures for the narrative, "diagonals bisect each other in a parallelogram," using the triangle congruency method at the pre-interview.

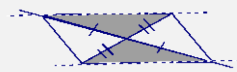
When Sam discussed the diagonals of the parallelogram, she talked about diagonals creating two pairs of congruent triangles. After my prompt for substantiation, Sam identified one such pair of congruent triangles, and then identified two corresponding sides and two corresponding angles from the two triangles to verify their congruency:

64a. Sam	Because I previously established that, it is given that these are parallel sides…	Sam added two marks: 
64b. Sam	And, these angles are equal and when lines intersect…	Sam added two angle signs: 
64c. Sam	… it's essentially the same intersection, translated to a new position…	Sam drew extended lines: 
64d. Sam	… I was suggesting that this angle is the same as this angle here.	Sam added an arrowhead on the two extended sides, and two angle signs: 
64e. Sam	… And that likewise, the complementary angles, the smaller angle that makes it add up to 180 degrees…	Sam identified two angles that form a striaght angle: 
64f. Sam	… is the same over here…	Sam identified another two angles that form a straight angle: 
64g. Sam	… So, now I know that the angle here of this triangle is equivalent to the angle here of this triangle…	Pointing at the alternating interior angles: 
64h. Sam	… and this side length is, the same of this side length… So, I've already shown how a side length and an angle match of each…	Referring to the two sides: 
64i. Sam	… And then diagonals bisect themselves equally. I can't really prove that, but I'm suggesting that this side length is the same as this side length…	Sam added two marks on the diagonal: 
64j. Sam	…this triangle is equivalent to this triangle here.	The shaded area indicates two congruent triangles: 

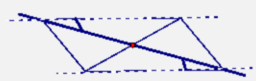
Sam's substantiation included two parts: the first was the substantiation of the equivalence of alternating interior angles [64b-64g], and the second was the verification of congruent triangles. The first part of substantiation, "this angle is equivalent to this angle" (i.e., alternating interior angles), was intuitive and self-evident. To show that opposite angles are equivalent in a parallelogram [64b], Sam used an instinctive process of translating the intersection to a new position [64c], and then "suggested" that the corresponding alternating exterior angles were equivalent [64d], another intuitive act. The second part of substantiation involved the verification of the two congruent triangles that she identified. During the process of verification, Sam did not use measurement tools to measure the angles and sides (an object level of verification) to check equivalence, but instead chose three elements of the triangles to verify congruent triangles abstractly. However it is important to note that Sam's choice of these three elements (angle, side, side) for verification of congruent triangles was incorrect, because this criterion does not guarantee congruent triangles. I conclude that Sam's substantiation was a combination of an objective level of substantiation (e.g., these angles are equal), and a meta level of verification (e.g., two triangles are congruent), even though her choice of the elements for verification was not entirely correct.

Ten weeks later, I interviewed Sam again, and the same tasks were performed. At the post-interview Sam was able to use triangle congruency to substantiate most of her declarations stating that "opposite angles are equivalent," "opposite sides are equivalent" and "diagonals bisect each other" in a parallelogram. She chose three appropriate elements, such as Side-Angle-Side and Angle-Side-Angle, to verify congruent triangles, and she was comfortable using the triangle congruency method. The following response illustrates Sam's substantiation that "diagonals bisect each other":

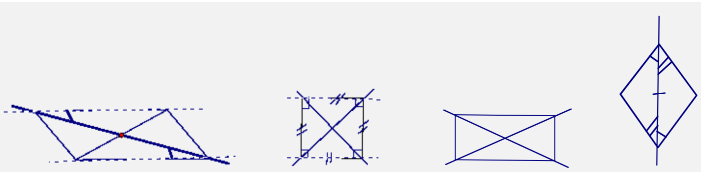
54a. Sam	… I'm looking at this triangle as compared to this one here…	Pointing at the shaded area: 
54b. Sam	And I know that these two angles are congruent…	Sam marked the angle signs on the two angles in the shaded triangles: 
54c. Sam	…And between these parallel lines, and now this diagonal,	Pointing at the two parallel lines and one diagonal: 
54d. Sam	… these angles are also congruent.	Sam marked the angle signs on the two angles of shaded triangles: 

54e. Sam	… So, by the triangle test, angle, side, angle, these two triangles are congruent.	Pointing at the corresponding angles, sides and angles of shaded triangles: 
54f. Sam	…which means that this side corresponds with this side and that this side corresponds with that side. That's probably the most roundabout way to find that answer.	Pointing at each half of the diagonals: 

In the preceding substantiation Sam first verified that the “two triangles are congruent” [54e] using the angle-side-angle criterion. She identified the exact three elements (i.e., two angles and their included side) needed for verification; and used the endorsed narrative “the two triangles are congruent” to construct a new narrative that “diagonals bisect each other,” by saying “this side corresponds with this side….” [54f] as a result of congruent triangles. Sam also made no intuitive claim about the equivalence of alternating interior angles at the post-interview, as she clearly explained:

46d. Sam	…And, we know that between parallel lines, if you take a third line and cross both lines, then it will have angles that are congruent. In this case, this angle and this angle. [Sam extended the two parallel lines, and marked angle signs on the two alternating interior angles] 

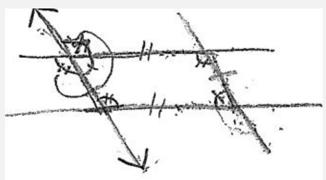
During the post-interview, Sam applied the same substantiation to other similar situations. For instance, when I asked Sam why diagonals bisect each other in a rectangle, she responded, “the same as what I did in parallelogram, I already established that.” When I asked Sam at the end of Task Two whether it was true that in all parallelograms the diagonals bisect each other, Sam responded, “Yes, that’s true” and then shared her thinking about this conclusion:



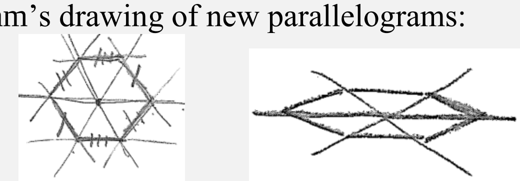
148. Sam	…because when you draw the diagonals in a figure, there is an intersection point and it divides the figure into four triangles. And, regardless of the figure, if it's a parallelogram, these two triangles will be congruent and these two triangles will be congruent [pointing at the two pairs of congruent triangles in the rectangle] So, it can be found that in congruent triangles, corresponding sides will be equal [therefore diagonals bisect each other in all these cases].

In summary, I conclude that there was a change in Sam's routine procedures, from using transformations as actions on the geometric figures to substantiate the equivalence of the angles at the object level, to using mathematical axioms to substantiate the same claim at the meta level. I am convinced that Sam was more rigorous at the post-interview, when she made the choices of the three elements needed for verification of congruent triangles, than at the pre-interview.

I have described Sam's change in routine procedures of substantiation and the changes in her routine procedures of constructing new narratives; I now briefly describe the changes in Sam's use of mathematical terminology. Recall Sam's routine of sorting for Task One. It is important to note the natures of the tasks designed for the interviews were limited and pre-constructed. For example, when Sam identified geometric figures among given figures in Task One, the pool of choices was limited to eighteen figures and those figures were pre-drawn. Consequently, Sam's misunderstandings about some of the geometric figures were not detected in Task One. It was in Task Two, when I asked Sam to draw two different parallelograms at the Pre-Interview, that I began to understand Sam's misconstrued definition of *parallelogram*. Her definition was quite different from what I expected:

2. Sam	*Sam's drawing*:  *Note: Sam drew a parallelogram first, and extended sides of the parallelogram later*
3. Interviewer	Why is this a parallelogram?
4a. Sam	I believe that this is a parallelogram because I drew it so that this side… would be parallel to this side [pointing at the two longer sides of the parallelogram]…
4b. Sam	… and this side would be parallel with this side [pointing at the two shorter sides of the parallelogram]

Later I asked Sam to draw a new parallelogram different from the one she drew, and she provided the following responses:

86. Sam	Sam's drawing of new parallelograms:  *Note: Sam drew a hexagon first, and she extended sides of the hexagon later*
87. Interviewer	Why is this a parallelogram?
88. Sam	I think it's a parallelogram… because all the sides are parallel to another side.
89. Interviewer	Why is it a different parallelogram?
90. Sam	It's different…because there are more sides and because the angles are different.

These conversations present an interpretive description of *parallelogram* as Sam used that word at the pre-interview. During our earlier conversation, I asked Sam what a parallelogram was, and she responded, “It is any figure that has at least one pair of parallel sides. I think trapezoid [pointing at N, a right trapezoid] is considered a parallelogram.” When I asked Sam to write down the definition, she wrote, “A parallelogram is a figure with all sides being pairs of parallel line segments,” which was inconsistent with her verbal statement. Neither Sam’s written narrative nor her verbal narrative about parallelograms mentioned the necessary condition of a parallelogram being a quadrilateral. Because of this missing condition, Sam chose a hexagon as an example of a different parallelogram. When identifying and defining parallelograms, Sam focused on the necessary condition of parallel sides. At the pre-interview, Sam’s concept of a parallelogram was unclear, as she expressed, “I actually don’t know if parallelograms are strictly four-sided figures… or many shapes should be parallelograms.”

Sam’s use of the word *parallelogram* (see Figure 12) signifies a collection of figures that share this visual property of parallel sides. Based on Sam’s definition, this collection of figures could include figures that have one pair of parallel sides such as *trapezoids*, two pairs of parallel sides such as *parallelograms*, or figures that can have more than two pairs of parallel sides such as *hexagons*. We notice that rectangles and squares are not included in the family tree of parallelograms. According to what I observed during the pre-interview, Sam did not include rectangles and squares as parallelograms, but rather considered them as a separate group of figures that have right angles.

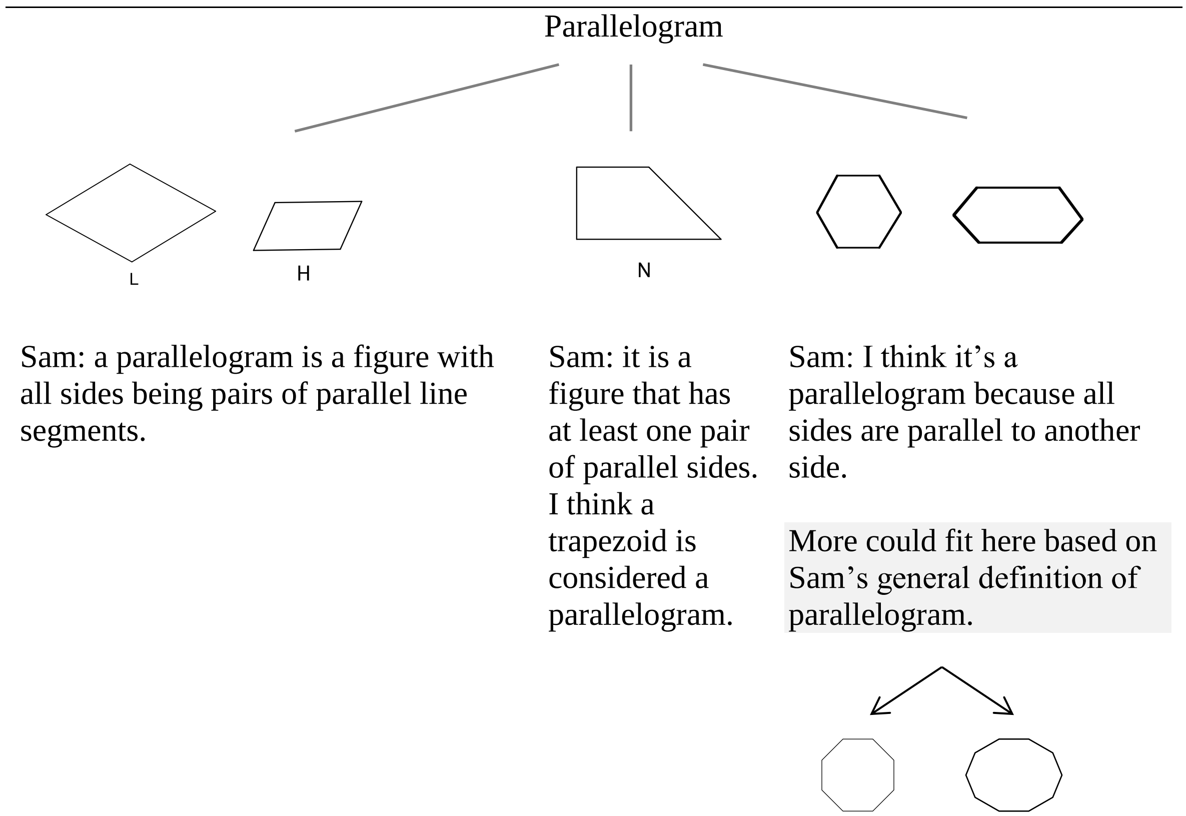
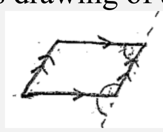
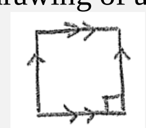


Figure 12. Sam’s use of the word parallelogram at the pre-interview.

At the post-interview the most important change in Sam's word use is in her use of the word *parallelogram*. Although Sam showed very similar routine procedures when identifying geometric figures in both interviews, her concept of a *parallelogram* was different from that of the Pre-interview. For example, when I asked Sam to draw two different parallelograms in Task Two, she drew a parallelogram and a square:

2. Sam	Sam's drawing of a parallelogram 
3. Interviewer	Why is this a parallelogram?
4. Sam	Because it has four sides and each opposing side is parallel to one another.
60. Sam	Sam's drawing of a different parallelogram 
61. Interviewer	Why is this a parallelogram?
62. Sam	It's a square… it has four sides of equal measure and all angles are 90 degrees.
63. Interviewer	Why is this different from the one you drew?
64. Sam	This one is different because all the angles in this figure are equal.

Sam's use of the word *parallelogram* changed with regard to adding a necessary condition of "four-sided" figure to another necessary condition of "parallel sides" (she mentioned parallel sides at the pre-interview). Sam considered rectangles and squares as figures with 90-degree angles and as parallelograms. It is evident that Sam's use of the word *parallelogram* signified a collection of figures sharing this common descriptive narrative, "a four-sided figure with two sets of parallel sides" at the post-interview.

It is notable that Sam included all quadrilaterals with two sets of parallel sides in this family tree of parallelograms, relating these quadrilaterals because "they have two sets of parallel sides." Sam did not provide any explicit information about how these figures were related other than being parallelograms. For example, Sam grouped rhombi together with parallelograms because all rhombi have two sets of parallel sides; however there were no connections made between rhombi and squares, although Sam defined a rhombus as a "four-sided figure with all side length equal in measure." Moreover, Sam did not mention any relations between squares and rectangles other than that they had four right angles. Sam had a good grasp of the concept of parallelograms in general, but her understanding of the hierarchy of parallelograms was missing, or not clearly demonstrated in the post-interview. Figure 13 illustrates Sam's understanding of definition of a *parallelogram* at the post-interview.

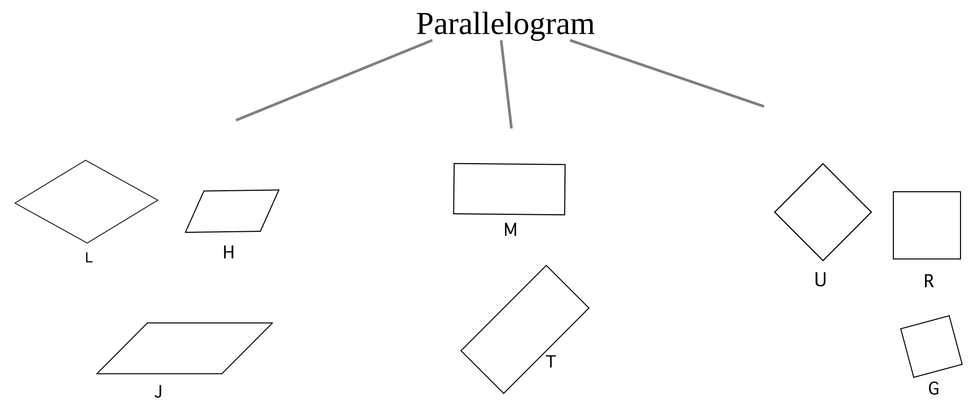


Figure 13. Sam's use of the word parallelogram at the post-interview.

To sum up, Sam's van Hiele pretest and posttest responses suggested that her thinking operated at Level 2 (descriptive). However, two main changes were found in Sam's geometric discourse, a change in word use and a change in reasoning. Sam had developed competence in using definitions to identify and group polygons, though with no hierarchy of classification, and had developed some informal deductive reasoning as her geometric thinking moved towards Level 3. I am not trying to contradict the findings from Sam's paper-pencil pretest and posttest with her interview results, but rather to integrate the results and to treat her thinking more dynamically. Her progress illustrates a student's geometric thinking developing continuously within Level 2 and in transition between Level 2 and Level 3, as she was more competent in using definitions to name polygons, and her routines of substantiation began to operate at a meta level in using definitions and axioms to construct mathematical proofs.

Discussion

The van Hieles wished to note language differences and different linguistic symbols at each level, in the study of language in geometric thinking, but were never explicit about it. The language of mathematics I wish to discuss here does not refer to a list of vocabulary words or grammar rules, but rather to the communicative competence necessary and sufficient for competent participation in mathematical discourse.

The van Hiele descriptions of the levels focus largely on how a student reasons about geometric figures in a language for instance, in response to being asked what is and is not a rectangle, applying a definition. What is missed or not clearly emphasized is the meaning of a mathematical term when used by a student. When I consider each van Hiele level as its own geometric discourse with characteristics of word use, narratives, routines and visual mediators, I regard word use as all-important, revealing facts concerning how a concept is formed. In this study, students' use of the word "parallelogram" provided significant information about how the concept of parallelogram is understood at different van Hiele levels among different students. A careful analysis of students' mathematical word use in geometric discourse sheds light on how words are used and whether the words are used correctly for the sake of learning and communication.

Discursive routines do not determine students' actions, but only constrain what they can reasonably say or do in a given situation, as negotiated conventions. However, discursive routines offer valuable information about what students do and say as courses of action to make conjectures and justifications in a geometric discourse. I find it very useful to see the details of students' routines of identifying, defining and justifying when working on a task about geometric figures and their properties, where the roles of definitions are demonstrated at the first three van Hiele levels. I also find it revealing to see the details of students' geometric reasoning across van Hiele levels through the development of geometric discourses. Battista (2007) argues about the validity of the reasoning, which involves the accuracy and precision of students' identifications, descriptions, conceptions, explanations, justifications, and points out that "there is a lack of distinction between *type of reasoning* and *qualitatively different levels in the development of reasoning*"(p. 853) throughout the van Hiele studies. For instance, in this study, Amy used direct recognition as a type of reasoning that is strictly based on intuition, and used the same type of reasoning to refer to a period of development of geometry thinking when her thinking was dominated by direct recognition. One challenge regarding the van Hiele theory is to sort out the van Hiele levels related to types of reasoning and/or the levels of reasoning; of course, "the devil is in the details" (p. 854). Sfard's discursive framework takes greater consideration of the details of what students usually say and do when working on a geometric task, and it adds more information with regard to the levels of geometric thinking.

Conclusions

This study focused on students' geometric discourse, and how this discourse helps us to learn more about their thinking. However, one open question asks what this mathematics discourse looks like when a student works on different mathematical tasks that include different content domains of mathematics, and how the subsets of mathematics discourse interact with each other. As mentioned previously, for those interested in geometry or teaching geometry, an investigation using a discursive lens into students' use of mathematical terminology in geometry would be a next step. This analysis could also be extended to other mathematical topics. More discussions regarding classroom interactions are needed. What can we do to help students use mathematical terminology more precisely for the sake of communication and development of a mathematical concept?

We need to develop frameworks for analyzing activities from both textbooks and classrooms, and to identify mathematical activities that help students move from an object level discourse to a meta level discourse. The rise and popularity of computer software has created a new learning environment for students and presented important instructional and learning tools in the school curriculum. Many researchers and curriculum developers want to design pre-constructed activities using software, and they hope that these activities will serve as mediators to help students learn geometry. In response, there is a need to develop instruments to examine these activities, with the goal of helping students develop more advanced levels of thinking. Finally, we need to revisit the van Hiele levels with multiple lenses, in order to acquire a better picture of human thinking, and to improve communication through classroom interaction, and to help teachers better facilitate classroom discussions at various levels and in various contexts.

References

Battista, M.T. (2007). The development of geometric and spatial thinking. In F. K. Lester (Ed.), *Second Handbook of Research on Mathematics Teaching and Learning* (Vol. 2, pp. 483-908). Charlotte, NC: Information Age Publishing.

Burger, W. F., & Shaughnessy, J. M. (1986). Characterizing the van Hiele levels of development in geometry. *Journal for Research in Mathematics Education, 17*, 31-48.

Gutierrez, A., Jaime, A., & Fortuny, J. M. (1991). An alternative paradigm to evaluate the acquisition of the van Hiele levels. *Journal for Research in Mathematics Education, 22*, 237-251.

Mayberry, J. (1983). The van Hiele levels of geometric thought in undergraduate preservice teachers. *Journal for Research in Mathematics Education, 14*, 58-69.

Moschkovich, J. (2010). Languages and learning mathematics: resources, challenges and issues for research. In J. Moschkovich (Ed.), *Language and mathematics education: Multiple perspectives and directions for research* (pp. 1-28). Charlotte, NC: Information Age Publishing.

Parker, T. H. & Baldridge, S. J. (2007). *Elementary geometry for teachers* (3rd ed.). Okemos: Sefton-Aash Publishing.

Sfard, A. (2002). Learning mathematics as developing a discourse. In R. Speiser, C. Maher, &C. Walter (Eds.), *Proceedings of the 21st Conference of PME-NA*. Columbus, Ohio: Clearing House for Science, Mathematics and Environmental Education, 23-44.

Sfard, A. (2008). *Thinking as Communicating: human development, the growth of discourses, and mathematizing*: Cambridge.

Usiskin, Z. (1982). *Van Hiele levels and achievement in secondary school geometry* (Final report of the Cognitive Development and Achievement in Secondary School Geometry Project: ERIC Document Reproduction Service No. ED 220 288). Chicago: University of Chicago.

van Hiele, P. M. (1959/1985). The child's thought and geometry. In D. Fuys, D. Geddes & R. Tischler (Eds.), *English translation of selected writings of Dina van Hiele-Geldof and Pierre M. Van Hiele* (pp. 243-252). Brooklyn, NY: Brooklyn College, School of Education.

van Hiele, P. M. (1986). *Structure and insight: A theory of mathematics education*. Orlando, FL: Academic Press.

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