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TOPOLOGICAL CRITERIA FOR SCHLICHTNESS

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Abstract We give two sufficient criteria for schlichtness of envelopes of holomorphy in terms of topology. These are weakened converses of results of Kerner and Royden. Our first criterion generalizes a result of Hammond in dimension 2. Along the way, we also prove a generalization of Royden's theorem.

Keywords: envelope of holomorphy; schlichtness; covering space

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Let $\Omega \subseteq \mathbb{C}^n$ be a domain. The *envelope of holomorphy* of Ω is a pair $(\tilde{\Omega}, \pi)$ consisting of a connected Stein manifold $\tilde{\Omega}$ and a locally biholomorphic map $\pi\colon \tilde{\Omega} \to \mathbb{C}^n$, together with a holomorphic inclusion $\alpha\colon \Omega \to \tilde{\Omega}$, characterized by the following properties: $\pi \circ \alpha$ is the identity, and each holomorphic function f on Ω has a unique holomorphic extension F_f on $\tilde{\Omega}$, with $f = F_f \circ \alpha$. Let $\Omega' = \pi(\tilde{\Omega})$ and let $i = \pi \circ \alpha\colon \Omega \to \Omega'$. The envelope of holomorphy $(\tilde{\Omega}, \pi)$ is *schlicht* if $\pi\colon \tilde{\Omega} \to \Omega'$ is biholomorphic. One would like to give conditions on Ω to have a schlicht envelope of holomorphy.

Two results of Kerner and Royden lead to necessary conditions. Kerner [**5**] has shown that $\alpha_*\colon \pi_1(\Omega) \to \pi_1(\tilde{\Omega})$ is surjective. Royden [**8**] has shown that $\alpha^*\colon H^1(\tilde{\Omega}; \mathbb{Z}) \to H^1(\Omega; \mathbb{Z})$ is injective. It follows trivially that if $(\tilde{\Omega}, \pi)$ is schlicht, so $\tilde{\Omega} = \Omega'$, then $i_*\colon \pi_1(\Omega) \to \pi_1(\Omega')$ is surjective and $i^*\colon H^1(\Omega'; \mathbb{Z}) \to H^1(\Omega; \mathbb{Z})$ is injective.

Neither of these conditions is sufficient, by a result of Fornæss and Zame [**1**] (see [**2**, § 3]). Following an idea of Hammond [**2**], one may seek sufficient conditions by adjoining to the surjectivity of i_* (or injectivity of i^*) the assumption that $\pi\colon \tilde{\Omega} \to \Omega'$ is a covering space. This strong assumption is still reasonable, as covering maps certainly occur among envelopes of holomorphy; indeed, Fornæss and Zame show in [**1**] that for any covering map $\pi\colon \tilde{\Omega} \to \Omega'$ there is a domain $\Omega \subseteq \Omega'$ with envelope of holomorphy $(\tilde{\Omega}, \pi)$.

Specifically, Hammond has shown that, in dimension $n = 2$, if $i_*\colon \pi_1(\Omega) \to \pi_1(\Omega')$ is surjective and $\pi\colon \tilde{\Omega} \to \Omega'$ is a covering map, then $(\tilde{\Omega}, \pi)$ is schlicht. We give an elementary proof of Hammond's theorem in all dimensions $n \geqslant 2$. In addition, we give a sufficient condition for schlichtness in terms of the injectivity of i^* on cohomology, again assuming π is a covering map. Along the way, we give an alternative proof of Royden's theorem, which also extends it to coefficient groups other than $\mathbb{Z}$.

Theorem 1. *If π is a covering map and $i_*\colon \pi_1(\Omega) \to \pi_1(\Omega')$ is surjective, then $(\tilde{\Omega}, \pi)$ is schlicht.*

This extends the theorem of Hammond for dimension $n = 2$. Hammond's proof relies on a result of Jupiter [**4**], which is special to dimension 2.

Proof. The number of sheets of the covering map π is equal to the index of $\pi_*(\pi_1(\tilde{\Omega}))$ in $\pi_1(\Omega')$ (see, for example, [**3**, Proposition 1.32]). The surjectivity of $i_* = \pi_* \circ \alpha_*$ implies that π_* is surjective. Hence, the index of the image subgroup is 1, so $\pi\colon \tilde{\Omega} \to \Omega'$ is 1-sheeted, i.e. a homeomorphism. Since π is a holomorphic homeomorphism, it is biholomorphic and so $\tilde{\Omega}$ is schlicht. □

Compare with the more technical proof in [**2**].

The cohomology in Royden's result is Čech cohomology with coefficients in the sheaf of locally constant $\mathbb{Z}$-valued functions. Since our spaces are manifolds, Čech cohomology coincides with singular cohomology (with coefficients in $\mathbb{Z}$); see, for example, [**6**, Theorem 73.2]. Recall also that by the universal coefficient theorem, $H^1(X; G) = \operatorname{Hom}(\pi_1(X), G)$ for a path-connected space X and abelian coefficient group G [**3**, p. 198].

Before we go on, observe that this proves Royden's theorem as a consequence of Kerner's theorem and extends it to other coefficient groups.

Theorem 2 (Royden). *For any abelian group G, $\alpha^*\colon H^1(\Omega; G) \to H^1(\tilde{\Omega}; G)$ is injective.*

Proof. Since $\alpha_*\colon \pi_1(\Omega) \to \pi_1(\tilde{\Omega})$ is surjective,

$$\alpha^*\colon \operatorname{Hom}(\pi_1(\Omega), G) \to \operatorname{Hom}(\pi_1(\tilde{\Omega}), G)$$

is injective and these Hom groups coincide with $H^1(\Omega; G)$, $H^1(\tilde{\Omega}; G)$. □

Royden proves this for $G = \mathbb{Z}$ using Čech cohomology, in particular the exponential short exact sequence (hence the restriction to $G = \mathbb{Z}$). No such result holds for higher cohomology groups [**1**, Theorem 4].

Now, we aim to give a sufficient criterion for schlichtness by assuming $i^*\colon H^1(\Omega'; G) \to H^1(\Omega; G)$ is injective for every abelian group G, and that π is a covering map. Our strategy is to deduce that $\pi_*\colon \pi_1(\tilde{\Omega}) \to \pi_1(\Omega')$ is surjective, as in the proof of Theorem 1. This would follow if we could deduce that $i_*\colon \pi_1(\Omega) \to \pi_1(\Omega')$ is surjective, but, in general, injectivity of $\operatorname{Hom}(A, G) \to \operatorname{Hom}(B, G)$ does not imply surjectivity of $B \to A$. The problem is that if the image of B is a proper subgroup which is not contained in any proper normal subgroup, then there is no non-zero $f\colon A \to G$ vanishing on the image of B. For example, let $\mathfrak{S}_3$ be the symmetric group on three letters and let $B = \mathbb{Z}/2\mathbb{Z}$ be the subgroup generated by a transposition. If $f\colon \mathfrak{S}_3 \to G$ is any group homomorphism such that the restriction $f \mid B$ is zero, then f itself is zero.

We must solve this problem by adjoining a hypothesis to ensure that every proper subgroup of $\pi_1(\Omega')$ is contained in a proper normal subgroup. However, this alone is not enough. For, suppose that $B \subset \pi_1(\Omega')$ is a proper subgroup, contained in a proper normal subgroup N. We get a non-zero homomorphism $f\colon \pi_1(\Omega') \to G = \pi_1(\Omega')/N$, namely the quotient map, whose restriction to $B \subseteq N$ is zero, so $\mathrm{Hom}(\pi_1(\Omega'), G) \to \mathrm{Hom}(B, G)$ is not injective. This will prove the theorem we want, but only if G is abelian, so we can identify these Hom groups with singular cohomology.

So we need to know that every proper subgroup of $\pi_1(\Omega')$ is not only contained in a proper normal subgroup, but in one such subgroup N whose quotient $G = \pi_1(\Omega')/N$ is abelian.

Fortunately, this condition is more natural than it sounds. It holds if $\pi_1(\Omega')$ is nilpotent, as in that case every maximal proper subgroup is normal and has prime index (see [**7**, Theorem 5.40]).

We get the following.

Theorem 3. *If π is a covering map, $\pi_1(\Omega')$ is nilpotent and $i^*\colon H^1(\Omega'; G) \to H^1(\Omega; G)$ is injective for every abelian group G, then $(\tilde{\Omega}, \pi)$ is schlicht.*

Proof. Since $i^* = \alpha^* \circ \pi^*$ is injective, π^* is injective as well. Via π_*, we regard $\pi_1(\tilde{\Omega})$ as a subgroup of $\pi_1(\Omega')$. Recall that if H is any nilpotent group, then every maximal proper subgroup N of H is normal and has prime index, and, in particular, H/N is abelian. If $\pi_1(\tilde{\Omega}) \subsetneqq \pi_1(\Omega')$, there exists a maximal subgroup $\pi_1(\tilde{\Omega}) \subseteq N \subsetneqq \pi_1(\Omega')$ and hence a surjection $\pi_1(\Omega') \to G = \pi_1(\Omega')/N$ to an abelian group with $\pi_1(\tilde{\Omega})$ mapping to zero. This surjection is non-zero and lies in the kernel of

$$\pi^*\colon H^1(\Omega'; G) = \mathrm{Hom}(\pi_1(\Omega'), G) \to \mathrm{Hom}(\pi_1(\tilde{\Omega}), G) = H^1(\tilde{\Omega}; G)$$

for the abelian group $G = \pi_1(\Omega')/N$, contradicting the injectivity of π^*.

It follows that $\pi_1(\tilde{\Omega}) = \pi_1(\Omega')$. As before, this implies that π is a degree 1 covering map, and hence a biholomorphism. □

Solvability would not be enough, as shown by the example of $\mathbb{Z}/2\mathbb{Z} \subset \mathfrak{S}_3$. This would not only obstruct the proof given above, but would actually lead to a counter-example to the version of the statement, with solvable in place of nilpotent.

Example 4. Recall that Artin's braid group on three strands, denoted B_3, is the fundamental group of the complement of the braid arrangement A_2 in $\mathbb{C}^3$, the union of the three hyperplanes defined by $(y-x)(z-x)(z-y) = 0$. Quotienting by the small diagonal, the line $x = y = z = 0$, B_3 is the fundamental group of $\Omega' \subset \mathbb{C}^2$, the complement of the union of three lines through the origin in $\mathbb{C}^2$. Let $B_2 \subset B_3$ be a subgroup corresponding to two of the strands, so $B_2 \cong \mathbb{Z}$ has index 3 in B_3 and is not normal. There exists a covering space $\tilde{\Omega} \to \Omega'$ such that $\pi_1(X) = B_2 \subset B_3$. Since U is a Stein manifold, so is $\tilde{\Omega}$ [**9**]. By [**1**, Theorem 5], there exists a domain $\Omega \subset \Omega'$ with envelope of holomorphy $\tilde{\Omega}$. This is not schlicht, but for every abelian group G, $\mathrm{Hom}(B_3, G) \to \mathrm{Hom}(B_2, G)$ is injective.

More generally, let H be any finitely presented group. H is the fundamental group of a 2-complex, which may be embedded in $\mathbb{R}^5$, or, for that matter, $\mathbb{C}^3$; then, a tubular neighbourhood Ω' of this complex (in $\mathbb{C}^3$) still has $\pi_1(\Omega') = H$. Any subgroup $K \subset H$

occurs as the fundamental group of a covering space $\tilde{\Omega} \to \Omega'$. Again, $\tilde{\Omega}$ is Stein since Ω' is, and there exists a domain $\Omega \subset \Omega'$ with envelope of holomorphy $\tilde{\Omega}$.

It is not necessary to assume that i^* is injective when coefficients are taken in any abelian group G. It would be enough to assume that i^* is injective when coefficients are taken in any finite cyclic group, in any abelian quotient G of $\pi_1(\Omega')$ or even just in a single abelian quotient $G = \pi_1(\Omega')/N$ for some proper normal subgroup N containing $\pi_1(\tilde{\Omega})$.

If, in addition, $\pi\colon \tilde{\Omega} \to \Omega'$ is a normal covering space, then $\pi_1(\tilde{\Omega}) \subseteq \pi_1(\Omega')$ is a normal subgroup and we can take G to be an abelian quotient of $\pi_1(\Omega')/\pi_1(\tilde{\Omega})$, which is the group of deck transformations.

Corollary 5. *Suppose π is a normal covering map with deck transformation group H. If there exists a non-zero abelian quotient G of H such that $i^*\colon H^1(\Omega'; G) \to H^1(\Omega; G)$ is injective, then $(\tilde{\Omega}, \pi)$ is schlicht.*

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