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Weak Covering Properties and Selection Principles

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WEAK COVERING PROPERTIES AND SELECTION PRINCIPLES

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Abstract. No convenient internal characterization of spaces that are productively Lindelöf is known. Perhaps the best general result known is Alster's internal characterization, under the Continuum Hypothesis, of productively Lindelöf spaces which have a basis of cardinality at most $\aleph_1$. It turns out that topological spaces having Alster's property are also productively weakly Lindelöf. The weakly Lindelöf spaces form a much larger class of spaces than the Lindelöf spaces. In many instances spaces having Alster's property satisfy a seemingly stronger version of Alster's property and consequently are productively X, where X is a covering property stronger than the Lindelöf property. This paper examines the question: When is it the case that a space that is productively X is also productively Y, where X and Y are covering properties related to the Lindelöf property.

1. Introduction

A topological space is said to be Lindelöf if each of its open covers contains a countable subset that covers the space. Though this class of spaces has been extensively studied there are still several easy to state problems that have not been resolved. Call a space X *productively Lindelöf* if $X \times Y$ is a Lindelöf space whenever Y is a Lindelöf space.

In the quest to find an internal characterization of the productively Lindelöf spaces K. Alster identified the following conditions: Call a family $\mathcal{F}$ of G_δ subsets of a space X a G_δ *compact* cover if there is for each compact subset K of X a set $F \in \mathcal{F}$ such that $K \subseteq F$. A space is said to be an *Alster space* if each G_δ compact cover of the space has a countable subset covering the space[1]. Alster (and independently [7]) proved that if X is an Alster space then it is productively Lindelöf, and $X^{\aleph_0}$ is Lindelöf.

Problem 1 (K. Alster)**.** *Is every productively Lindelöf space an Alster space?*

A significant body of partial results has developed around this problem, yet no definitive answer is known to it.

There are several weakenings of the Lindelöf property that have been investigated because of their natural occurrence in some mathematical contexts. The corresponding product theory for these is not as extensively developed as for Lindelöf spaces. Product theoretic questions may be more manageable for the corresponding weakened analogues of the selective versions of the Lindelöf property.

In another direction, a number of selective versions of the Lindelöf property and weakenings of it have been investigated because of their relevance to several other mathematical problems. It has been found that some questions about Lindelöf spaces are "easier" for these more restricted classes. It is natural to inquire whether the corresponding product-theoretic problem for these narrower classes of spaces is more manageable. Some such questions have been raised: For example: In [34] and [35] the notion of a *productively Menger* space is considered and in [3] the notion of a *productively FC-Lindelöf space* is introduced.

And thirdly, solving versions of a problem still unresolved for Lindelöf spaces by strengthening the hypotheses to selective versions while weakening the conclusions to weak covering properties, may yield some insights on the original problem. Progress on the internal characterization problem may also yield insights on an older problem of E. Michael:

Problem 2 (E.A. Michael)**.** *If X is a productively Lindelöf space, then is $X^{\aleph_0}$ a Lindelöf space?*

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[1]This definition is not identical to the definition in [7], but is equivalent to it, and is in fact the property (*) defined by Alster on p. 133 of [2].

These are the motivations for our paper. The paper is organized as follows. In Section 2 we introduce some basic notation and terminology. In Section 3 we give a number of examples of when products fail to have some of the properties we are investigating. Section 4 focuses on the question of characterizing the productively Lindelöf spaces. We raise the question of when P and Q are covering properties of topological spaces, is a space that is productively P also productively Q? In Section 5 we incorporate the weak covering properties into the investigation.

2. Some notation and terminology

A space is said to be *weakly Lindelöf* if each of its open covers contains a countable subset for which the union is dense in the space. A space is said to be *almost Lindelöf* if each of its open covers contains a countable subset for which the set of closures of elements of the countable set is a cover of the space. Both of these properties are weaker than the Lindelöf property, and we have the following implications: Lindelöf $\Rightarrow$ almost Lindelöf $\Rightarrow$ weakly Lindelöf. For spaces with the T_3 separation property almost Lindelöf also implies Lindelöf. Aside from this there are no other implications among these three properties. We will focus on the "weakly" properties in this paper, leaving the "almost" properties for another time.

Next we describe selective versions of these covering properties. We use the following notation for three of several upcoming relevant classes of families of open sets of a given topological space:

$\mathcal{O}$	The collection of open covers
$\mathcal{D}$	$\{\mathcal{U} : (\forall U \in \mathcal{U})(U \text{ open}) \text{ and } (\bigcup\mathcal{U} \text{ dense in } X)\}$

Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of an infinite set. Then $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the following hypothesis:

For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that, for each n, $B_n \in A_n$ and $\{B_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

Thus, $\mathsf{S}_1(\mathcal{O},\mathcal{O})$ denotes the classical *Rothberger* property. We shall call spaces with the property $\mathsf{S}_1(\mathcal{O},\mathcal{D})$ *weakly Rothberger*.

The symbol $\mathsf{S}_{fin}(\mathcal{A},\mathcal{B})$ denotes the hypothesis

For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that, for each n, $B_n \subseteq A_n$ is finite, and $\bigcup\{B_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ denotes the classical *Menger* property, while $\mathsf{S}_{fin}(\mathcal{O},\mathcal{D})$ denotes the *weakly Menger* property.

Several additional families $\mathcal{A}$ and $\mathcal{B}$ of topologically significant objects will be introduced as needed during the rest of the paper. Our conventions for the rest of the paper are: By "space" we mean a topological space. Unless other separation axioms are indicated specifically, we assume all spaces to be infinite and T_1. Undefined notation and terminology will be as in [12].

3. Possibilities of failure for products

Towards investigating the product theory as outlined above, we consider if it is possible for certain products to fail having a covering property. We asked above, for example, if there could be a Rothberger space whose product with the space of irrational numbers is not weakly Lindelöf. First, we settle that it is at least possible that the product of two Rothberger spaces can fail to be a weakly Lindelöf space. We give two examples of how this could be. Both are consistency results.

Theorem 1. *It is consistent, relative to the consistency of* ZFC*, that there are Rothberger spaces* X *and* Y *for which* $X \times Y$ *is not weakly Lindelöf.*

Proof. In [15] Hajnal and Juhasz give examples X and Y of Lindelöf spaces for which $X \times Y$ is not weakly Lindelöf[2]. These examples are constructed in ZFC. Now consider these two ground model examples in the generic extension obtained by adding $\kappa > \aleph_0$ Cohen reals. Since X and Y are Lindelöf in the ground model, Theorem 11 of [31] implies that X and Y are Rothberger in the generic extension. Since $X \times Y$ is not weakly Lindelöf in the ground model, Theorem 1 of [4] implies that $X \times Y$ is not weakly Lindelöf in the generic extension. $\square$

[2] A nice exposition of this example can be found in [33], Example 3.25.

Our second example is a little stronger than the one just given. A *Souslin line* is a complete dense linearly ordered space X which is not separable but every family of disjoint intervals is countable. SH, the *Souslin Hypothesis*, states that there are no Souslin lines. SH is independent of ZFC. Theorem 1 is proven using forcing. One may ask to what degree axiomatic circumstances determine whether a product fails to have the covering property of its factor spaces. Shelah [32] proved that in the generic extension obtained by adding a Cohen real there is a Souslin line. Thus, the following Theorem 2 improves Theorem 1. In the proof of this theorem we use the following notation. If $(L, <)$ is a linearly ordered set there are three topologies considered on it: We denote the topological space by L if the topology is generated by sets of the form (a, b) where $a < b$ are elements of L. When the topology is generated instead by sets of the form $[a,\, b)$, the topological space is denoted by the symbol L^+. When the topology is generated by sets of the form $(a, b]$, then the topological space is denoted by the symbol L^-.

Theorem 2 ($\neg$SH)**.** *There are Rothberger spaces X and Y such that $X \times Y$ is not weakly Lindelöf.*

Proof. Let L be a Souslin line. We may assume that L has no nonempty open intervals that are separable. Then L^+ as well as L^- are Lindelöf spaces ([33], Lemma 3.31). But L^+ is a refinement of the standard topology on L, and so L is Lindelöf.

Claim 1: L^+ (and similarly L^-) is a Rothberger space[3].

Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of L^+. By Lemma 3.31 in [33] we may assume that each $\mathcal{U}_n$ consists of countably many open intervals of form $[a^n_k, b^n_k)$, $k \in \mathbb{N}$.

Consider $(\mathcal{U}_{2\cdot n} : n \in \mathbb{N})$. The set $A := \overline{\{a^{2\cdot n}_k : k \in \mathbb{N}\} \bigcup \{b^{2\cdot n}_k : k \in \mathbb{N}\}}$ is countable (lest L has a separable uncountable interval) nowhere dense in L. Choose for each $x \in L \setminus A$ an open interval $I_x \subseteq L \setminus A$ with $x \in I_x$. Note that for each n there is a k with $I_x \subseteq (a^{2\cdot n}_k,\, b^{2\cdot n}_k)$ (since $I_x \cap A = \emptyset$).

Choose from each $\mathcal{U}_{2\cdot n+1}$ an element $J_{2\cdot n+1}$ such that $A \subseteq \bigcup_{n\in\mathbb{N}} J_{2\cdot n+1}$. Now $K := L \setminus (\bigcup_{n\in\mathbb{N}} J_{2\cdot n+1})$ is a closed subset of L and so Lindelöf. Moreover $\{I_x : x \in K\}$ is an open cover of K, and so has a countable subcover, say $\{I_{x_n} : n \in \mathbb{N}\}$. Now choose for each n a k_n such that $I_{x_n} \subseteq J_{2\cdot n} = [a^{2\cdot n}_{k_n},\, b^{2\cdot n}_{k_n}) \in \mathcal{U}_{2\cdot n}$. Then the sequence $(J_n : n \in \mathbb{N})$ is a cover of L^+, and for each n, $J_n \in \mathcal{U}_n$. This completes the proof of the claim. As L is not separable, [33], Lemma 3.33 shows that $L^+ \times L^-$ is not weakly Lindelöf. $\square$

Our third example is in a different direction: Spaces that are weakly Rothberger in finite powers need not have a weakly Menger product. Since the topological sum of two Rothberger spaces is Rothberger, and their product is a closed subspace of the square of their topological sum, the negation of Souslin's Hypothesis implies that there is a Rothberger space whose square is not weakly Lindelöf. But if two Rothberger spaces are weakly Rothberger in their finite powers, must their product be weakly Rothberger?

Theorem 3 (CH)**.** *There are weakly Rothberger spaces X and Y such that*

(1) *Each (finite or infinite) power of X and of Y is weakly Rothberger, and*
(2) *$X \times Y$ is not weakly Menger.*

The proof of Theorem 3 is developed through a few propositions. Recall that for topological space $(X, \mathcal{T})$, $\mathsf{PR}(X)$ denotes the collection of nonempty finite subsets of X. For $S \in \mathsf{PR}(X)$ and an open set $V \subseteq X$, $[S, V]$ denotes $\{T \in \mathsf{PR}(X) : S \subseteq T \subseteq V\}$. The collection of subsets of the form $[S, V]$ of $PR(X)$ is a basis for a topology, denoted $\mathsf{PR}(\mathcal{T})$, on $\mathsf{PR}(X)$. Then $(\mathsf{PR}(X), \mathsf{PR}(\mathcal{T}))$ is the Pixley-Roy space of X If X has a countable base, then $\mathsf{PR}(X)$ is a union of countably many sets, each with the finite intersection property; this implies that $\mathsf{PR}(X)$ has countable cellularity. But countable cellularity is equivalent to: each element of $\mathcal{D}$ has a countable subset which is in $\mathcal{D}$.

For topological spaces X and Y, the space $X \oplus Y$ denotes the topological sum of X and Y.

Proposition 4. *Let X and Y spaces. Then $\mathsf{PR}(X) \times \mathsf{PR}(Y)$ is homeomorphic to $\mathsf{PR}(X \oplus Y)$.*

Proof. The function $\Phi : \mathsf{PR}(X) \times \mathsf{PR}(Y) \to \mathsf{PR}(X \oplus Y)$ defined by $\Phi((F, G)) = F \cup G$ is one-to-one and onto, continuous and open, and thus a homeomorphism. $\square$

At this point it is convenient to introduce another family of open covers: A family $\mathcal{F}$ of subsets of an infinite set S is said to be an *ω-cover*[4] of S if S is not a member of $\mathcal{F}$, yet for each finite subset F of S there

[3]Towards proving this we refine the argument from page 19 of [28].

[4]Note that we are deviating from standard usage of the term ω-cover: We do not require that the cover be an open cover of a space.

is a member U of $\mathcal{F}$ such that $F \subseteq U$. Let X be a topological space.

$$\Omega := \{\mathcal{U} \in \mathcal{O} : \mathcal{U} \text{ is an } \omega\text{-cover of } X\}.$$

Proposition 5. [29] (CH) *There are separable metric spaces X and Y each with the property $\mathsf{S}_1(\Omega,\Omega)$, but their topological sum $Z = X \oplus Y$ does not have $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$.*

Proof. In [29] CH is used to construct subsets X and Y of ${}^{\omega}\mathbb{Z}$ such that each has the property $\mathsf{S}_1(\Omega,\Omega)$, but $(X \cup Y) \oplus (X \cup Y) =^{\omega} \mathbb{Z}$.

As was noted in Theorem 3.9 of [18], a topological space has the Menger property in all finite powers if, and ony if, it has the property $\mathsf{S}_{fin}(\Omega,\Omega)$. Since $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ is preserved by continuous images, closed subsets, and countable unions, since $\mathsf{S}_{fin}(\Omega,\Omega)$ is equivalent to $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$ in all finite powers, and since $\mathsf{S}_{fin}(\Omega,\Omega)$ is preserved by closed subsets, finite powers, and continuous images, and since ${}^{\omega}\mathbb{Z}$ (which is homeomorphic to the set of irrational numbers) does not have $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$, it follows that $(X \cup Y)^2$ does not have $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$. Thus neither $X \cup Y$ nor $X \times Y$ has $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$. It follows that $X \oplus Y$ does not have the Menger property. $\square$

Theorem 6 (Daniels, [11] Theorem 5B)**.** *If Z is a metrizable space with the property $\mathsf{S}_1(\Omega,\Omega)$, then $\mathsf{PR}(Z)$ is weakly Rothberger in each (finite or infinite) power.*

Thus Theorem 6 implies that each of $\mathsf{PR}(X)$ and $\mathsf{PR}(Y)$ is weakly Rothberger in all powers. Next apply the following theorem to $X \oplus Y$:

Theorem 7 (Daniels, [11] Theorem 2A)**.** *If for a space Z, $\mathsf{PR}(Z)$ is weakly Menger, then each finite power of Z is Menger.*

Thus Theorem 7 states that if $\mathsf{PR}(X)$ has property $\mathsf{S}_{fin}(\mathcal{O},\mathcal{D})$, then X has property $\mathsf{S}_{fin}(\Omega,\Omega)$: For metrizable spaces the converse is implied by Theorem 6.

It follows that $\mathsf{PR}(X \oplus Y)$ is not weakly Menger. But then Proposition 4 implies that $\mathsf{PR}(X) \times \mathsf{PR}(Y)$ is not weakly Menger. This completes the proof of Theorem 3. $\square$

As far as ZFC results are concerned, there is also the following result by Todorčevic. First we introduce another family of open covers: A family $\mathcal{F}$ of subsets of an infinite set S is said to be a γ-*cover* of S if for each $x \in S$ the set $\{F \in \mathcal{F} : x \notin F\}$ is finite and $\mathcal{F}$ is infinite. Then for a topological space X we define

$$\Gamma := \{\mathcal{U} \in \mathcal{O} : \mathcal{U} \text{ is an } \gamma\text{-cover of } X\}.$$

Following Gerlits and Nagy, call a space which satisfies $\mathsf{S}_1(\Omega,\Gamma)$ is a γ-*space* [13].

Theorem 8 (Todorčević, [36], Theorem 8)**.** *There are T_3 γ-spaces X and Y such that $X \times Y$ is not Lindelöf.*

And finally for this section:

Theorem 9. *It is consistent, relative to the consistency of* ZFC*, that there is a T_3-Rothberger space whose square is not Lindelöf.*

Proof. Let $\mathbb{S}$ denote the *Sorgenfrey line*, the topological space obtained from refining the standard topology on the real line by also declaring each interval of the form $[a, b)$ open. It is well known that $\mathbb{S}$ is a T_3 Lindelöf space while $\mathbb{S} \times \mathbb{S}$ is not Lindelöf, but still T_3. By Theorem 11 of [31], if κ is an uncountable cardinal and if $\mathbb{C}(\kappa)$ denotes the Cohen forcing notion for adding κ Cohen reals, then in the generic extension the ground model Sorgenfrey line is a Rothberger space. But proper forcing preserves not being Lindelöf, and $\mathbb{S}$ in the generic extension by $\mathbb{C}(\kappa)$, the square of the ground model copy of $\mathbb{S}$ is not Lindelöf. Since T_3 is preserved, it follows that the square of a (almost) Rothberger space need not be (almost) Lindelöf. $\square$

As an aside to the proof of Theorem 9: In Lemma 17 of [5] it was shown that $\mathbb{S}$ does not have the property $\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$, and since $\mathbb{S}$ is T_3, this means that $\mathbb{S}$ is not almost Menger. As noted in the proof of Theorem 9, in the generic extension by uncountably many Cohen reals, the ground model version of $\mathbb{S}$ is Rothberger and thus Menger. Thus, proper forcing does not preserve being not Menger.

Also note that the ground model Sorgenfrey line remains a separable space in the generic extension, and thus a weakly Rothberger space in a strong sense: TWO has a winning strategy in the game $\mathsf{G}_1^{\omega}(\mathcal{O},\mathcal{D})$.

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4. Productively Lindelöf spaces

For a topological property Q that is inherited by closed sets we shall say that a space X is *productively* Q if for any space Y which has property Q, also $X \times Y$ has property Q. Note that if a space X is productively Q, then each finite power of X is also productively Q.

Much work has been done on characterizing the members of the class of productively Lindelöf spaces. We expand the basic problem of characterizing the class of productively Lindelöf spaces to also characterizing the classes of spaces that are productive for covering properties that are relatives of the Lindelöf property. Some basic problems emerge: Assume that we have topological properties Q and R where Q implies R. When is it the case that:

(1) If a space is productively R, then it is productively Q?
(2) If a space is productively Q, then it is productively R?
(3) If the product of X with every space of property Q has property R, then is X productively R?

Compact spaces are productively Lindelöf but need not be Rothberger spaces, and thus need not be productively Rothberger. Thus, spaces that are productive for one class of Lindelöf spaces need not be productive for another.

Alster's Theorems and a property of Alster.

In [2] Alster proves the following interesting theorem:

Theorem 10 (Alster)**.** *Consider the following statements about a space X:*

(1) *X is an Alster space.*
(2) *X is productively Lindelöf.*

Then (1) implies (2). If X is a space of weight at most $\aleph_1$, and if CH *holds, then also (2) implies (1).*

Alster [2], Lemma 1, also proved that the Alster spaces give an affirmative answer to Problem 2.

Theorem 11 (Alster)**.** *If X is an Alster space, then $X^{\aleph_0}$ is a Lindelöf space.*

The following classes of covers are central to Alster's analysis of the productively Lindelöf spaces:

$\mathcal{G}_K$: The family consisting of sets $\mathcal{U}$ where X is not in $\mathcal{U}$, each element of $\mathcal{U}$ is a G_δ set, and for each compact set $C \subset X$ there is a $U \in \mathcal{U}$ such that $C \subseteq U$.

$\mathcal{G}$: The family of all covers $\mathcal{U}$ of the space X for which each element of $\mathcal{U}$ is a G_δ-set.

In [7] Theorem 4.5 it is proved that the product of finitely many Alster spaces is again an Alster space. On account of this fact it is useful to also consider the following class of covers of spaces:

$\mathcal{G}_\Omega$: This is the set of covers $\mathcal{U} \in \mathcal{G}$ for which X is not in $\mathcal{U}$, but for each finite set $F \subset X$ there is a $U \in \mathcal{U}$ such that $F \subseteq U$.

Observe that $\mathcal{G}_K$ is a subset of $\mathcal{G}_\Omega$.

The connection of Alster's condition to selection principles will now be determined through a sequence of Lemmas, culminating in Theorem 15:

Lemma 12. *For a topological space X the following are equivalent:*

(1) *X is an Alster space.*
(2) *X satisfies the selection principle* $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G})$.
(3) *X satisfies the selection principle* $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Omega)$.

Proof. (1)⇒(2): Suppose that X is an Alster space and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of elements of $\mathcal{G}_K$. Define

$$\mathcal{U} = \{\bigcap_{n\in\mathbb{N}} U_n : (\forall n)(U_n \in \mathcal{U}_n)\}.$$

Then $\mathcal{U}$ is a member of $\mathcal{G}_K$. Since X is Alster, choose a countable subset $(V_n : n \in \mathbb{N})$ of $\mathcal{U}$ which is a cover of X. For each n write

$$V_n = \bigcap_{k\in\mathbb{N}} U_k^n$$

where for each k, U_k^n is an element of $\mathcal{U}_k$. Finally for each n set $W_n = U_n^n$, an element of $\mathcal{U}_n$. Then $\{W_n : n \in \mathbb{N}\}$ is a cover of X and thus a member of $\mathcal{G}$.

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Proof of (2)⇒(1): Let a member $\mathcal{U}$ of $\mathcal{G}_K$ be given. For each n, set $\mathcal{U}_n = \mathcal{U}$. Then apply $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G})$ to this sequence and select for each n a $U_n \in \mathcal{U}_n$ such that $\{U_n : n \in \mathbb{N}\}$ is a cover of X.

Proof of (2)⇒(3):

Claim 1: X satisfies the selection principle $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G})$ if and only if X^2 satisfies the selection principle $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G})$ (hence every finite power of X satisfies $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G})$).

Proof. Follows since the finite power of Alster space is an Alster space.

Claim 2: Each finite power of X satisfies the selection principle $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G})$ if and only if X satisfies the selection principle $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Omega)$.

Proof. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of elements of $\mathcal{G}_K$. Then $(\mathcal{W}_n : n \in \mathbb{N})$, where $\mathcal{W}_n = \{(U_n)^\kappa : U \in \mathcal{U}_n\}$, is a sequence of elements of $\mathcal{G}_K$ of X^κ. Then we can select by hypothesis $(U_n)^k \in \mathcal{W}_n$ such that $\{(U_n)^k : n \in \mathbb{N}\}$ is a cover of X^k. Consider now a finite subset $F = \{x_1, \cdots, x_k\}$ of X. We consider F like a point of X^k, say $z = (x_1, \cdots, x_k)$. So there is an element $(U_n)^k$ such that $z \in (U_n)^k$. Then there is an element $U_n \in \mathcal{U}_n$ such that $F \subset U_n$. It follows that $\{U_n : n \in \mathbb{N}\}$ witnesses that X satisfies $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Omega)$. The converse is obvious. This completes the proof.

Proof of (3)⇒(2): Obvious. □

A space X is has the *Hurewicz property* if for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of open covers of X there is a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$ and for each $x \in X$, for all but finitely many n, x belongs to $\bigcup \mathcal{V}_n$. The Alster property implies the following strengthening of the Hurewicz property:

Lemma 13. *If X is a space that has property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G})$, then there is for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{G}_K$ a sequence $(\mathcal{V}_n : n \in \mathbb{N})$ such that for each n we have $\mathcal{V}_n \subseteq \mathcal{U}_n$, $|\mathcal{V}_n| \leq n$, and for each $x \in X$, for all but finitely many n, $x \in \bigcup \mathcal{V}_n$.*

Proof. Let a sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{G}_K$ be given. For each n define $\mathcal{W}_n$ to be the set $\mathcal{W} = \{\bigcap_{k \in \mathbb{N}} U_k : (\forall k)(U_k \in \mathcal{U}_k)\}$. Applying $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G})$ to the sequence $(\mathcal{W}_n : n \in \mathbb{N})$ we find for each n a $W_n \in \mathcal{W}_n$ such that for each $x \in X$ there is an n with $x \in W_n$.

For each n write $W_n = \bigcap \{U_k^n : k \in \mathbb{N}\}$ where for each n and k we have $U_k^n \in \mathcal{U}_k$. Then, for each k, set

$$\mathcal{V}_k = \{U_k^1, \cdots, U_k^k\},$$

a finite subset of $\mathcal{U}_k$. Note that if $x \in X$ is an element of W_n, then for each $k \geq n$ we have $x \in \bigcup \mathcal{V}_k$. □

For each n, let T_n be the n-th triangular number[5]. Using the technique in the proof of the previous Lemma, we find

Lemma 14. *If X is a space that has property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G})$, then there is for each sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{G}_K$ a sequence $(U_n : n \in \mathbb{N})$ such that for each n we have $U_n \in \mathcal{U}_n$, and for each $x \in X$, for all but finitely many n,*

$$x \in \bigcup_{T_n < j \leq T_{n+1}} U_j.$$

Proof. Let a sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{G}_K$ be given. For each n define

$$\mathcal{V}_n = \{\bigcap_{T_n < i \leq T_{n+1}} U_i : (\forall i)(T_n < i \leq T_{n+1})(U_i \in \mathcal{U}_i)\}.$$

Now apply the conclusion of the previous lemma to the sequence $(\mathcal{V}_n : n \in \mathbb{N})$ to find for each n a set $\mathcal{W}_n \subseteq \mathcal{V}_n$ of cardinality n such that for each x, for all but finitely many n, $x \in \bigcup \mathcal{W}_n$. Each element of $\mathcal{W}_n$ is of the form $U^j_{T_n+1} \cap \cdots \cap U^j_{T_{n+1}}$ where $1 \leq j \leq n$ and each U_i^j is an element of $\mathcal{U}_i$.

Now for each m choose $V_m \in \mathcal{U}_m$ as follows: Find the largest n with $T_n < m$ and then identify j with $1 \leq j \leq n+1$ with $m = T_n + j$, and put $V_m = U^j_{T_n+j}$. □

$\mathcal{G}^{gp}$: This is the set of covers $\mathcal{U} \in \mathcal{G}$ for which there is a (disjoint) partition $\mathcal{U} = \bigcup_{n \in \mathbb{N}} \mathcal{U}_n$ such that each $\mathcal{U}_n$ is finite, and for each $x \in X$ for all but finitely many n, x is in $\bigcup \mathcal{U}_n$.

Theorem 15. *For a topological space X the following are equivalent:*

(1) *X is an Alster space.*

[5] $T_n = 1 + 2 + \cdots + n$.

(2) *X satisfies the selection principle* $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G})$.
(3) *X satisfies the selection principle* $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Omega)$.
(4) *X satisfies the selection principle* $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}^{gp})$.
(5) *X satisfies the selection principle* $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Omega^{gp})$.
(6) *X satisfies the selection principle* $\mathsf{S}_{fin}(\mathcal{G}_K, \mathcal{G})$.
(7) *X satisfies the selection principle* $\mathsf{S}_{fin}(\mathcal{G}_K, \mathcal{G}^{gp})$.
(8) *X satisfies the selection principle* $\mathsf{S}_{fin}(\mathcal{G}_K, \mathcal{G}_\Omega^{gp})$.

Corollary 16. *If X is an Alster space then X has a Hurewicz space in all finite powers.*

Strengthening Alster's property: $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$.

Of several standard ways in which to strengthen the Alster property, consider the following one. Define

$\mathcal{G}_\Gamma$: This is the set of covers $\mathcal{U} \in \mathcal{G}$ which are infinite, and each infinite subset of $\mathcal{U}$ is a cover of X.

Then $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$ is a formal strengthening of the Alster property. It is not clear if this formal strengthening is in fact a real strengthening. Several known examples of productively Lindelöf spaces are so because they have this stronger version of Alster's property. We review some of these:

A: An easy consequence of one of Alster's results gives:

Theorem 17 (CH)**.** *Assume that X is a space of weight at most $\aleph_1$, and that each compact subset of X is a G_δ set. If X has the property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G})$, then X has the property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$.*

Proof. It follows immediately from Alster's Theorem, Theorem 10, that if X is a space of weight at most $\aleph_1$ in which each compact set is G_δ and if X is productively Lindelöf, then CH implies that X is σ-compact (as the set of compact subsets of X is a cover of X by G_δ sets of the required kind). Thus, under CH, every productively Lindelöf space of weight at most $\aleph_1$ for which each compact subspace is a G_δ set has the property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$. □

B: A topological space is said to be a P-*space* if each intersection of countably many open sets is open. Galvin (see [13]) pointed out that any Lindelöf P-space is a γ-space. Thus, as the topology of a P-space is the G_δ topology, we find:

Lemma 18 (Galvin)**.** *Each Lindelöf* P*-space has the property* $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$.

In [19] Proposition 2.1 the authors show that the product of a Lindelöf P-space with any Lindelöf space is a Lindelöf space. This result now follows from (1) ⇒ (2) of Theorem 10, and Lemma 18. A classical theorem of Noble [24] states that the product of countably many Lindelöf P-spaces is still Lindelöf. This result now follows from Theorem 11 and Lemma 18. Additionally it is also known that:

Theorem 19 (Misra, [23])**.** *A Lindelöf* P*-space is a productively* P *space.*

C: A space is *scattered* if each nonempty subspace has an isolated point.

Theorem 20 (Gewand [14], Theorem 2.2)**.** *If X is a scattered Lindelöf space, then in the G_δ topology X is a Lindelöf* P*-space.*

Corollary 21. *A scattered Lindelöf space satisfies* $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$.

It follows that Lindelöf scattered spaces are productively Lindelöf, and that the countable power of such a space is Lindelöf.

D: A space is σ-compact if it is the union of countably many compact subsets.

Theorem 22. *Each σ-compact space has the property* $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$.

Proof. Let X be σ-compact and write $X = \bigcup_{n\in\mathbb{N}} K_n$ where for each n we have $K_n \subseteq K_{n+1}$ and K_n is compact. Let a sequence $(\mathcal{U}_n : n \in \mathbb{N})$ of elements of $\mathcal{G}_K$ be given. For each n, choose $U_n \in \mathcal{U}_n$ with $K_n \subseteq U_n$. Then each U_n is a G_δ subset of X, and for each $x \in X$, for all but finitely many n, $x \in U_n$. □

This implies the well-known fact that σ-compact spaces are productively Lindelöf. Via Theorem 11 we also find that the countable power of a σ-compact space is Lindelöf.

We now explore the productive properties of spaces with this stronger version of the Alster property.

Theorem 23. *If X is a topological space with property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$, then X is:*
1. *productively Lindelöf.*
2. *productively Menger.*
3. *productively Hurewicz.*

Proof. We show that X is productively Hurewicz. The proof that X is productively Menger is similar.

Let Y be a Hurewicz space, and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $X \times Y$. We may assume that each $\mathcal{U}_n$ is closed under finite unions. For each compact subset K of X, and for each n, find a G_δ set $\phi_n(K) \supset K$ such that for each $y \in Y$ there is an open set $U \in \mathcal{U}_n$ with $\phi_n(K) \times \{y\} \subseteq U$.

This defines, for each n, a set $\mathcal{G}_n = \{\phi_n(K) : K \subset X \text{ compact}\} \in \mathcal{G}_K$. Applying $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$ to $(\mathcal{G}_n : n \in \mathbb{N})$, we find for each n a compact set $K_n \subset X$ such that $\{\phi_n(K_n) : n \in \mathbb{N}\}$ is a member of $\mathcal{G}_\Gamma$.

Now for each n define

$$\mathcal{H}_n = \{V \subset Y : V \text{ open and there is a } U \in \mathcal{U}_n \text{ with } \phi_n(K_n) \times V \subseteq U\}.$$

Apply the fact that Y is a Hurewicz space to the sequence $(\mathcal{H}_n : n \in \mathbb{N})$: For each n choose a finite set $\mathcal{J}_n \subseteq \mathcal{H}_n$ such that for each $y \in Y$, for all but finitely many n, $y \in \cup \mathcal{J}_n$.

Finally, for each n choose for each $H \in \mathcal{J}_n$ a $U_H \in \mathcal{U}_n$ with $\phi_n(K_n) \times H \subseteq U_H$, and put $\mathcal{V}_n = \{U_H : H \in \mathcal{J}_n\}$. Then we have for each n that $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and for each $(x, y) \in X \times Y$, for all but finitely many n, (x, y) is a member of $\bigcup \mathcal{V}_n$. □

There are many ZFC examples of Lindelöf P-spaces. For example in [30] it is shown that for each uncountable cardinal number κ there is a $\mathsf{T}_{3\frac{1}{2}}$ Lindelöf P -group of cardinality κ on which TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\Omega, \Gamma)$. Since the topology is the G_δ topology, this means that in these examples TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{G}_K, \mathcal{G}_\Gamma)$.

Now we need two more classes of open covers: A family $\mathcal{F}$ of subsets of an infinite set S is said to be a *large cover* of S if for each $x \in S$ the set $\{F \in \mathcal{F} : x \in F\}$ is infinite.

A family $\mathcal{F}$ is a *groupable* cover if, and only if, there is a partition $\mathcal{F} = \bigcup_{n<\infty} \mathcal{F}_n$ where the $\mathcal{F}_n$'s are finite and disjoint from each other, such that each point in the space belongs to all but finitely many of the sets $\bigcup \mathcal{F}_n$.

We associate the following symbols with classes of open covers that are large or groupable:

$$\Lambda := \{\mathcal{U} \in \mathcal{O} : \mathcal{U} \text{ is an } \textit{large}\text{-cover of } X\}.$$

$$\mathcal{O}^{gp} := \{\mathcal{U} \in \mathcal{O} : \mathcal{U} \text{ is groupable in } X\}.$$

A space X is called a *Gerlits-Nagy space* if it satisfies the selection principle $\mathsf{S}_1(\Omega, \mathcal{O}^{gp})$ [20]. Each γ-space is a Gerlits -Nagy space, and each Gerlits-Nagy space is a Rothberger space.

Theorem 24. *If X is a Rothberger space with property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$, then X is:*
1. *productively Rothberger.*
2. *productively Gerlits-Nagy.*

Proof. (of (1)) Let a Rothberger space Y be given, and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $X \times Y$. As X is Rothberger, each of its compact subsets is Rothberger. Write $\mathbb{N} = \bigcup_{n \in \mathbb{N}} S_n$ where the S_n's are infinite and pairwise disjoint.

Fix n, and for each compact subset C of X, and for each $y \in Y$, choose a finite sequence $U_{i_1}, \cdots, U_{i_k}$ where
1. $i_1 < \cdots < i_k$ are elements of S_n and
2. U_{i_j} is an element of $\mathcal{U}_{i_j}$, $1 \le j \le k$, and
3. $C \times \{y\} \subseteq U_{i_1} \cup \cdots \cup U_{i_k}$.

This is possible as compact subsets of X are Rothberger spaces.

Thus, for fixed n, for each compact subset C of X we find that $C \times Y$ is Rothberger, thus Lindelöf, and we can find a G_δ subset $\phi_n(C)$ of X such that $C \subseteq \phi_n(C)$, and for each $y \in Y$ there is a sequence $i_1 < \cdots < i_k$ of elements of S_n such that $\phi_n(C) \times \{y\}$ is a subset of $U_{i_1} \cup \cdots \cup U_{i_k}$ as above. But then $\mathcal{G}_n = \{\phi_n(C) : C \subset X \text{ compact}\}$ is a member of $\mathcal{G}_K$ for X.

Apply $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$ to the sequence $(\mathcal{G}_n : n \in \mathbb{N})$ and select for each n a $G_n \in \mathcal{G}_n$ such that for each $x \in X$, for all but finitely many n, $x \in G_n$.

For fixed n, choose for each $y \in Y$ an open set $U_y^n \subset Y$ such that $y \in U_y^n$ and there is a sequence $i_1 < \cdots < i_k$ of elements of S_n such that $G_n \times U_y^n$ is a subset of $U_{i_1} \cup \cdots \cup U_{i_k}$ as above. Then $\mathcal{H}_n = \{U_y^n : y \in Y\}$ is an open cover of Y.

Next apply the fact that Y is Rothberger to the sequence $(\mathcal{H}_n : n \in \mathbb{N})$, and choose for each n an $H_n \in \mathcal{H}_n$ such that for each $y \in Y$ there are infinitely many n with $y \in H_n$.

Then for each n choose $i_1^n < \cdots < i_{k_n}^n \in S_n$ such that $G_n \times H_n \subseteq U_{i_1^n} \cup \cdots \cup U_{i_{k_n}^n}$.

It follows that there is a sequence of $U_n \in \mathcal{U}_n$ such that $\{U_n : n \in \mathbb{N}\}$ is an open cover of $X \times Y$.

The proof of (2) is similar, and left to the reader. □

It follows that Lindelöf P-spaces, Lindelöf scattered spaces, as well as σ-compact Rothberger spaces are not only productively Lindelöf, but also productively Menger, productively Hurewicz, productively Rothberger and productively Gerlits-Nagy.

Regarding the hypothesis of Theorem 24 that X should be Rothberger: Note that the hypothesis $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G})$ implies $\mathsf{S}_1(\mathcal{O}_K, \mathcal{O})$, which implies $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$. In Example 2 of [6] it was shown that CH implies the existence of a subspace X of $\mathbb{R}^{\mathbb{N}}$ which is not countable dimensional and yet satisfies $\mathsf{S}_1(\mathcal{O}_K, \Gamma)$. Such an X cannot be a Rothberger space, since metrizable Rothberger spaces are zero dimensional. One may consider weakening the hypothesis that X is Rothberger to the hypothesis that each compact subset of X is Rothberger. However, this is not a weakening of the hypotheses, since:

Lemma 25. *If X is a space such that each compact subspace is Rothberger, and $\mathsf{S}_1(\mathcal{O}_K, \mathcal{O})$ holds of X then X is a Rothberger space.*

Similarly, one can show:

Lemma 26. *If X is a space such that each compact subspace is Rothberger, and $\mathsf{S}_1(\mathcal{O}_K, \mathcal{O}^{gp})$ holds of X then X is a Gerlits-Nagy space.*

We leave the proof of these lemmas to the reader. We do not know if a similar statement is true about γ-spaces:

Problem 3. *Is it true that if each compact subset of X is a Rothberger space and X has the property $\mathsf{S}_1(\mathcal{O}_K, \Gamma)$, then X has the property $\mathsf{S}_1(\Omega, \Gamma)$?*

What we can prove is:

Lemma 27. *If X is a space such that each compact subspace is finite, and $\mathsf{S}_1(\mathcal{O}_K, \Gamma)$ holds of X then X is a $\mathsf{S}_1(\Omega, \Gamma)$-space.*

We now record the few results we have on productively γ spaces.

Lemma 28. [18] *Every open ω-cover of $X \times Y$ is refined by one whose elements are of the form $U \times V$ where $U \subseteq X$ and $V \subseteq Y$ are open.*

Theorem 29. *If X is a space satisfying $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$ and if each compact subset of X is finite, then X is $\mathsf{S}_1(\Omega, \Gamma)$-productive.*

Proof. We know that X is productively Lindelöf. Let a γ-space Y be given, and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open ω-covers of $X \times Y$.

Claim $X \times Y$ is Lindelöf in each finite power.

For γ-spaces it is well known that each finite power of a γ-space is a γ-space. Similarly for P-spaces, each finite power of a P-space is a P-space. Moreover, $(X \times Y)^n$ is homeomorphic to $X^n \times Y^n$. Note that since $X \times Y$ is a Lindelöf space in each finite power, we may assume, by the Proposition on p. 156 of [13], that each $\mathcal{U}_n$ is a countable set. For each n we may also assume by Lemma 28 that the elements of $\mathcal{U}_n$ are of the form $U \times V$ where U is open in X and V is open in Y. Thus, each $\mathcal{U}_n$ is of the form $\{U_k^n \times V_k^n : k \in \mathbb{N}\}$.

Fix n and fix a finite subset F of X. For each finite set $G \subset Y$ choose a $U_{k(F,G)}^n \times V_{k(F,G)}^n \in \mathcal{U}_n$ containing $F \times G$. The intersection $W^n(F) = \bigcap\{U_{k(F,G)}^n : G \subset Y \text{ finite}\}$ is a countable intersection as $\mathcal{U}_n$ is countable, and thus is an element of $\mathcal{G}_K$ for X. Likewise, $W(F) = \cap_{n\in\mathbb{N}} W^n(F)$ is and element of $\mathcal{G}_K$ for X, and for each finite $G \subset Y$ and each n, $W(F) \times V_{k(F,G)}^n$ contains $F \times G$.

Note that $\{W(F) : F \subset X \text{ finite}\}$ is an ω-cover of X. Since X is a Lindelöf space satisfying $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$, choose a countable set $\{F_n : n \in \mathbb{N}\}$ of finite subsets of X with $\{W(F_n) : n \in \mathbb{N}\}$ an element of $\mathcal{G}_\Gamma$.

Fix n: Then $\{W(F_n) \times V^n_k : k \in \mathbb{N} \text{ and } W(F_n) \subseteq U^n_k\}$ is an ω-cover of $W(F_n) \times Y$, and also $\mathcal{Z}_n = \{V^n_k : k \in \mathbb{N} \text{ and } W(F_n) \subseteq U^n_k\}$ is an ω-cover for Y.

Since Y is a γ-set, choose for each n a k_n such that $\{V^n_{k_n} : n \in \mathbb{N}\}$ is a γ-cover of Y. Then $\{W(F_n) \times V^n_{k_n} : n \in \mathbb{N}\}$ is a γ-cover for $X \times Y$, and so $\{U^n_{k_n} \times V^n_{k_n} : n \in \mathbb{N}\}$ is a γ-cover of $X \times Y$. □

Since in a T_2 Lindelöf P-space compact subsets are finite, it follows that T_2 Lindelöf P-spaces as well as T_2 Lindelöf scattered spaces are productively γ-spaces. As T_2 compact Rothberger spaces are scattered Lindelöf spaces, these are productively γ-spaces.

Corollary 30. *σ-compact Rothberger spaces are productively γ.*

Proof. Since X is σ-compact we write $X = \bigcup_{n\in\mathbb{N}} X_n$ where each X_n is a compact Rothberger space. Since the union of finitely many compact Rothberger spaces is a compact Rothberger space, we may assume that for each n we have $X_n \subseteq X_{n+1}$. A compact Rothberger space is a scattered Lindelöf space. Then for Y a γ-space, for each n $X_n \times Y$ is a γ-space. This gives a union of an increasing sequence of γ-spaces, and so by Jordan's theorem ([17], Corollary 14) is again a γ-space. □

By Corollary 15 of [30] there is for each infinite cardinal κ a σ -compact T_0 topological group of cardinality κ such that TWO has a winning strategy in the game $\mathsf{G}^{\omega}_1(\Omega, \Gamma)$. By Corollary 17 of [30] there is for each infinite cardinal number κ a T_0 topological group $(G, *)$ of cardinality κ which is a σ -compact Rothberger space in all finite powers.

Alster also showed that

Theorem 31 (Alster, [2], Theorem 4)**.** *Assume* CH. *If X is of weight at most $\aleph_1$ and has property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G})$, and if each compact subset of X is at most countable, then in the G_δ topology X is Lindelöf.*

It follows that in addition to be productively Lindelöf such spaces are also productively Menger-, Hurewicz-, Rothberger-, Gerlits-Nagy- and γ.

5. Productivity of weak covering properties.

Spaces that are productively weakly Lindelöf do not currently have as well developed a theory. A number of ways of generalizing the notion of an Alster space or its strengthenings as considered in the previous section suggest themselves, but we have had limited success in exploiting these to identify classes of spaces that are for example productively weakly Lindelöf spaces. In this section we report some of these results, and pose a number of questions whose answers may help identify criteria under which a space with a weak covering property is productively so.

The following two elementary properties of the notion of dense set will be used several times.

Lemma 32. *Let X be a topological space. If $Y \subset X$ is dense in X, and $D \subset Y$ is dense in Y, then D is dense in X.*

Lemma 33. *If $D \subset X$ is dense in X and $E \subset Y$ is dense in Y, then $D \times E$ is dense in $X \times Y$.*

The power of these two lemmas lie in the following:

Theorem 34. *Let X be a topological with dense subset D.*

1. *If D is productively weakly Lindelöf, so is X.*
2. *If D is productively weakly Menger, so is X.*
3. *If D is productively weakly Hurewicz, so is X.*
4. *If D is productively weakly Rothberger, so is X.*
5. *If D is productively weakly Gerlits-Nagy, so is X.*

Proof. We show the argument for productively weakly Rothberger, leaving the rest to the reader. Thus, let Y be a weakly Rothberger space and let D be productively weakly Rothberger. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $X \times Y$. Then the relativizations to $D \times Y$ is a sequence of open covers of $D \times Y$, since D is dense in X. Applying the fact that $D \times Y$ is weakly Rothberger we find in each $\mathcal{U}_n$ an element U_n such that $\bigcup_{n\in\mathbb{N}} U_n$ is dense in $D \times Y$, and so by Lemma 32 is dense in $X \times Y$. □

Also the following fact is useful:

Theorem 35. *Let (X, τ) be a topological and let τ' be a finer topology on X (i.e., $\tau \subset \tau'$).*

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(1) *If (X, τ') is productively weakly Lindelöf, so is (X, τ).*
(2) *If (X, τ') is productively weakly Menger, so is (X, τ).*
(3) *If (X, τ') is productively weakly Hurewicz, so is (X, τ).*
(4) *If (X, τ') is productively weakly Rothberger, so is (X, τ).**
(5) *If (X, τ') is productively weakly Gerlits-Nagy, so is (X, τ).**

In what follows we find conditions under which a space is productively weakly T, where T is one of the weak covering properties we are considering.

Weakly compact spaces.

The space (X, τ) is *weakly* compact if there is for each open cover of the space a finite subset with union dense in the space. Since the closure of a finite union of sets is the union of the finitely many sets individual closures, the weakly compact spaces coincide with the almost compact spaces, and in the context of T_2-spaces, coincide with the H-closed spaces.

Analogous to Tychonoff's theorem for compact spaces, one has:

Theorem 36 (Scarborough and Stone [27] Theorem 2.4)**.** *The product of weakly compact spaces is weakly compact.*

The argument in Proposition 1.9 of [9] shows

Theorem 37. *If X is a weakly compact space then X is productively weakly Lindelöf.*

This can be extended to the following:

Theorem 38. *Let X be a weakly compact space.*
(1) *X is productively weakly Menger.*
(2) *X is productively weakly Hurewicz.*

Since compact spaces are weakly compact, these results also imply that compact spaces are productively weakly Lindelöf, productively weakly Menger, and productively weakly Hurewicz.

Weak versions of the Alster property

Towards exploration of productiveness for the weaker versions of the covering properties we considered, we introduce the following family of subsets of a topological space:

$\mathcal{G}_D$ denotes the collection of sets $\mathcal{U}$ where each element of $\mathcal{U}$ is a G_δ set, and $\bigcup \mathcal{U}$ is dense in the space.

A space is said to be *weakly Alster* if each member of $\mathcal{G}_K$ has a countable subset which is a member of $\mathcal{G}_D$.

Using the proof technique of Theorem 4.5 of [7] we find:

Lemma 39. *If X weakly Alster and Y is weakly Lindelöf, then $X \times Y$ is weakly Lindelöf.*

Proof. Let $\mathcal{U}$ be an open cover of $X \times Y$. We may assume that $\mathcal{U}$ is closed under finite unions. For each compact set $C \subset X$ and for each $y \in Y$ we find an open set $U(C, y) \in \mathcal{U}$ such that $C \times \{y\} \subseteq U(C, y)$. For each y we find open sets $V(C, y) \subset X$ and $W(C, y) \subset Y$ such that $C \times \{y\} \subset V(C, y) \times W(C, y) \subset U(C, y)$.

By Theorem 37 $C \times Y$ is weakly Lindelöf. Thus the cover $\{W(C, y) : y \in Y\}$ of Y contains a countable subset with union dense in Y, say $\{W(C, y_n^C) : n \in \mathbb{N}\}$. Define $V(C) = \bigcap \{V(C, y_n^C) : n \in \mathbb{N}\}$, a G_δ subset of X containing the compact set $C \subseteq X$. But then $\mathcal{V} = \{V(C) : C \subset X \text{ compact}\}$ is a member of $\mathcal{G}_K$ for X. Since X is weakly Alster we find a countable subset $\{V(C_m) : m \in \mathbb{N}\}$ of $\mathcal{V}$ with union dense in X. But then $\{W(C_m, y_n^{C_m}) : m, n \in \mathbb{N}\} \subset \mathcal{U}$ is countable and its union is dense in $X \times Y$. □

Since an Alster space is a weakly Alster space, we find

Corollary 40 (CH)**.** *Every productively Lindelöf space of weight at most $\aleph_1$ is productively weakly Lindelöf.*

Proof. By Theorem 10, productively Lindelöf spaces of weight at most $\aleph_1$ are Alster spaces, and thus weakly Alster spaces. □

We don't know if the additional hypotheses are necessary in Corollary 40:

Problem 4. *Is every productively Lindelöf space productively weakly Lindelöf?*

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Problem 5. *Is every productively weakly Lindelöf space a weakly Alster space?*

An argument similar to the one in Theorem 39 shows

Theorem 41. *If X and Y are weakly Alster, then $X \times Y$ is weakly Alster.*

Additionally, analogous to Lemma 12 and Theorem 15 we find

Lemma 42. *For a topological space X the following are equivalent:*
1. *X is a weakly Alster space.*
2. *X satisfies the selection principle $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_D)$.*
3. *X satisfies the selection principle $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_{D_\Omega})$.*
4. *X satisfies the selection principle $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_{D^{gp}})$.*
5. *X satisfies the selection principle $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_{D^{gp}_\Omega})$.*
6. *X satisfies the selection principle $\mathsf{S}_{fin}(\mathcal{G}_K, \mathcal{G}_D)$.*
7. *X satisfies the selection principle $\mathsf{S}_{fin}(\mathcal{G}_K, \mathcal{G}_{D^{gp}})$.*
8. *X satisfies the selection principle $\mathsf{S}_{fin}(\mathcal{G}_K, \mathcal{G}_{D^{gp}_\Omega})$.*

We also leave to the reader the proof of

Lemma 43. *If the T_2-space X has a dense subspace which is a weakly Alster space, then X is a weakly Alster space.*

A stronger version of weakly Alster: $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_{D_\Gamma})$

$\mathcal{G}_{D_\Gamma}$ is the family of infinite sets $\mathcal{U}$ where each member of $\mathcal{U}$ is a G_δ subset of X, and for each nonempty open subset U of X, $\{V \in \mathcal{U} : U \cap V = \emptyset\}$ is finite.

A space with the property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$ has the property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_{D_\Gamma})$. Indeed:

Lemma 44. *If a space X has a dense subset which has the property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$, then X has the property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_{D_\Gamma})$.*

Proof. Let X be a space which has a dense subspace Y which has the property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$. If $(\mathcal{U}_n : n \in \mathbb{N})$ is a sequence of elements of $\mathcal{G}_K$ for X, then choose for each $n \in \mathbb{N}$ an element $U_n \in \mathcal{U}$ such that $\{U_n : n \in \mathbb{N}\}$ is an element of $\mathcal{G}_\Gamma$ for Y. Since Y is dense in X, $\{U_n : n \in \mathbb{N}\}$ is an element of $\mathcal{G}_{D_\Gamma}$ for X. □

Theorem 45. *If X has the property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_{D_\Gamma})$, then it is*
1. *productively weakly Menger.*
2. *productively weakly Hurewicz.*

Proof. We give the argument for weakly Menger productive. Thus, let Y be a weakly Menger space, and let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $X \times Y$. We may assume that each $\mathcal{U}_n$ is closed under finite unions.

For each n: For each compact subset C of X find for each $y \in Y$ a set $U_n(C, y) \in \mathcal{U}_n$ such that $C \times \{y\} \subseteq U_n(C, y)$. Then for each n we find open subsets $V_n(C, y) \subset X$ and $W_n(C, y) \subset Y$ such that

$$C \times \{y\} \subseteq V_n(C, y) \times W_n(C, y) \subset U_n(C, y).$$

For each C define $\mathcal{U}_n(C) = \{W_n(C, y) : y \in Y\}$, an open cover of Y.

Partition $\mathbb{N}$ into countably many infinite subsets S_m, $m \in \mathbb{N}$.

Since Y is weakly Menger we find for each m and for each $n \in S_m$ a finite set $F(C, n) \subset Y$ such that $\bigcup\{W_n(C, y) : y \in F(C, n),\ n \in S_m\}$ is dense in Y in the following sense: For each nonempty open $T \subset Y$ there are infinitely many $n \in S_m$ for which $W_n(C, y) \cap T \neq \emptyset$ for some $y \in F(C, n)$.

Then for each C define $V(C) = \bigcap_{n\in\mathbb{N}} \bigcap\{V_n(C, y) : y \in F(C, n)\}$, a G_δ subset of X that contains C. Then $\mathcal{V} = \{V(C) : C \subset X \text{ compact}\}$ is a member of $\mathcal{G}_K$ for X. Applying the fact that X has the property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_{D_\Gamma})$, we find a countable set $\{V(C_m) : m \in \mathbb{N}\}$ such that for each nonempty open set $S \subset X$, for all but finitely many n, $S \cap V(C_m) \neq \emptyset$.

For each n find the m with $n \in S_m$ and define $\mathcal{V}_n = \{V_n(C_m, y) \times W_n(C_m, y) : y \in F(C_m, n)\}$. Then each $\mathcal{V}_n$ is a refinement of a finite subset of $\mathcal{U}_n$. We must show that $\bigcup_{n\in\mathbb{N}}(\bigcup \mathcal{V}_n)$ is dense in $X \times Y$. Thus, let $U \times V$ be a nonempty open subset of $X \times Y$. Choose N so large that for all $m \geq N$ we have $V(C_m) \cap U \neq \emptyset$.

For such an m there are infinitely many $n \in S_m$ such that $W_n(C_m, y) \cap V \neq \emptyset$ for some $y \in F(C_m, n)$. Since $V(C_m)$ is a subset of $V_n(C_m)$, it follows that for infinitely many n there are $y \in F(C_m, n)$ for which $V_n(C_m) \times W_n(C_m, y) \cap U \times V \neq \emptyset$. □

As in the case of Lindelöf productiveness, a proof of the following conjectures may depend on a characterization of the properties of being productively Menger or productively Hurewicz.

Conjecture 1. *If a space is productively Menger then it is productively weakly Menger. If a space is productively Hurewicz then it is productively weakly Hurewicz.*

Theorem 46. *If X is a Rothberger space with the property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_{D_\Gamma})$, then it is productively weakly Rothberger.*

Proof. Let Y be a weakly Rothberger space. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $X \times Y$. We may assume that for each n the elements of $\mathcal{U}_n$ are of the form $U \times V$. Partition $\mathbb{N}$ into infinitely many infinite subsets S_m, $m \in \mathbb{N}$. Then partition each S_m into infinitely many infinite pairwise disjoint subsets $S_{m,k}$, $k \in \mathbb{N}$.

For each m, and for each compact subset C of X, do the following. For each k and for each $y \in Y$ choose $i_i < i_2 < \cdots < i_t$ from $S_{m,k}$ such that for each j, $y \in V_{i_j}$, and such that $C \subseteq U_{i_1} \cup \cdots \cup U_{i_t}$, where now for each j V_{i_j} corresponds with U_{i_j}. This is possible since C, a compact subset of X is a Rothberger space. Define $V_{m,k}(y, C)$ to be the the intersection of these V_{i_j}, and $U_{m,k}(y, C)$ to be the union of these U_{i_j}. Then $\mathcal{W}_{m,k}(C) = \{V_{m,k}(y, C) : y \in Y\}$ is an open cover of Y for each m and k. Applying the fact that Y is weakly Rothberger to $(\mathcal{W}_{m,k} : k \in \mathbb{N})$, choose for each k a $y_k \in Y$ such that for each open set $V \subset Y$ there are infinitely many k with $V_{m,k}(y_k, C) \cap V \neq \emptyset$. Define $V_m(C)$ to be the set $\bigcap_{k\in\mathbb{N}} U_{m,k}$.

Then for each compact set $C \subset X$ define $V(C) = \bigcap_{m\in\mathbb{N}} V_m(C)$, a G_δ subset of X which contains C. Now apply $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_{D_\Gamma})$ to the member $\{V(C) : C \subset X \text{ compact}\}$ of $\mathcal{G}_K$. We find a sequence $(V(C_m) : m \in \mathbb{N})$ of G_δ sets such that for each nonempty open subset U of X, for all but finitely many n, $V(C_n) \cap U$ is nonempty. Then the sequence $(V_n(C_n) : n \in \mathbb{N})$ has the same properties.

Now consider the sets $U_{m,k}(y_k, C_m) \times V_{m,k}(y_k, C_m)$ for $k, m \in \mathbb{N}$. For each nonempty open $U \times V \subset X \times Y$ there is an N such that for all $m > N$, $V(C_m) \cap U$ is nonempty, whence $U_{m,k} \cap C_m$ is nonempty for all k But for such an m, for infinitely many k also $V_{m,k}(y_k, C_m) \cap V$ is nonempty. Thus, for infinitely many m and k, $U_{m,k}(y_k, C_m) \times V_{m,k}(y_k, C_m)$ has nonempty intersection with $U \times V$. But now $U_{m,k}(y_k, C_m)$ is of the form $U_{i_1} \cup \cdots \cup U_{i_t}$ while $V_{m,k}(y_k, C_m)$ is of the form $V_{i_1} \cap \cdots \cap V_{i_t}$, where $i_1 < \cdots < i_t$ are from $S_{m,k}$ and $U_{i_j} \times V_{i_j}$ is an element of $\mathcal{U}_{i_j}$.

It follows that there is a sequence of $S_n \in \mathcal{U}_n$ such that for each nonempty open $U \times V \subset X \times Y$, here are infinitely many n with $S_n \cap U \times V$ nonempty. □

Corollary 47. *Every space which has a dense σ-compact subset has property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_{D_\Gamma})$.*

Proof. A σ-compact space has the property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_\Gamma)$. □

Corollary 48. *Every separable space has the property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_{D_\Gamma})$.*

Proof. Let X be a separable space, and let D be a countable dense subset of X. Then D is a dense σ-compact subset of X. □

Corollary 49. *Every separable space is productively weakly Lindelöf, productively weakly Menger and productively weakly Hurewicz.*

Proof. Theorem 45 and Corollary 48. □

Corollary 50. *For each cardinal number κ, $\mathbb{R}^\kappa$ is productively weakly Lindelöf, productively weakly Menger, productively weakly Hurewicz and productively weakly Rothberger.*

Proof. We only consider the case when κ in infinite. By Proposition 4 of [10],

$$\{f \in \mathbb{R}^\kappa : |\{i : f(j) \neq 0\}| < \aleph_0\}$$

is a dense σ-compact subset of $\mathbb{R}^\kappa$. By Lemma 39, Corollary 47 and Theorem 45 $\mathbb{R}^\kappa$ has the claimed properties.

Since $\mathbb{Q}$ is σ-compact and Rothberger, Proposition 4 of [10] implies that the subset

$$S_\kappa = \{f \in \mathbb{Q}^\kappa : |\{\alpha \in \kappa : f(\alpha) = 0\}| < \aleph_0\}$$

of $\mathbb{Q}^\kappa$ is σ-compact. By Corollary 15 of [30], TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\Omega,\Gamma)$ in S_κ, and thus S_κ is a γ-space. It is evident that S_κ is a dense subset of $\mathbb{R}^\kappa$.

But S_κ is dense in $\mathbb{R}^\kappa$ and so as S_κ is productively weakly Rothberger by Theorem 46, also $\mathbb{R}^\kappa$ is productively weakly Rothberger. □

Separable spaces are weakly Alster, and in fact have several additional properties: For example a separable space is also weakly Rothberger. Note that a compact space could be weakly Rothberger without being Rothberger: The closed unit interval is a compact separable space but is not a Rothberger space.

Corollary 51. *If X is a separable space then X is productively weakly Rothberger.*

Proof. Every countable space has the property $\mathsf{S}_1(\mathcal{G}_K,\mathcal{G}_\Gamma)$, and thus the property $\mathsf{S}_1(\mathcal{G}_K,\mathcal{G}_{D_\Gamma})$. Since a countable space is Rothberger, Theorem 46 implies that any countable space is productively weakly Rothberger. Apply Theorem 35 to conclude that any separable space is productively weakly Rothberger. □

Lemma 52. *Compact Rothberger spaces are productively weakly Rothberger.*

Proof. Let X be a compact Rothberger space and let Y by weakly Rothberger. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of open covers of $X \times Y$. We may assume each $\mathcal{U}_n$ consist of sets of the form $U \times V$. Write $\mathbb{N} = \bigcup_{n\in\mathbb{N}} S_n$ where the S_n's are infinite and pairwise disjoint.

Fix n as well as $y \in Y$, and choose a finite sequence $U_{i_1},\cdots,U_{i_k}$ where

(1) $i_1 < \cdots < i_k$ are elements of S_n;
(2) U_{i_j} is an element of $\mathcal{U}_{i_j}$, $1 \le j \le k$;
(3) $X = U_{i_1} \cup \cdots \cup U_{i_k}$;
(4) $y \in V_{i_1} \cap \cdots \cap V_{i_k}$ (and define $V_n(y)$ to be this latter intersection).

This defines for each n an open cover $\mathcal{H}_n = \{V_n(y) : y \in Y\}$ of Y. Now apply the fact that Y is weakly Rothberger to the sequence $(\mathcal{H}_n : n \in \mathbb{N})$ and select for each n an $H_n \in \mathcal{H}_n$ such that for each open set $V \subseteq Y$, for infinitely many n, $V \cap H_n$ is nonempty. This produces a sequence of elements $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\bigcup\{U_n : n \in \mathbb{N}\}$ is dense in $X \times Y$. □

Problem 6. *Is it true that if X has the property $\mathsf{S}_1(\mathcal{G}_K,\mathcal{G}_{D_\Gamma})$ and each compact subset of X is weakly Rothberger, then X is productively weakly Rothberger?*

Using techniques from the previous section we can prove:

Lemma 53. *If each compact subspace of X is a weakly Rothberger space and if X satisfies $\mathsf{S}_1(\mathcal{O}_K,\mathcal{D})$, then X it is weakly Rothberger.*

For a space X define an element $\mathcal{U}$ of $\mathcal{D}$ to be *groupable* if there is a partition $\mathcal{U} = \bigcup\{\mathcal{U}_n : n \in \mathbb{N}\}$ where each $\mathcal{U}_n$ is a finite set, and for each nonempty open $U \subseteq X$, for all but finitely many n we have $U \cap \bigcup\mathcal{U}_n$ is nonempty. Then define

$$\mathcal{D}^{gp} = \{\mathcal{U} \in \mathcal{D} : \mathcal{U} \text{ is groupable}\}.$$

A space is weakly Gerlits-Nagy space if it satisfies $\mathsf{S}_1(\Omega,\mathcal{D}^{gp})$.

Lemma 54. *Compact Gerlits-Nagy spaces are productively weakly Gerlits-Nagy.*

Proof. Let X be a compact Gerlits-Nagy space and let Y be weakly Gerlits-Nagy. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of ω-covers of $X \times Y$. We may assume each $\mathcal{U}_n$ consists of sets of the form $U \times V$. Write $\mathbb{N} = \bigcup_{n\in\mathbb{N}} S_n$ where the S_n's are infinite and pairwise disjoint.

Fix n and F, a finite subset of Y, and choose a finite sequence $U_{i_1},\cdots,U_{i_k}$ where:

(1) $i_1 < \cdots < i_k$ are elements of S_n;
(2) U_{i_j} is an element of $\mathcal{U}_{i_j}$, $1 \le j \le k$;
(3) $X = U_{i_1} \cup \cdots \cup U_{i_k}$;
(4) $F \subseteq V_{i_1} \cup \cdots \cup V_{i_k} = V_n(F)$, say.

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This defines for each n an ω-cover $\mathcal{H}_n = \{V_n(F) : F$ a finite subset of $Y\}$ of Y .

Now apply the fact that Y is weakly Gerlits-Nagy to the sequence $(\mathcal{H}_n : n \in \mathbb{N})$ and select for each n an $H_n \in \mathcal{H}_n$ and select an increasing sequence $m_1 < m_2 < \cdots < m_k < \cdots$ in $\mathbb{N}$ such that for each open set $V \subset Y$, for all but finitely many k, $V \cap \bigcup\{H_j : m_k \leq j < m_{k+1}\}$ is nonempty.

This produces a sequence of elements $W_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{W_n : n \in \mathbb{N}\}$ is an element of $\mathcal{D}^{gp}$ for $X \times Y$. $\square$

Problem 7. *Is each compact weakly Rothberger space a weakly Gerlits-Nagy space? (or maybe a weakly γ-space?)*

We also don't know if weakly Lindelöf P-spaces have stronger properties in terms of products. They are at least productively weakly Lindelöf as we now show:

Lemma 55. *If X is a weakly Lindelöf P-space, then it has property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_D)$.*

Proof. For each $n \in \mathbb{N}$ let a set $\mathcal{U}_n \in \mathcal{G}_K$ be given. Since X is a P-space, $\mathcal{U}_n$ is an open cover of X. For each compact subset C of X choose for each n a $U_n(C) \in \mathcal{U}_n$ with $C \subseteq U_n(C)$, and then define $V(C) = \bigcap_{n\in\mathbb{N}} U_n(C)$. Then $V(C)$ is a G_δ subset of X and $\mathcal{V} = \{V(C) : C \subset X$ compact$\}$ is an open cover of X as X is a P-space. Since X is weakly Lindelöf, choose a countable subset $\{V(C_n) : n \in \mathbb{N}\}$ of $\mathcal{U}$ which has the property that there is for each nonempty open $V \subset X$ an infinite number of n with $V \cap V(C_n) \neq \emptyset$. For each n, choose $U_n = U_n(C_n) \in \mathcal{U}_n$. Then $\{U_n : n \in \mathbb{N}\}$ is a member of $\mathcal{G}_D$. $\square$

It also follows that weakly Lindelöf P-spaces are weakly Rothberger. Since the finite product of P-spaces are P-spaces, Lemma 55 and Lemma 39 imply that the product of finitely many weakly Lindelöf P-spaces is a weakly Lindelöf P-space.

Problem 8. *Let X be a weakly Lindelöf P-space.*

(a) *Does X then have the property $\mathsf{S}_1(\mathcal{G}_K, \mathcal{G}_{D_\Gamma})$?*
(b) *Is X productively weakly-Rothberger?*
(c) *Is X weakly-Hurewicz?*

For the family $\mathcal{D}$ we introduce

$$\mathcal{D}_\Gamma := \{\mathcal{U} \in \mathcal{D} : \mathcal{U} \text{ infinite and each infinite subset of } \mathcal{U} \text{ is in } \mathcal{D}\}.$$

A space is *weak γ-space* if it satisfies $\mathsf{S}_1(\Omega, \mathcal{D}_\Gamma)$. A space is *weakly Gerlits-Nagy space* if it satisfies $\mathsf{S}_1(\Omega, \mathcal{D}^{gp})$.

Lemma 56. *Compact γ-spaces are productively weakly γ-spaces.*

Proof. Let X be a compact γ-pace and let Y be a weakly γ-space. Let $(\mathcal{U}_n : n \in \mathbb{N})$ be a sequence of ω-covers of $X \times Y$. We may assume each $\mathcal{U}_n$ consist of sets of the form $U \times V$. Write $\mathbb{N} = \bigcup_{n\in\mathbb{N}} S_n$ where the S_n's are infinite and pairwise disjoint.

Fix n as well as F finite subset of Y, and choose a finite sequence $U_{i_1}, \cdots, U_{i_k}$ where

(1) $i_1 < \cdots < i_k$ are elements of S_n;
(2) U_{i_j} is an element of $\mathcal{U}_{i_j}$, $1 \leq j \leq k$;
(3) $X = U_{i_1} \cup \cdots \cup U_{i_k}$;
(4) $F \subseteq V_{i_1} \cap \cdots \cap V_{i_k} = V_n(F)$, say.

This defines for each n an ω-cover $\mathcal{H}_n = \{V_n(F) : F$ finite subset of $Y\}$ of Y. Now apply the fact that Y is weakly γ-space to the sequence $(\mathcal{H}_n : n \in \mathbb{N})$ and select for each n an $H_n \in \mathcal{H}_n$ such that $\{H_n :: n \in \mathbb{N}\} \in \mathcal{D}_\Gamma$. This produces a sequence of elements $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{U_n : n \in \mathbb{N}\}$ is a member of $\mathcal{D}_\Gamma$ for $X \times Y$. $\square$

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