STABLY FREE MODULES OVER THE KLEIN BOTTLE

by

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ABSTRACT

This paper is concerned with constructing countably many, non-free stably free modules for the Klein bottle group. The work is based on the papers "Stably Free, Projective Right Ideals" by J.T. Stafford (1985) and "Projective, Nonfree Modules Over Group Rings of Solvable Groups" by V. A. Artamonov (1981). Stafford proves general results that guarantee the existence of non-free stably frees for the Klein bottle group but has not made the argument explicit. Artamonov allows us to construct infinitely many non-free stably free modules. This paper will also construct presentations and sets of generators for these modules. This paper concludes with applications for the Klein bottle group and the Homotopy Classification Problem.

TABLE OF CONTENTS

CHAPTER 1

INTRODUCTION

Projective modules over group algebras play a key role in many aspects of geometry and topology. The Quillen-Suslin theorem (1976) [15], [18] states that polynomial algebras do not admit non-free projective modules, which provides a proof to Serre's Conjecture. Swan [19] points out that the Quillen-Suslin theorem extends also to group algebras associated with free Abelian groups.

What about group algebras of virtually free Abelian groups; that is, groups that have a free Abelian group as a subgroup of finite index? One of the simplest examples of such a group is the fundamental group of the Klein bottle, $G = \langle x, y \mid x^{-1}yx = y^{-1}\rangle$. The subgroup $\langle y, x^2\rangle$ in G is free Abelian of rank 2; thus, by Swan, it has no non-free projective $k\langle y, x^2\rangle$-modules. This subgroup is also index 2 in G. Lewin [13] has shown that the Klein bottle admits non-free projective $\mathbb{Q}G$-modules.

Let k be an arbitrary ring. Then, in light of the presentation of G, $G = \langle\, x, y \mid yx = xy^{-1}\,\rangle$, one can view the group ring kG as the skew-Laurent polynomial ring $kG = R[\,x, x^{-1}; \sigma\,]$, where $R = k\,[y, y^{-1}]$ and one has the relation $rx = xr^{\sigma}$ for $r \in R$ and $\sigma : R \to R$ is the automorphism induced by $y \mapsto y^{-1}$.

This presentation also allows us to view G as a semidirect product of infinite cyclic groups, namely $G = \mathbb{Z} \rtimes_{\sigma} \mathbb{Z}$, where $\sigma : \mathbb{Z} \to \mathbb{Z}$ is defined by $\sigma : y \mapsto y^{-1}$ and $\mathbb{Z}$ is written multiplicatively. With this perspective in mind, the only attribute of G

keeping it from not being $\mathbb{Z} \times \mathbb{Z}$, which is free Abelian, is the relation $yx = xy^{-1}$. This tiny bit of noncommutativity guarantees the existence of non-free projective modules over the Klein bottle.

Definition 1.1. Let R be an arbitrary ring. An R-module P is said to be **stably free** if there exists $m, n \in \mathbb{N}$, such that $P \oplus R^m \cong R^n$. When R satisfies IBN, we define the **rank** of P to be $n - m$.

By definition, stably free modules are necessarily projective. Hence, the existence of a non-free, stably free module implies that there are non-free, projectives. In fact, the result by Lewin shows that the Klein bottle has a non-free, stably free $\mathbb{Q}G$-module. Furthermore, for the Klein bottle group, $K_0(\mathbb{Z}G) = 0$ (see [8]), which implies that all projective modules over $\mathbb{Z}G$ are stably free.

The intent of this paper is to explore more deeply the nature of these non-free, stably free modules over the Klein bottle. Another brilliant paper that constructs non-free, stably free modules is written by Stafford [17]. Stafford showed that if G is a non-Abelian, poly-(infinite) cyclic group and k is a Noetherian domain, then the ring kG has a non-free, stably free right ideal [17, Theorem 2.12]. The presentation of the Klein bottle group shows that it is included in this class of groups.

This paper will explicitly construct a non-free, stably free kG-module over the Klein bottle by using the construction given by Stafford. We will call this module K. The paper will then construct a presentation and set of generators for K. The paper will also concern itself with these same questions and constructions for higher sums of K. From Stafford's work, we can determine that K must be non-free, stably free. This paper will also show that $K \oplus K$ is stably free and has a free factor. Using arguments by Artamonov, we can generalize the construction by Stafford to create countably infinite, non-isomorphic, non-free, stably free modules over kG.

For this paper, we will try to stay in a general setting. That is, we will work with the ring kG, where k is a Noetherian domain and G is a poly-(infinite) cyclic group, even though our concern is in the case in which $k = \mathbb{Z}$ and $G = \langle\, x, y \mid x^{-1}yx = y^{-1} \,\rangle$. We then may apply our results to the case of the Klein bottle in a hope of getting closer to answering questions from geometry and topology, such as the Geometric Realization or Homotopy Classification.

CHAPTER 2

PRESENTATIONS OF STABLY FREE MODULES

2.1 Presentation of K for $K \oplus kG \cong kG^2$

Definition 2.1. Let R be a ring (associative with unity). Then, $R[x, x^{-1}]$ denotes the ring consisting of all finite sums of the form

$$\sum_{i \in \mathbb{Z}} x^i r_i,$$

with $r_i \in R$ and multiplication defined as normal polynomial multiplication. This is the ring of **Laurent polynomials** over R.

Let $\sigma : R \to R$ be a ring automorphism. Then, $R[x, x^{-1}; \sigma]$ denotes the ring consisting of all Laurent polynomials with multiplication defined by the relation $rx = xr^{\sigma}$.[1] This is the ring of **skew-Laurent polynomials** over R.

Let G be a semidirect product of a finitely generated, free Abelian group H by an infinite cyclic group. Thus, $G = H \rtimes \mathbb{Z}$. Say that H is generated by the set $\{y_1, \ldots y_n\}$ for $n \geqslant 1$ and the cyclic group has generator x, with both groups written multiplicatively.

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[1]For the purposes of this paper, we prefer that all modules be right, which is why we write the coefficient on the right.

As a semidirect product, there is a need to reference the group homomorphism $\mathbb{Z} \to \mathrm{Aut}(H)$. The domain is an infinite cyclic group (which is thus free), so we need only define the image of the single generator x. Define this particular automorphism to be σ. With this in mind, we really mean that $G = H \rtimes_\sigma \mathbb{Z}$. To ensure that G be non-Abelian, we require that $\sigma \neq \mathbb{1}_H$, which implies that $\exists\, y_i$, such that $y_i^\sigma \neq y_i$. Without the loss of generality, assume that $y_1^\sigma \neq y_1$.

Example 2.2. If $G = \mathbb{Z}\langle y \rangle \rtimes_\sigma \mathbb{Z}$, then $\sigma \in \mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$. Since $\sigma \neq \mathbb{1}$, $\sigma(y) = y^{-1}$ and G is the Klein bottle group. In this case, σ has order 2. ∎

Example 2.3. In general, since H is a finitely generated, free Abelian group and $\sigma \in \mathrm{Aut}(H)$, σ is generated by Nielsen automorphisms. For $H = \langle y_1,\ y_2,\ y_3 \rangle$[2], $\mathrm{Aut}(H) = \mathrm{GL}_3(\mathbb{Z})$. Let σ be the automorphism induced by $y_1 \mapsto y_2 y_1$ and by leaving y_2 and y_3 fixed. This is an automorphism of H with infinite order, since H is torsion free.

Now let σ be induced by the permutation $(y_1\ \ y_2\ \ y_3)$. This is an automorphism of order 3. ∎

Let k be any commutative Noetherian domain (e.g. $\mathbb{Z}$), and form kG, the group ring with coefficients in k. When we consider the assumptions we put on G as above, we may also view kG as a skew-Laurent polynomial extension of the subring kH; that is, $kG = kH[\,x, x^{-1}; \sigma\,]$ with $r^\sigma = x^{-1} r x$ for all $r \in kH$. Thus, kG is also a Noetherian domain [9, Cor 1.15] but is not commutative because G is non-Abelian.

Let us choose $r, s \in kG$, such that if $y_1^\sigma \neq y_1^{-1}$, then let $r = 1 + y_1$; otherwise let $r = 1 + y_1 + y_1^3$. Let $s = r^{\sigma^{-1}}$ in either case. Then, we define the set

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[2]The notation $\langle \ldots \rangle$ represents the module generated by the given set.

$$K = \{\, f \in kG \;\mid\; rf = (x+s)g,\; g \in kG \,\} \cong \langle r \rangle \cap \langle x+s \rangle. \tag{2.1}$$

The isomorphism claimed in (2.1) is obvious since kG is a domain. Since kG is an Ore domain [12, p. 304], we know that K is nontrivial. In particular, K is a proper right ideal of kG.

Let $f \in K$. Then, $\exists g \in kG$, such that $rf = (x+s)g$. We claim that the function $\theta : K \to kG$, such that $f \mapsto g$ is a kG-map. Let $g, g' \in kG$, such that $rf = (x+s)g = (x+s)g'$. This implies that $(x+s)g - (x+s)g' = (x+s)(g-g') = 0$. Now, $(x+s) \neq 0$, so $g - g' = 0 \Rightarrow g = g'$, since kG is a domain. Thus, $\theta : f \mapsto g$ is well-defined. Now let $rf = (x+s)g$ and $rf' = (x+s)g'$. Then, $r(f+f') = rf + rf' = (x+s)g + (x+s)g' = (x+s)(g+g')$. Thus, $\theta : f + f' \mapsto g + g'$, and the function is additive. Let $\alpha \in kG$. Then, $r(f\alpha) = (rf)\alpha = ((x+s)g)\alpha = (x+s)(g\alpha)$. So, $\theta : f\alpha \mapsto g\alpha$, and the function is a kG-map.

The above statement gives us immediately that the following function is also a kG-map. Define $i : K \to kG \oplus kG$ as

$$f \mapsto \begin{pmatrix} f \\ -g \end{pmatrix}.$$

Also, define $\pi : kG \oplus kG \to kG$ as

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \mapsto rm_1 + (x+s)\,m_2.$$

Theorem 2.4. *The sequence* $0 \longrightarrow K \xrightarrow{\;i\;} kG \oplus kG \xrightarrow{\;\pi\;} kG \longrightarrow 0$ *is exact.*

Proof. As in any proof of exactness, there are four things to show.

1. (i is injective.)

   Let $i(f) = (\,0 \;\; 0\,)^T$. Then, particularly, the first coordinate is 0. But the first coordinate of $i(f)$ is simply f. Thus, $f = 0$.

   Note that the map θ is also one-to-one, which is sufficient to show that i is injective. Specifically, if $rf = (x+s)0 = 0$, then $f = 0$ because kG is a domain.

2. ($\operatorname{im} i \subseteq \ker \pi$.)

   $$\pi i(f) = \pi \begin{pmatrix} f \\ -g \end{pmatrix} = rf - (x+s)g = 0.$$

3. ($\ker \pi \subseteq \operatorname{im} i$.)

   Let $\pi \begin{pmatrix} f \\ g \end{pmatrix} = 0$. Then, $rf + (x+s)g = 0 \Rightarrow rf = -(x+s)g = (x+s)(-g)$.

   So, $f \in K$. And, $\begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f \\ -(-g) \end{pmatrix} = i(f) \in \operatorname{im} i$.

4. (π is surjective.)

   Let $\alpha \in kG$. Then,

   $$\begin{aligned} \pi \begin{pmatrix} sx^{-2}\alpha \\ (x^{-1} - rx^{-2})\alpha \end{pmatrix} &= r(sx^{-2})\alpha + (x+s)(x^{-1} - rx^{-2})\alpha \\ &= (rsx^{-2} + 1 - xrx^{-2} + sx^{-1} - srx^{-2})\alpha \\ &= (1 - sxx^{-2} + sx^{-1})\alpha \\ &= \alpha. \end{aligned}$$

   In particular, when $\alpha = 1$, we have

$$r(sx^{-2}) + (x+s)(x^{-1} - rx^{-2}) = 1. \tag{2.2}$$

□

Now, the above sequence is exact. But kG is free and hence projective. Thus, the above sequence is split. To find this splitting, I refer the reader to the method mentioned in [10, p.2]. Define $\mathcal{s} : kG \to kG \oplus kG$ as

$$\mathcal{s} : \alpha \longmapsto \begin{pmatrix} (sx^{-2})\alpha \\ (x^{-1} - rx^{-2})\alpha \end{pmatrix}.$$

Now,

$$\begin{aligned} \pi\mathcal{s}(\alpha) &= \pi \begin{pmatrix} (sx^{-2})\alpha \\ (x^{-1} - rx^{-2})\alpha \end{pmatrix} \\ &= r(sx^{-2})\alpha + (x+s)(x^{-1} - rx^{-2})\alpha \\ &= (r(sx^{-2}) + (x+s)(x^{-1} - rx^{-2}))\alpha \\ &= \alpha. \end{aligned}$$

Thus, $\mathcal{s}$ is a splitting of π; that is, $\pi\mathcal{s} = \mathbb{1}_{kG}$.

The existence of $\mathcal{s}$ guarantees that $kG \oplus kG$ projects onto K and that $kG \oplus kG \cong K \oplus kG$. This fact gives us an immediate, yet very important corollary to Theorem 2.4.

Corollary 2.5. *K is a stably free kG-module; that is, $K \oplus kG \cong kG \oplus kG$.*

Define $p : kG \oplus kG \to K$ as

$$\vec{m} \mapsto i^{-1}(\vec{m} - \mathcal{s}\pi\vec{m}).$$

Please note that $\pi(\vec{m} - \mathscr{s}\pi\vec{m}) = \pi\vec{m} - \pi\mathscr{s}\pi\vec{m} = \pi\vec{m} - (\mathbb{1}_{kG})\pi\vec{m} = \pi\vec{m} - \pi\vec{m} = 0$. So, $\vec{m} - \mathscr{s}\pi\vec{m} \in \ker\pi = \operatorname{im} i$, by exactness. The injectivity of i guarantees that p is well-defined.

For completeness, we give the following theorem.

Theorem 2.6. *The sequence* $0 \longrightarrow kG \xrightarrow{\mathscr{s}} kG \oplus kG \xrightarrow{p} K \longrightarrow 0$ *is exact.*

Proof. We follow the same proof template as before.

1. ($\mathscr{s}$ is injective.)

   Let $\mathscr{s}(\alpha) = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$. Then, $\begin{pmatrix} (sx^{-2})\alpha \\ (x^{-1} - rx^{-2})\alpha \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \Rightarrow sx^{-1}\alpha = 0$ $\Rightarrow \alpha = 0$, since $sx^{-1} \neq 0$ and kG is a domain.

2. ($\operatorname{im}\mathscr{s} \subseteq \ker p$.)

   $p\mathscr{s}(\alpha) = i^{-1}(\mathscr{s}\alpha - \mathscr{s}\pi\mathscr{s}\alpha) = i^{-1}(\mathscr{s}\alpha - \mathscr{s}(\mathbb{1})\alpha) = i^{-1}(\mathscr{s}\alpha - \mathscr{s}\alpha) = 0.$

3. ($\ker p \subseteq \operatorname{im}\mathscr{s}$.)

   Let $p(\vec{m}) = 0$. Then, $p(\vec{m}) = i^{-1}(\vec{m} - \mathscr{s}\pi\vec{m}) = 0 \Rightarrow \vec{m} - \mathscr{s}\pi\vec{m} = 0$ $\Rightarrow \vec{m} = \mathscr{s}(\pi\vec{m}) \in \operatorname{im}\mathscr{s}$.

4. (p is surjective.)

   $pi(f) = i^{-1}(i(f) - \mathscr{s}\pi i(f)) = i^{-1}(i(f) - \mathscr{s}(0)f) = i^{-1}if = f$. Thus, i is a splitting of p. In particular, this shows that p is surjective; that is, $p(if) = f$ for all $f \in K$. □

Hence, we see the following diagram with exactness in both directions.

$$\begin{tikzcd} 0 \arrow[r, leftrightarrow] & K \arrow[r, bend left, "i"] & kG^2 \arrow[l, bend left, "p"] \arrow[r, bend left, "\pi"] & kG \arrow[l, bend left, "s"] \arrow[r, leftrightarrow] & 0 \end{tikzcd}$$

So it is well established by the previous theorems that K is stably free, but we can say more about the freedom of K.

Definition 2.7. For a ring R, let $S = R[\,x, x^{-1};\, \sigma\,]$, the skew Laurent extension of R, and let $r, s \in S$. We say that the pair (r, s) has **Property 2.7** iff

1. r is not a unit in S.
2. $rS + (x + s)\,S = S$; that is, $\{r,\ x + s\}$ spans S.
3. $sr^{\sigma} \notin rR$ (a weaker statement than linear independence of $\{sr^{\sigma},\ r\}$).

Theorem 2.8 (Stafford, Thm 1.2 [17])**.** *Let R be a commutative Noetherian domain. Suppose $S = R[\,x, x^{-1};\, \sigma\,]$ is a skew Laurent extension of R with elements $r, s \in R$, such that (r, s) satisfy property 2.7. Then, the S-module $K = \{\, f \in S \ \mid\ rf \in \langle x + s \rangle S\,\}$ is a non-free, stably free right ideal of S, satisfying $K \oplus S \cong S \oplus S$.*

Proof. We mention only the sketch of Stafford's proof. We have already shown that K is stably free by the same method used by Stafford. Now, S is an Ore domain. Using the Ore condition on S, we can construct two elements in K that can be generated by a single element only if $sr^{\sigma} \in rR$. Thus, we assume $sr^{\sigma} \notin rR$ for the sake of contradiction. □

Corollary 2.9. *K is a non-free, stably free kG-module.*

Proof. All we need to show is that (r, s), as defined on page 5, satisfy property 2.7. We only need to separate the cases for the third condition.

Every unit in a (skew-) Laurent ring is necessarily a monomial. Hence, r isn't a unit.

Saying the set $\{r, x+s\}$ generates kG is the same thing as saying the map π is surjective. In particular, surjectivity comes from equation (2.2).

If $r = 1 + y_1$, then $\langle r \rangle$ is a prime ideal in a commutative ring. Now, $s, r^\sigma \notin \langle r \rangle$. Therefore, $sr^\sigma \notin \langle r \rangle$.

Otherwise, $r = 1 + y_1 + y_1^3$ and

$$\begin{aligned} sr^\sigma &= \left(1 + y_1^{-1} + y_1^{-3}\right)^2 \\ &= \left(y_1^6 + 2y_1^5 + y_1^4 + 2y_1^3 + 2y_1^2 + 1\right) y_1^{-6}. \end{aligned}$$

Since y_1^{-6} is a unit, it suffices to show that

$$\left(1 + y_1 + y_1^3\right) \nmid \left(y_1^6 + 2y_1^5 + y_1^4 + 2y_1^3 + 2y_1^2 + 1\right).$$

Using polynomial division, we see that

$$y_1^6 + 2y_1^5 + y_1^4 + 2y_1^3 + 2y_1^2 + 1 = \left(y_1^3 + y_1 + 1\right)\left(y_1^3 + 2y_1^2 - 1\right) + y_1 + 2.$$

Since the remainder is nonzero, the condition holds. $\square$

We now reach our next major result. The exact sequence given in Theorem 2.6 gives us a presentation of K.

$$\begin{aligned} K &= \left\langle\, e_1,\ e_2 \mid \partial(1)\,\right\rangle \\ &= \left\langle\, e_1,\ e_2 \;\middle|\; e_1 s x^{-2} + e_2(x^{-1} - r x^{-2})\,\right\rangle \end{aligned}$$

$$= \langle\, e_1,\ e_2 \mid e_1 s + e_2(x-r)\,\rangle\,. \tag{2.3}$$

Now, K can be presented with two generators and one relator. Let $e_1 = (1\ \ 0)^T$ and $e_2 = (0\ \ 1)^T$; that is, let them be the natural generators of $kG \oplus kG$. We are interested in what these generators look like when mapped into K. Since $\{e_1,\ e_2\}$ generate $\mathbb{Z}G \oplus \mathbb{Z}G$, the generators of K are simply $p(e_1)$ and $p(e_2)$. Namely,

$$\begin{aligned}
p(e_1) &= i^{-1}(e_1 - \mathscr{s}\pi e_1) \\
&= i^{-1}(e_1 - \mathscr{s}r) \\
&= i^{-1}(e_1 - (sx^{-2}re_1 + (x^{-1} - rx^{-2})re_2)) \\
&= 1 - (sx^{-2})r \qquad (2.4) \\
&= (x^2 - sr^{\sigma^2})x^{-2}. \\
p(e_2) &= i^{-1}(e_2 - \mathscr{s}\pi e_2) \\
&= i^{-1}(e_2 - \mathscr{s}(x+s)) \\
&= i^{-1}(e_2 - (sx^{-2}(x+s)e_1 + (x^{-1} - rx^{-2})(x+s)e_2)) \\
&= -(sx^{-2})(x+s) \qquad (2.5) \\
&= (sx + sr^{\sigma})(-x^{-2}).
\end{aligned}$$

So,

$$\begin{aligned}
K &= \langle\, 1 - (sx^{-2})r,\ \ -(sx^{-2})(x+s)\,\rangle \qquad (2.6) \\
&= \langle\, x^2 - sr^{\sigma^2},\ sx + sr^{\sigma}\,\rangle.
\end{aligned}$$

In light of equation (2.6), since $kG \oplus kG$ is free and we know where the generators are sent, we can redefine the map p without reference to π or $\mathscr{s}$.

$$p : kG \oplus kG \longrightarrow K : \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \longmapsto (1 - (sx^{-2})r)m_1 - (sx^{-2})(x + s)m_2.$$

It is also worthy to note that $p(e_1)s + p(e_2)(x - r) = (1 - (sx^{-2})r)s - (sx^{-2})(x + s)(x - r) = 0$. Unfortunately, the relation (2.3) does not hold for the other generating set listed above. We leave these simple computations to the reader.

We are now at liberty to rewrite the previous maps in terms of matrix multiplication, which is often convenient when working with free modules. Thus, we have

$$\begin{aligned} i &= \begin{pmatrix} 1 \\ -\theta \end{pmatrix} \\ \mathscr{s} &= \begin{pmatrix} sx^{-2} \\ x^{-1} - rx^{-2} \end{pmatrix} \\ \pi &= \begin{pmatrix} r & x + s \end{pmatrix} \\ p &= \begin{pmatrix} 1 - (sx^{-2})r & -(sx^{-2})(x + s) \end{pmatrix} \end{aligned}$$

Next we would like to construct an explicit isomorphism between $kG \oplus kG$ and $K \oplus kG$. We can do this using the maps from above. In particular, let

$$\Phi : kG \oplus kG \to K \oplus kG : \vec{m} \mapsto \begin{pmatrix} p(\vec{m}) \\ \pi(\vec{m}) \end{pmatrix}$$

and let

$$\Psi : K \oplus kG \to kG \oplus kG : \begin{pmatrix} f \\ a \end{pmatrix} \mapsto if + \mathscr{s}a.$$

Then,

$$\begin{aligned}
\Psi\Phi\vec{m} &= \Psi \begin{pmatrix} p(\vec{m}) \\ \pi(\vec{m}) \end{pmatrix} \\
&= ip\vec{m} + \mathscr{s}\pi\vec{m} \\
&= i(\, i^{-1}(\vec{m} - \mathscr{s}\pi\vec{m})\,) + \mathscr{s}\pi\vec{m} \\
&= (\vec{m} - \mathscr{s}\pi\vec{m}) + \mathscr{s}\pi\vec{m} \\
&= \vec{m}.
\end{aligned}$$

Thus, $\Psi\Phi = \mathbb{1}_{kG^2}$.

And,

$$\begin{aligned}
\Phi\Psi \begin{pmatrix} f \\ a \end{pmatrix} &= \Phi(if + \mathscr{s}\alpha) \\
&= \begin{pmatrix} p(if + \mathscr{s}\alpha) \\ \pi(if + \mathscr{s}\alpha) \end{pmatrix} \\
&= \begin{pmatrix} pif + p\mathscr{s}\alpha \\ \pi if + \pi\mathscr{s}\alpha \end{pmatrix} \\
&= \begin{pmatrix} (\mathbb{1})f + (0)\alpha \\ (0)f + (\mathbb{1})\alpha \end{pmatrix} \\
&= \begin{pmatrix} f \\ \alpha \end{pmatrix}.
\end{aligned}$$

So, $\Phi\Psi = \mathbb{1}_{K \oplus kG}$.

Proposition 2.10. *Φ is an isomorphism with $\Phi^{-1} = \Psi$.*

We may also express the isomorphisms Φ and Ψ in terms of matrix multiplication, namely

$$
\begin{aligned}
\Phi &= \begin{pmatrix} 1 - (sx^{-2})r & -(sx^{-2})(x+s) \\ r & x+s \end{pmatrix} \\
\Psi &= \begin{pmatrix} i & \mathscr{s} \end{pmatrix} \\
&= \begin{pmatrix} 1 & sx^{-2} \\ -\theta & x^{-1} - rx^{-2} \end{pmatrix}
\end{aligned}
$$

The columns of the matrix of Φ give us a free basis of $K \oplus kG$, namely

$$
\left\{ \begin{pmatrix} 1 - (sx^{-2})r \\ r \end{pmatrix}, \begin{pmatrix} -(sx^{-2})(x+s) \\ x+s \end{pmatrix} \right\} \tag{2.7}
$$

We now show that this free basis does indeed span $K \oplus kG$; that is, the span of basis (2.7) includes the natural generating set of $K \oplus kG$,

$$
\left\{ \begin{pmatrix} 1 - (sx^{-2})r \\ 0 \end{pmatrix}, \begin{pmatrix} -(sx^{-2})(x+s) \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right\}.
$$

Please note that

$$\begin{pmatrix} p(e_1) \\ r \end{pmatrix}(sx^{-2}) + \begin{pmatrix} p(e_2) \\ x+s \end{pmatrix}(x^{-1} - rx^{-2}) \;=\; \begin{pmatrix} p(e_1)(sx^{-2}) + p(e_2)(x^{-1} - rx^{-2}) \\ r(sx^{-2}) + (x+s)(x^{-1} - rx^{-2}) \end{pmatrix} \;=\; \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

by the relation on K (2.3) and by equation (2.2).

Hence, the span of (2.7) contains the element $(0\ \ 1)^T$, which generates the second summand of $K \oplus kG$. So we can clear any element out of the second coordinate that we want. If we do this to the elements of (2.7), then we get generating set (2.6) for K. Therefore, (2.7) is a basis for $K \oplus kG$.

Example 2.11. The fundamental group of the Klein bottle is $G = \langle\, x,\ y \mid x^{-1}yxy \,\rangle$. The relator $x^{-1}yxy$ naturally gives us the construction of the homomorphism σ; that is, $y^\sigma = x^{-1}yx = y^{-1}$. Hence, we may view G as a semidirect product of infinite cyclic groups. So, $G = \mathbb{Z} \rtimes_\sigma \mathbb{Z}$.

Then, using the notation from the beginning of the section, we have $y = y_1$ and $H = \langle y \rangle$. So we want to use the ring $\mathbb{Z}G = \mathbb{Z}\langle y \rangle[\,x, x^{-1};\, \sigma\,]$. Since $y^\sigma = y^{-1}$, we define $r = 1 + y + y^3$. Then, $s = 1 + y^{-1} + y^{-3}$. Thus, the module

$$K = \langle 1 + y + y^3 \rangle \cap \langle x + 1 + y^{-1} + y^{-3} \rangle$$

is a non-free, stably free right ideal over $\mathbb{Z}G$ by Theorem 2.8. As a presentation, we have

$$K = \left\langle\, e_1,\ e_2 \;\middle|\; e_1(1 + y^{-1} + y^{-3}) + e_2(x - 1 - y - y^3) \,\right\rangle$$

by (2.3). Likewise, K is generated by the set

$$\begin{aligned}\{x^2 - sr,\ sx + s^2\} &= \{x^2 - (1 + y^{-1} + y^{-3})(1 + y + y^3),\\ &\qquad x(1 + y + y^3) + (1 + y^{-1} + y^{-3})^2\}\\ &= \{x^2 - y^3 - y^2 - y - 3 - y^{-1} - y^{-2} - y^{-3},\\ &\qquad xy^3 + xy + x + 1 + 2y^{-1} + y^{-2} + 2y^{-3} + 2y^{-4} + y^{-6}\}\end{aligned}$$

according to (2.6). ■

2.2 Presentation of $K \oplus K$

In Section 2.1, we constructed a presentation of K, a list of its generators, and an isomorphism connecting $K \oplus kG$ with $kG \oplus kG$. This provides a general template for stably free modules. Assuming a splitting is known, this process can be applied to any stably free module. What interests us will be higher sums of K. This section deals with constructing presentations for $K \oplus K$ and Section 2.4 will deal with $\bigoplus_\ell K$.

By equation, (2.3) we have a natural presentation of $K \oplus K$, namely

$$K \oplus K = \langle\, e_1,\ e_2,\ e_3,\ e_4 \mid e_1 s + e_2(x - r),\ e_3 s + e_4(x - r)\,\rangle\,.$$

So, $K \oplus K$ can be generated by 4 elements in the same way that K can be generated by 2. Thus,

$$K \oplus K = \left\langle \begin{pmatrix} x^2 - sr^{\sigma^2} \\ 0 \end{pmatrix}, \begin{pmatrix} xr + sr^{\sigma} \\ 0 \end{pmatrix}, \begin{pmatrix} 0 \\ x^2 - sr^{\sigma^2} \end{pmatrix}, \begin{pmatrix} 0 \\ xr + sr^{\sigma} \end{pmatrix} \right\rangle.$$

We know that K is a stably free module. Hence, we see that

$$K^2 \oplus kG = K \oplus (K \oplus kG) \cong K \oplus (kG^2) = (K \oplus kG) \oplus kG \cong kG^3.$$

Thus, $K \oplus K$ is stably free as well.

But since $K \oplus K \oplus kG \cong kG^3$, $K \oplus K$ can be generated by 3 elements. We wish to pursue these generators further. Consider the sequence of maps

$$kG \oplus kG \oplus kG \xrightarrow{\varphi_1} (K \oplus kG) \oplus kG \xrightarrow{\varphi_2} K \oplus (kG \oplus K) \xrightarrow{q} K \oplus K$$

with

$$\varphi_1 : \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \longmapsto \begin{pmatrix} p \begin{pmatrix} m_1 & m_2 \end{pmatrix}^T \\ \pi \begin{pmatrix} m_1 & m_2 \end{pmatrix}^T \\ m_3 \end{pmatrix},$$

$$\varphi_2 : \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \longmapsto \begin{pmatrix} m_1 \\ \pi \begin{pmatrix} m_2 & m_3 \end{pmatrix}^T \\ p \begin{pmatrix} m_2 & m_3 \end{pmatrix}^T \end{pmatrix},$$

and

$$q : \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \longmapsto \begin{pmatrix} m_1 \\ m_3 \end{pmatrix}.$$

Let p_2 be their composition; that is, $p_2 = q\varphi_2\varphi_1$. Hence, p_2 takes the form

$$p_2 : \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \xmapsto{\varphi_1} \begin{pmatrix} p\,(\, m_1 \; m_2 \,)^T \\ \pi\,(\, m_1 \; m_2 \,)^T \\ m_3 \end{pmatrix} \xmapsto{\varphi_2} \begin{pmatrix} p\,(\, m_1 \; m_2 \,)^T \\ \pi\,(\, \Pi_{1,2} \; m_3 \,)^T \\ p\,(\, \Pi_{1,2} \; m_3 \,)^T \end{pmatrix} \xmapsto{q} \begin{pmatrix} p\,(\, m_1 \; m_2 \,)^T \\ p\,(\, \Pi_{1,2} \; m_3 \,)^T \end{pmatrix}$$

with $\Pi_{1,2} = \pi \left(m_1 \;\; m_2\right)^T$.

Now, let

$$0 \longrightarrow \ker p_2 \longrightarrow kG^3 \xrightarrow{p_2} K \oplus K \longrightarrow 0$$

be short exact, and let $(m_1 \quad m_2 \quad m_3)^T \in \ker p_2$. Then, $p(m_1 \quad m_2)^T = 0$ and $p(\Pi_{1,2} \;\; m_3)^T = 0$. This implies that $(m_1 \;\; m_2)^T$, $(\Pi_{1,2} \;\; m_3)^T \in \ker p = \operatorname{im} \mathscr{s}$.

Thus, $\exists \alpha, \beta \in kG$, such that

$$\begin{pmatrix} m_1 \\ m_2 \end{pmatrix} = \begin{pmatrix} (sx^{-2})\alpha \\ (x^{-1} - rx^{-2})\alpha \end{pmatrix} = \mathscr{s}\alpha$$

and

$$\begin{pmatrix} \Pi_{1,2} \\ m_3 \end{pmatrix} = \begin{pmatrix} (sx^{-2})\beta \\ (x^{-1} - rx^{-2})\beta \end{pmatrix} = \mathscr{s}\beta.$$

But $(sx^{-2})\beta = \Pi_{1,2} = \pi \left(m_1 \;\; m_2\right)^T = \pi(\mathscr{s}\alpha) = \alpha$. Thus,

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} = \begin{pmatrix} (sx^{-2})(sx^{-2})\beta \\ (x^{-1} - rx^{-2})(sx^{-2})\beta \\ (x^{-1} - rx^{-2})\beta \end{pmatrix} = \begin{pmatrix} (sr^{\sigma}x^{-4})\beta \\ (rx^{-3} - rr^{\sigma}x^{-4})\beta \\ (x^{-1} - rx^{-2})\beta \end{pmatrix}$$

Now the kernel of p_2 is generated by a single element. Now $K \oplus K \oplus \ker p_2 \cong kG^3$ since $K \oplus K$ is projective. Thus, $\ker p_2$ is also projective. Indeed, it is a cyclic submodule of a free module, and thus is isomorphic to kG as we would expect. Now define the map $\mathscr{s}_2 : kG \to kG^3$ as

$$\partial_2 : \beta \mapsto \begin{pmatrix} (sx^{-2})(sx^{-2})\beta \\ (x^{-1} - rx^{-2})(sx^{-2})\beta \\ (x^{-1} - rx^{-2})\beta \end{pmatrix}$$

Therefore,

$$\begin{aligned} K \oplus K &= \langle\, e_1,\ e_2,\ e_3 \mid \partial_2(1)\,\rangle \\ &= \Big\langle\, e_1,\ e_2,\ e_3 \;\Big|\; e_1(sx^{-2})(sx^{-2}) + e_2(x^{-1} - rx^{-2})(sx^{-2}) + e_3(x^{-1} - rx^{-2})\,\Big\rangle \\ &= \Big\langle\, e_1,\ e_2,\ e_3 \;\Big|\; e_1 sr^{\sigma} + e_2(rx - rr^{\sigma}) + e_3(x^3 - rx^2)\,\Big\rangle\,. \end{aligned} \tag{2.8}$$

Notice the similarity with presentation (2.3) on page 12.

The map p_2 provides us some more information than just a new presentation. Like we did with K, the projection p_2 can give us the 3 generators of $K \oplus K$. Let e_1, e_2, and e_3 be the natural basis of kG^3. Then, $K \oplus K$ is generated by $p_2(e_1), p_2(e_2)$, and $p_2(e_3)$, but in more detail we have

$$\begin{aligned} p_2(e_1) &= \begin{pmatrix} p\begin{pmatrix} 1 \\ 0 \end{pmatrix} & p\begin{pmatrix} \pi\begin{pmatrix} 1 & 0 \end{pmatrix}^T \\ 0 \end{pmatrix} \end{pmatrix}^T \\ &= \begin{pmatrix} p\,(e_1) & p\,(re_1) \end{pmatrix}^T = \begin{pmatrix} p\,(e_1) & p\,(e_1)\,r \end{pmatrix}^T \\ &= \begin{pmatrix} 1 - (sx^{-2})r \\ (1 - (sx^{-2})r)r \end{pmatrix} \\ &= \begin{pmatrix} x^2 - sr^{\sigma^2} \\ rx^2 - s(r^{\sigma^2})^2 \end{pmatrix} x^{-2} \end{aligned} \tag{2.9}$$

$$
\begin{aligned}
p_2(e_2) &= \left(p\begin{pmatrix} 0 \\ 1 \end{pmatrix} \quad p\begin{pmatrix} \pi\begin{pmatrix} 0 & 1 \end{pmatrix}^T \\ 0 \end{pmatrix} \right)^T \\
&= \begin{pmatrix} p\,(e_2) & p\,((x+s)e_1) \end{pmatrix}^T = \begin{pmatrix} p\,(e_2) & p\,(e_1)\,(x+s) \end{pmatrix}^T \\
&= \begin{pmatrix} -(sx^{-2})(x+s) \\ (1-(sx^{-2})r)(x+s) \end{pmatrix} \qquad (2.10) \\
&= \begin{pmatrix} -sx - sr^{\sigma} \\ x^3 + sx^2 - sr^{\sigma^2}x - sr^{\sigma}r^{\sigma^2} \end{pmatrix} x^{-2}
\end{aligned}
$$

$$
\begin{aligned}
p_2(e_3) &= \left(p\begin{pmatrix} 0 \\ 0 \end{pmatrix} \quad p\begin{pmatrix} \pi\begin{pmatrix} 0 & 0 \end{pmatrix}^T \\ 1 \end{pmatrix} \right)^T \\
&= \begin{pmatrix} 0 & p\,(e_2) \end{pmatrix}^T \\
&= \begin{pmatrix} 0 \\ -(sx^{-2})(x+s) \end{pmatrix} \qquad (2.11) \\
&= \begin{pmatrix} 0 \\ sx + sr^{\sigma} \end{pmatrix} x^{-2}
\end{aligned}
$$

So,

$$
K \oplus K = \left\langle \begin{pmatrix} 1-(sx^{-2})r \\ (1-(sx^{-2})r)r \end{pmatrix}, \begin{pmatrix} -(sx^{-2})(x+s) \\ (1-(sx^{-2})r)(x+s) \end{pmatrix}, \begin{pmatrix} 0 \\ -(sx^{-2})(x+s) \end{pmatrix} \right\rangle
$$

$$= \left\langle \begin{pmatrix} x^2 - sr^{\sigma^2} \\ rx^2 - s(r^{\sigma^2})^2 \end{pmatrix}, \begin{pmatrix} -sx - sr^{\sigma} \\ x^3 + sx^2 - sr^{\sigma^2}x - sr^{\sigma}r^{\sigma^2} \end{pmatrix}, \begin{pmatrix} 0 \\ sx + sr^{\sigma} \end{pmatrix} \right\rangle \tag{2.12}$$

Please note that

$$\begin{pmatrix} p(e_1) \\ p(e_1)r \end{pmatrix} sr^{\sigma} + \begin{pmatrix} p(e_2) \\ p(e_1)(x+s) \end{pmatrix} (x-r)r^{\sigma} + \begin{pmatrix} 0 \\ p(e_2) \end{pmatrix} (x-r)x^2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

but the relation does not hold for the other listed generating set of $K \oplus K$.

We would like now to construct an isomorphism between kG^3 and $K^2 \oplus kG$. The bulk of the work has already been done. Recall the maps φ_1 and φ_2 from the beginning of the section. Their composite with also the map q gave us the projection p_2. Not surprisingly, the composition of φ_1 with φ_2 gives us the isomorphism. More specifically, let

$$\Phi_2 : kG^3 \longrightarrow K^2 \oplus kG,$$

such that $\Phi_2 = \varphi_2\varphi_1$; that is,

$$\Phi_2 : \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \longmapsto \begin{pmatrix} p\,(m_1 \;\; m_2)^T \\ \pi\,(\Pi_{1,2} \;\; m_3)^T \\ p\,(\Pi_{1,2} \;\; m_3)^T \end{pmatrix}$$

with $\Pi_{1,2} = \pi\,(m_1 \;\; m_2)^T$.

We can represent the maps $\varphi_1, \varphi_2,$ and q in matrix form. This will give us immediately a matrix representation of p_2, s_2, and Φ_2. First, recognize that

$$\varphi_1 = \begin{pmatrix} (1-(sx^{-2})r) & -(sx^{-2})(x+s) & 0 \\ r & x+s & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$\varphi_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & r & x+s \\ 0 & (1-(sx^{-2})r) & -(sx^{-2})(x+s) \end{pmatrix}$$

and

$$q = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Therefore,

$$\Phi_2 = \varphi_2\varphi_1 = \begin{pmatrix} 1-(sx^{-2})r & -(sx^{-2})(x+s) & 0 \\ rr & r(x+s) & x+s \\ (1-(sx^{-2})r)r & (1-(sx^{-2})r)(x+s) & -(sx^{-2})(x+s) \end{pmatrix},$$

$$p_2 = q\Phi_2 = \begin{pmatrix} 1-(sx^{-2})r & -(sx^{-2})(x+s) & 0 \\ (1-(sx^{-2})r)r & (1-(sx^{-2})r)(x+s) & -(sx^{-2})(x+s) \end{pmatrix},$$

and

$$\mathscr{s}_2 = \begin{pmatrix} (sx^{-2})(sx^{-2}) \\ (x^{-1}-rx^{-2})(sx^{-2}) \\ (x^{-1}-rx^{-2}) \end{pmatrix}.$$

The columns of the matrix of Φ_2 gives us a free basis of $K^2 \oplus kG$, namely

$$\left\{ \begin{pmatrix} p(e_1) \\ p(e_1)r \\ rr \end{pmatrix}, \begin{pmatrix} p(e_2) \\ p(e_1)(x+s) \\ r(x+s) \end{pmatrix}, \begin{pmatrix} 0 \\ p(e_2) \\ x+s \end{pmatrix} \right\}. \tag{2.13}$$

In the same fashion as the end of Section 2.1, we can construct coefficients for this basis that show the natural generating set for $K^2 \oplus kG$ is included in its span. As all the necessary steps were detailed in the end of Section 2.1, we need not include its generalization here and leave to the reader the privilege of checking this claim.

We can show that the new spanning set resulting from p_2 does in fact generate the previous one. Since the columns of p_2 correspond to the generating set (2.12), it suffices to show that the generating set on page 17 is contained in the image of p_2. Specifically, the following equations hold:

$$\begin{pmatrix} p(e_1) & p(e_2) & 0 \\ p(e_1)r & p(e_1)(x+s) & p(e_2) \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -(sx^{-2})(x+s) \end{pmatrix},$$

$$\begin{pmatrix} p(e_1) & p(e_2) & 0 \\ p(e_1)r & p(e_1)(x+s) & p(e_2) \end{pmatrix} \begin{pmatrix} sx^{-2} \\ x^{-1} - rx^{-2} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 1-(sx^{-2})r \end{pmatrix},$$

$$\begin{pmatrix} p(e_1) & p(e_2) & 0 \\ p(e_1)r & p(e_1)(x+s) & p(e_2) \end{pmatrix} \begin{pmatrix} p(e_1) \\ -\theta p(e_1) \\ 0 \end{pmatrix} = \begin{pmatrix} 1-(sx^{-2})r \\ 0 \end{pmatrix},$$

$$\begin{pmatrix} p(e_1) & p(e_2) & 0 \\ p(e_1)r & p(e_1)(x+s) & p(e_2) \end{pmatrix} \begin{pmatrix} p(e_2) \\ -\theta p(e_2) \\ 0 \end{pmatrix} = \begin{pmatrix} -(sx^{-2})(x+s) \\ 0 \end{pmatrix}.$$

The verifications of these equations are trivial and left for the reader.

Before we end this section, we would like to mention that though this section is analogous to Section 2.1 with respect to $K \oplus K$, this process is not unique. At one point, we had to make a choice which, even though trivial, changed the results we got. Had we chosen differently, our results would differ accordingly. Let us be more specific.

There was a choice in the map when constructing p_2. Remember that

$$p_2 \left(m_1 \;\; m_2 \;\; m_3\right)^T = \begin{pmatrix} p\left(m_1 \;\; m_2\right)^T \\ p\left(\Pi_{1,2} \;\; m_3\right)^T \end{pmatrix},$$

with $\Pi_{1,2} = \pi(m_1 \;\; m_2)^T$. There are several natural ways to define p_2 alternatively. Excluding simple permutations of coordinates, only one other alternative remains. Our placing $\Pi_{1,2}$ in the first coordinate of p was an arbitrary choice. By no means is p a symmetric function. Hence, the map $\widetilde{p}_2 : kG^3 \to K \oplus K$ defined as

$$\widetilde{p}_2 \left(m_1 \;\; m_2 \;\; m_3\right)^T = \begin{pmatrix} p\left(m_1 \;\; m_2\right)^T \\ p\left(m_3 \;\; \Pi_{1,2}\right)^T \end{pmatrix}$$

is distinct from p_2. In terms of matrix multiplication, this turns out to mean that $p_2 = q\varphi_2\varphi_1$ and $\widetilde{p}_2 = q\varphi_1\varphi_2$. Thus, we can define another isomorphism $\widetilde{\Phi}_2 : kG^3 \to K \oplus K$, such that $\widetilde{\Phi}_2 = \varphi_1\varphi_2$; that is,

$$\widetilde{\Phi}_2 : \begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} \longmapsto \begin{pmatrix} p\left(m_1 \;\; m_2\right)^T \\ \pi\left(m_3 \;\; \Pi_{1,2}\right)^T \\ p\left(m_3 \;\; \Pi_{1,2}\right)^T \end{pmatrix}$$

and another inclusion map $\widetilde{\jmath}_2 : kG \to kG^3$.

In matrix form, we see that

$$\begin{aligned}\widetilde{\Phi}_2 &= \varphi_1\varphi_2 \\ &= \begin{pmatrix} p(e_1) & p(e_2)r & p(e_2)(x+s) \\ r & (x+s)r & (x+s)(x+s) \\ 0 & p(e_1) & p(e_2) \end{pmatrix},\end{aligned}$$

$$\begin{aligned}\widetilde{p}_2 &= q\widetilde{\Phi}_2 \\ &= \begin{pmatrix} p(e_1) & p(e_2)r & p(e_2)(x+s) \\ 0 & p(e_1) & p(e_2) \end{pmatrix},\end{aligned}$$

and

$$\widetilde{\jmath}_2 = \begin{pmatrix} (sx^{-2}) \\ (sx^{-2})(x^{-1} - rx^{-2}) \\ (x^{-1} - rx^{-2})^2 \end{pmatrix}.$$

The map $\widetilde{\jmath}_2$ gives us a new presentation of $K \oplus K$. Therefore,

$$\begin{aligned} K \oplus K &= \langle\, e_1,\ e_2,\ e_3 \mid \widetilde{\jmath}_2(1)\,\rangle \qquad (2.14)\\ &= \left\langle\, e_1,\ e_2,\ e_3 \;\middle|\; e_1(sx^{-2}) + e_2(sx^{-2})(x^{-1} - rx^{-2}) + e_3(x^{-1} - rx^{-2})^2 \,\right\rangle \\ &= \left\langle\, e_1,\ e_2,\ e_3 \;\middle|\; e_1(sx^2) + e_2(sx - sr^{\sigma^2}) + e_3(x^2 - (r + r^{\sigma})x + rr^{\sigma^2}) \,\right\rangle \end{aligned}$$

And lastly, the columns vectors of $\widetilde{p}_2$ give us a new generating set for $K \oplus K$, namely

$$K \oplus K = \left\langle \begin{pmatrix} 1-(sx^{-2})r \\ 0 \end{pmatrix}, \begin{pmatrix} -(sx^{-2})(x+s)r \\ 1-(sx^{-2})r \end{pmatrix}, \begin{pmatrix} -(sx^{-2})(x+s)(x+s) \\ -(sx^{-2})(x+s) \end{pmatrix} \right\rangle \tag{2.15}$$

2.3 The Map $\pi_2 : kG^3 \longrightarrow kG$

Using the definitions of the maps defined in Section 2.2, we see the following sequence of kG-modules:

$$0 \longrightarrow kG \xrightarrow{\mathscr{s}_2} kG^3 \xrightarrow{p_2} K \oplus K \longrightarrow 0. \tag{2.16}$$

Exactness of this sequence is immediate from Section 2.2.

We take a step back into a more general setting. Let R be a ring and let $\phi : R^n \to R^m$ be an epimorphism. Then, $\ker \phi$ is a stably free R-module. In fact, every stably free R-module is the kernel of such a map for appropriate n and m. We already saw, in Section 2.1, the epimorphism $\pi : kG^2 \to kG$ has $\ker \pi \cong K$.

In Section 2.2, we showed that $K \oplus K$ is stably free. Then, there must exist a map $\pi_2 : kG^3 \to kG$, such that $\ker \pi_2 \cong K \oplus K$. Also, it must be true that $\pi_2 \mathscr{s}_2 = \mathbb{1}_{kG}$, or in other words, $\mathscr{s}_2$ is a splitting of π_2.

We already have a potential candidate. We have an isomorphism from $\Phi_2 : kG^3 \to K^2 \oplus kG$. If we compose this isomorphism with the canonical projection onto its free summand, then we have a map $\pi_2 : kG^3 \to kG$ such that

$$\pi_2 \left(m_1 \;\; m_2 \;\; m_3\right)^T = r^2 m_1 + r(x+s)m_2 + (x+s)m_3,$$

or as a matrix,

$$\pi_2 = \begin{pmatrix} r^2 & r(x+s) & x+s \end{pmatrix}.$$

Next we need to check if the sequence splits.

$$
\begin{aligned}
\pi_2 \mathscr{s}_2(1) &= \pi_2 \begin{pmatrix} (sx^{-2})(sx^{-2}) \\ (x^{-1} - rx^{-2})(sx^{-2}) \\ x^{-1} - rx^{-2} \end{pmatrix} \\
&= r^2(sx^{-2})(sx^{-2}) + r(x+s)(x^{-1} - rx^{-2})(sx^{-2}) + (x+s)(x^{-1} - rx^{-2}) \\
&= r[r(sx^{-2}) + (x+s)(x^{-1} - rx^{-2})](sx^{-2}) + (x+s)(x^{-1} - rx^{-2}) \\
&= r(sx^{-2}) + (x+s)(x^{-1} - rx^{-2}) \\
&= 1
\end{aligned}
$$

Let $\begin{pmatrix} f_1 & f_2 \end{pmatrix}^T \in K^2$. Then, we may construct the map $i_2 : K^2 \to kG^3$ as

$$
\begin{aligned}
i_2 \begin{pmatrix} f_1 \\ f_2 \end{pmatrix} &= \begin{pmatrix} i f_1 \\ 0 \end{pmatrix} + \begin{pmatrix} \mathscr{s} f_2 \\ -\theta f_2 \end{pmatrix} \\
&= \begin{pmatrix} f_1 \\ -\theta f_1 \\ 0 \end{pmatrix} + \begin{pmatrix} (sx^{-2}) f_2 \\ (x^{-1} - rx^{-2}) f_2 \\ -\theta f_2 \end{pmatrix}
\end{aligned}
$$

Theorem 2.12. *The diagram*

$$
\begin{tikzcd}
0 \arrow[r, leftrightarrow] & K^2 \arrow[r, bend left, "i_2"] & kG^3 \arrow[l, bend left, "p_2"] \arrow[r, bend left, "\pi_2"] & kG \arrow[l, bend left, "\mathscr{s}_2"] \arrow[r, leftrightarrow] & 0
\end{tikzcd}
$$

is split exact in both directions. In particular, $K \oplus K \cong \ker \pi_2$.

Proof. We have already shown everything we need except $\operatorname{im} i_2 = \ker \pi_2$ and $\ker i_2 = 0$. Since i, $\mathscr{s}$, and θ are all injective, i_2 is injective too.

We need only show that $i_2 = (1 - \mathscr{s}_2\pi_2)p_2^{-1}$. The map on the right-hand side of the equation is easily seen to be the appropriate map that completes the diagram above (and is well-defined since the sequence is split exact). This implies $\operatorname{im} i_2 = \ker \pi_2$. Thus, it suffices to show that the two maps agree on a generating set of $K \oplus K$, namely $\{p_2(e_1),\ p_2(e_2),\ p_2(e_3)\}$.

The calculations turn out to be routine, yet tedious. Hence, we leave the computation for the reader to verify. The evaluations will indeed prove that i_2 and $(1 - \mathscr{s}_2\pi_2)p_2^{-1}$ agree on a generating set. Since they are both homomorphisms, $i_2 = (1 - \mathscr{s}_2\pi_2)p_2^{-1}$ on all of $K \oplus K$. $\square$

In an analogous fashion to Section 2.1, we can construct the inverse isomorphism Ψ_2 of Φ_2 using the maps i_2 and $\mathscr{s}_2$, namely

$$\begin{aligned}
\Psi_2 &= \begin{pmatrix} i_2 & \mathscr{s}_2 \end{pmatrix} \\
&= \begin{pmatrix} 1 & sx^{-2} & (sx^{-2})(sx^{-2}) \\ -\theta & (x^{-1} - rx^{-2}) & (x^{-1} - rx^{-2})(sx^{-2}) \\ 0 & -\theta & x^{-1} - rx^{-2} \end{pmatrix}.
\end{aligned}$$

This map is the natural inverse of

$$\begin{aligned}
\Phi_2 &= \begin{pmatrix} p_2 & \pi_2 \end{pmatrix} \\
&= \begin{pmatrix} p(e_1) & p(e_2) & 0 \\ p(e_1)r & p(e_1)(x+s) & p(e_2) \\ rr & r(x+s) & x+s \end{pmatrix}.
\end{aligned}$$

Instead of using the isomorphism Φ_2, we could construct $\widetilde{\pi}_2$ from the isomorphism

$\widetilde{\Phi}_2$. So, $\widetilde{\pi}_2 : kG^3 \to kG$ is the matrix

$$\widetilde{\pi}_2 = \begin{pmatrix} r & (x+s)r & (x+s)^2 \end{pmatrix}.$$

And therefore, $K \oplus K \cong \ker \widetilde{\pi}_2$ by the same argument as Theorem 2.12.

2.4 Presentation of $\bigoplus K$

In Section 2.2, we started off by showing that $K \oplus K$ was stably free. In general, we see this holds for $\bigoplus_\ell K$.

Proposition 2.13. *Let P be a stably free R-module, such that $P \oplus R^m \cong R^n$. Then, $\bigoplus_\ell P$ is stably free and $P^\ell \oplus R^m \cong R^{\ell(n-m)+m}$.*

Proof. We will prove this fact by induction. For the base step, $P^1 \oplus R^m \cong R^n = R^{1(n-m)+m}$.

For the inductive hypothesis, let $P^i \oplus R^m \cong R^{i(n-m)+m}$ for all $i \leqslant \ell$. Then,

$$\begin{aligned}
P^{\ell+1} \oplus R^m &= (P \oplus P^\ell) \oplus R^m \\
&= P \oplus (P^\ell \oplus R^m) \\
&\cong P \oplus (R^{\ell(n-m)+m}) \\
&= (P \oplus R^m) \oplus R^{\ell(n-m)} \\
&\cong (R^n) \oplus R^{\ell(n-m)} \\
&= R^{\ell(n-m)+n} \\
&= R^{(\ell+1)n-\ell m+(m-m)} \\
&= R^{(\ell+1)n-(\ell+1)m+m} \\
&= R^{(\ell+1)(n-m)+m}
\end{aligned}$$

Hence, $\bigoplus_\ell P$ is stably free. $\square$

This proposition implies that $\bigoplus_\ell K$ is a stably free module. There is another immediate and useful corollary of Proposition 2.13 that applies to a more general setting.

Corollary 2.14. *Let P and Q be stably free R-modules. Then, $P \oplus Q$ is stably free. In general, if (P_i) is a finite list of stably free R-modules, then $\bigoplus_i P_i$ is stably free.*

Proof. Now, there exists positive integers m_1, m_2, n_1, and n_2, such that $P \oplus R^{m_1} \cong R^{n_1}$ and $Q \oplus R^{m_2} \cong R^{n_2}$. Then, a quick modification of the proof of Prop. 2.13 says that $P \oplus Q \oplus R^{max\{m_1,m_2\}}$ is free. The stable freedom of $\bigoplus_i P_i$ is seen from an obvious induction argument. $\square$

We now will present a method that constructs an isomorphism between kG^{n+1} and $K^n \oplus kG$, call it Φ_n. We have seen from Sections 2.1 and 2.2 that an isomorphism in matrix form is sufficient to find a generating set and presentation of K^n. That is, once we have found a matrix representation of Φ_n, we know that the column vectors form a free basis of $K^n \oplus kG$. After we kill off the free summand of $K^n \oplus kG$, call this new matrix p_n, the column vectors give us a generating set for K^n. Therefore, K^n can always be generated by $n+1$ many elements. And the kernel of p_n will be generated by a single element (since kG is cyclic and the kernel is isomorphic to kG). Thus, the generator of $\ker p_n$ will be the single relator of the presentation of K^n. Also, as we saw in Section 2.3, we can construct a projection $\pi_n : kG^{n+1} \to kG$ by composing Φ_n with the projection onto the single free summand of the image. Then, $K^n \cong \ker \pi_n$.

Let the $(n+1) \times (n+1)$ matrix $\varphi_{(n,m)}$ be defined as

$$\varphi_{(n,m)} = \begin{pmatrix} 1 & \cdots & 0 & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & 0 & 0 & \cdots & 0 \\ 0 & \cdots & 0 & p(e_1) & p(e_2) & 0 & \cdots & 0 \\ 0 & \cdots & 0 & r & x+s & 0 & \cdots & 0 \\ 0 & \cdots & 0 & 0 & 0 & 1 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & 0 & 0 & 0 & \cdots & 1 \end{pmatrix}$$

where $p(e_1)$ appears in the coordinate (m,m). Let $\mathbb{1}_n$ be the $n \times n$ identity matrix. Then, we can abbreviate the above matrix as

$$\varphi_{(n,m)} = \begin{pmatrix} \mathbb{1}_{m-1} & 0 & 0 \\ 0 & \Phi & 0 \\ 0 & 0 & \mathbb{1}_{n-(m+1)} \end{pmatrix},$$

where 0 represents the appropriate rectangular zero matrix. Thus, in summary, the map $\varphi_{(n,m)}$ applies the isomorphism Φ to the m-th and $(m+1)$-th coordinates and leaves all other coordinates fixed.

We then define the map $\Phi_n : kG^{n+1} \to K^n \oplus kG$ to be

$$\Phi_n = \varphi_{(n,n-1)} \circ \ldots \circ \varphi_{(n,1)}. \tag{2.17}$$

Let $p_n : kG^{n+1} \to K^n$ be the map Φ_n composed with the projection onto K^n, and let $\pi_n : kG^{n+1} \to kG$ be the map Φ_n composed with the projection onto kG.

Theorem 2.15. *The map $\Phi_n : kG^{n+1} \to K^n \oplus kG$ is an isomorphism.*

Proof. Now, $\varphi_{(n,i)}$ is an isomorphism for $1 \leqslant i < n$ by induction on n. Now, Φ_n is a composition of isomorphisms. □

Corollary 2.16.

1. *The column vectors of Φ_n form a basis of $K^n \oplus kG$.*

2. *p_n is a epimorphism onto K^n, and the column vectors of p_n form a generating set of K^n.*

3. *$\ker p_n$ is cyclic.*

4. *K^n is an $(n+1)$-generator, 1-relator module.*

5. *$K^n \cong \ker \pi_n$.*

Proof.

1. This is immediate since Φ_n is an isomorphism.

2. The composition of an isomorphism with a surjection is certainly an epimorphism. The columns of p_n generate its image, and it is surjective.

3. Φ_n is an isomorphism, so $\ker \Phi_n = 0$. The kernel of the projection is kG. Thus, the kernel of their composition is isomorphic is kG.

4. p_n is an epimorphism from a free module kG^{n+1}. Thus, K^n is generated by $n+1$ elements. $\ker p_n$ is cyclic, so call its generator $\mathscr{s}_n$. Therefore, a presentation of K^n looks like
$$K^n = \langle\, e_1,\ e_2,\ \ldots,\ e_{n+1} \ \mid\ \mathscr{s}_n(1)\,\rangle \tag{2.18}$$

5. $\mathscr{s}_n$ is a splitting of π_n. Therefore, there exists a map i_n, such that the diagram

$$\begin{tikzcd} 0 \arrow[r, leftrightarrow] & K^n \arrow[r, bend left, "i_n"] & kG^{n+1} \arrow[l, bend left, "p_n"] \arrow[r, bend left, "\pi_n"] & kG \arrow[l, bend left, "\mathscr{s}_n"] \arrow[r, leftrightarrow] & 0 \end{tikzcd}$$

is short exact in both directions. Therefore, $K^n \cong \operatorname{im} i_n = \ker \pi_n$. □

Before we close the section, we mention that neither the maps Φ_2 or $\widetilde{\Phi}_2$ were constructed in the same fashion as (2.17). Please refer to pages 23 and 26 for a reminder. Notice that the first matrix in the product for Φ_2 has permuted the rows of $\varphi_{(2,2)}$. We ask the reader to recalculate Φ_2 using definition (2.17) and check that the two matrices are the same up to permutation. Thus, the slightly different constructions actually give the same presentation and generating set for $K \oplus K$.

An advantage of the way we constructed Φ_2 in Section 2.2 is that the reversal of the matrices created an alternative isomorphism $\widetilde{\Phi}_2$. The permutation $\varphi_{(2,1)}\varphi_{(2,2)}$ only is a map onto $K^2 \oplus kG$ if and only if $rK \oplus (x+s)K = K$. Hence, if we first permute rows or columns of $\varphi_{(n,j)}$, then permutations of the $\varphi_{(n,i)}$ will create different isomorphisms $kG^{n+1} \to K^n \oplus kG$. We illustrate this in the next example.

Example 2.17. Consider the case $n = 3$. Then, the matrices $\varphi_{(3,i)}$ are as follows:

$$\varphi_{(3,1)} = \begin{pmatrix} p(e_1) & p(e_2) & 0 & 0 \\ r & x+s & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\varphi_{(3,2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p(e_1) & p(e_2) & 0 \\ 0 & r & x+s & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

$$\varphi_{(3,3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & p(e_1) & p(e_2) \\ 0 & 0 & r & x+s \end{pmatrix}$$

And then,

$$\Phi_3 = \varphi_{(3,3)} \circ \varphi_{(3,2)} \circ \varphi_{(3,1)} = \begin{pmatrix} p(e_1) & p(e_2) & 0 & 0 \\ p(e_1)r & p(e_1)(x+s) & p(e_2) & 0 \\ p(e_1)r^2 & p(e_1)r(x+s) & p(e_1)(x+s) & p(e_2) \\ r^3 & r^2(x+s) & r(x+s) & (x+s) \end{pmatrix}.$$

For another isomorphism, let

$$\varphi^{'}_{(3,2)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & p(e_1) & p(e_2) & 0 \\ 0 & 0 & 0 & 1 \\ 0 & r & x+s & 0 \end{pmatrix}$$

$$\varphi'_{(3,3)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & r & x+s \\ 0 & 0 & p(e_1) & p(e_2) \end{pmatrix}$$

Then we define

$$\Phi'_3 = \varphi'_{(3,2)} \circ \varphi'_{(3,3)} \circ \varphi_{(3,1)} = \begin{pmatrix} p(e_1) & p(e_2) & 0 & 0 \\ p(e_1)r & p(e_1)(x+s) & p(e_2)r & p(e_2)(x+s) \\ 0 & 0 & p(e_1) & p(e_2) \\ r^2 & r(x+s) & (x+s)r & (x+s)^2 \end{pmatrix} \blacksquare$$

2.5 Generalization to Poly-(Infinite) Cyclic Groups

Back in Section 2.1, we defined the group G as $G = H \rtimes \mathbb{Z}$ with H a finitely generated, free Abelian group. Hence, G is poly-(infinite) cyclic. We plan now to extend the domain of G to any non-Abelian, poly-(infinite) cyclic group.

Since G is poly-(infinite) cyclic, there exists a group series of the form

$$G_0 \triangleleft G_1 \triangleleft \ldots \triangleleft G_n$$

with $G_n = G$, G_0 is trivial, and $G_i/G_{i-1} \cong \mathbb{Z}$ for each $0 < i \leqslant n$. Since G is non-Abelian, there exists a minimal $j \in (0, n]$, such that G_j is non-Abelian. In particular, G_{j-1} is free Abelian. Let the generators of G_{j-1} be $y_1, y_2, \ldots y_{j-1}$ and let $x \in G_j$, such that $x + G_{j-1}$ generates G_j/G_{j-1}. Therefore, the sequence

$$0 \longrightarrow G_{j-1} \longrightarrow G_j \longrightarrow \langle x \rangle \longrightarrow 0$$

is short exact. Since $\langle x \rangle \cong \mathbb{Z}$, which is a free group, the sequence is split and $G_j \cong G_{j-1} \rtimes \mathbb{Z}$. Therefore, K is a non-free, stably free, right ideal of kG_j.

We next assume for the sake of induction that kG_i has a non-free, stably free, right ideal P_i for $i < \ell$. Consider the ring kG_ℓ. It can naturally be viewed as a left $kG_{\ell-1}$-module. In particular, kG_ℓ is a flat $kG_{\ell-1}$-module. Thus, the ideal $P_i(kG_\ell) \cong P_i \otimes_{kG_{\ell-1}} kG_\ell$, and this second module is stably free. The module is non-free by [17, Prop. 2.3]. Therefore, by induction, kG has a non-free, stably free, right ideal.

The above induction argument is due to [17, Theorem 2.12] without much deviation. Its inclusion here is meant only to give us the tools we need to write a presentation for this constructed module. Let us start by examining the module one step above the base case. We know that K is a non-free, stably free, right ideal for the ring kG_j, where j was minimal. By the above proof, $K(kG_{j+1})$ is a non-free, stably free, right ideal for kG_{j+1}. We next compare the definitions of these two ideals. By (2.1),

$$K = \{\, f \in S = kG_j \;\mid\; rf = (x+s)g, \;\; g \in S \,\}$$

and

$$K \otimes_{kG_j} kG_{j+1} = \{\, f \in S = kG_{j+1} \;\mid\; rf = (x+s)g, \;\; g \in S \,\}\,.$$

We notice that the only difference in the definitions of these two sets is the change of the ring S. If we allow S to change as we change rings by the functor $\otimes_{kG_i} kG_{i+1}$, then we may also call the resulting right ideal K. Therefore, K is a non-free, stably free, right ideal of kG, where G is a poly-(infinite) cyclic group.

The change of rings does not change the presentation of K or its generators either. We close this section with a theorem that summarizes the above statements.

Theorem 2.18. *Let G be a non-Abelian, poly-(infinite) cyclic group and let k be a commutative Noetherian domain. Let S be the group ring $S = kG$. Then, the right ideal $K = \{\, f \in S \;\mid\; rf = (x+s)g,\;\; g \in S \,\}$ is a non-free, stably free S-module with presentation $K = \langle\, e_1, e_2 \mid e_1 s + e_2(x - r) \,\rangle$ and generated by the set $\left\{ x^2 - sr^{\sigma^2},\; sx + sr^{\sigma} \right\}$. Furthermore, Theorem 2.15 and its corollary extend to include the group ring S.*

CHAPTER 3

FREE SUMMANDS OF $K \oplus K$

3.1 Properties of Stably Free Modules

We saw from Proposition 2.13 and Corollary 2.14 that the direct sums of stably free modules are stably free. Another question to ask is whether the direct sum of stably free modules is free. Obviously, if the sum contains enough free summands, then the sum will be free. But how about $K \oplus K$? We saw that this module is stably free but is it free? If $K \oplus K$ is not free, then how about K^3? In particular, does there exists a natural number n, such that K^n is free but for all $i < n$, K^i is not free? The following two examples show that either case can occur.

Example 3.1. In a paper by Evans [7], he constructs a ring R and a sequence of modules $\{M_n\}$ with the properties that M_n is stably free for each n, M_n has rank n, and M_n has no free direct summands. This last property shows that M_n is not free. Thus, in his ring R, there exist finitely generated, non-free, stably free modules of arbitrary rank. ∎

Example 3.2. On the other hand, a paper by Brown, Lenagan, and Stafford [3] shows an opposite result. Let G be the fundamental group of the Klein bottle. Then, by application of Brown, Lenagan, and Stafford's Theorems 4.9, 4.10, and 4.11,[1] it can be shown that any stably free $\mathbb{Z}G$-module with rank larger than or equal to 4 is necessarily free. ∎

Thus, for the integral group ring over the Klein bottle, we have an explicit bound on the rank of non-free, stably free modules. For $\mathbb{Z}G$, we know that K^4 is free from above. What about K^2 or K^3? The conclusion of this chapter will be to show that for the Klein bottle, K^3 is free, but like the previous chapter, we prefer to work in a more general setting and mention the Klein bottle as a corollary. We take this portion now to develop a few criteria that would guarantee the possible freedom of $K \oplus K$.

Now, if $K \oplus K$ is free, there exists some positive integer ℓ, such that $K \oplus K \cong kG^\ell$. We claim that $\ell = 2$, for $kG^3 \cong K \oplus K \oplus kG \cong (kG^\ell) \oplus kG = kG^{\ell+1}$. Then, by the IBN, $\ell + 1 = 3$. A simple generalization of this argument tells us that for a ring R satisfying the IBN, if a stably free R-module P is free, then the notions of rank for stably free modules and rank for free modules coincide.

In particular, if $K \oplus K$ is free, we see that $K \oplus K \cong K \oplus kG$. Thus, to show the freedom of $K \oplus K$, it suffices to prove the last isomorphism. We actually will consider a slightly more general isomorphism. Namely, is it possible that there exists a module L, such that $K \oplus K \cong L \oplus kG$, and if so, what does this tell us about the freedom of $K \oplus K$?

Throughout the rest of this section, we assume R to be a right Noetherian ring. In addition, we assume all R-modules to be right modules unless otherwise stated.[2]

---

[1] Brown, Lenagan, and Stafford don't work with the Klein bottle but instead with polycyclic-by-finite group rings. The Klein bottle is just a special case.

[2] We assume that R is right Noetherian simply because in terms of free modules, R is well behaved. That is, every right Noetherian ring is stably finite [12, p.7], which means that if $N \oplus R^n \cong R^n$, then $N = 0$ for any n. This assumption will be used throughout the work but usually in the background

Proposition 3.3. *Let P be a stably free R-module. If P has a free summand, that is, $P \cong Q \oplus R$ for some R-module Q, then Q is stably free and there exists an integer ℓ, such that $\bigoplus_\ell P$ is free.*

Proof. Since P is stably free, there exist $m, n \in \mathbb{Z}$, such that $P \oplus R^m \cong R^n$. Therefore, $Q \oplus R^{m+1} \cong (Q \oplus R) \oplus R^m \cong (P) \oplus R^m \cong R^n$. Thus, Q is stably free.

Now we shall prove the second statement,

$$\begin{aligned} P^{m+1} &\cong (Q \oplus R)^{m+1} \\ &= Q^{m+1} \oplus R^{m+1} \\ &\cong R^{(m+1)(n-(m+1))+(m+1)} && \text{by Prop. 2.13} \\ &= R^{(m+1)(n-(m+1)+1)} \\ &= R^{(m+1)(n-m)} \quad \square \end{aligned} \tag{3.1}$$

Example 3.4. Since k is a commutative Noetherian ring, kG is a Noetherian ring [14, Corollary 10.2.8]. According to equation (3.1) in Proposition 3.3, if $K \oplus K \cong L \oplus kG$ for some kG-module L, then $(K \oplus K)^2 = K^4$ is free. The details here are not difficult to work out. If $L \oplus kG^2 \cong kG^3$, then $(K \oplus K) \oplus (K \oplus K) \cong (L \oplus kG) \oplus (L \oplus kG) = L \oplus (L \oplus kG \oplus kG) \cong L \oplus (kG \oplus kG \oplus kG) \cong kG^4$. Thus, K^4 is free given the assumption.

We can do better than the last statement. $K \oplus K \oplus K \cong K \oplus (L \oplus kG) = L \oplus (K \oplus kG) \cong L \oplus (kG \oplus kG) \cong kG^3$. Thus, $K \oplus K \oplus K$ would be free if the

---

without mention. Stable finiteness also implies the Invariant Basis Number. Proofs of all these statements can be found in Lam [12, §1].

assumption holds. Thus, the integer mentioned in the proof of Proposition 3.1 might not necessarily be the smallest number for which the sum is free. ■

Once again, let us assume that $K \oplus K$ has a free factor. As the above example shows, $K \oplus K \oplus K$ is free of rank 3. From this it is clear to see that all higher sums of K must also be free. Note for all $\ell \geqslant 4$ that $\ell - 3 \geqslant 1$ and $K^\ell = K^3 \oplus K^{\ell-3} \cong kG^3 \oplus K^{\ell-3} \cong kG^\ell$. This unfortunately does not give enough information to conclude whether $K \oplus K$ is free or not. If another appropriate assumption is made on L, we can prove the freedom of $K \oplus K$.

Proposition 3.5. *Let P be an R-module, such that $P \oplus R \cong R^2$. If $P \oplus P \cong Q \oplus R$ for some R-module Q and there exists an epimorphism $q : Q \to P$, then the epimorphism is an isomorphism and $P \oplus P \cong R \oplus R$.*

Proof. By our assumption, there is an exact sequence

$$0 \longrightarrow \ker q \longrightarrow Q \xrightarrow{q} P \longrightarrow 0.$$

But P is a projective module, which implies that this sequence splits. Thus, we have that $Q \cong \ker q \oplus P$. Hence,

$$\begin{aligned} P \oplus P &\cong Q \oplus R \\ &\cong (\ker q \oplus P) \oplus R \\ &\cong \ker q \oplus (R^2) \end{aligned}$$

Remember that $P \oplus P$ is stably free. Then,

$$R^3 \cong (P \oplus P) \oplus R$$

$$\cong (\ker q \oplus R^2) \oplus R$$
$$\cong \ker q \oplus R^3.$$

Thus, $R^3 \cong \ker q \oplus R^3$. But R is Noetherian, and hence $\ker q = 0$.[3] This gives us immediately that $P \cong Q$ and more importantly $P \oplus P \cong Q \oplus R \cong P \oplus R \cong R \oplus R$. □

We next provide another property L might exhibit that will imply that $K \oplus K$ is free.

Proposition 3.6. *If P is a one-relator, projective R-module, then $P \oplus R$ is free.*

Proof. Let $P = \langle\, e_1,\ e_2,\ \ldots,\ e_n \mid \sum e_i m_i \,\rangle$ with $m_1 \neq 0$. The presentation of P gives rise to the following exact sequence,

$$0 \longrightarrow \ker q \longrightarrow R^n \xrightarrow{q} P \longrightarrow 0.$$

Since, P is projective, the sequence is split and $R^n \cong P \oplus \ker q$. Since $m_1 \neq 0$ by assumption, $\ker q \neq 0$. Also, $\ker q$ is generated by a single element. Now, say $\ker q = aR$. Then, multiplication by a for $R \to aR$ is an epimorphism and is not an isomorphism only if a is a torsion element. But free modules and hence projective modules are torsion-free. Thus, a projective cyclic module must be free. Therefore, $P \oplus R \cong R^n$. □

Corollary 3.7. *If $P \oplus P \cong Q \oplus R$ and $Q = \langle\, e_1, e_2 \mid e_1 m_1 + e_2 m_2 \,\rangle$, then $P \oplus P \cong R \oplus R$.*

Proof. By Proposition 3.6, $Q \oplus R \cong R^2$. But $P \oplus P \cong Q \oplus R$ by assumption. □

[3] If a ring is stably finite and P is a stably free (or a projective) module, then $(N \oplus P \cong P) \Rightarrow (N = 0)$.

3.2 Free Kernels of $K \oplus K$

At this point, if we could project $K \oplus K$ onto a free module, then we could apply the above theory about free summands. No free image is, from our current point of view, obvious. Though, a free kernel can easily be constructed from the presentation of $K \oplus K$ (2.8). Our hope would be that the resulting exact sequence would split. That could be done by showing the image to be projective. This is our motivation in the next bit of work.

For notational convenience, we define the following terms. Let

$$R_1 = \begin{pmatrix} sr^\sigma & xr^\sigma - rr^\sigma & x^3 - x^2 r^{\sigma^2} \end{pmatrix}^T$$

and

$$R_2 = \begin{pmatrix} 0 & 0 & r \end{pmatrix}^T.$$

Define $L_0 = kG^3/\langle R_1,\ R_2\rangle = \langle\, e_1,\ e_2,\ e_3 \mid e_1(s)r^\sigma + e_2(x-r)r^\sigma + e_3(x-r)x^2,\ \ e_3 r\,\rangle$, and hence we have the short exact sequence

$$0 \longrightarrow \langle R_2\rangle \longrightarrow kG^3/\langle R_1\rangle \longrightarrow L_0 \longrightarrow 0.$$

Please note, that since the set $\{R_1, R_2\}$ is obviously right-linearly independent, the map $\langle R_2\rangle \to kG^3/\langle R_1\rangle$ is in fact injective. Also, note that $kG^3/\langle R_1\rangle \cong K \oplus K$.

Hence, if this sequence is split, then we have proven that $K \oplus K$ has a free summand. But remember from Proposition 3.3 that if $K \oplus K \cong L_0 \oplus kG$, then L_0 is projective. Thus, we claim it is equivalent to check whether L_0 is projective. If L_0 is projective, then the sequence is split. If L_0 is not, then the sequence cannot split, otherwise we would contradict Proposition 3.3.

Consider the term

$$
\begin{aligned}
(R_1x^{-2}+R_2)x^{-1} &= \left[\begin{pmatrix} sr^\sigma \\ (x-r)r^\sigma \\ (x-r)x^2 \end{pmatrix} x^{-2} + \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}\right] x^{-1} \\
&= \left[\begin{pmatrix} sr^\sigma x^{-2} \\ (x-r)r^\sigma x^{-2} \\ (x-r) \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ r \end{pmatrix}\right] x^{-1} \\
&= \begin{pmatrix} sr^\sigma x^{-2} \\ (x-r)r^\sigma x^{-2} \\ x \end{pmatrix} x^{-1} \\
&= \begin{pmatrix} sr^\sigma x^{-3} \\ (x-r)r^\sigma x^{-3} \\ 1 \end{pmatrix}
\end{aligned}
$$

Hence, $\begin{pmatrix} sr^\sigma x^{-3} & (x-r)r^\sigma x^{-3} & 1 \end{pmatrix}^T \in \langle R_1, R_2\rangle\, kG$, which means the following equation holds in L_0.

$$
-e_3 = e_1(s)(r^\sigma x^{-3}) + e_2(x-r)(r^\sigma x^{-3}) = (e_1(s)+e_2(x-r))(r^\sigma x^{-3}) \tag{3.2}
$$

So this substitution may be used in the presentation of L_0. Then, L_0 can be generated by two elements e_1 and e_2, and the relations can be rewritten without e_3. When we substitute (3.2) into the relations of L_0, they coincide as a single relation. As the algebra is simpler, we will work out the second relation first.

So,

$$\begin{aligned} e_3 r = 0 \;\; &\Rightarrow \;\; -\left[e_1 s + e_2(x-r)\right] r^{\sigma} x^{-3} r = 0 \\ &\Rightarrow \;\; \left[e_1 s + e_2(x-r)\right] r^{\sigma} r^{\sigma^3} x^{-3} = 0 \\ &\Rightarrow \;\; \left[e_1 s + e_2(x-r)\right] r^{\sigma} r^{\sigma^3} = 0 \end{aligned}$$

and

$$\begin{aligned} & e_1(s r^{\sigma}) + e_2(x r^{\sigma} - r r^{\sigma}) + e_3(x^3 - x^2 r^{\sigma^2}) = 0 \\ \Rightarrow \;\; & \left[e_1(s) + e_2(x-r)\right] r^{\sigma} + e_3(x-r)x^2 = 0 \\ \Rightarrow \;\; & \left[e_1(s) + e_2(x-r)\right] r^{\sigma} - \left[(e_1(s) + e_2(x-r))(r^{\sigma} x^{-3})\right](x-r)x^2 = 0 \\ \Rightarrow \;\; & \underbrace{\left[e_1(s) + e_2(x-r)\right] r^{\sigma}} - \underbrace{\left[e_1(s) + e_2(x-r)\right] r^{\sigma}}(x^{-3})(x-r)x^2 = 0 \\ \Rightarrow \;\; & \underbrace{\left[e_1(s) + e_2(x-r)\right] r^{\sigma}}\left[1 - (x^{-3})(x-r)x^2\right] = 0 \\ \Rightarrow \;\; & \left[e_1(s) + e_2(x-r)\right] r^{\sigma}\left[1 - (x^{-3} x x^2 - x^{-3} r x^2)\right] = 0 \\ \Rightarrow \;\; & \left[e_1(s) + e_2(x-r)\right] r^{\sigma}\left[1 - (1 - r^{\sigma^3} x^{-1})\right] = 0 \\ \Rightarrow \;\; & \left[e_1(s) + e_2(x-r)\right] r^{\sigma}\left[r^{\sigma^3} x^{-1}\right] = 0 \\ \Rightarrow \;\; & \left[e_1(s) + e_2(x-r)\right] r^{\sigma} r^{\sigma^3} = 0 \end{aligned}$$

Therefore, L_0 has another presentation, namely

$$L_0 = \left\langle e_1,\ e_2 \;\middle|\; (e_1(s) + e_2(x-r))\, r^{\sigma} r^{\sigma^3} \right\rangle.[4]$$

---

[4]We could have defined R_2 with $x^{-2} r^{\sigma^2}$ in the third coordinate instead of r. This is clear when one recognizes this new R_2 is just the original R_2 but multiplied by the unit x^2 on the right. The appropriate algebraic procedure would reveal that equation (3.2) would still hold with the new R_2 and hence reveal the same presentation as above.

When looking at this presentation of L_0, I hope the reader notices the strong resemblance it has to the presentation of K (2.3). In a natural way, we see that L_0 projects onto K. (Now the reader sees the motivation for Proposition 3.5.) Represent both modules by their presentations. Let

$$R_0 = (\, s \quad x - r \,)^T .$$

Then, the sequence

$$0 \longrightarrow N \longrightarrow L_0 \longrightarrow K \longrightarrow 0$$

is short exact, where the surjection is the map $m + \langle R_0 r^{\sigma} r^{\sigma^3} \rangle kG \mapsto m + \langle R_0 \rangle kG$ (this is well-defined, since $\langle R_0 r^{\sigma} r^{\sigma^3} \rangle kG \subseteq \langle R_0 \rangle kG$) and N is its kernel. Now, K is projective, so $L_0 \cong N \oplus K$.

For the sake of contradiction, we assume that L_0 is projective. Then, we have $K \oplus K \cong L_0 \oplus kG$. Then, Proposition 3.5 implies that $L_0 \cong K$, and especially $N = 0$. But under the surjection $R_0 \mapsto 0$ and $R_0 \notin \langle R_0 r^{\sigma} r^{\sigma^3} \rangle kG$, since r is a non-unit. So, $N \neq 0$. $\Rightarrow\Leftarrow$. Thus, L_0 cannot be projective,[5] which implies that L_0 is not a summand of $K \oplus K$.

Another guess for a factorization of $K \oplus K$ would be to kill off the third generator in the presentation (2.8). Let

$$R_3 = (\, 0 \quad 0 \quad 1 \,)^T .$$

---

[5]There is a much easier argument for showing that L_0 is not projective. By examining the presentation of L_0, the reader will recognize that R_0 is a torsion element in L_0. Now, projective modules are necessarily torsion-free. Therefore, L_0 cannot be projective. The author prefers the original argument as it could also be adapted for a more general setting where torsion elements may not exist.

Then, let $L_1 = (K \oplus K)/\langle R_3 \rangle = kG^3/\langle R_1, R_3 \rangle$. L_1 clearly has the presentation $L_1 = \langle\, e_1,\ e_2 \mid (e_1 s + e_2(x-r))r^\sigma \,\rangle$. L_1 can then be projected onto K in a natural way. But the same argument used on L_0 shows that L_1 cannot be projective and hence is not a summand of $K \oplus K$. Then why even mention it?

Let T be the kernel of the composition of the maps $K \oplus K \to L_1 \to K$. Please note that this map is surjective. Thus, the short sequence

$$0 \longrightarrow T \longrightarrow K \oplus K \longrightarrow K \longrightarrow 0 \tag{3.3}$$

is exact. This sequence resembles the other short exact sequence of $K \oplus K$

$$0 \longrightarrow K \longrightarrow K \oplus K \longrightarrow K \longrightarrow 0$$

with the obvious choice for inclusion and projection. We are certainly interested now in this module T.

Now, K is projective. So, sequence (3.3) splits and $K \oplus K \cong T \oplus K$. Thus, $T \oplus kG^2 \cong kG^3$, which implies that T is stably free and hence projective.

A presentation for T can be easily seen. Define the maps

$$K \oplus K = kG^3/\langle R_1 \rangle \xrightarrow{\phi} kG^3/\langle R_1, R_3 \rangle = L_1 = kG^2/\langle R_0 r^\sigma \rangle \xrightarrow{\psi} kG^2/\langle R_0 \rangle = K$$

Then, $T = \ker \psi\phi$. Thus, their composition is the map

$$\begin{pmatrix} m_1 \\ m_2 \\ m_3 \end{pmatrix} + \langle R_1 \rangle \overset{\phi}{\longmapsto} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \langle R_0 r^\sigma \rangle \overset{\psi}{\longmapsto} \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} + \langle R_0 \rangle.$$

Therefore, an element of the kernel is of the form $(\, s\alpha \quad (x-r)\alpha \quad \beta\,)^T$. T is then generated by two elements. T is a submodule of $K \oplus K$ and hence inherits its relation. As the first two coordinates are generated by a single element, we can collapse the relation of $K \oplus K$ and discover the following presentation for T. [6]

$$T = \left\langle e_1,\ e_2 \;\middle|\; e_1 r^{\sigma} + e_2(x^3 - x^2 r^{\sigma^2}) \right\rangle$$

$$= \langle\, e_1,\ e_2 \mid e_1 r^{\sigma} + e_2(x-r)x^2 \,\rangle \tag{3.4}$$

Therefore, by Proposition 3.6, $T \oplus kG \cong kG^2$.

Many natural questions now arise about the module T, a few of which we address here. Is T free? Is $T \cong K$? Or is T a new non-isomorphic stably free kG-module. Certainly one and only one of these questions have an affirmative answer. An affirmative response for the first question would answer our question about the freedom of $K \oplus K$. The second question would provide a new presentation of K. Or a yes for the last question would allow us to start examining T in the same manner we have for K. Alas, the answer to the second question is yes.

Theorem 3.8. *The kG-module T is isomorphic to K, and* $K = \left\langle e_1,\ e_2 \;\middle|\; e_1 r^{\sigma} + e_2(x^3 - x^2 r^{\sigma^2}) \right\rangle$.

Proof. We begin by rewriting the relation of T.

---

[6] We remark at this time that in light of equation (3.2), the composition derived from the sequence of maps

$$K \oplus K \to L_0 \to K$$

is the same map as $\psi\phi$ defined above. We leave it to the reader to verify this claim.

$$
\begin{aligned}
& e_1 r^{\sigma} + e_2(x^3 - x^2 r^{\sigma^2}) = 0 \\
& \Rightarrow e_1 r^{\sigma} + e_2(x - r)x^2 = 0 \\
& \Rightarrow e_1 r^{\sigma} x^{-2} + e_2(x - r) = 0 \\
& \Rightarrow e_1 x^{-2} s + e_2(x - r) = 0 \\
& \Rightarrow e_1 x^{-2}(s x^{-2}) + e_2(x^{-1} - r x^{-2}) = 0
\end{aligned}
$$

Thus, by equation (2.2), we see that

$$
(rx^2)(x^{-2}(sx^{-2})) + (x+s)(x^{-1} - rx^{-2}) = 1 \tag{3.5}
$$

Then, let $T_0 = \{f \in kG \mid (rx^2)f = (x+s)g,\ g \in kG\} \cong \langle rx^2 \rangle \cap \langle x + s \rangle$.

So, the short sequence

$$
0 \longrightarrow T_0 \longrightarrow kG \oplus kG \longrightarrow kG \longrightarrow 0
$$

is exact where the inclusion map is the analog of the map i and the projection map is the analog of the map π but with multiplication by r replaced by rx^2. Then, equation (3.5) gives us the appropriate splitting of this sequence. Hence, $T_0 = \langle e_1,\ e_2 \mid e_1 x^{-2}(sx^{-2}) + e_2(x^{-1} - rx^{-2}) \rangle \cong \langle e_1,\ e_2 \mid e_1 r^{\sigma} + e_2(x^3 - x^2 r^{\sigma^2}) \rangle = T$.

Now, the modules $\langle r \rangle$ and $\langle rx^2 \rangle$ are identical since x^2 is a unit. Therefore, $T \cong \langle rx^2 \rangle \cap \langle x + s \rangle = \langle r \rangle \cap \langle x + s \rangle \cong K$. □

3.3 Epimorphisms of Free Modules $R^n \to R^m$

In light of the proof of Theorem 3.8, the new presentation of K is nothing spectacular at all. In fact, any mystery that the kernel T offered would have completely vanished had the author not insisted in writing the relator and generators of K as polynomials of positive degree. Notice that back in Section 2.1 we factored x^{-2} out of the relator and generators of K and multiplied them by x^2. Since x^2 is a unit, this did not change the presentation or spanning set of K. Had we used the actual images of the p and s when constructing T, then it would have been immediate that $T \cong K$.

But the creation of the module T does provide us a way of constructing new presentations of the module K. As was briefly mentioned in Section 2.3, every stably free R-module can be characterized as a kernel of some epimorphism $R^n \to R^m$. We saw easily that $K \cong \ker \pi$. Define the map $\pi' : kG^2 \to kG$, such that

$$\pi' = \begin{pmatrix} rx^2 & x+s \end{pmatrix}.$$

Then, it is clear that $T \cong \ker \pi'$. In particularly, $K \cong \ker \pi'$. So, there exists two distinct epimorphisms for which K is a kernel. We also saw in Section 2.3 that $K \oplus K \cong \ker \pi_2 \cong \ker \widetilde{\pi}_2$. This motivates our next definition.

Definition 3.9. Let R be a ring and let $f, g : R^n \to R^m$ be R-module epimorphisms. Then, we say that f and g are conjugate, denoted by $f \sim g$, iff there exists isomorphisms φ, ψ, such that the following diagram commutes.

$$\begin{tikzcd} R^n \arrow[r, "f"] \arrow[d, "\varphi"'] & R^m \arrow[d, "\psi"] \\ R^n \arrow[r, "g"] & R^m \end{tikzcd}$$

In other words, $\varphi f = g\psi$.[7]

It can readily be seen that

$$\pi = \pi' \begin{pmatrix} x^{-2} & 0 \\ 0 & 1 \end{pmatrix}.$$

Thus, $\pi \sim \pi'$. In general, the next proposition shows that the conjugacy classes defined above correspond directly with the isomorphism classes of stably free modules.

Proposition 3.10. *Let R be a ring. Any two $f, g : R^n \to R^m$ epimorphisms are conjugate if and only if* $\ker f \cong \ker g$.

Proof. ($\Rightarrow$) If $f \sim g$, then $\exists \varphi, \psi$ isomorphisms, such that $\varphi f = g\psi$. So, $\ker f = \ker \varphi f = \ker g\psi \cong \ker g$, since φ, ψ are injective.
($\Leftarrow$) Let $\varphi : \ker f \to \ker g$ be an isomorphism. Since f and g are surjective, $\operatorname{im} f = \operatorname{im} g = R^m$. Since R^n is free, there exists a map $\psi : R^n \to R^n$, such that the following diagram commutes.

$$\begin{tikzcd}
 & R^n \arrow[dl, dotted, "\psi"'] \arrow[d, "f"] & \\
R^n \arrow[r, "g"] & R^m \arrow[r] & 0
\end{tikzcd}$$

Since $\ker f$ is a summand of R^n, let C be the complement of $\ker f$ and let ψ' be the restriction $\psi \mid_C$.

[7] Please note that the above definition defines an equivalence relation.

Let F be the map $F = \varphi \oplus \psi'$. Next consider the diagram

$$\begin{tikzcd}
0 \arrow[r] & \ker f \arrow[r] \arrow[d, "\varphi"] & R^n \arrow[r, "f"] \arrow[d, "F"] & R^m \arrow[r] \arrow[d, "\mathbb{1}"] & 0 \\
0 \arrow[r] & \ker g \arrow[r] & R^n \arrow[r, "g"] & R^m \arrow[r] & 0
\end{tikzcd}$$

The diagram commutes and the two rows are exact. Thus by the 5-lemma, F is an isomorphism. Therefore, $f \sim g$. $\square$

The proof of Proposition 3.10 leads us to believe that we may simplify the definition of $\sim$.

Corollary 3.11. *$f \sim g$ if and only if there exists an automorphism $\varphi : R^n \to R^n$, such that $f = g\varphi$.*

Let P be any kG-module, such that $K \cong P$. Since P is stably free, there exists an epimorphism $\rho : kG^2 \to kG$, such that $\pi \sim \rho$. Then by Corollary 3.11, there exists a map φ, such that $\pi = \rho\varphi$. Then, we have $\rho\,(\varphi\mathscr{s}) = (\rho\varphi)\,\mathscr{s} = \pi\mathscr{s} = \mathbb{1}$. Thus, $\varphi\mathscr{s}$ is a splitting of ρ. And as we saw before, $\varphi\mathscr{s}$ is in fact the single relator of P, and K can be presented as

$$K \cong \langle\, e_1,\ e_2 \mid \varphi\mathscr{s} \,\rangle\,.$$

Corollary 3.11 shows us that this is the only way to construct new presentations of K.[8]

[8]This is assuming we do not want to change the size of the generating set.

Also, by Corollary 3.11, there exists an automorphism $\varphi : kG^3 \to kG^3$, such that $\pi_2 = \widetilde{\pi}_2\varphi$. Or, if we think of these maps as matrices, there exists a nonsingular matrix φ such that

$$\begin{pmatrix} r^2 & r(x+s) & x+s \end{pmatrix} = \begin{pmatrix} r & (x+s)r & (x+s)^2 \end{pmatrix} \varphi.$$

Corollary 3.12. *Let R be a ring and let $f : R^n \to R^m$ be an epimorphism. Let $P = \ker f$. Then, P is free if and only if f can be lifted to an isomorphism $\widetilde{f} : R^n \to R^{m+r}$, such that $f = q\widetilde{f}$, where $q : R^{m+r} \to R^m$ is the canonical projection and $n = m + r$.*

Proof.
($\Leftarrow$) If $f = q\widetilde{f}$, then $\ker f = \ker q\widetilde{f} \cong \ker q = R^r$.
($\Rightarrow$) If $\ker f$ is free, then $\ker f \cong R^r$ for $r = n - m$. Then, $\ker q = R^r \cong \ker f$. Thus, $q \sim f$. So by Corollary 3.11, there exists an automorphism $\widetilde{f}$, such that $f = q\widetilde{f}$. □

So by Corollary 3.12, $\pi \nsim (\,1\;\;0\,)$. Yet if $K \oplus K$ is free, then $\pi_2 \sim (\,1\;\;0\;\;0\,)$. Also, if $K \oplus K \cong L \oplus kG$ for some module L, then $\pi_3 \sim (\,1\;\;0\;\;0\;\;0\,)$.

3.4 Schanuel's Lemma and $K \oplus K$

Let us begin this section by mentioning an important result from module theory commonly referred to as Schanuel's Lemma.

Lemma 3.13. *Given exact sequences of modules*

$$0 \longrightarrow N \longrightarrow P \longrightarrow M \longrightarrow 0$$

$$0 \longrightarrow N' \longrightarrow P' \longrightarrow M \longrightarrow 0$$

where P is projective, then the sequence

$$0 \longrightarrow N \longrightarrow P \oplus N' \longrightarrow P' \longrightarrow 0$$

is short exact. Particularly, if P' is projective, then $N \oplus P' \cong N' \oplus P$.

The result of Schanuel's lemma looks a lot like what we were looking for in Section 3.2. Namely, if we can choose appropriately the modules L and M and the corresponding maps, such that the sequences

$$\begin{cases} 0 \longrightarrow kG \xrightarrow{j} K \longrightarrow M \longrightarrow 0 \\ 0 \longrightarrow K \xrightarrow{j'} L \longrightarrow M \longrightarrow 0 \end{cases}$$

or

$$\begin{cases} 0 \longrightarrow K \xrightarrow{j} kG \longrightarrow M \longrightarrow 0 \\ 0 \longrightarrow L \xrightarrow{j'} K \longrightarrow M \longrightarrow 0 \end{cases}$$

are exact, then we may apply Schanuel's lemma and have $K \oplus K \cong L \oplus kG$.

Let us consider the first pair of sequences. The map $j : kG \to K$ is necessarily injective. K is torsion-free, which means the generator of kG can be sent to any nonzero element in K. Let $j(1) = p(e_1)a + p(e_2)b$, where $a, b \in kG$ with $a \neq 0$ (remember that $\{p(e_1),\ p(e_2)\}$ generate K). By exactness, $M \cong K/j(kG)$. This would give us the presentation $M = \langle\, e_1,\ e_2 \mid e_1 s + e_2(x - r),\ e_1 a + e_2 b \,\rangle$.

We can guarantee exactness for the accompanying sequence by defining $L = K \oplus M$. Therefore, by Lemma 3.13, the sequence

$$0 \longrightarrow kG \longrightarrow K \oplus K \longrightarrow L \longrightarrow 0$$

is short exact. Thus, it suffices to show that L is projective. Projective modules are torsion-free, which implies that necessarily $b \neq 0$.

Assume for the sake of contradiction that L is projective. Then, by Lemma 3.13, $K \oplus K \cong L \oplus kG$. Since $L = K \oplus M$, there exists a projection $q : L \to K$. Therefore, by Proposition 3.5, $L \cong K$. So, $K \cong K \oplus M$. Now the stably finite property applies toward stably free modules too.

$$\begin{aligned} K \oplus kG &\cong K \oplus M \oplus kG \\ kG^2 &\cong kG^2 \oplus M \\ &\Rightarrow M = 0 \end{aligned}$$

But $M = K/j(kG)$. Therefore, $j(kG) = K$. So, K is cyclic $\Rightarrow\Leftarrow$. Therefore, the module $L = K \oplus M$ cannot be a summand of $K \oplus K$. Uncertain if there is another module L, such that the sequence

$$0 \longrightarrow K \longrightarrow L \longrightarrow M \longrightarrow 0$$

is exact, we focus our efforts on the second possibility mentioned at the beginning of the section.

Consider the short exact sequence

$$0 \longrightarrow K \longrightarrow kG \longrightarrow kG/K \longrightarrow 0$$

where the injective map is the natural inclusion and the surjective map is also canonical. Also, consider the short sequence

$$0 \longrightarrow L \longrightarrow K \xrightarrow{\nu} kG/K \longrightarrow 0$$

where $L = \ker \nu$ and $\nu : K \to kG/K$ is some surjective map. Does such an epimorphism exist?

Our first hope would be that there exists a surjective map $K \to kG$. Then, the composition map between this surjection and the canonical map $kG \to kG/K$ would suffice for ν. But the following proposition shows that such an epimorphism does not exist.

Proposition 3.14. *Let R be a stably finite ring. If for any R-module P, $P \oplus R^m \cong R^n$, then any epimorphism $P \to R^{n-m}$ must be an isomorphism.*

Proof. Let $q : P \to R^{n-m}$ be a surjective R-map. Then, there exists an exact sequence

$$0 \longrightarrow \ker q \longrightarrow P \xrightarrow{q} R^{n-m} \longrightarrow 0$$

But R^{n-m} is free and hence the sequence is split. So, $P \cong \ker q \oplus R^{n-m}$.

Now,

$$\begin{aligned} & P \oplus R^m \cong (\ker q \oplus R^{n-m}) \oplus R^m \\ & \Rightarrow R^n \cong \ker q \oplus R^n \\ & \Rightarrow \ker q = 0 \quad \text{by stable finiteness} \end{aligned}$$

Therefore, q is an isomorphism. $\square$

The theme of Proposition 3.14 is that any epimorphism from a stably free module of finite rank onto a free module of equal rank is injective.

As a result, since $K \oplus kG \cong kG^2$, Proposition 3.14 says that no epimorphism $K \to kG$ exists. Also, Proposition 3.14 tells us that if there exists an epimorphism $K \oplus K \to kG \oplus kG$, then $K \oplus K$ is free.

Now, K is a submodule of kG, so the composition of an injective map $K \to kG$ with the natural map $kG \to kG/K$ could possibly produce the ν we are looking for. If we choose the identity map $K \to kG$, the composition would be the zero map, which is definitely not onto. Hence, we must investigate a more exotic embedding $K \to kG$. In order to construct such an embedding, we deviate briefly from our current work and return to a result from Section 2.1. We remind the reader about the map $\theta : K \to kG$. θ is an injective map, such that $\theta : f \mapsto g$ whenever $rf = (x+s)g$. Since θ is an injective map, $K \cong \operatorname{im}\theta$. For notational purposes, let $K^\theta = \theta(K) = \operatorname{im}\theta$. Hence, $K \cong K^\theta$.

Now, $K^\theta = \{\, g \in kG \;\mid\; rf = (x+s)g,\ f \in kG \,\}$. This is of course a "dual" definition to that of K. Naturally, $K^\theta \cong \langle r \rangle \cap \langle x+s \rangle$. As each of these three right ideals are isomorphic to each other, presentation (2.3) holds as a module presentation for each of the three ideals.

Certainly, the generators of these ideals differ. As we already know the generators for K (2.6), the generators of $\langle r \rangle \cap \langle x+s \rangle$ are easy to compute. Indeed, it is trivial to verify that multiplication by r is an isomorphism from K onto $\langle r \rangle \cap \langle x+s \rangle$. Therefore,

$$\begin{aligned} \langle r \rangle \cap \langle x+s \rangle &= \langle\, r(1-(sx^{-2})r),\ -r(sx^{-2})(x+s) \,\rangle \\ &= \langle\, rx^2 - srr^{\sigma^2},\ srx + srr^{\sigma} \,\rangle \end{aligned} \tag{3.6}$$

The generators of K^θ can also be found like this, namely $K^\theta = \langle\, p(e_1)^\theta,\ p(e_2)^\theta \,\rangle$. Fortunately, θ can easily be computed and hence we can give more explicit generators of K^θ. To calculate f^θ, the most natural way is to multiply rf and factor the product by $x+s$ on the left. There exists a natural generalization of the long polynomial division algorithm that respects factoring on the left or right. We could compute

$p(e_1)^\theta$ and $p(e_2)^\theta$ using this algorithm but we have already done the work above.

We remind the reader that the term $\vec{m} - \mathcal{s}\pi\vec{m}$ for $\vec{m} \in kG^2$ is in the image of i. Thus, $\forall \vec{m} \in kG^2$, $\exists f \in K$, such that

$$\vec{m} - \mathcal{s}\pi\vec{m} = \begin{pmatrix} f & -f^\theta \end{pmatrix}^T = i(f)$$

Hence, back on page 12, when we calculated $p(e_1)$ and $p(e_2)$ we also evaluated the values for $p(e_1)^\theta$ and $p(e_2)^\theta$. Therefore,

$$\begin{aligned} K^\theta &= \langle\, (x^{-1} - rx^{-2})r,\ (x^{-1} - rx^{-2})(x+s) - 1 \,\rangle \qquad (3.7) \\ &= \langle\, r^\sigma x^{-1} - rr^{\sigma^2} x^{-2},\ -rr^\sigma x^{-2} \,\rangle \\ &= \langle\, r^\sigma x - rr^{\sigma^2},\ rr^\sigma \,\rangle \end{aligned}$$

The reader can verify that $p(e_1)^\theta s + p(e_2)^\theta (x - r) = 0$.

Let us now return to our discussion concerning Schanuel's Lemma. Define the map $\nu : K \to kG/K$ as

$$\nu : f \longmapsto f^\theta + K.$$

Now the question we cannot help but ask is whether ν is surjective. If we place an assumption on σ, then the answer is yes.

Theorem 3.15. *Let σ be an automorphism of order 2. Then, there exists a kG-module L, such that $K \oplus K \cong L \oplus kG$.*

Proof. We begin the proof by showing that the map ν is surjective.

According to equation (3.7), $rr^\sigma x^{-2} = (-1)(-rr^\sigma x^{-2}) \in K^\theta$. Also, $r(rr^\sigma x^{-2}) \neq (x+s)g$ for any g since $rr^\sigma x^{-2}$ is a monomial in terms of x. So, $rr^\sigma x^{-2} \notin K$.

Consider the term $1-(rr^{\sigma}x^{-2})$. We show now that it is contained in K.

$$\begin{aligned}
r(1-rr^{\sigma}x^{-2}) &= r(1-r^{\sigma^2}sx^{-2}) \quad \text{by our assumption on the order of } \sigma \\
&= r - srr^{\sigma^2}x^{-2} \\
&= (rx^2 - srr^{\sigma^2})x^{-2} \\
&= (x^2r^{\sigma^2} - srr^{\sigma^2})x^{-2} \\
&= (x^2 - sr)r^{\sigma^2}x^{-2} \\
&= (x^2 - sx + sx - sr)x^{-2}r \\
&= (x+s)(x-r)x^{-2}r \\
&= (x+s)(x^{-1} - rx^{-2})r
\end{aligned}$$

Therefore, $1-rr^{\sigma}x^{-2} \in K$, or $1+K = rr^{\sigma}x^{-2}+K$.[9]

Next, $-p(e_2) = (-1)p(e_2) \in K$ and, as was shown above (2.5), $\theta(-p(e_2)) = rr^{\sigma}x^{-2}$. Thus, $\nu(-p(e_2)) = rr^{\sigma}x^{-2}+K = 1+K$. Now, kG/K is a cyclic module generated by $1+K$. Furthermore, $1+K \in \operatorname{im}\nu$. Therefore, ν is surjective.

Let $L = \ker\nu$. Thus, the short sequences

$$0 \longrightarrow K \longrightarrow kG \longrightarrow kG/K \longrightarrow 0$$
$$0 \longrightarrow L \longrightarrow K \longrightarrow kG/K \longrightarrow 0$$

are exact. Now kG and K are projective. Therefore, by Schanuel's Lemma, $K \oplus K \cong L \oplus kG$. □

[9] Since σ is order 2, $1-rr^{\sigma}x^{-2} = 1-(sx^{-2})r = p(e_1) \in K$. The above argument was presented this way to show at least one explicit example of this type of computation.

Example 3.16. Let $G = \langle\, x,\ y \ \mid\ xyx^{-1} = y^{-1}\,\rangle$; that is, G is the fundamental group of the Klein Bottle, and consider the ring $\mathbb{Z}G$. Note that σ in an automorphism of order 2.

By Theorem 3.15, there exists an kG-module L, such that $K \oplus K \cong L \oplus kG$. Interesting enough,

$$L = \{\, f \in K \ \mid\ \theta f \in K\,\} \cong K \cap K^{\theta}, \tag{3.8}$$

the intersection of two non-free, stably free ideals. Therefore, by Example 3.4, K^3 and all higher sums of K for the Klein bottle must be free. ∎

For the case of the Klein bottle, we see that $K \oplus K$ splits off a free summand as mentioned in the previous example. Yet our assumption on σ seems quite restrictive, and we wonder if $K \oplus K$ has a free summand for any choice of σ.

Question 3.17. Can the assumptions about the map σ in Theorem 3.15 be removed or at least relaxed to include maps of any finite order?

CHAPTER 4

INFINITELY MANY STABLY FREE MODULES

Up until this point, we have fixed a value for r; that is, if $y_1^\sigma \neq y_1^{-1}$, then $r = 1 + y_1$, else $r = 1 + y_1 + y_1^3$. Then, we choose $s = r^{\sigma^{-1}}$. Our choices for r and s were entirely determined as the simplest choices that satisfy property 2.7. We now will use this chapter to explore what changes for K as r changes.[1]

If we deviate from our original choice for r, then there are a lot of choices. The goal of this chapter is to show that there are infinitely many different choices for r and s, such that (r, s) satisfy property 2.7 yet the resulting kG-modules $\langle r\rangle \cap \langle x + s\rangle$ are non-isomorphic. The process of constructing these (r, s) pairs and showing the ideals are not isomorphic is due to a modification of a proof by V. A. Artamonov [1, Theorem 1].

We will continue to let $s = r^{\sigma^{-1}}$. Hence, the change solely rests on how we modify r. If $y_1^\sigma \neq y_1^{-1}$, then let $r_n = 1 + ny_1$ for $n \in \mathbb{Z}$ and $n \neq 0$. Else let $r_n = 1 + ny_1 + ny_1^3$. Hence, $s_n = 1 + ny_1^{\sigma^{-1}}$ or $1 + ny_1^{-1} + ny_1^{-3}$, respectively. Define

$$K_n = \{\, f \in kG \mid r_n f = (x + s_n)g,\ g \in kG \,\} \cong \langle r_n\rangle \cap \langle x + s_n\rangle. \tag{4.1}$$

[1]We mention that the equations, formulae, theory, presentations, and generators mentioned in Chapters 2 and 3 are defined and listed in terms of r and s. This means that even if r and s are changed, then everything from Sections 2 and 3 will still apply to the module K generated by this new r and s, so long as (r, s) satisfies property 2.7.

Note that when $n = 1$, we have that $K = K_1$, the same module defined in Section 2.1.

By our choice of r_n,[2] we therefore have another application of Theorem 2.8.

Corollary 4.1. *Let p be the characteristic of k and let $p \nmid n \in \mathbb{Z}$. Then, K_n is a non-free, stably free right ideal of kG.*

Proof. The proof is almost the exact same as the proof of Corollary 2.9; that is, we must show that (r_n, s_n) satisfy property 2.7. In particular, r_n is not monomial since $p \nmid n$. Since $s_n = r_n^{\sigma^{-1}}$, equation (2.2) still holds.

For the last condition, if $r_n = 1 + ny_1$, then $\langle r_n \rangle$ is a prime ideal. Hence, $s_n r_n^\sigma \notin \langle r_n \rangle$.

Otherwise, $r_n = 1 + ny_1 + ny_1^3$ and

$$\begin{aligned} s_n r_n^\sigma &= \left(1 + ny_1^{-1} + ny_1^{-3}\right)^2 \\ &= \left(y_1^6 + 2ny_1^5 + n^2 y_1^4 + 2ny_1^3 + 2n^2 y_1^2 + n^2\right) y_1^{-6}. \end{aligned}$$

So, it suffices to show that

$$\left(1 + ny_1 + ny_1^3\right) \nmid \left(y_1^6 + 2ny_1^5 + n^2 y_1^4 + 2ny_1^3 + 2n^2 y_1^2 + n^2\right).$$

Using polynomial division, we see that

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[2]Our choice for r_n is certainly not unique. In Artamonov's paper, $r_n = ny_1 - 1$ when $y_1^\sigma \neq y_1^{-1}$ and the case when $y_1^\sigma = y_1^{-1}$ is ignored completely. We defined r_n the way we did simply so that r_1, s_1, and K_1 would coincide with the r, s, and K defined in Section 2.1.

We also mention that even in the case when $y_1^\sigma = y_1^{-1}$, the pair $(1 + ny_1,\ 1 + ny_1^{\sigma^{-1}}) = (1 + ny_1,\ 1 + ny_1^{-1})$ satisfies property 2.7 for all integers $n \not\equiv 0, 1$, or -1 in k. This fact is easy to verify. The reason that we continue to separate the two cases when $n \neq \pm 1$ will become apparent in the forthcoming proof.

$$y_1^6+2ny_1^5+n^2y_1^4+2ny_1^3+2n^2y_1^2+n^2 = \left(ny_1^3+ny_1+1\right)\left(\frac{1}{n}y_1^3+2y_1^2+\frac{n^2-1}{n}y_1-\frac{1}{n^2}\right)$$

$$+(n^2-1)y_1^2+\frac{2-n^2}{n}y_1+\frac{n^4+1}{n^2}.$$

Since the remainder is nonzero $\forall n \in \mathbb{Z}$, the condition holds. □

Corollary 4.1 shows us that there are infinitely many different ways to construct non-free, stably free ideals of kG. So does kG have a lot of stably free modules? We have infinitely many different representations of stably free modules, but how many isomorphism classes does Corollary 4.1 give us? Remember that in Section 3.4 we talked about 3 distinct right ideals of kG that were all isomorphic to K_1, namely,

$$\begin{aligned} K_1 &\cong \{\, f \in kG \mid rf = (x+s)g,\ g \in kG \,\} \\ &\cong \{\, g \in kG \mid rf = (x+s)g,\ f \in kG \,\} \\ &\cong \langle r \rangle \cap \langle x+s \rangle \end{aligned}$$

These three kG-modules are all representations of the same isomorphism class. Remember also in Section 3.2, we constructed a module T as the kernel of an exotic map, which also turned out to be isomorphic to K. Also, as we mentioned in Section 3.3, for each matrix in $\mathrm{GL}_2(kG)$, we get a new presentation of K. So how do we know for $m \neq n$ whether $K_m \cong K_n$ or $K_m \ncong K_n$? In particular, how do we know whether $K_n \cong K$ or not? By the forementioned proof of Artamonov, we will construct a set $Q \subseteq \mathbb{N}$, such that for all $p, q \in Q$, $K_p \ncong K_q$.

This brings us to the main result of this chapter. Artamonov's paper was written before Stafford's, and Artamonov's assumptions on the group ring are much more complicated. Stafford's assumptions greatly simplify the proof, but appropriate alterations from Artamonov's original proof are needed for our context. Hence, we provide the altered proof here but mention that the essence of the proof remains true to Artamonov's original. We begin with a definition to simplify the assumptions of the theorem.

Definition 4.2. A set $Q \subseteq \mathbb{N}$ satisfies **Property 4.2** iff

1. Q is infinite.
2. Every element of Q is a prime number.
3. If $Q = \{q_1, q_2, \ldots\}$, then for any $k > 1$ and $q_k \in Q$, $q_k \equiv 1 \mod (q_1 q_2 \ldots q_{k-1})$.

Lemma 4.3. *There exists a set Q satisfying property 4.2.*

Proof. We mention first Dirichlet's Theorem on Primes in Arithmetic Progressions [16], which states that for any $a, b \in \mathbb{Z}$, such that $\gcd(a, b) = 1$, there are infinitely many primes p, such that $p \equiv a \mod b$. Specifically, when $a = 1$, Dirichlet's Theorem tells us that there is always a prime number p, such that $p \equiv 1 \mod m$ for any integer m.

Let Q be a set of all prime numbers q_i defined recursively as: let $q_1 = 2$ and, for all q_i defined for $i \leqslant n$, let q_{n+1} be the smallest prime, such that $q_{n+1} \equiv 1 \mod (q_1 \ldots q_n)$. For example, $q_1 = 2$, $q_2 = 3$, $q_3 = 7$, $q_4 = 43$, etc.

This process never terminates by Dirichlet's theorem. Therefore, Q is a countably infinite subset of $\mathbb{N}$. □

In a natural way, we see a map from $\mathbb{Z}$ into k. That is, if $j : \mathbb{Z} \to k$ is the unique unital ring homomorphism, then $j(\mathbb{Z})$ is a subring of k. Thus, k contains an isomorphic copy of $\mathbb{Z}_p$, where p is the characteristic of the ring and $\mathbb{Z}_0 = \mathbb{Z}$. Now if we assume that $j(Q)$ is an infinite subset of k, then k necessarily has characteristic 0. Therefore, $\mathbb{Z} \cong j(\mathbb{Z})$. So by an abuse of notation, we may say that $\mathbb{Z} \leqslant k$ and $Q \subset k$.

Theorem 4.4. *Let G be a non-Abelian, poly-(infinite) cyclic group. Let k be a Noetherian commutative domain with a subset Q of non-units satisfying property 4.2. Then, there exists at least countably many non-isomorphic non-free, stably free kG-modules.*

Proof. The plan of this proof is simple. We know by Corollary 4.1 that K_q is a non-free, stably free right ideal of kG for all $q \in Q$. We next need to show that for distinct $p, q \in Q$ the modules K_p and K_q are non-isomorphic. We will prove this last statement by showing that for distinct p and q, K_p and K_q differ on an invariant.

Let (q) be the principal ideal generated by q in k. Since q is not a unit by assumption, (q) is a proper ideal of k. Furthermore, in a commutative ring every proper ideal is contained in a maximal ideal, but all maximal ideals are prime. So, there exists a prime ideal $\mathfrak{p}_q$ that contains the element q for each element of Q. By Property 4.2, for any $p, q \in Q$, such that $p \neq q$, we have that $p \notin \mathfrak{p}_q$.

We define the quotient $K_{q,p}$ to be

$$K_{q,p} = K_q/\mathfrak{p}_p K_q \tag{4.2}$$

We claim that $K_{q,p}$ is a stably free $(k/\mathfrak{p}_p)G$-module.

Let the map $\pi_{(q)} : kG \oplus kG \to kG$ be defined as

$$\pi_{(q)} : \begin{pmatrix} m_1 \\ m_2 \end{pmatrix} \mapsto r_q m_1 + (x + s_q) m_2 \tag{4.3}$$

By property 2.7, this map is an epimorphism. More importantly, by the same arguments used in Theorem 2.4, we see that $K_q \cong \ker \pi_{(q)}$.

Next let $\pi_{q,p}$ be the quotient map defined by dividing $\mathfrak{p}_q$ out of $\pi_{(q)}$; that is, $\pi_{q,p} : (k/\mathfrak{p}_p)G \oplus (k/\mathfrak{p}_p)G \longrightarrow (k/\mathfrak{p}_p)G$, such that

$$\pi_{q,p} : \begin{pmatrix} m_1 & \bmod \mathfrak{p}_p \\ m_2 & \bmod \mathfrak{p}_p \end{pmatrix} \mapsto r_q m_1 + (x + s_q)m_2 \quad \bmod \mathfrak{p}_p$$

Then certainly the kernel of $\pi_{q,p}$ will be $K_{q,p}$. Thus, $K_{q,p}$ is a stably free $(k/\mathfrak{p}_p)G$-module.

Consider the case when $p = q$; that is, consider $K_{q,q}$. So, multiples of q are considered 0 in the quotient. Then, we have $r_q \equiv 1$ and $s_q \equiv 1$, both modulo $\mathfrak{p}_q$. Thus, the map $\pi_{q,q}$ simplifies to be

$$\pi_{q,q} : \begin{pmatrix} m_1 & \bmod \mathfrak{p}_p \\ m_2 & \bmod \mathfrak{p}_p \end{pmatrix} \mapsto m_1 + (x + 1)m_2 \quad \bmod \mathfrak{p}_p.$$

More particularly,

$$K_{q,q} = \{\, f \in (k/\mathfrak{p}_q)G \mid f = (x+1)g,\ g \in (k/\mathfrak{p}_q)G \,\} = \langle\, x + 1 \rangle,$$

where the last set is the right ideal in $(k/\mathfrak{p}_q)G$ generated by $x + 1$. Hence, $K_{q,q}$ is a cyclic module. Therefore, $K_{q,q}$ is free.

Next, consider the case when $p < q$. By our construction of the set Q, $q \equiv_{\mathfrak{p}_p} 1$. Hence, if $y_1^{\sigma} \neq y_1^{-1}$, then $r_q \equiv_{\mathfrak{p}_p} 1 + y_1$ and $s_q \equiv_{\mathfrak{p}_p} 1 + y_1^{\sigma^{-1}}$, else $r_q \equiv_{\mathfrak{p}_p} 1 + y_1 + y_1^3$ and $s_q \equiv_{\mathfrak{p}_p} 1 + y_1^{-1} + y_1^{-3}$. In particular, $r_q \equiv_{\mathfrak{p}_p} r$ and $s_q \equiv_{\mathfrak{p}_p} s$. Now, $\mathfrak{p}_p$ is a prime ideal in k. So, $k/\mathfrak{p}_p$ is a Noetherian domain. We chose r and s specifically so that (r, s) satisfies property 2.7 for any Noetherian domain by Corollary 2.9. Therefore, $K_{q,p}$ is a non-free, stably free module by Theorem 2.8.

Now, let $p < q \in Q$. If $K_q \cong K_p$, then $K_{q,p} \cong K_{p,p}$. But we just showed that $K_{p,p}$ is free but $K_{q,p}$ is not. Therefore, $K_q \ncong K_p$. □

Example 4.5. Let G be the fundamental group of the Klein bottle and let $k = \mathbb{Z}$. By Lemma 4.3, $\mathbb{Z}$ has a subset Q satisfying property 4.2. Since $\mathbb{Z}$ is a unique factorization domain, all the ideals generated by prime numbers are prime ideals. Hence, $\mathfrak{p}_p = (p)$ and $k/\mathfrak{p}_p = \mathbb{Z}_p$. Therefore, by Theorem 4.4, $\mathbb{Z}G$ has infinitely many non-isomorphic, non-free, stably free, right ideals. ■

Example 4.6. Again let G be defined as above, but let $k = \mathbb{Z}[i]$, i.e., the Gaussian integers. The only units in $\mathbb{Z}[i]$ are 1, -1, i, and $-i$. Thus, the set Q constructed in Lemma 4.3 still satisfies property 4.2. But even though Q is a set of rational prime numbers, not all the elements of Q are prime elements in $\mathbb{Z}[i]$.

For example, $2 = (1+i)(1-i)$. So, 2 is reducible. We can still construct an explicit ideal $\mathfrak{p}_2$. Let $\mathfrak{p}_2 = (1 + i)$. Therefore, $\mathbb{Z}[i]G$ has infinitely many non-isomorphic, non-free, stably free, right ideals.

In general, we see that in a UFD it is easy to construct explicit ideals $\mathfrak{p}_q$ for each $q \in Q$. Let $q = up_1 \dots p_n$, where u is a unit and p_i is a prime in k. Then, $q \in (p_1)$, so let $\mathfrak{p}_q = (p_1)$. ■

We notice that the above theorem does not apply when k is a field. It may also be difficult to construct a set Q satisfying property 4.2 if k is a domain with a lot of units. In particular, we see that if k contains the prime field $\mathbb{Q}$, then no subset of k will satisfy property 4.2.

We know by Corollary 2.8 and Corollary 4.1 that kG has non-free, stably free modules even when k is a field. We also know from the Quillen-Suslin theorem, that unless G is non-Abelian, kG has no non-free, projective modules. This tells us that the more "nice" structure we put on the group, the fewer non-free, projectives we get from the group ring. It is not crazy to think also that if k has too much "nice" structure, then maybe kG has fewer non-free, projectives. It gives us a very natural question.

Question 4.7. Can the results of Theorem 4.4 be extended to all Noetherian domains, including fields or rings with nonzero characteristic?

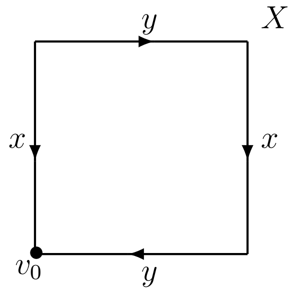
CHAPTER 5

APPLICATIONS TO THE KLEIN BOTTLE

5.1 Homotopy Classification vs. Geometric Realization

Definition 5.1. Let G be a group. A (G, n)**-complex** is a n-dimensional CW-complex with fundamental group G and vanishing higher homotopy groups up to dimension $n - 1$.[1]

Let X be the CW-complex generated from the fundamental square



that is, X is the Klein bottle, and let $G = \pi_1(X,\ v_0) \cong \langle x,\ y \ \mid\ x^{-1}yx = y^{-1} \rangle$. Thus, X is a $(G, 2)$-complex. In fact, any presentation of G gives us a $(G, 2)$-complex.

Let the single 2-cell of X be denoted by A and oriented clockwise. Then, it is easy to see that X is covered by $\mathbb{R}^2$, the real plane tessellated by this fundamental square. Now, $\mathbb{R}^2$ is a contractible space. Thus, X is aspherical.

Our concern here is to consider all possible homotopy types for finite $(G, 2)$-complexes. This investigation is analogous to the work done by Dunwoody ([5], [6]),

[1] We note that a $(G, 2)$-complex is simply a 2-complex with fundamental group G.

Dunwoody and Berridge ([2]), Harlander and Jensen ([10],[11]), and several others in the case of the trefoil knot group.

Definition 5.2. Let Y be a finite n-dimensional CW-complex. Then, the **Directed Euler Characteristic** of Y, denoted $\chi(Y)$, is the alternating sum $\sum_{k=0}^{n}(-1)^{n-k}c_k$, where c_k is the number of k-cells in Y.

From Homology theory, we see that $\chi(Y)$ is bounded below by the sum $\sum_{k=0}^{n}(-1)^{n-k}\dim H_k(G,\mathbb{Q})$. Thus, we may define $\chi_{\min}(G) = \inf\{\chi(Y)\}$, where the infimum ranges over all $(G,2)$-complexes Y.

In particular, since X is aspherical, the homology of G and X coincide and $\chi_{\min}(G) = \chi(X) = 0$. It turns out that up to homotopy, there exists a unique $(G,2)$-complex with minimum Euler Characteristic [10, Theorem 3]. Hence, the Klein bottle is the only complex on the minimal Euler Characteristic level.

It is easy to construct $(G,2)$-complexes with higher Euler characteristic. Let $X_n = X \vee (\bigvee_n \mathcal{S}^2)$; that is, X_n is the join between X and a bouquet of n-many 2-spheres. Note that $X_0 = X$. Since $\mathcal{S}^2$ is simply connected, $\pi_1(X_n) = G$. Also, joining a 2-sphere onto a complex increases the number of 2-cells by one. Thus, $\chi(X_n) = n$ and $X_n \not\simeq X_m$ if $n \neq m$.

Thus, for any 2-dimensional group H, we can construct infinitely many different homotopy classes of $(H,2)$-complexes. The question we would like to answer is how many homotopy types sit on each Euler characteristic level. The answer for the minimal level was given in [10] as mentioned above. For the trefoil group, distinct homotopy types were constructed on Euler characteristic level one above the minimal. For the case of the Klein bottle, we want to construct a $(G,2)$-complex with Euler characteristic 1 but homotopically distinct from X_1. We can do this at least algebraically.

Definition 5.3. An **algebraic** (G, n)**-complex** is an exact sequence

$$\mathcal{C}_* \quad : \quad F_n \to \ldots \to F_0 \to \mathbb{Z} \to 0$$

where the F_i are finitely generated, free $\mathbb{Z}G$-modules for $0 \leqslant i \leqslant n$. The **Directed Euler Characteristic** of $\mathcal{C}_*$, denoted $\chi(\mathcal{C}_*)$, is the alternating sum $\sum_{k=0}^{n}(-1)^{n-k}c_k$, where c_k is the rank of F_k in $\mathcal{C}_*$.

Every presentation of G gives rise to an algebraic $(G, 2)$-complex. As mentioned above, since

$$X = \left|\langle x,\ y \ \mid \ x^{-1}yx = y^{-1}\rangle\right|,$$

X has the cellular chain complex

$$\mathcal{C}_*(X) \quad : \quad \mathbb{Z}\langle A\rangle \longrightarrow \mathbb{Z}\langle x,\ y\rangle \longrightarrow \mathbb{Z}\langle v_0\rangle \to 0.$$

Then, its Cayley complex $\widetilde{X}$ is the real plane tessellated by squares. Let $\widetilde{v_0}$ be a lift of v_0 and let $\widetilde{x},\ \widetilde{y},\ \widetilde{A}$ be lifts of $x,\ y,\ A$, respectively, based at $\widetilde{v_0}$.

Since the group of deck transformations on $\widetilde{X}$ is isomorphic to G, there is a one-to-one correspondence with the lifts of v_0 and elements of G. This correspondence gives a left free G-action on the 0-cells of $\widetilde{X}$ in an obvious way. Also, G freely left-acts on the 1-cells of $\widetilde{X}$ by permuting the boundary of the path. Recursively, G freely left-acts on the higher n-cells by permuting the boundary. In order to consider a right-action, we may simply read the words in G right to left instead of left to right.

The cellular chain complex of $\widetilde{X}$ is then of the form

$$\mathcal{C}_*(\widetilde{X}) \quad : \quad \mathbb{Z}\langle\widetilde{A_i}\rangle \longrightarrow \mathbb{Z}\langle\widetilde{x}_i,\ \widetilde{y}_i\rangle \longrightarrow \mathbb{Z}\langle\widetilde{v}_i\rangle \to 0,$$

where $\widetilde{v_i},\ \widetilde{x_i},\ \widetilde{y_i},\ \widetilde{A_i}$ represents all the possible lifts of $v_0,\ x,\ y,\ A$, respectively. Since G acts freely on the cells, this action induces a free $\mathbb{Z}G$-action onto the complex. Thus, we have that the augmented cellular chain complex

$$\mathcal{C}_*(\widetilde{X}) \quad : \quad \mathbb{Z}G \xrightarrow{\partial_2} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$$

is an algebraic $(G,2)$-complex, where ε is the augmentation map, ∂_1 is the induced map from $\widetilde{x} \longmapsto \widetilde{v_0}(x-1)$ and $\widetilde{y} \longmapsto \widetilde{v_0}(y-1)$, and ∂_2 is the Fox derivative.

For the case of X_1,

$$X_1 = \left|\langle x,\ y \ \mid\ x^{-1}yx = y^{-1},\ 1\rangle\right| = X \vee \mathcal{S}^2,$$

and hence has the cellular complex

$$\mathcal{C}_*(X_1) \quad : \quad \mathbb{Z}\langle A, \mathcal{S}^2\rangle \longrightarrow \mathbb{Z}\langle x,\ y\rangle \longrightarrow \mathbb{Z}\langle v\rangle \rightarrow 0.$$

Therefore, its Cayley complex $\widetilde{X_1}$ is the real plane with a 2-sphere glued at each lift of v_0. Thus, the augmented cellular chain complex of $\widetilde{X_1}$ gives rise to the algebraic complex

$$\mathcal{C}_*(\widetilde{X_1}) \quad : \quad \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\partial_2 \oplus 0} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

Notice that the second homotopy group $\pi_2(X_1) \cong H_2(\widetilde{X_1}) = \ker(\partial_2 \oplus 0) = \mathbb{Z}G$.

By the same argument, we see that $\pi_2(X_n) \cong H_2(\widetilde{X_n}) = \mathbb{Z}G^n$ by Hurewicz's isomorphism. It is also clear that a $(G,2)$-complex with Euler characteristic n will generate an algebraic $(G,2)$-complex with algebraic Euler characteristic n. In other words, the definitions of geometric and algebraic Euler characteristic coincide.

We now will construct an algebraic $(G,2)$-complex with no obvious geometric interpretation. Let $\mathcal{K}_*$ be the exact sequence

$$\mathcal{K}_* \quad : \quad \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\partial_2 \circ \pi} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

We notice that for this complex, $H_2(\mathcal{K}_*) = K \ncong \mathbb{Z}G = H_2(\mathcal{C}_*(X_1))$. Therefore, $\mathcal{K}_*$ and $\mathcal{C}_*(X_1)$ are chain-homotopically distinct. In fact, by Theorem 4.4, we have the following result.

Theorem 5.4. *There exist infinitely many, chain-homotopically distinct, algebraic $(G,2)$-complexes with Euler characteristic 1, where G is the fundamental group of the Klein bottle.*

An important problem in Homotopy Theory is the Geometric Realization Problem, which asks whether every algebraic complex is in fact the cellular chain complex of an universal cover. In other words, we saw that every CW-complex admits an algebraic complex via the augmented cellular chain complex of its universal cover. Can the process always be reversed? It is well known that the answer is affirmative for every dimensional excluding dimension 2, where the question is still open. A method for geometric realization for dimension greater than 2 is presented in [10, Theorem 4].

Question 5.5. Is there a geometric realization to $\mathcal{K}_*$?

If the answer to Question 5.5 turns out to be negative, then $\mathcal{K}_*$ is a counterexample to the Geometric Realization Problem. Under certain circumstances, this problem is equivalent to several other problem in Topology and Group Theory, such as Wall's

$D(2)$ problem. Hence, $\mathcal{K}_*$ would provide a means for constructing counterexamples for these problems.

If $\mathcal{K}_*$ can be geometrically realized, then there would be two homotopically distinct, geometric $(G, 2)$-complexes with Euler characteristic 1; that is, Euler characteristic one above the minimal level. Since the constructions of K and K_n are extremely similar, any realization of K will probably lead to a realization of K_n. Hence, we conjecture that there would exist infinitely many, homotopically distinct, geometric $(G, 2)$-complexes with Euler characteristic 1. In particular, there would exist infinitely many presentations of G that produce different homotopy types. Thus, a way of possibly proving that $\mathcal{K}_*$ is not geometric for some K_n is to prove that the presentations of G give only finitely many homotopy types, possibly considering Nielsen equivalences.

From a mathematical viewpoint, the author would be happy with either answer to Question 5.5, so long as it can be answered.

5.2 The Freedom of $K \oplus K$

The closest we can get in this paper to answering whether $K \oplus K$ is free or not is that it is almost free. That is, for the case of the Klein bottle group G, $K \oplus K \cong L \oplus \mathbb{Z}G$, where $L \cong K \cap K^\theta$. It is the hope of the author that $K \oplus K$ will be revealed to be not free.

If $K \oplus K$ is not free, then we may construct the algebraic complex

$$\mathcal{K}_*^2 \quad : \quad \mathbb{Z}G^3 \xrightarrow{\partial_2 \circ \pi_2} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

Computing the homology, we see that $H_2(\mathcal{K}_*^2) = K \oplus K \ncong \mathbb{Z}G^2 = H_2(\mathcal{C}_*(X_2))$. Hence, there would exist chain-homotopically distinct algebraic complexes two levels above the minimal Euler characteristic for the Klein bottle group.

Also, any proof that determines the freedom of $K \oplus K$ will most likely determine the freedom of $K_n \oplus K_n$. Thus, if $K \oplus K$ is non-free, then we conjecture again $K_n \oplus K_n$ will not be free. Hence, we could construct infinitely many, non-isomorphic, non-free, stably free kG-modules of rank 2.

If $\mathcal{K}_*$ can be geometrically realized, then $\mathcal{K}_*^2$ can be realized as well. Thus, there would exist homotopically distinct geometric complexes two levels above the minimal Euler characteristic for the Klein bottle group. This would then answer the question posed by Harlander and Jensen at the end of [10].

In addition, we know that $L \oplus \mathbb{Z}G^2 \cong \mathbb{Z}G^3$. So, if $K \oplus K$ is non-free, then $L \oplus \mathbb{Z}G \ncong \mathbb{Z}G^2$ and L is a non-free, stably free $\mathbb{Z}G$-module requiring a minimum of two additional free summands to be free. Such a module has not yet been constructed to the author's knowledge.

5.3 The Algebraic Euler Characteristic Tree for the Klein Bottle

Our goal will be to eventually identify how many homotopy types exists on each geometric Euler characteristic level. We have a good grasp for what happens for the algebraic $(G, 2)$-complexes.

First, let us remember that each stably free $\mathbb{Z}G$-module is the kernel of an epimorphism $\mathbb{Z}G^n \to \mathbb{Z}G^m$. In particular, these epimorphisms can be represented as matrices with elements from $\mathbb{Z}G$. Now, $\mathbb{Z}$ and G are both countable sets. Thus, $\mathbb{Z}G$

is also countable and the set of all finite $\mathbb{Z}G$-matrices is also countable. We summarize this in the following proposition.

Proposition 5.6. *There are at most countably many stably free kG-modules when kG is countable.*

In a similar train of thought, we may count the number of algebraic (G, n)-complexes. We can consider each algebraic complex as a pair of $(n+1)$-tuples. The first $(n+1)$-tuple is the list of corresponding ranks of free modules. The second is a list of boundary maps $F_i \to F_{i-1}$, for $0 \leqslant i \leqslant n$ and $F_{-1} = \mathbb{Z}$. Now we may consider each map as a matrix. The collection of $(n+1)$-tuples of a countable set is countable, and the collection of matrices is countable if the set of scalars is countable, like above. We again summarize.

Proposition 5.7. *There are only countably many algebraic (G, n)-complexes when G is countable.*

We note that if $\mathcal{C}_*$ is an algebraic $(G, 2)$-complex, then its homology is trivial except possibly dimension 2. We know that $H_2(\mathcal{C}_*)$ is a stably free module. Now, $\chi(\mathcal{C}_*)$ already tells us a lot about $H_2(\mathcal{C}_*)$.

Theorem 5.8. *Let $\mathcal{C}_*$ be an algebraic $(G, 2)$-complex. Then, the rank of $H_2(\mathcal{C}_*)$ is equal to $\chi(\mathcal{C}_*)$. In particular, if $\chi(\mathcal{C}_*) = 0$, then $H_2(\mathcal{C}_*)$ is trivial.*

Proof. Let $\mathcal{C}_*$ be an algebraic complex with $\chi(\mathcal{C}_*) = n$. Thus, there exists maps and ranks n_0, n_1, n_2, such that

$$\mathcal{C}_* \quad : \quad \mathbb{Z}G^{n_2} \longrightarrow \mathbb{Z}G^{n_1} \longrightarrow \mathbb{Z}G^{n_0} \longrightarrow \mathbb{Z} \longrightarrow 0$$

is exact. We compare this complex with the algebraic complex of X_n,

$$\mathcal{C}_*(\widetilde{X_n}) \quad : \quad \mathbb{Z}G^{n+1} \longrightarrow \mathbb{Z}G \oplus \mathbb{Z}G \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0.$$

By the Euler characteristic, we know that $n_0 - n_1 + n_2 = n$.

If we join on the kernels of the second boundary, we have the exact sequences

$$\begin{array}{ll} \mathcal{C}_* : & 0 \longrightarrow H_2\left(\mathcal{C}_*\right) \longrightarrow \mathbb{Z}G^{n_2} \longrightarrow \mathbb{Z}G^{n_1} \longrightarrow \mathbb{Z}G^{n_0} \longrightarrow \mathbb{Z} \longrightarrow 0 \\ \mathcal{C}_*(\widetilde{X_n}) : & 0 \longrightarrow \mathbb{Z}G^{n} \longrightarrow \mathbb{Z}G^{n+1} \longrightarrow \mathbb{Z}G^{2} \longrightarrow \mathbb{Z}G \longrightarrow \mathbb{Z} \longrightarrow 0. \end{array}$$

We may now use the Generalized Schanuel's Lemma (see [4, Lemma 4.4]) and see that

$$H_2\left(\mathcal{C}_*\right) \oplus \mathbb{Z}G^{n+1} \oplus \mathbb{Z}G^{n_1} \oplus \mathbb{Z}G \cong \mathbb{Z}G^{n} \oplus \mathbb{Z}G^{n_2} \oplus \mathbb{Z}G^{2} \oplus \mathbb{Z}G^{n_0}.$$

By IBN, we have that

$$\operatorname{rank} H_2\left(\mathcal{C}_*\right) + (n+1) + n_1 + 1 = n + n_2 + 2 + n_0.$$

Solving for $\operatorname{rank} H_2\left(\mathcal{C}_*\right)$, we can conclude

$$\begin{aligned} \operatorname{rank} H_2\left(\mathcal{C}_*\right) &= n + n_2 + 2 + n_0 - ((n+1) + n_1 + 1) \\ &= (n-n) + (2-1-1) + (n_0 - n_1 + n_2) \\ &= n \end{aligned}$$

In particular, if $\chi\left(\mathcal{C}_*\right) = 0$, then the rank $H_2\left(\mathcal{C}_*\right) = 0$. Thus, there exists a natural number m, such that,

$$\begin{aligned} \mathbb{Z}G^{m+0} &\cong H_2(\mathcal{C}_*) \oplus \mathbb{Z}G^m \\ \mathbb{Z}G^m &\cong H_2(\mathcal{C}_*) \oplus \mathbb{Z}G^m \\ &\Rightarrow H_2(\mathcal{C}_*) = 0. \quad \square \end{aligned}$$

We next describe each algebraic Euler characteristic level of the Klein bottle.

5.3.1 Minimal Euler Characteristic Level, $\chi(\mathcal{C}_*) = 0$

By Theorem 5.8, $\ker \partial_2 = 0$ for each complex on the minimal level. This means that, in fact, each complex on this level is a free resolution of $\mathbb{Z}$ over $\mathbb{Z}G$. But resolutions are unique up to chain-homotopy [4, Theorem 7.5]. Therefore, this level contains a unique complex corresponding to the algebraic complex of the Klein bottle.

5.3.2 Euler Characteristic Level One Above Minimal, $\chi(\mathcal{C}_*) = 1$

By Theorem 4.4, there are at least countably many non-isomorphic stably free $\mathbb{Z}G$-modules of rank 1. Now, there are only countably many algebraic $(G, 2)$-complexes total by Proposition 5.7, which implies that there are at most countably many on each level. Therefore, there are countably many chain-homotopy types on this level.

5.3.3 Euler Characteristic Level Two Above Minimal, $\chi(\mathcal{C}_*) = 2$

If $K \oplus K$ is non-free, then we will have chain-homotopically distinct complexes on this level. Most likely, if $K \oplus K$ is non-free, then we will have countably many chain-homotopy types on this level. At least we know this level is non-empty since it contains $\mathcal{C}_*(X_2)$.

5.3.4 Euler Characteristic Level Three Above Minimal, $\chi(\mathcal{C}_*) = 3$

Unfortunately, K^3 is free and we have no method here is determine the number of chain-homotopy types. The best we can say now is that it is non-empty like on level 2. Though, the author's guess is that this level will either contain countably many or a unique chain-homotopy type.

5.3.5 Euler Characteristic Level Four and Above, $\chi(\mathcal{C}_*) = 4$

By results from [3], any stably free $\mathbb{Z}G$-module of rank 4 or larger is free. Therefore, the homology of a complex on one of these levels in determined entirely by its Euler characteristic. Each of these levels are non-empty. Hence, if these levels contain distinct elements, another invariant than homology must be used to distinguish them. The author guesses that each of these levels are trivial.

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APPENDIX A

SPANNING PROPERTIES

Definition A.1. Let R be a ring and let V be a right R-module. Let $v_1, v_2, \ldots, v_n \in V$. Define the module

$$\langle v_1, v_2, \ldots, v_n \rangle R = v_1 R + v_2 R + \ldots + v_n R,$$

where $v_i R = \{ v_i r \mid r \in R \}$. We call this the **right-span** of $\{v_1, v_2, \ldots, v_n\}$. The set $\{v_1, v_2, \ldots, v_n\}$ **right-spans** V or is a **right-spanning set**[1] of V iff $\langle v_1, v_2, \ldots, v_n \rangle R = V$. The set $\{v_1, v_2, \ldots, v_n\}$ is **right-linearly independent**[2] iff the only solution to the equation $v_1 r_1 + v_2 r_2 + \ldots + v_n r_n = 0$ is $r_i = 0$ for all i.

The corresponding left definitions are defined mutatis mutandis.

The set $\{r,\ x + s\}$, as we saw in the proof of Theorem 2.4, has a very important property.

$$\begin{aligned} r(sx^{-2}) + (x+s)(x^{-1} - rx^{-2}) &= rsx^{-2} + 1 - xrx^{-2} + sx^{-1} - srx^{-2} \\ &= 1 - sxx^{-2} + sx^{-1} \end{aligned}$$

[1]As it is also popular in literature, the term generating may be used instead of spanning.

[2]It is noteworthy to mention that right-linearly independence is not equivalent to left-linear independence. I refer the curious reader to [12, p. 12] for an example of this phenomenon.

$$= \quad 1 \in \langle r,\ x+s\rangle kG$$

Thus, $\langle r,\ x+s\rangle kG = kG$, or we can say $\{r,\ x+s\}$ right-spans kG. This is of course the same thing as saying π is surjective.

From equation (2.2) on page 8, we see a similar spanning property about an equally-important set $\{s,\ x-r\}$. By equation (2.2), we have

$$rs + (x+s)(x-r) = x^2$$

Now x^2 is a unit in kG and by multiplying both sides of the equation by x^{-2}, we see that

$$(x^{-2}r)s + (x^{-1} + x^{-2}s)(x-r) = 1 \tag{A.1}$$

Thus, $\{s,\ x-r\}$ left-spans kG. We could also say since the set $\{sx^{-2},\ x^{-1} - rx^{-2}\}$ clearly left-spans by equation (2.2) and since the two sets differ by only associates that $\{s,\ x-r\}$ left-spans too.

The following theorem shows that certain assumptions on the automorphism σ guarantee that $\{s,\ x-r\}$ right-spans too.

Theorem A.2. *If σ is of finite order, then the set $\{s,\ x-r\}$ right-spans kG. In particular, $\{s,\ x-r\}$ is a double-sided spanning set.*

Proof. Let kG_{j-1} be defined as in Section 2.5 on page 36. So, $r \in kG_{j-1}$. Let $n \in \mathbb{Z}$. Consider the following term:

$$r^{\sigma^{n-1}} \left(\prod_{k=0}^{n-2} r^{\sigma^k} \right) x^{-n}$$

Since kG_{j-1} is a commutative ring, we may rearrange the r^{σ^k}, in particular

$$r^{\sigma^{n-1}} \left(\prod_{k=0}^{n-2} r^{\sigma^k} \right) x^{-n} = \left(\prod_{k=0}^{n-1} r^{\sigma^k} \right) x^{-n} \tag{A.2}$$

Likewise, consider another product:

$$(x-r) \left(x^{-1} + \sum_{i=2}^{n} \left(\prod_{j=1}^{i-1} r^{\sigma^j} \right) x^{-i} \right)$$

Obviously, by distribution, we have

$$(x-r) \left(x^{-1} + \sum_{i=2}^{n} \left(\prod_{j=1}^{i-1} r^{\sigma^j} \right) x^{-i} \right) = (1 - rx^{-1}) + \underbrace{x \sum_{i=2}^{n} \left(\prod_{j=1}^{i-1} r^{\sigma^j} \right) x^{-i}}_{*} \tag{A.3}$$

$$\underbrace{-\, r \sum_{i=2}^{n} \left(\prod_{j=1}^{i-1} r^{\sigma^j} \right) x^{-i}}_{**}$$

Some rearrangement of the terms $*$ and $**$ is necessary for simplicity. Let us begin with $*$.

$$\begin{aligned} x \sum_{i=2}^{n} \left(\prod_{j=1}^{i-1} r^{\sigma^j} \right) x^{-i} &= \sum_{i=2}^{n} x \left(\prod_{j=1}^{i-1} r^{\sigma^j} \right) x^{-i} \\ &= \sum_{i=2}^{n} \left(\prod_{j=1}^{i-1} r^{\sigma^{j-1}} \right) x x^{-i} \\ &= \sum_{i=2}^{n} \left(\prod_{j=0}^{i-2} r^{\sigma^j} \right) x^{-(i-1)} \end{aligned}$$

The second line results from the relation $xr = r^{\sigma^{-1}} x$ and the third line is a re-indexing of the product.

Next, we simplify $**$.

$$
\begin{aligned}
r \sum_{i=2}^{n} \left(\prod_{j=1}^{i-1} r^{\sigma^j} \right) x^{-i} &= \sum_{i=2}^{n} r \left(\prod_{j=1}^{i-1} r^{\sigma^j} \right) x^{-i} \\
&= \sum_{i=2}^{n} \left(\prod_{j=0}^{i-1} r^{\sigma^j} \right) x^{-i}
\end{aligned}
$$

Now, we change the indices of $*$ so that they coincide with those in $**$. So, we combine the sums and simplify the summands.

$$
\sum_{i=2}^{n} \left(\prod_{j=0}^{i-2} r^{\sigma^j} \right) x^{-(i-1)} - \sum_{i=2}^{n} \left(\prod_{j=0}^{i-1} r^{\sigma^j} \right) x^{-i} =
$$

$$
\sum_{i=2}^{n} \left(\left(\prod_{j=0}^{i-2} r^{\sigma^j} \right) x^{-(i-1)} - \left(\prod_{j=0}^{i-1} r^{\sigma^j} \right) x^{-i} \right) = \left(r x^{-1} - \left(\prod_{j=0}^{n-1} r^{\sigma^j} \right) x^{-n} \right) \tag{A.4}
$$

The last equation is a result of the telescoping summands.

We come now to the heart of the matter. By the previous results just shown, we know the following equation holds:

$$
r^{\sigma^{n-1}} \left(\prod_{k=0}^{n-2} r^{\sigma^k} \right) x^{-n} + (x-r) \left(x^{-1} + \sum_{i=2}^{n} \left(\prod_{j=1}^{i-1} r^{\sigma^j} \right) x^{-i} \right) = 1
$$

Let's see how this comes about in a little more detail.

$$
\begin{aligned}
& r^{\sigma^{n-1}} \left(\prod_{k=0}^{n-2} r^{\sigma^k} \right) x^{-n} + (x-r) \left(x^{-1} + \sum_{i=2}^{n} \left(\prod_{j=1}^{i-1} r^{\sigma^j} \right) x^{-i} \right) \\
= & \left(\prod_{k=0}^{n-1} r^{\sigma^k} \right) x^{-n} + (1 - r x^{-1}) + x \sum_{i=2}^{n} \left(\prod_{j=1}^{i-1} r^{\sigma^j} \right) x^{-i} - r \sum_{i=2}^{n} \left(\prod_{j=1}^{i-1} r^{\sigma^j} \right) x^{-i} \\
= & \left(\prod_{k=0}^{n-1} r^{\sigma^k} \right) x^{-n} + (1 - r x^{-1}) + \left(r x^{-1} - \left(\prod_{j=0}^{n-1} r^{\sigma^j} \right) x^{-n} \right)
\end{aligned}
$$

$$\begin{aligned}
&= 1 + \left(\left(\prod_{k=0}^{n-1} r^{\sigma^k} \right) x^{-n} - \left(\prod_{j=0}^{n-1} r^{\sigma^j} \right) x^{-n} \right) + (-rx^{-1} + rx^{-1}) \\
&= 1
\end{aligned}$$

where the first equality holds by (A.2) and (A.3), and the next equality holds by (A.4).

Remember $s = r^{\sigma^{-1}}$ by definition. Up until now, we have not used the assumption on σ. Let n be the order of σ since it is finite. Then, $s = r^{\sigma^{-1}} = r^{\sigma^{n-1}}$. Therefore, we have

$$s \left(\prod_{k=0}^{n-2} r^{\sigma^k} \right) x^{-n} + (x - r) \left(x^{-1} + \sum_{i=2}^{n} \left(\prod_{j=1}^{i-1} r^{\sigma^j} \right) x^{-i} \right) = 1 \tag{A.5}$$

Hence, $\langle s,\ x - r\rangle kG = kG$. □

Example A.3. Let σ be order 2, like in the case of the Klein bottle. Then, equation (A.5) looks as follows:

$$\begin{aligned}
& s \left(\prod_{k=0}^{2-2} r^{\sigma^k} \right) x^{-2} + (x - r) \left(x^{-1} + \sum_{i=2}^{2} \left(\prod_{j=1}^{i-1} r^{\sigma^j} \right) x^{-i} \right) \\
&= s \left(r^{\sigma^0} \right) x^{-2} + (x - r) \left(x^{-1} + \left(\prod_{j=1}^{2-1} r^{\sigma^j} \right) x^{-2} \right) \\
&= srx^{-2} + (x - r) \left(x^{-1} + \left(r^{\sigma^1} \right) x^{-2} \right) \\
&= srx^{-2} + (x - r) \left(x^{-1} + r^{\sigma} x^{-2} \right) \\
&= srx^{-2} + (1 + xr^{\sigma}x^{-2} - rx^{-1} - rr^{\sigma}x^{-2}) \\
&= 1 + (srx^{-2} - rr^{\sigma}x^{-2}) + (rxx^{-2} - rx^{-1}) \\
&= 1 + (srx^{-2} - r(s)x^{-2}) + (rx^{-1} - rx^{-1}) = 1 \blacksquare
\end{aligned}$$

Example A.4. For a little more complicated example, let us consider the case when

σ is order 4. Then, according to equation (A.5),

$$
\begin{aligned}
& s\left(\prod_{k=0}^{2} r^{\sigma^k}\right) x^{-4} + (x-r)\left(x^{-1} + \sum_{i=2}^{4}\left(\prod_{j=1}^{i-1} r^{\sigma^j}\right) x^{-i}\right) \\
= \;& s\left(r^{\sigma^0} r^{\sigma^1} r^{\sigma^2}\right) x^{-4} \\
& \qquad + (x-r)\left(x^{-1} + \left(\prod_{j=1}^{1} r^{\sigma^j}\right) x^{-2} + \left(\prod_{j=1}^{2} r^{\sigma^j}\right) x^{-3} + \left(\prod_{j=1}^{3} r^{\sigma^j}\right) x^{-4}\right) \\
= \;& r^{\sigma^3}\left(r r^{\sigma} r^{\sigma^2}\right) x^{-4} + (x-r)\left(x^{-1} + \left(r^{\sigma}\right) x^{-2} + \left(r^{\sigma} r^{\sigma^2}\right) x^{-3} + \left(r^{\sigma} r^{\sigma^2} r^{\sigma^3}\right) x^{-4}\right) \\
= \;& r r^{\sigma} r^{\sigma^2} r^{\sigma^3} x^{-4} + (1 - r x^{-1}) + (x r^{\sigma} x^{-2} - r r^{\sigma} x^{-2}) + (x r^{\sigma} r^{\sigma^2} x^{-3} - r r^{\sigma} r^{\sigma^2} x^{-3}) \\
& \qquad + (x r^{\sigma} r^{\sigma^2} r^{\sigma^3} x^{-4} - r r^{\sigma} r^{\sigma^2} r^{\sigma^3} x^{-4}) \\
= \;& r r^{\sigma} r^{\sigma^2} r^{\sigma^3} x^{-4} + (1 - r x^{-1}) + (r x^{-1} - r r^{\sigma} x^{-2}) + (r r^{\sigma} x^{-2} - r r^{\sigma} r^{\sigma^2} x^{-3}) \\
& \qquad + (r r^{\sigma} r^{\sigma^2} x^{-3} - r r^{\sigma} r^{\sigma^2} r^{\sigma^3} x^{-4}) = 1 \;\blacksquare
\end{aligned}
$$

The proof of Theorem A.2 was constructive in that we created explicit coefficients from kG. But it is clear that this argument cannot be generalized to include automorphisms of infinite order. In fact, a completely different plan of attack would be needed to extend the results of Theorem A.2.

Question A.5. Can the assumption on the map σ in Theorem A.2 be removed?