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Jonathan L. Brendefur
Boise State University

Kim Bunning
Boise State University

Walter G. Secada
University of Miami

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Using Solution Strategies to Examine and Promote High School Students' Understanding of Exponential Functions: One Teacher's Attempt[1]

Jonathan L. Brendefur, PhD[2], Boise State University
Kim Bunning, PhD[3], Boise State University
Walter G. Secada, PhD[4], University of Miami

Abstract

Much research has been conducted on how elementary students develop mathematical understanding and subsequently how teachers might use this information. This article builds on this type of work by investigating how one high-school algebra teacher designs and conducts a lesson on exponential functions. Through a lesson study format she studies with her colleagues how other algebra students have mathematically modeled a bacteria growth problem with no prior formal instruction. Analysis revealed that the teacher was able to use students' algebraic thinking to structure her class and begin promoting mathematical understanding. The implications for building on students' conceptions of algebra are explored throughout the paper.

Keywords: *Functions; learning progressions; lesson study; student thinking; teacher education*

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[2] J.L. Brendefur is professor of mathematics education in the Department of Curriculum and Instruction at Boise State University.

[3] K. Bunning is a regional mathematics specialist for the Initiative for Developing Mathematical Thinking at Boise State University.

[4] W.G. Secada is professor of Teaching and Learning and Senior Associate Dean of the School of Education at the University of Miami.

Introduction

High school mathematics curricula are permeated with the topic of 'functions' yet it remains a topic that high-achieving students understand as little more than graphs and equations (Norman, 1992; Oehrtman, Carlson, & Thompson, 2008; Schoenfeld, Smith, & Arcavi, 1993). Further, when it comes to solving real-world problems involving functions, high-school students tend to have deep misconceptions (Berman & Brendefur, 1998). As such, researchers and educators need to develop an understanding of how students reason informally and formally about functional relationships and, in turn, how to integrate this knowledge into classroom practice.

The purpose of this paper is to report how one teacher developed and utilized knowledge of students' thinking about functions in an attempt to support and promote conceptual understanding of functions in her Algebra I classroom. The teacher took part in a graduate level seminar during which participants attempted to integrate theoretical ideas of promoting conceptual understanding (Carpenter & Lehrer, 1999; Kazemi & Stipek, 2001; Rittle-Johnson, Siegler, & Alibali, 2001) with student ideas about functions (Schoenfeld et al., 1993; Vinner & Dreyfus, 1989; Williams, 1998).

Theoretical Perspectives

Reform efforts call for classrooms to be environments in which iterative processes of experimentation and creativity lead to mathematical understandings that are then examined, validated and concretized (Carpenter & Lehrer, 1999; Hiebert et al., 1997; NCTM, 1991, 2000; NGA & CSSO, 2011; NRC, 2005). This type of learning is generative in that students build a deeper and more connected understanding of mathematics, which over time, becomes self-sustaining. However, in order to foster the kind of understanding envisioned by the reform movement, teachers must have a deep understanding of mathematics themselves and formally structure what it means to learn, know and teach mathematics for understanding (Ball, Thames, & Geoffrey, 2008; Hiebert et al., 1997; NRC, 2005). In this particular case, teachers would need to have an understanding of students' informal ideas about exponential growth functions along with any potential misconceptions students may develop, and the best ways in which to guide students' development toward more formal and relational understandings of exponential growth functions.

Students' understanding of exponential functions

The central concern of this work is to better understand how teachers can utilize their knowledge of student thinking in order to promote student understanding and reasoning in algebra, specifically contexts that can be modeled by exponential growth functions. Understanding is a complex, dynamic state in which a student is able to connect a piece of knowledge to other, related pieces of knowledge (Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992; Hiebert et al., 1997). In this view, understanding means that a student's knowledge is organized into networks of mental representations in which the strength of the network is related to the number of connections linked to the common conceptual ideas (Carpenter & Lehrer, 1999).

Despite calls for students to develop flexible ways of thinking about functions (NCTM, 2000), most work in the existing literature on student knowledge of functions focuses upon students' acquisition of formal knowledge of functions and the procedural skills necessary to construct and manipulate conventional representations of functions (Dreyfus & Eisenberg, 1982; Oehrtman et al., 2008; Schoenfeld et al., 1993; Vinner & Dreyfus, 1989). For example, the research often describes students' knowledge of functions as fragmented and inflexible in utilizing different representations of functions (Confrey, 1988; Dreyfus & Eisenberg, 1982; Oehrtman et al., 2008; Rizzuti & Confrey, 1988); it focuses on students' concept image of functions and the extent to which that concept image is aligned, or misaligned, with a formal definition of functions (Vinner & Dreyfus, 1989); and it debates students' inabilities to reify, or perceive functions as higher-level, abstract objects (Reed, 2006; Sfard, 1992; Slavit, 1997). These finding do have consequential implications for teaching since teachers have to understand the limits of students' procedural and conceptual thinking.

In an effort to focus on students' understanding of functions, we relied on the notion of quantitative reasoning for two reasons. First, quantitative reasoning is grounded in problem situations that are "verbal descriptions of situations constituted by interrelated quantities" (Smith & Thompson, 2007, p. 102). These problems enable students to conceptualize the problem elements in ways that are related to the problem situation rather than only as quantities and operations – thus encouraging a focus on students' notions of the functional relationships arising from the problem situation. Second, requiring students to focus on the relationships among quantities instead of procedural operations or formal notations can provide rich descriptions of students' intuitive understandings of the functional relationships inherent in the problem situation and students' abilities to formalize their thinking (Smith & Thompson, 2007; Thompson & Thompson, 1995). As Confrey & Smith (1994) point out, when students are encouraged to "generate functional relationships by acting within contextual situations and by using multiple representations in both creating and representing their solution processes, legitimate and diverse ways of thinking about functions are created" (p. 32).

In this paper, we used the setting of bacterium growth to spur students' informal thinking about exponential growth (see Figure 1 for the complete task). The first three questions are similar to the task Confrey and Smith (1994) used to study exponential growth. However, our paper focuses on the strategies employed by students as they attempted to answer questions four and five; that is, what happens when the growth of the bacteria is slowed.

Knowledge required for teaching mathematics

Over the last two decades, there have been numerous articles focusing on teachers' content and pedagogical content knowledge (Ball, Lubienski, & Mewborn, 2001; Ball et al., 2008; Silverman & Thompson, 2008). Many of these studies demonstrate that a teacher's instructional practice is influenced by the teacher's knowledge (Ball et al., 2001; Borko et al., 1992; Grossman, Wilson, & Shulman, 1989). While secondary mathematics teachers' content knowledge is usually judged as adequate for teaching, teaching for understanding requires more than simply well-developed content knowledge (Ball & Bass, 2000; Hill & Ball, 2004). In addition, this body of research has revealed the relationship between teachers' knowledge and their instructional practices is more complex and multifaceted than initially hypothesized. There is a difference between knowing the mathematics and knowing how to use it in practice. In fact, "astonishingly, little empirical evidence exists to link teachers' content knowledge to their students' learning" (Ball & Bass, 2000, p. 94). One important aspect of our work is to connect teachers' knowledge of mathematics to their understanding of how students' come to understand mathematics, and then, to link these ideas to their pedagogical practices.

Teachers' knowledge in action - pedagogical moves

All forms of knowledge must be synthesized during the act of teaching. One potential representation of this synthesis is Simon's (1995) notion of a hypothetical learning trajectory (HLT). In Simon's description, HLT's "refer to the teacher's prediction as to the path by which learning might proceed . . . A hypothetical learning trajectory provides the teacher with a rationale for choosing a particular instructional design" (Simon, 1995, p. 135). He goes on to describe how design decisions may be based on teachers' best guesses of how learning might proceed. According to Simon (Simon & Tzur, 2004; 1995) HLT's are made up of three components: the learning goal that defines the direction, the learning activities, and the hypothetical learning process - a prediction of how the students' thinking and understanding will evolve in the context of the learning activities. Similarly, Clements and Sarama (2004) described hypothetical learning trajectories as "…descriptions of children's thinking and learning in a specific mathematical domain and a related, conjectured route through a set of instructional tasks designed to engender those mental processes or actions hypothesized to move children through a developmental progression of levels of thinking" (p. 83). They go on to say that learning trajectories are created with the intent of supporting children's achievement of specific goals in that mathematical domain. A complete learning trajectory then includes the learning goal, the developmental progressions of learning and thinking, and a sequence of instructional activities that may be used to support the children's achievement toward the goals (Clements & Sarama, 2004).

We believe learning trajectories can be used to help teachers identify potential solution strategies and misconceptions, create useful tasks, design instructional sequences, and develop questions that might further student thinking. We refer to this process as "pedagogical moves." For example, considering how to pose a particular task, how to group students in ways that will enhance their interactions, how to ask particular types of questions, and how to effectively facilitate a whole class discussion are ways of enhancing teachers' various types of knowledge. In this sense, when teachers wrestle with and create learning trajectories, their knowledge of mathematics and mathematics pedagogy are enhanced.

In the following sections, we examine how a group of teachers collectively developed a hypothetical learning trajectory by considering the interactions and participation of the teachers in a graduate seminar. We then look at how one teacher attempted to utilize elements of the trajectory in her classroom, examining carefully the effectiveness of her pedagogical moves. That is, we focus our attention on how this particular teacher decided on "pedagogical moves" that were, at least to some degree, anticipated through the process of constructing a hypothetical learning trajectory.

Methods and Data Sources

In this section, we first describe briefly the strategies employed by 75 ninth-grade students enrolled in a beginning-algebra course as they solved the bacterium growth problem and briefly discuss how the participants in the graduate level seminar used these strategies to develop a hypothetical learning trajectory. We, then, follow one graduate student as she taught a lesson on exponential functions to her ninth grade algebra I class.

Study of student strategies

This two-year investigation focused on high school students' understandings of functional relationships. The settings were classrooms in which the use of context was promoted in order to enhance students' mathematical development. In this study, we consider the students' responses to a problem involving bacterial growth. Algebra I students were given a problem in which 100 Salmonella bacteria were on a piece of meat and that each hour the number of bacteria doubled. They were then asked to determine when there would be more than 1000 bacteria. Next, they were asked to graph this information and to determine the growth rate. The situation in the context – bacteria doubling every hour – was then modified to bacteria growing at half that rate by placing the meat in a refrigerator. Students were then asked to create representations based on this new information.

Teachers' development and creation of pedagogical moves

The bulk of this current paper is dedicated to examining how one teacher, Ms Carson, used knowledge of students' thinking about exponential growth functions in order to promote relational understanding in her Algebra I classroom. She was a participant in a graduate seminar course and was chosen because she represented a teacher with solid mathematical knowledge – a mathematics major and a graduate student working on her masters in mathematics. The goals of the seminar course were to develop an understanding of how students' think about and use algebraic reasoning, especially in the context of exponential functions in order to examine their roles in promoting algebraic reasoning and to create instructional sequences to be piloted in their own Algebra I and II classrooms. The seminar course was run like a Japanese lesson study (Lewis, 2002; Stewart & Brendefur, 2005; Stigler & Hiebert, 1999), in which the teachers in the class studied the terrain of the content, examined their own curriculum and instructional practices, and, then, created extended lessons to implement, examine, and modify.

At one point during the seminar the graduate students solved the Bacteria Task (see Figure 1 below) and examined the high-school students' solution strategies (Secada & Brendefur, 2000), which are highlighted in the next section. They studied the multiple approaches students used to solve the exponential growth problem and how students justified their thinking. This information was then used to build a three-day lesson to engage students in thinking and learning about exponential growth functions. To highlight the instructional implications of this knowledge, one of these teachers, as mentioned, subsequently volunteered to implement her three-day plan in which project staff (two of the authors) observed. Project staff made daily visitations to interview the teacher before and after the lessons and to observe the classroom episodes while the teacher implemented the lesson.

Figure 1. Bacteria in Food[2] Task

The bacteria salmonella often causes food poisoning. At 35°C, a single bacterium divides every hour. Suppose there are 100 salmonella bacteria in a portion of food and the temperature is 35°C.

1. Complete the table below to find out after how many hours there will be more than 1,000 bacteria present in the food.

Time (in hours)	0	1	2	3	4	5
Number of Bacteria	100					

2. Make a graph showing the growth pattern of the bacteria in the above table.

3. What is the growth factor for the situation described in problems 1 and 2?

   Suppose that by refrigerating the food, you are able to slow the growth of bacteria to exactly one-half of the growth rate in the previous situation.

4. Make a graph showing this new growth pattern. Draw the graph on the same coordinate grid system you used in problem 2.
5. Estimate the growth factor for the new growth pattern described in problem 4. Explain how you found your answer.

In order to examine how teachers might use student thinking to guide instruction, various sources of data were collected and analyzed using principles of qualitative research. Sources included interviews (audiotaped) and observations (audiotaped and field notes). The data were analyzed using ethnographic research techniques (Erickson, 1986). In an iterative process, the data were coded and organized around an inductive approach to identify and understand themes and relationships within the data (Thomas, 2006). Two readers read the set of data and created a

[2] Adapted from the 8th grade unit 'Growth' in the middle school curriculum Mathematics in Context.

set of primary codes, mostly from the literature review. Then, secondary codes within the primary codes were created based on a more thorough read of the data. General assertions were made through induction for each of these codes or categories, which were then confirmed and disconfirmed by a third reader.

Results of the Seminar Discussions: Building an HLT

One of the authors, who was the instructor of the seminar course, presented results of 75 high school students' work on the bacteria problem to the teachers in the graduate seminar course, which the teacher in this study participated. Below is commentary on how the participants in the seminar used this information to create various aspects of hypothetical learning trajectories, including pedagogical moves. The bulk of the results section however, is dedicated to examining the results of the observations of Ms. Carson's classroom.

Study of student strategies results

In the seminar, Ms. Carson and the other graduate students studied high-school students' written problem-solving strategies (Secada & Brendefur, 2000) and attempted to identify the mathematics that undergird these strategies. By doing so, they were able to reveal a more detailed view of students' solutions than simply whether their answers were right or wrong. Moreover, consistent with Confrey and Smith's (1994) findings, they were able to organize student strategies into categories which, even absent detailed interview protocols, were possible to describe in such a way that they could hypothesize what the students were thinking as they solved the task using a particular strategy. In addition, the student strategies made intuitive sense, and they demonstrate an internal coherence and logic based on the context of the problem.

Unlike Confrey & Smith's (1994) findings, teachers categorized student strategies into five distinct methods (reported in more detail below). This difference can more than likely be attributed to the differences in the structure of the task presented in this study compared to the relatively simple task in the Confrey and Smith (1994) study. As noted earlier, one purpose for examining students' responses was to create a possible learning progression that could be subsequently used by teachers to inform curricular and instructional decisions – one goal of the seminar.

Double the time strategy

This method focused on the time it takes for the bacteria to double. In other words, once the food is put into the refrigerator it takes the bacteria twice as long to double. Based on the results of the student work, 4 of the 75 high-school students used this approach.

During the lesson study seminar, the researchers and teachers focused on possible students' conceptions that may have contributed to the construction of such a model. The notion that when the food is refrigerated the bacteria's growth rate slows, generating the same number of bacteria as before, but at twice the time, appeared to be a simple but surprisingly complex way of examining the growth when compared to other methods. Only a few of teachers admitted thinking about this approach before observing students' work. In the discussion we asked, "What possible conceptions or lack of conceptions did the students have to create the two tables (see Figure 2) [one for the original set of data and one for the second set of data points which only included y values for every other x value, 0, 2, 4, . . .]?" The participating teachers' initial reactions were that the algebra curriculum they were using did not engage students in problems that encouraged them to think about realistic contexts and how a particular context might affect a graph representing that situation.

Figure 2. Double the time strategy

Time (in hours)	**0**	**1**	**2**	**3**	**4**	**5**	**6**

Bacteria (35^0 C)	100	200	400	800	1600	3200	6400
Bacteria (refrigerator)	100		200		400		800

Although this might have been the case, we focused the discussion again on possible student conceptions. The discussion turned toward the ideas students need to construct graphs, such as an understanding of intervals and interpolation. The participants decided that the second student was possibly divorcing the context from the procedure. The teachers then hypothesized that this student was using procedural knowledge of finding a midpoint (an averaging technique) as a way to produce numbers 15, 300, and 600 for the odd number times in the problem. This gave rise to a discussion of the pedagogical move of asking students why they used a particular procedure and asking them to make sense of the numbers as related to the context of the problem.

Another pedagogical move discussed by the participants was the idea of pairing up students who used a quantitative strategy with students who used a descriptive strategy. The seminar participants conceded this might allow a deeper discussion among students about what the graph was representing and, conversely, how a graph that accurately represented the phenomenon could be constructed. The participants decided these two pedagogical moves might be useful in encouraging students to develop deeper understandings of exponential growth functions, in particular how to represent this particular context.

Divide the number of bacteria by two strategy

In the second distinct method, high school students interpreted the "one-half the growth rate" to mean that the original amount of bacteria should be ½ of the amount they found in the first part of the task. So they divided the original number of bacteria for each hour by 2 (see

Figure 3). Almost half the students solved the problem this way, reporting that at time 0 there would be 50 bacteria.

The teachers were noticeably bothered by this strategy. One responded, “This is an easy way to look at the problem. All they did was divide the number above by 2”. Again, we focused on why students might approach the problem in this way and what the next step should be instructionally when this occurs – building a hypothetical learning trajectory. It was decided that most students might have observed the numbers in the table for the original problem and then divided that number by 2.

In any case, the lesson study participants proposed the students using this strategy should be asked to develop an equation, generating these numbers. It was decided this pedagogical move might encourage students to reconcile the value of 50 bacteria for the start time. The students might wonder whether “50 bacteria” contradicts the context of the problem, then what might be wrong and what should be changed. The participants expressed their belief that once students discussed this idea, the next pedagogical move might be to have students pair up with other students that have different graphs or equations and describe the similarities and differences and reasons for them. The participants decided these two pedagogical moves might help reconcile students’ misconception.

Figure 3. One-half the growth rate

Time (in hours)	**0**	**1**	**2**	**3**	**4**	**5**	**6**
Bacteria (35^0 C)	100	200	400	800	1600	3200	6400
Bacteria (refrigerator)	50 (100)	100	200	400	800	1600	3200

1.5 method solution strategy

The third method, and second most common, involved changing the base from 2 to 1.5. Students solved the problem by adding half the base population to itself in order to obtain the next entry. Some students even created a quasi-recursive formula to describe the pattern, while other students created the following equation: $y = 100*1.5^x$.

In the lesson study discussion, the teachers wanted students using this method to discuss a) what do you do with the values that have decimals, b) how this equation was similar and different to their initial equation for the first part of the problem (in order to discuss the 2^x versus the 1.5^x) and c) how their equation fits the problem statement: “you are able to slow the growth of bacteria to exactly one-half of the growth rate in the previous situation. They decided to encourage students to discuss the mathematical differences in their solutions. In particular, the seminar teachers wanted students to describe how their mathematical models represented the task and how their models represented continuous versus discrete data.

Results and Descriptions of Classroom Episodes

It was our intent during the lesson study seminar to engage teachers in discussions that integrated theories for promoting student understanding (Carpenter & Lehrer, 1999) with the examination of student work in order to help structure curricular and instructional decisions (proposed pedagogical moves). This section highlights the pedagogical moves of one of the graduate seminar participants, Ms. Carson.

Ms. Carson’s pedagogical moves

The teacher began the lesson with a brief introduction to the problem. She did not clarify or make reference to the term "growth factor" before students were encouraged to work independently on the problem. "Be sure you answer every question. And do the best you can with the math knowledge you have up to date. I will give you 15 more minutes to work on this."

The students began working individually on the tasks as outlined by the bacteria worksheet. During this time, Ms. Carson walked around the room, observing students' work and taking note of each student's strategy to be used to pair up the students. She noted that most of the students were using the cut in half, the 1.5, and linear strategies: an approach where students took the growth rate and fixed it at a constant rate, typically 50, 100, or 200.

Once the individual work time had ended, Ms. Carson paired the students based on their solution strategies. She encouraged them to discuss the similarities and differences between each other's strategies and whether one or both made mathematical sense:

> I want you to share what you did. Share what you did and try to come up with these similarities and the differences. Okay? Maybe that might help you write some of the stuff down that you talked about in your groups. Okay? I want you to share your solutions in what you did and why you did it, why you thought you did it, why you should do it that way. And talk about why they might be different and why you are interpreting everything. Okay?

During this time Ms. Carson went from group to group clarifying the directions and asking students to articulate how they solved the problem and what were the similarities and differences between the two strategies.

After 20 minutes, Ms. Carson brought the class back together and asked them to prepare on a sheet of paper their table and graphs as well as a list of the similarities and differences they had found for problems 4 and 5 – the questions that refer to the slowed growth rate. Students were then dismissed.

The following day, students returned to work with the same partners. Each pair was given an overhead and asked to present solutions and share their ideas with the whole class. Six student strategies were then chosen for presentation and discussion during the remainder of the period. The chosen methods included a version of the "1.5 Method", two versions of the "Cut the Number of Bacteria in Half Method" (presented by two different students), one solution that invoked the "Double the Time Strategy" as well as an additional solution that used a variation of this strategy, and finally, a version of the "Step Function Method." Pairs were chosen, based on the different types of strategies they used to solve the problem, to present their ideas.

Whole class discussion of the 1.5 method

Ms. Carson began by asking for volunteers from one of the six pairs that were asked to write their work on an overhead (see Figure 4 for a list of the student solution strategies). Tyler volunteered to put up his overhead with the following table. Also on the overhead is his example 100/2 = 50, so 100 + 50 = 150.

Figure 4. Students' solution strategies

			Tyler's Solution Strategy – 1.5				
Time	**0**	**1**	**2**	**3**	**4**	**5**	
Bacteria – 1	100	200	400	800	1600	3200	
Bacteria – 2	100	150	225	337.5	506.25	759.375	
			Tony's Solution Strategy – Divide by 2				
Bacteria – 1	100	200	400	800	1600	3200	
Bacteria – 2	100	100	200	400	800	1600	
			Ann's Solution Strategy – Divide by 2				
Bacteria – 1	100	200	400	800	1600	3200	
Bacteria – 2	50	100	200	400	800	1600	
			Alex's Solution Strategy – 1.5				
Time	**0**	**1**	**2**	**3**	**4**	**5**	
Bacteria – 1	100	200	400	800	1600	3200	
Time	**0**	**1.5**	**3**	**4.5**	**6**	**7.5**	
Bacteria – 2	100	200	400	800	1600	3200	
			Chris' Solution Strategy – Divide time				
Time	**0**	**1**	**2**	**3**	**4**	**5**	**6**
Bacteria – 1	100	200	400	800	1600	3200	6400

Bacteria – 2	100	150	200	300	400	600	800
Alicia's Solution Strategy – Step function							
Time	**0**	**1**	**2**	**3**	**4**	**5**	**6**
Bacteria – 1	100	200	400	800	1600	3200	6400
Bacteria – 2	100	100	200	200	400	400	800

Tyler began by describing how he solved the problem, but before he could complete his explanation, the Ms. Carson asked him to clarify:

Teacher: *So how are you interpreting what half meant?*

Tyler: *Like half as much. I took the number of the previous hour, then divided that by 2. I then took that number and added it to the next number and so on.*

Teacher: *Did anybody else do something similar to what Tyler did up here? [Omar raises his hand.] Omar, what did you do?*

Omar: *I took the previous number of bacteria times 0.5 and then added the previous number of bacteria, the first one, it is 100 times point 5, that is 50, and then add another 100, which gives me 150 bacteria.*

Teacher: *That is how it was similar. Was there anything different from what you did that Tyler did?*

Omar: *Not really.*

At this point, Jerritt raised his hand, eager to enter the conversation.

Teacher: *Jerritt?*

Jerritt: *Well, I did it, I took the number before and then multiplied it by 1.5 and then got 150.*

Teacher: *Okay. So, 1.5 where the other two were half and he had point 5. How is that similar or how is that different?*

Jerritt: *It is the same as dividing, like if you divided 100 in half and then you add a hundred to that, it is the same as multiplying it by 1.5.*

At this point, the teacher appeared to be satisfied with the discussion points and decided to move on to the next method.

Whole class discussion of divide by two: Strategy #1

Tony was asked to present his strategy to the class. He brought up his overhead and began to explain how he solved the problem. Tony explained that he took the original 100 bacteria and placed that in the column for 0 hours in order to begin working with the second graph. He then took the 200 bacteria in the original problem and divided it by 2. He explained that he continued this approach, dividing by 2 to generate the rest of the numbers.

Tony: *It doubles each time, the first time I had 100 and 200 from my other graph and then I took 100 and put a 0 as 100 and then one because it was 200 before, I just made it 100 so I cut it in half each time.*

Teacher: *So, your original you cut that in half?*

Tony: *Yes.*

The teacher presses other students to make sense of Tony's idea compared to Tyler, who used the 1.5 strategy.

Teacher: *How is that similar, is there anything similar?*

Student: *Not sure.*

Teacher: *Does anyone think there is anything similar in what they are doing there or are they different?*

Student: *They are different. The second one is basically cutting it in half.*

Teacher: *So cutting it in half is not the same way? So your interpretation of that slowing of the growth rate makes it different? So how did you guys interpret what happens?*

The students do not respond to the question, so Ms. Carson moves on to the next solution strategy.

Whole class discussion of divide by two: Strategy #2

Another student, Ann, used a similar strategy to Tony's but did not resolve the issue of starting with 100. In the following passages, Ms. Carson attempted to address this through a series of questions to Ann.

Teacher: *Ann, you did this, too, right?*

Teacher: *I want you to talk a little bit about your numbers for the second problem.*

Ann: *I did the same as Tony. I divided each number of bacteria in table 1 and it to form the new table.*

Teacher: *So did you actually do the same thing? Was everything the same?*

Ann: *Yeah.*

Teacher: *Everything except for the zero.*

Ann: *Yeah, I didn't know what to do, so I halved all of them.*

Teacher: *Okay, so you just weren't real sure but you thought everything was in halves?*

Ann: *Yeah.*

Teacher: *And why did those people tell you it had to be in half?*

Ann: *I'm not sure.*

Teacher: *Anybody do it similarly to this or a little differently? Zach?*

This series of questions might have enabled Ann to realize that 50 did not make sense given the context of the problem. However, the teacher did not press Ann to make sense of this conflict herself. Rather, Ms. Carson called on another student – Zach, who had solved the problem using the same strategy – to assist Ann. Unfortunately, Zach simply agreed with Ann's statement and offered no explanation as to why 50 was chosen over 100, other than that it fit with the halving process.

Whole class discussion of a variation of the double in time strategy

Alex was then asked to share his strategy, which was a variation of the double in time strategy Ms. Carson was familiar with from the work she had completed in the graduate seminar. However, Alex's strategy incorporated a new twist to the strategy – one in which Ms. Carson apparently had not seen.

Alex: *I read it as the bacteria divide every hour and now it took ½ that time before they doubled again.*

Teacher: *So instead of cutting the bacteria in half you took the time? Is that what you did Omar?*

Omar: *Yah.*

Teacher: *And you said Alex did something similar but yet it was a little different, so. What did you do, Alex?*

Alex: *Well, I changed the time.*

The students seemed to be confused at this point. Although both strategies involved 1.5, Alex's strategy focused on taking ½ the time to mean that initially it took the bacteria 1 hour to double so now it would take another ½ hour to double. Hence, it would take 1.5 hours for the bacteria to double – instead of one hour. Omar, on the other hand, had kept the time constant and decided the bacteria would grow by 50%. Thus, in his calculations, the number of bacteria would be the original amount times 1.5. The teacher encouraged the students to clarify their thought processes.

Teacher: *And while you are putting it up there, could you at the same time explain to everyone what you were doing and if they had questions or comments for you, they can ask? You guys, I want you to analyze what Alex is doing up here and try to come up with what he is doing, why he is doing it, and he will share and let's talk about this one for a while.*

Alex: *Well, what I did is I took the hours and divided in half … whatever popped into my head.*

Teacher: *So, you looked at it in terms of the time instead of what it was doing to the bacteria? And the other people, were you guys thinking about what it was doing to the bacteria? Am I right in that or no?*

Alex: *Yah.*

Interestingly, the conversation ended here, with no resolution of the problem. Ms. Carson then moved on to the next set of strategies.

Whole class discussion of the double in time strategy

Chris and Alicia were paired together and asked to present their strategies together on the overhead. Chris began the conversation by explaining how he used the double in time strategy to get the number of bacteria for the even numbered hours. To figure out the odd number hours, he explained how he split the difference between the number of bacteria for each consecutive pair of even numbered hours.

Chris: *The way I figured it out was I knew that originally for zero hours there would be 100 bacteria, so for 0 it is 100 for 2 → 200; 4 → 400; 6 → 800.*

Teacher: *And why did you do that?*

Chris: *Because then I would be, it would be dividing like all the time on a gradual line and not at a specific time.*

Teacher: *And what were you telling me that you actually, you used some formula to get you there?*

Chris: *Oh, I tried to, but I didn't get it.*

Teacher: *But what did you try?*

Chris: *I had a lot of stuff written down.*

Teacher: *I know you mentioned something to me about how you were trying to do a mid-point.*

Chris: *Oh, yeah. This was the mid-point between these two so that right here and between here was the problem with 100 bacteria and then this over here is 200 and so if you go up here [pointing the point at (2, 200)] and it is in between. Equally in this slope is the number of bacteria for 1 hour. And in-between this and this [pointing again at (0, 100) and (2, 200)]. So, that would be 150.*

Teacher: *Okay.*

Chris: *Same thing for this one. [pointing at (2, 200) and (4, 400)]. So, that number would be between 200 and 400, so it is 300.*

Teacher: *Okay. Anyone have any questions for him on what he was thinking about or what he did?*

There was a pause, but no hands were raised. The teacher then asked Alicia to present her strategy, which invoked the Step Function Method.

Whole class discussion of the step function method

Alicia used a step function to represent the growth of bacteria. Ms. Carson had decided to pair Chris and Alicia together because each had the same number of bacteria for the even hours, but different numbers for the odd hours.

Teacher: *So Alicia, why don't you share what you did?*

Alicia: *Oh, I had basically the same except I just thought that instead of dividing every hour, it would divide every two hours. I think that most of the bacteria will only divide every 2 hours. Therefore none of the bacteria will divide after 1 hour.*

Teacher: *So what is happening to your bacteria over those two hours? When are they dividing?*

Alicia: *Every two hours.*

Teacher: *Every two hours? They are all sitting and waiting until two hours come up and then they all divide?*

Alicia: *Yeah.*

The teacher at this point attempted to use Chris's idea to help Alicia understand the division of a bacterium.

Teacher: *And Chris, what do you think was happening to your bacteria?*

Chris: *They are dividing all those times so you get more bacteria.*

Teacher: *When we were taking half the time, you interpreted it as it was slowing down, the time it took for the bacteria to double. Is that what you guys were interpreting? So Alex how is yours similar to this?*

At this point, Ms. Carson attempted to draw closure to the lesson.

Teacher: *So you guys have listened to a lot of different strategies. A lot of you guys, I think you guys did the same thing, is there someone who is right? Are you all right? What do you guys think? Is there one right answer?*

Student: *No.*

Teacher: *Okay, so elaborate on that, Kelly.*

Kelly: *Well, we did it a different way. And some people like took the half to mean something else. Like you wanted it to like cut in half or whatever.*

Teacher: *What were we interpreting differently?*

Kelly: *Well, some looked at it like cutting the bacteria and some people looked at it like how you divide the time.*

Teacher: *Anybody else? [Pause] What do you think? Omar, what do you think?*

Omar: *I guess the idea of the person trying to find a role in the way that they like … equations … way too early. The first time I looked at the first graph I saw, I went up by I saw 0 and 100 so I thought 1 and 200, I just saw that. And then 2 and 300 and then increase it by one. But then after I read over the question again and got some help and saw that it is actually supposed to double it.*

Teacher: *So are you looking at it as if I were to hand this problem out to you guys again that all of you would do it differently?*

Students: *Um hum. [General agreement.]*

Omar: *Probably. That is only what I think.*

Teacher: *Why do you think you would interpret it different later?*

Student: *I don't know.*

Teacher: *Ann, what did you want to say?*

Ann: *I don't think that this is big enough [pointing to her second graph]*.

Once the students had shared their ideas, Ms Carson attempted to get the students to go back to the problem and examine what it was asking, that is, to refer to the context of bacteria growth. For example, she encouraged them to examine what was doubling and what was splitting in half.

Teacher: *What usually happens when cells divide? Can you explain it to us?*

Student: *They separate into two wholes. They are exactly identical in size. … so they are not reattached and then like half a bacteria and … They are just the same … So I know there is something wrong with my first, I mean, my second graph here. I have like .25, … but I know that there can't be like half a bacteria. I guess I had to round up or round down. [Describing her 1.5 strategy.]*

Teacher: *Explain what it means after one hour for that very first part? Why do you have 200?*

Student: *Because I rounded it like all of them in half and so like divide and so there will be 200, there will be twice that and it just keeps doubling.*

Teacher: *So what is the growth rate for the first one?*

Student: *By 100 and then by 200 and then by 400.*

Teacher: *I think you said it but what do other people think? What is the growth rate of that first problem? How are the bacteria growing? Mathematically, what are you doing?*

Student: *Multiplying by 2.*

Teacher: *Multiplying by 2. Does everybody agree with that?*

Students: *[General agreement]*

Teacher: *Did anybody come up with an equation for that?*

Students: *[Students begin talking to themselves.]*

Teacher: *Take two minutes with the people you are paired with and try to come up with the equation for the first one so every time you put in one of the variables, the time, you will get the correct number of bacteria.*

The second day ended with students trying to write an equation for the growth rate of problem 1. They were asked to think about this for homework and come the next day with a response.

Concluding the whole class discussion

During the third day students were asked to present their equations for the first problem. Most of the students were able to generate the equation $y = 100*2^{x}$. Ms. Carson then led the class through a short discussion about growth rate and what the second part of the task was asking them to do. She explained to them that the question was asking them to take one half the growth rate and, based on the initial equation, the growth rate referred to the number of bacteria for a given unit of time: the x-values. Students were then asked to generate the new equation and present it to the class. Many of the students were still not able to generate the correct equation, $y = 100*2^{(1/2)x}$, but were able to follow one students' explanation. The students were then given a similar problem, but starting with 1 bacterium instead of 100. They were then asked to solve the same set of problems, creating tables, graphs, and equations to depict this new phenomenon.

Discussion

It is clear from examining the classroom vignettes that this teacher attempted to use the ideas that were developed during the graduate seminar to inform and guide her pedagogical moves. For example, the teacher paired students together who had attempted similar strategies in order to encourage a mathematical discussion that might lead each student to a deeper and more mathematically sound solution. Despite her attempts, the transcripts provide evidence of the contrary. Many students did not change their strategies nor were they able to explain in great detail how they made sense of the problem or why they chose a particular method over another. It is important, then, to examine the pedagogical moves of the teacher, how they were enacted during the lesson, and whether they encouraged or hindered students' development of conceptual understanding. In other words, were Ms. Carson's pedagogical moves drawn from her knowledge of the student strategies or were they simply the "scripted moves" suggested during the lesson study seminar?

We explore this general proposition in the ensuing paragraphs through a close examination of four pedagogical moves employed by Ms. Carson. To frame this discussion, we consider the five mental activities through which understanding emerges put forth by Carpenter & Lehrer (1999): constructing relationships, extending and applying mathematical knowledge, reflecting about experiences, articulating what one knows, and making mathematical knowledge one's own (p. 20).

Building student understanding

Constructing relationships implies that students use their prior knowledge to make sense of a problem and then build onto that existing knowledge by making connections (Carpenter & Lehrer, 1999). Further, students begin to construct relationships when asked to solve problems that encourage them to use their informal mathematical ideas in conjunction with prior mathematical knowledge (Confrey & Smith, 1994; Thompson & Thompson, 1995). This was one of the focal points of the work in the seminar. The participants were encouraged to examine the ways in which students develop understandings of exponential growth functions in order to create hypothetical learning trajectories.

In this case, Ms. Carson's students were encouraged to solve a novel, realistic problem to tap into and expand their current algebraic understanding of exponential functions. However, it appeared that the students in Ms. Carson's class, at least by the time they began to solve the refrigerated part of the problem, had divorced the realistic constraints of the problem from their mathematical thinking. For example, many of the students took the phrase, "you are able to slow the growth of bacteria to exactly one-half of the growth rate in the previous situation" to mean divide by 2. Based on their traditional training in mathematics classrooms, many of the students attempted to divide something by 2; not knowing exactly what that something was or should be in relation to the context of the problem, bacteria growth. Hence, for many students who used the divide by two strategy, they did exactly that and divided each of the numbers for the bacteria in the initial table by 2. Some of the students left the original number of bacteria as 100 but many, not deviating from the mathematical procedure, left it as 50. The first set of students used the idea of divided by 2, but knew that it did not make sense for the problem because they had to start with 100. The other students employed the notion that rules in mathematics cannot be abandoned.

Despite the teachers participating in the seminar had anticipated that students might divorce the context of the problem from the mathematics, key aspects of the hypothetical learning trajectory were based on the notion that students would use the context to help them make sense of the mathematics. Hence, the act of divorcing the context of the problem from the mathematics may have led Ms. Carson's students to struggle in ways she had not anticipated and, in turn, may have led to Ms. Carson's struggles with effectively implementing key aspects of her lesson plan. She did not encourage her students to make sense of the context of the problem. The potential result is students came to view this task as just another situation in which they should apply a mathematical rule with which they were already familiar without consideration of the context.

The second activity to support understanding is reflecting on experiences. In mathematics reflection can be thought of as reexamining one's ideas to make sense of the problem. In this case, Ms. Carson's pedagogical move of asking students to relate their own strategies to other students' strategies could be viewed as a way to encourage reflection. Ms. Carson asked students repeatedly to examine the similarities and differences between their own and their partner's strategies and among the different strategies presented during day two to the whole class. The pedagogical move the teacher employed was meant to extend their mathematical knowledge by asking them to reflect upon their solution strategies and those of their peers. Based on the transcripts, however, it appears that the opportunity may have been there, but there was no clear evidence students actually used each other's ideas to expand their own understanding.

The third activity required in order to build understanding is for students to extend and apply their mathematical knowledge. Here, students must be afforded opportunities to determine how mathematical ideas are related. The more these relationships are understood, the more deeply embedded students' understanding of the situation is. The bacteria problem was novel and, therefore, demanded students use their previous mathematical understanding in conjunction with their informal understandings of bacterium growth in order to solve the problem.

Students were also asked to return to their method and decide on its accuracy and relevance. Ms. Carson supported developing understanding through not only the use of a novel task, but through the use of the pair activity where she asked students to study the way they solved the problem and to compare it to their partner's way of solving the problem. However, because she failed to press them to explain the conceptual basis for their thinking to their peers, students gained little understanding through the process of sharing strategies with others. Ms. Carson did not press them to make meaningful mathematical connections between their solution strategies and their partners. Although many students wrote down differences on their papers, it appeared as though those differences were procedural, not conceptual. Where the lesson may have broken down in terms of promoting reflection and mathematical understanding was in pressing students to make sense of their own methods in relation to other students' methods.

The fourth activity required for developing understanding is for students to articulate their ideas. This means students should share their ideas "verbally, in writing, or through some other means like pictures, diagrams, or models" (p. 22). Ms. Carson attempted to include all three forms to a certain degree. Students were to solve the problem, which asked students to represent the phenomena through a table, a graph, and an equation. The students were also asked to share and explain the strategy they used. While a few students were able to explain procedurally how they solved the problem, few explained their mathematical ideas conceptually or were pressed to do so.

The last activity to build understanding is to make the mathematical knowledge one's own. This relates to having a disposition toward learning that enables one to "develop a personal investment in building knowledge" (p. 23). In Ms. Carson's class, students were asked to author their own strategies. In fact, Ms. Carson attempted to take the authority away from herself and place the authority in the mathematics and in the students' explanations. However, students were not asked to defend their reasoning in any mathematically significant way. We did not uncover evidence that Ms. Carson pushed students to further their thinking by asking them why they used a particular procedure or encouraging them to write an equation or create a graph, all of which were discussed in the graduate seminar as potential pedagogical moves. By not pressing students in these ways, Ms. Carson did not create the necessary opportunities to encourage her students to identify with their own strategies and, thus, the mathematics. In other words, they failed to take ownership of their mathematical knowledge.

Scripted moves versus pedagogical moves with purpose

One way to examine the lesson is through a comparison of Ms. Carson's pedagogical moves. For example, asking students to explain their ideas could be considered to be an appropriate pedagogical move. However, if teachers only ask students to explain their ideas without pressing them to *defend* their ideas mathematically, then it is considered a "scripted move." Asking students to reiterate the process or procedure they went through to solve the problem is another example of a pedagogical move that we would consider to be 'scripted', while encouraging students to share their understanding of how mathematical models and ideas are related would be considered a pedagogical move with purpose. Instead of ignoring or correcting students' mistakes, a pedagogical move with purpose would encourage students to make sense of the error by examining it as a positive attribute or possible correct conception for a related activity. Finally, instead of working alone on problems and then being asked to check for the right answer, a pedagogical move with purpose would be to encourage students to reach a mathematical consensus with their peers.

In reviewing the transcript of Ms. Carson's lesson, it is evident she employed many of the pedagogical moves that had been discussed during the lesson study group. She asked repeatedly for students to share their ideas: "Just tell us what you did." and "Alicia, why don't you share what you did?" For each of the six episodes, students shared how they solved the problem. At this point, Ms. Carson could have pressed them to explain why they solved the problem that way or asked them how their method addressed the slowing of the bacteria growth rate. Instead, Ms.

Carson tended to repeat their method, for instance, "So your original, you cut that in half?" which left the student to respond by answering yes or no. At this point, we notice that Ms. Carson is beginning to ask important questions, but has not yet recognized the *purpose* behind the questions she was asking - to further their mathematical thinking. For Ms. Carson, the purpose may have been to simply share student thinking rather than press students to defend their ideas mathematically. This is a process that many teachers have to go through before they begin changing their instruction to include a deep pressing of the conceptual mathematical ideas. Furthermore, teachers may be able to confront the limitations of class discussions in building mathematical proficiency only after substantial effort has been expended to facilitate classroom discourse. Once teachers have examined the learning process students follow through rich discussions, they are more likely to find opportunities to use students' conversations as the origin of a more rigorous, active classroom structure in which all students participate in mathematical problem-solving; not only those who choose to share their ideas.

Next, Ms. Carson's pattern was to ask the class as a whole whether this strategy was similar or different to any past strategies: "How is that similar, is there anything similar?" "Does anyone think there is anything similar in what they are doing there or are they different?" This pedagogical move was an attempt to promote deeper conversations and to encourage students to defend their strategies or note relationships in the mathematics. However, two things occurred. First, many times the questions Ms. Carson raised were restated from an open-ended question to a closed, yes/no, question. Second, she did not hold students accountable for explaining the mathematical relations among the strategies. They were able to respond, "I don't know." "I'm not sure." "They were different." Asking students to explain how their strategies were similar or different was again a pedagogical move that had been discussed by the teachers during the lesson study seminar. However, it appeared as though Ms. Carson did not implement it with purpose, making it simply a 'scripted move' rather than one with purpose. In the end, she did not encourage students to consider the mathematical similarities and differences between their various strategies, and thus, did promote understanding. Again, this is important for researchers and professional developers to understand – it appears that most teachers will go through this stage of asking important questions, but not following up with the more conceptual or big ideas (Tirosh & Graeber, 2003).

There were also many opportunities to explore the students' misconceptions. For instance, Ms. Carson asked Tony and then Ann to present their strategies for cutting the number of bacteria in half. Tony explained the bacteria doubled each hour for the first part of the problem. He went on to say that for the second part of the problem he just cut the number in half. At this time Ms. Carson asked Tony how his strategy was similar or different than the 1.5 strategy Omar had just shared. Tony responded by saying, "Not sure." At this point, Ms. Carson asked the same question to the whole class to determine whether they could respond. After one student repeated what Tony had said, Ms. Carson attempted to answer the question herself, "So your interpretation of that slowing of the growth rate makes it different?" At this point, Ms. Carson interjected a new idea, focusing on the interpretation of the growth rate. When she asked the next question, "So how did you guys interpret what happens?" none of the students responded. At this point, Ms. Carson might not have been able or was confused as how to scaffold student thinking about the problem. Instead she went on to ask another student, Ann, who solved the problem in a similar way to explain her method. Again, the questions Ms. Carson asked were answered by saying, "I did it the same as Tony. I divided each number of bacteria in table 1 and it to form the new table." "Yeah, I didn't know what to do, so I halved all of them."

"Yeah." "I'm not sure." In this case, Ms. Carson attempted to extricate this conceptual error with little apparent success.

Conclusions and Implications

Examining the pedagogical moves that were proposed by the lesson study seminar participants and how Ms. Carson enacted those moves provided ways through which we could begin to examine the development of this lesson with regards to teaching and learning mathematics. We conclude by sharing the following thoughts as both implications of this research, and possible launching points for further inquiry in this area of research.

First, developing a deeper understanding of the knowledge that is required for teachers to create, consider and enact pedagogical moves with purpose is paramount. Notable in particular is the importance of developing knowledge of mathematics and students and knowledge of mathematics and teaching so that pedagogical moves with purpose can be employed in order to elicit and develop student understandings. Ms. Carson's attempts at helping her students build conceptual understanding, which included asking students to create their own methods for solving the problem, asking them to articulate their ideas to other students, and asking them to examine similarities and differences among strategies, were, at this point, 'scripted moves.' Initially, as teachers begin to change their practices, this procedural response is typical (Tirosh & Graeber, 2003). We saw her thinking and teaching as evolving from scripted moves towards making pedagogical moves with purpose.

These issues point toward several areas in which continued inquiry might help us further our understandings of issues raised in this article. First, it appears that being part of a lesson study team focused on studying how to promote student understanding related to algebra is not sufficient for substantive changes to occur in the classroom. That is, collectively developing a hypothetical learning trajectory is not enough. Teachers must be able to effectively enact the pedagogical moves proposed as part of these learning trajectories. This leads to our second question. What knowledge is required in order for teachers to be able to enact pedagogical moves with purpose? This study raises the question of how to best get teachers to change their instructional practices. If teachers are asked to promote conceptual understanding how can they develop the knowledge to be able to enact pedagogical moves with purpose? The types of knowledge required in order to support teachers as they attempt to enact pedagogical moves and the impact of this kind of teaching on students' mathematical understandings, achievement, and dispositions needs to be examined further. Studies like the one presented in this paper are important in that they provide insight into the ways in which even experienced teachers approach the process of promoting conceptual understanding. This study in particular sheds light on one aspect of the reform movement – using student strategies to help promote student understanding and the difficulties in doing so. Having a trajectory in mind of how students' solve certain problems and the mathematical conceptions and misconceptions is important, but knowing how to anticipate their ideas and how to press their thinking while in the classroom is equally critical to ensuring improved mathematical pedagogy, and, thus, increasing student understanding of the mathematics.

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