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Weak Covering Properties and Infinite Games

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ABSTRACT. We investigate game-theoretic properties of selection principles related to weaker forms of the Menger and Rothberger properties. For appropriate spaces some of these selection principles are characterized in terms of a corresponding game. We use generic extensions by Cohen reals to illustrate the necessity of some of the hypotheses in our theorems.

1. INTRODUCTION

A *Lindelöf space* is a topological space for which each open cover contains a countable subset covers the space. Some unresolved or undecidable problems about Lindelöf spaces have definite answers for spaces having a stronger version of the Lindelöf property. For example: Cardinality questions that are undecidable for classes of Lindelöf spaces have been resolved for the corresponding classes of *compact* spaces. While it is not known if regular T_3 Lindelöf spaces are D-spaces[1] it is known that each T_1 *Menger* space (a selective version of the Lindelöf property and defined later in our paper) is a D-space [1]. Whereas the Lindelöf property is not preserved by the product construction, compactness is.

In some cases where the Lindelöf property (or one of its strengthenings) is not preserved by a topological construction, it has been found that a weaker version of the Lindelöf property is preserved. Also, it has been found that some theorems true for Lindelöf spaces are in fact true for a wider class of spaces that have a weak version of the Lindelöf property. We consider two such weakenings of the Lindelöf property that have emerged in the literature.

To define these, let $\mathcal{O}$ denote the collection of all open covers of a space, let $\mathcal{D}$ denote the collection of families of open sets with union dense in the space, and let $\overline{\mathcal{O}}$ denote the collection of families $\mathcal{U}$ of open subsets of the space for which $\{\overline{U} : U \in \mathcal{U}\}$ covers the space[2]. A space is said to be *weakly Lindelöf* if each open cover contains a countable subset with union dense in the space. Thus, weakly Lindelöf means that each element of $\mathcal{O}$ has a countable subset which is a member of $\mathcal{D}$. The notion of weakly Lindelöf appears to have been introduced in the 1959 paper [10] of Frolik. The name "weakly Lindelöf" however seems to have been coined by Hager and Mrowka (see the introductory paragraph of [6]). A thorough introduction to weakly Lindelöf spaces can be found in [6]. A topological space X is said to be *almost Lindelöf* if there is for each open cover $\mathcal{U}$ of X a countable subset $\mathcal{V}$ such that $\{\overline{V} : V \in \mathcal{V}\}$ is a cover of X. The notion of an almost Lindelöf space

[1]As we do not need the concept of a D-space elsewhere in our paper, it is left undefined.

[2]The symbol $\overline{V}$ denotes the closure of the set V. The notation $\overline{\mathcal{O}}$ is due to Boaz Tsaban.

was introduced in the 1984 paper [36] by Willard and Dissanyake. The following implications hold among these three properties:

$$\text{Lindelöf} \Rightarrow \text{almost Lindelöf} \Rightarrow \text{weakly Lindelöf}.$$

Moreover, almost Lindelöf T_3-spaces are Lindelöf.

Various selective versions of the Lindelöf property, for example the Menger property or the Rothberger property (defined below), have characterizations in terms of infinite games. These game characterizations are powerful tools to derive other mathematical properties of these classes of spaces. We investigate the possibility of characterizing certain selective versions of the almost Lindelöf and the weakly Lindelöf properties by infinite games. To indicate to what extent some of the hypotheses of some of our results are necessary we also explore the preservation of selective versions of these properties under generic extensions of the universe. Behavior of these properties under topological constructions will be addressed in the paper [3].

2. Definitions

Recall the general framework for describing selection hypotheses (as in [15] and [23]): Let $\mathbf{N}$ denote the set of positive integers. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of an infinite set. Then $\mathsf{S}_1(\mathcal{A}, \mathcal{B})$ denotes the following hypothesis:

> For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that, for each n, $B_n \in A_n$ and $\{B_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

Then $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ denotes the *Rothberger* property.

The symbol $\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$ similarly denotes the hypothesis

> For each sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(B_n : n \in \mathbb{N})$ such that, for each n, $B_n \subseteq A_n$ is finite, and $\bigcup\{B_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$ denotes the *Menger* property.

Our conventions for the rest of the paper are: By "space" we mean a topological space. Unless stronger separation axioms are indicated specifically, we assume all spaces to be infinite and T_1. Undefined notation and terminology will be as in [9].

3. Preserving Lindelöf like properties in Cohen real generic extensions

It is well known that

Theorem 1. *Let $\mathbb{P}$ be a proper partial order and let X be a space.*

(1) *If X is non-Lindelöf, then $\mathbf{1}_{\mathbb{P}} \Vdash$ "$\check{X}$ is not Lindelöf".*
(2) *If X is not almost Lindelöf, then $\mathbf{1}_{\mathbb{P}} \Vdash$ "$\check{X}$ is not almost Lindelöf".*
(3) *If X is not weakly Lindelöf, then $\mathbf{1}_{\mathbb{P}} \Vdash$ "$\check{X}$ is not weakly Lindelöf".*

Proof. These statements follow directly from Proposition 4.1 of [14]: Consider a ground model that satisfies ZFC. If $\mathbb{P}$ is a proper partially ordered set in this ground model and S is a ground model set, then for a countable set Y that is a subset of S but a member of the generic extension, there is a countable set Z that is a subset of S and is a member of the ground model, and contains Y. $\square$

The Lindelöf property is preserved by adding Cohen reals or random reals (see for example [30]), but need not be preserved by countably closed forcing [33] or

even certain ccc forcing extensions [12]. More information about these issues can be found in [16], [30] and [33]. We now establish some corresponding results for the almost Lindelöf and the weakly Lindelöf properties.

For κ a fixed uncountable cardinal, $\mathbb{C}(\kappa)$ denotes the partially ordered set for adding κ Cohen reals generically: The underlying set, $\mathsf{Fn}(\kappa \times \omega, \omega)$, of elements of $\mathbb{C}(\kappa)$ is

$$\{p \subset (\kappa \times \omega) \times \omega : p \text{ a finite function with domain a subset of } \kappa \times \omega\}$$

For p and q in $\mathsf{Fn}(\kappa \times \omega, \omega)$ we write $p < q$ if $q \subset p$. It is well known that forcing with $\mathbb{C}(\kappa)$ preserves cardinals and cofinalities, and that in the resulting model $2^{\aleph_0} \geq \kappa$. Since $\mathbb{C}(\kappa)$ is a proper partially ordered set, Theorem 1 applies to it.

Definition 2. *For a positive integer n, an n-dowment is a family $\mathcal{L}_n$ of finite antichains of $\mathbb{C}(\kappa)$ such that:*

  (1) *For each maximal antichain $\mathbb{A} \subset \mathbb{C}(\kappa)$ there is an $L \in \mathcal{L}_n$ such that $L \subseteq \mathbb{A}$*
  (2) *For each $p \in \mathbb{C}(\kappa)$ such that $|dom(p)| < n$ and for every collection $L_1, L_2, \cdots L_n \in \mathcal{L}_n$ there are $q_1 \in L_1, \cdots, q_n \in L_n$, and there is $r \in \mathbb{C}(\kappa)$, such that $r \leq p$ and $r \leq q_i$ for each $i \leq n$.*

The notion of an n-dowment as well as a proof of the existence of n-dowments is treated in Section 1 of [8].

Preserving weakly Lindelöf

Theorem 3. *Let $\kappa > \aleph_0$ be a cardinal. Let X be a topological space. If X is weakly Lindelöf, then in $\mathsf{V}^{\mathbb{C}(\kappa)}$ X is weakly Lindelöf.*

Proof. For each n let $\mathcal{L}_n$ be an n-dowment for $\mathbb{C}(\kappa)$. Let τ denote the (ground model) topology of X. Let $\dot{\mathcal{U}}$ be a $\mathbb{C}(\kappa)$-name such that

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{``}\dot{\mathcal{U}} \subseteq \check{\tau} \text{ is an open cover of } \check{X}\text{''}.$$

For each x choose a maximal antichain $\mathbb{A}_x \subseteq \mathbb{C}(\kappa)$ and ground model open sets $V_{p,x}$, $p \in \mathbb{A}_x$, such that $x \in V_{p,x}$ and $p \Vdash \text{``}\check{V}_{p,x} \in \dot{\mathcal{U}}\text{''}$. For each n choose $\mathbb{A}_{n,x} \in \mathcal{L}_n$ with $A_{n,x} \subseteq A_x$ and define $V_{n,x} = \cap\{V_{p,x} : p \in \mathbb{A}_{n,x}\}$. Then $V_{n,x}$ is open, and $\mathcal{U}_n = \{V_{n,x} : x \in X\}$ is an open cover of X in the ground model.

Since X is weakly Lindelöf, choose a countable set $\mathcal{V}_n \subseteq \mathcal{U}_n$ such that $\bigcup \mathcal{V}_n$ is dense in X.

Claim: $\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{``}(\forall V \in \check{\tau})(\exists n)(\exists U \in \check{\mathcal{V}}_n)(\exists \check{W} \in \dot{\mathcal{U}})(V \cap U \neq \emptyset \text{ and } U \subseteq \check{W})\text{''}.$

Fix $V \in \tau$ nonempty and fix an element $q \in \mathbb{C}(\kappa)$. Choose n so that $|dom(q)| \leq n$. Since $\bigcup \mathcal{V}_n$ is dense in X, choose $V_{n,x} \in \mathcal{V}_n$ such that $\emptyset \neq V \cap V_{n,x}$. Choose $p \in \mathbb{A}_{n,x}$ such that for an $r \in \mathbb{C}(\kappa)$ we have $r \leq q, p$. Then

$$r \Vdash \text{``}\check{V} \cap \check{V}_{n,x} \neq \emptyset \text{ and } \check{V}_{n,x} \subseteq \check{V}_{p,x} \in \dot{\mathcal{U}}\text{''}$$

Thus the set of $r \in \mathbb{C}(\kappa)$ forcing the statement in the claim is dense in $\mathbb{C}(\kappa)$. The claim follows.

Now let G be a $\mathbb{C}(\kappa)$-generic filter and let V be a ground model open set. In the generic extension we find an n and a $U \in \mathcal{V}_n$ and a $W \in \dot{\mathcal{U}}_G$ with $V \cap U \neq \emptyset$ and $U \subseteq W$. Thus, choose for each n and each $U \in \mathcal{V}_n$ for which this is possible, a $W_U \in \dot{\mathcal{U}}_G$ with $U \subseteq W_U$. Then $\{W_U : (\exists n)(U \in \mathcal{V}_n)\}$ is a countable subset of $\dot{\mathcal{U}}_G$ and has dense union in X. $\square$

Preserving almost Lindelöf

Since T_3 almost Lindelöf spaces are Lindelöf, and since the Lindelöf property of the T_3 product space $^{\omega_1}2$ is not preserved by forcing with countably closed partially ordered sets, the almost Lindelöf property is not preserved by countably closed forcing. This phenomenon can also occur in generic extensions obtained by forcing with certain countable chain condition partially ordered sets. However, generic extensions by Cohen reals preserve the property of being almost Lindelöf.

Lemma 4. *Let $\kappa > \aleph_0$ be a cardinal number and let X be a topological space and let $U \subseteq X$ be an open subset of X. Then*

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{``} \overline{\check{U}} = \check{\overline{U}} \text{''}.$$

Proof. Note that the ground model open subsets of X is a basis for the topology of X in the generic extension. This implies that the ground model closure of a ground model set is equal to the closure of that ground model set in the generic extension. This argument applies to all generic extensions, not only the Cohen real extension. □

Using Lemma 4 and minor modifications in the proof of Theorem 3, one also obtains the following preservation theorem:

Theorem 5. *Let $\kappa > \aleph_0$ be a cardinal. If a topological space X is almost Lindelöf, then in $\mathsf{V}^{\mathbb{C}(\kappa)}$ X is almost Lindelöf.*

4. The Rothberger property and weakenings

For the properties considered in this section the following table contains the property name, its symbolic definition and, where available, a reference for where the property was introduced:

Property name	Definition	Source
Rothberger	$\mathsf{S}_1(\mathcal{O}, \mathcal{O})$	[22]
Almost Rothberger	$\mathsf{S}_1(\mathcal{O}, \overline{\mathcal{O}})$	[26], p. 251
Weakly Rothberger	$\mathsf{S}_1(\mathcal{O}, \mathcal{D})$	[7], p. 98

The implications

$$\text{Rothberger} \Rightarrow \text{Almost Rothberger} \Rightarrow \text{weakly Rothberger}$$

are irreversible, but almost Rothberger T_3-spaces are Rothberger spaces. It is clear that the Rothberger property implies the Lindelöf property, the almost Rothberger property implies the almost Lindelöf property, and the weakly Rothberger property implies the weakly Lindelöf property. The weakly Rothberger property and the almost Rothberger property have not been as extensively studied as the Rothberger property. The related properties $\mathsf{S}_1(\mathcal{D}, \mathcal{D})$, $\mathsf{S}_1(\overline{\mathcal{O}}, \mathcal{D})$ and $\mathsf{S}_1(\overline{\mathcal{O}}, \overline{\mathcal{O}})$ have received some attention, but we will not report on these properties here.

The symbol $\mathsf{G}_1^\omega(\mathcal{A}, \mathcal{B})$ denotes the following game: Players ONE and TWO play ω innings: In inning $n < \omega$ ONE first selects an O_n from $\mathcal{A}$, and then TWO responds with a $T_n \in O_n$. A play

$$O_0, T_0, \cdots, O_n, T_n, \cdots$$

is won by TWO if $\{T_n : n < \omega\} \in \mathcal{B}$. Otherwise, ONE wins.

For a specific instance of these games either TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{A}, \mathcal{B})$, or ONE has a winning strategy in the game, or else neither player has a winning strategy in the game. In the last case the game is said to be *undetermined*.

The game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{O})$ was introduced by Galvin [11]. In [11] Galvin considers winning strategies for TWO in the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{O})$ and he proves:

Theorem 6 (Galvin). *Let X be a Lindelöf space such that each one-element subset of X is a G_δ subset of X. Then the following are equivalent:*

(1) *TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{O})$ on X.*
(2) *X is a countable set.*

In [20] Pawlikowski proved the following fundamental theorem, characterizing the Rothberger property in terms of games:

Theorem 7 (Pawlikowski). *For a space X the following are equivalent:*

(1) *X has property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$.*
(2) *ONE has no winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{O})$.*

Thus, in the class of Rothberger spaces whose singleton subsets are G_δ sets, the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{O})$ is undetermined if, and only if, the space is uncountable.

Tkachuk considered the game of length ω for the weakly Rothberger property in [35], denoted there as the game Θ_*. Games seem to be unexamined for the almost Rothberger property. We now explore to what extent analogues of Theorem 6 and Theorem 7 hold for the weakly Rothberger spaces and the almost Rothberger spaces.

Games and the weakly Rothberger property

The following observation is essentially Proposition 2.6(iii) of [35]:

Lemma 8. *If X has a dense subspace Y and TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$ on Y, then TWO has a winning strategy in $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$ on X.*

It follows that on each separable space TWO has a winning strategy in $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$. We do not currently have a general characterization of spaces for which TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$. Theorem 9 below is the most general result we know that is for weakly Rothberger spaces an analogue of Theorem 6 for Rothberger spaces. Theorem 9 has been obtained earlier by Tkachuk and can be deduced from [35] Theorem 2.11(i) plus Theorem 3.3(2). Since Tkachuk derives this for a game dual of $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$ (Theorem 2.11(i) and then introduces and proves the duality (Theorem 3.3(2)), we decided to include a new proof customized to our context here. Also note that although the blanket assumption in [35] is that spaces are at least Tychonoff, the arguments in [35] also work for the wider class of T_3 spaces.

Theorem 9 (Tkachuk). *If X is a first-countable T_3 space the following are equivalent:*

(1) *X is separable.*
(2) *TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$.*

Proof. The implication (1)$\Rightarrow$(2) follows from the preceding remarks. We must prove that (2)$\Rightarrow$ (1). Thus, let X be a first countable topological space in which TWO has a winning strategy, σ, in the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$. Let $\prec$ be a well-order of X. Let θ be an infinite regular cardinal which is so large that X, its topology, $\mathcal{O}$, $<\omega\mathcal{O}$, $\mathcal{D}$, $\prec$, σ, and for each $x \in X$ a neighborhood base $(U_n(x) : n \in \mathbb{N})$ of $\{x\}$, are elements of H_θ. We may assume that $\mathcal{N} = \{(U_n(x) : n \in \mathbb{N}) : x \in X\}$ is a member of H_θ. Let $(\mathcal{M}, \in_\mathcal{M})$ be a countable elementary submodel of $(\mathsf{H}_\theta, \in_\theta)$ such that each of the objects above is an element of $\mathcal{M}$.

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Claim 1: For each finite sequence $(\mathcal{U}_1, \cdots, \mathcal{U}_k)$ of open covers of X there is a point $x \in X$ such that for each neighborhood U of x there is an open cover $\mathcal{U}$ such that $U = \sigma(\mathcal{U}_1, \cdots, \mathcal{U}_k, \mathcal{U})$.

The argument for the proof of this claim is originally due to Galvin [11], and goes as follows: For suppose the contrary. Then choose for each $x \in X$ a neighborhood U_x such that for each open cover $\mathcal{U}$ of X, $U_x \neq \sigma(\mathcal{U}_1, \cdots, \mathcal{U}_k, \mathcal{U})$. But then as $\mathcal{V} = \{U_x : x \in X\}$ is an open cover of X, we have that for some $y \in X$, $\sigma(\mathcal{U}_1, \cdots, \mathcal{U}_k, \mathcal{V}) = U_y$. This contradicts the selection of U_y, completing the proof of Claim 1.

The sentence in Claim 1 holds in $(\mathsf{H}_\theta, \in_\theta)$ and all its parameters are in $\mathcal{M}$, so the statement holds in $(\mathcal{M}, \in_\mathcal{M})$. Thus, choose for each finite sequence ν of open covers of X an x_ν the $\prec$-least element of X satisfying Claim 1. Note that x_ν also satisfies Claim 1 in $(\mathsf{H}_\theta, \in_\theta)$, and is in H_θ also the $\prec$-first such element. The set $D = \{x_\nu : \nu \in {}^{<\omega}\mathcal{O}\}$ is definable from $\prec$, σ, X and ${}^{<\omega}\mathcal{O}$, all parameters in $\mathcal{M}$, and thus is a member of $\mathcal{M}$.

Enumerate $\mathcal{O} \cap \mathcal{M}$ bijectively as $(O_n : n \in \mathbb{N})$. Now $D \cap \mathcal{M}$ is the countable set $\{x_\nu : \nu \in {}^{<\omega}\{O_0, O_1, \cdots, O_n, \cdots\}\}$.

Claim 2: $D \cap \mathcal{M}$ is dense in X.

For suppose the contrary, and choose a nonempty open set U for which $\overline{U} \cap \mathcal{M} \cap D$ is the empty set. The latter is possible since X is T_3.

Since $x_\emptyset$ is a member of $\mathcal{M}$, and $\mathcal{N} = \{(U_n(x) : n \in \mathbb{N}) : x \in X\}$ is a member of $\mathcal{M}$, also $(U_n(x_\emptyset) : n \in \mathbb{N})$ (which is definable from $\mathcal{N}$ and $x_\emptyset$, both parameters in $\mathcal{M}$) is in $\mathcal{M}$. But then $\{U_n(x_\emptyset) : n \in \mathbb{N}\}$ is an element, and subset of, $\mathcal{M}$. Since $x_\emptyset$ is not in $\overline{U}$, fix an m_1 with $U_{m_1}(x_\emptyset) \cap \overline{U} = \emptyset$. Then by Claim 1 $\mathcal{M}$ witnesses that there is an open cover $\mathcal{U}(\in \mathcal{M})$ of X such that $U_{m_1}(x_\emptyset) = \sigma(\mathcal{U})$. Thus, choose n_1 so that $U_{m_1}(x_\emptyset) = \sigma(O_{n_1})$.

Next, consider x_{n_1}. By the same considerations as above there is a neighborhood $U_{m_2}(x_{n_1})$ disjoint from $\overline{U}$, and an n_2 such that for the open cover O_{n_2} of X, $U_{n_2}(x_{n_1}) = \sigma(O_{n_1}, O_{n_2})$. Then apply these considerations to x_{n_1, n_2} to choose a neighborhood $U_{m_3}(x_{n_1, n_2})$ disjoint from $\overline{U}$, and an open cover O_{n_3} of X so that $U_{m_3}(x_{n_1, n_2}) = \sigma(O_{n_1}, O_{n_2}, O_{n_3})$, and so on. Proceeding like this we obtain a σ-play

$$O_{n_1}, \sigma(O_{n_1}), O_{n_2}, \sigma(O_{n_1}, O_{n_2}), \cdots, O_{n_k}, \sigma(O_{n_1}, \cdots, O_{n_k}), \cdots$$

for which $\overline{U} \cap \cup_{k \in \mathbb{N}} \sigma(O_{n_1}, \cdots, O_{n_k})$ is the empty set. Since $\overline{U}$ has nonempty interior this contradicts the fact that σ is a winning strategy of TWO in $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$.

This completes the proof of $(2) \Rightarrow (1)$. $\square$

A. Bella informed us in a personal communication that he has further generalized Theorem 9 as follows:

Theorem 10 (Bella). *Let X be a first countable space. The following assertions are equivalent:*

(1) *there exists a countable set $D \subseteq X$ such that for any neighborhood assignment $\phi : D \to \tau(X)$, where $x \in \phi(x)$ for each $x \in D$, we have $X = \overline{\bigcup \phi(D)}$.*
(2) *TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$.*

Note that in Bella's Theorem, the hypothesis used in Theorem 9 that X is T_3 has been eliminated, a significant strengthening. Also note that condition imposed on the countable set D in (1) of Bella's Theorem is equivalent to the condition

that the set D is dense. Thus, (1) of Bella's Theorem is a nice reformulation of separability, revealing the analogy with Theorem 14.

Next we explore the possibility of an analogue of Theorem 7 for the weakly Rothberger property. In a number of specific examples it has been proven that the property $\mathsf{S}_1(\mathcal{O}, \mathcal{D})$ is equivalent to ONE not having a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$. This raises the question of when the property $\mathsf{S}_1(\mathcal{O}, \mathcal{D})$ is equivalent to player ONE not having a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$. Here is a partial result in this direction. The Menger property, studied later in this paper, is needed in Theorem 11 below. The fact we need in the proof of Theorem 11, due to Hurewicz, is that a space has the Menger property if, and only if, ONE does not have a winning strategy in the game $\mathsf{G}_{fin}^\omega(\mathcal{O}, \mathcal{O})$. This game is played as follows: ONE and TWO play an inning per $n < \omega$. In the n-th inning ONE first chooses an open cover O_n of the space, and then TWO responds by choosing a finite subset T_n of O_n. A play $O_0, T_0, \cdots, O_m, T_n, \cdots$ is won by TWO if $\bigcup_{n<\omega} T_n$ is an open cover of the space. Else, ONE wins.

Theorem 11. Let X be a Menger space. The following are equivalent:

(1) X is weakly Rothberger.
(2) ONE has no winning strategy in $G_1^\omega(\mathcal{O}, \mathcal{D})$.

Proof. We must show that if a space has property $S_1(\mathcal{O}, \mathcal{D})$, then ONE has no winning strategy in the game $G_1^\omega(\mathcal{O}, \mathcal{D})$. The argument used here is due to Pawlikowski [20] for Rothberger spaces. We give some of the details for the convenience of the reader.

Since X is assumed to be a Menger space, it is a Lindelöf space. Let F be a strategy for ONE in the game $G_1^\omega(\mathcal{O}, \mathcal{D})$. We may assume that in each inning F calls on ONE to play a countable element of $\mathcal{O}$. Define the array U_σ, $\sigma \in {}^{<\omega}\mathbb{N}$, as follows: $(U_n : n \in \mathbb{N})$ enumerates ONE's first move, $F(\emptyset)$. For n_1, $(U_{(n_1,n)} : n \in \mathbb{N})$ enumerates $F(U_{n_1})$. For n_1, n_2, $(U_{(n_1,n_2,n)} : n \in \mathbb{N})$ enumerates $F(U_{n_1}, U_{(n_1,n_2)})$, and so on. This array has the property that for each σ the set $\{U_{\sigma \frown n} : n \in \mathbb{N}\}$ is in $\mathcal{O}$.

For fixed m and $j \in \mathbb{N}$ and ρ a function from $\{1, ..., j^m\}$ to $\mathbb{N}$, define the set

$$U_\rho(m, j) = \bigcap_{\sigma \in {}^m\{1,...,j\}} (\bigcup \{U_{\sigma \frown \rho\lceil i} : i \le j^m\})$$

and then for fixed m and j define

$$\mathcal{U}(m, j) = \{U_\rho(m, j) : \rho \text{ a function from } \{1, ..., j^m\} \text{ to } \mathbb{N}\}.$$

Then each $\mathcal{U}(m, j) \in \mathcal{O}$.

Claim: There exist increasing sequences $\{j_n : n \in \mathbb{N}\}$ and $\{m_n : n \in \mathbb{N}\}$ such that for each $x \in X$ and for each n there is a function σ from $\{1, ..., m_{n+1} - m_n\}$ to j_{n+1} for which $x \in U_\sigma(m_n, j_n)$.

Proof of the claim. We observe that X is Menger and this implies that ONE does not have a winning strategy in the corresponding game $G_{fin}^\omega(\mathcal{O}, \mathcal{O})$ using the following strategy, G.

For a first move ONE puts $j_1 = m_1 = 1$, and plays $G(\emptyset) = \mathcal{U}(m_1, j_1)$. For a response $T_1 \subseteq \mathcal{U}(m_1, j_1)$ by TWO, ONE first does the following computations:

$m_2 = m_1 + j_1^{m_1}$, and $j_2 > j_1$ is at least the maximum of all values of σ's for which $U_\sigma(m_1, j_1)$ is in T_1. Then ONE plays $G(T_1) = \mathcal{U}(m_2, j_2)$.

For a response $T_2 \subseteq G(T_1)$ by TWO, ONE again first computes the numbers m_3 and j_3 according to the rules that $m_3 = m_2 + j_2^{m_2}$, and $j_3 > j_2$ is at least the maximum of all values of σ's for which $U_\sigma(m_2, j_2)$ is in T_2, and so on.

Look at a G-play $G(\emptyset), T_1, G(T_1), T_2, G(T_1, T_2), \ldots$ which is lost by ONE. Then $\bigcup_{n \in \mathbb{N}} T_n \in \mathcal{O}$, and we find increasing sequences $(j_n : n \in \mathbb{N})$ and $(m_n : n \in \mathbb{N})$ such that for each n:

(1) $m_{n+1} = m_n + j_n^{m_n}$;
(2) $G(T_1, \ldots T_n) = \mathcal{U}(m_{n+1}, j_{n+1})$;
(3) j_{n+1} is at least as large as the value of an σ for which $U_\sigma(m_n, j_n)$ is in T_n.

It follows that the m_n's and j_n's have the required properties, completing the proof of the claim.

With the sequences $(j_n : n \in \mathbb{N})$ and $(m_n : n \in \mathbb{N})$ fixed, define next for each n the family $\mathcal{W}_n$ as follows: For every sequence $k_1 < \ldots < k_n$ from $\mathbb{N}$, and for any $\sigma_1, \ldots, \sigma_n$ where each σ_i is an $\{1, \ldots, j_{k_i+1}\}$-valued function with domain $m_{k_i+1} - m_{k_i}$, define

$$\mathcal{W}(k_1, \ldots, k_n, \sigma_1, \ldots, \sigma_n) = \bigcap_{i \le n} U_{\sigma_i}(m_{k_i}, j_{k_i}).$$

$\mathcal{W}_n$ consists of all sets of the form $W(k_1, \ldots, k_n, \sigma_1, \ldots, \sigma_n)$.

Since each $\mathcal{W}_n$ is in $\mathcal{O}$, the selection hypothesis $S_1(\mathcal{O}, \mathcal{D})$ applied to $(\mathcal{W}_n : n \in \mathbb{N})$ gives for each n a set $S_n = W(k_1^n, \ldots, k_n^n, \sigma_1^n, \ldots, \sigma_n^n)$ such that $\{S_n : n \in \mathbb{N}\}$ is in $\mathcal{D}$.

Recursively choose for each n an $\ell_n \in \{k_1^n, \ldots, k_n^n\} \setminus \{\ell_i : i < n\}$. For each n define $\rho_n = \sigma_{i_n}^n$ where i_n is such that $\ell_n = k_{i_n}^n$.

From the definitions we see that for each n, $S_n \subseteq U_{\rho_n}(m_{\ell_n}, j_{\ell_n})$.

If we now define $f : \mathbb{N} \to \mathbb{N}$ so that, for each n, $f(m_{\ell_n} + i) = \rho_n(i)$, whenever $i \le m_{\ell_n+1} - m_{\ell_n}$, we find that the play

$$F(\emptyset), U_{f(1)}, F(U_{f(1)}), U_{f(1), f(2)}, F(U_{f(1)}, U_{f(1), f(2)}) \ldots$$

is won by TWO. $\square$

To what extent is the hypothesis that a space is a Menger space needed in Theorem 11? Towards an answer to this question we show that in some models of set theory there are many non-Lindelöf (and thus non-Menger) spaces where ONE does not have a winning strategy in $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$. Our proof for one of the main ingredients of the argument is modeled on the arguments in [28].

Theorem 12. *Let $\kappa > \aleph_0$ be a cardinal. If a topological space X is weakly Lindelöf, then in $\mathsf{V}^{\mathbb{C}(\kappa)}$ ONE has no winning strategy in $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$ on X.*

Proof. Let $\dot{\sigma}$ be a $\mathbb{C}(\kappa)$ name such that $\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash$ "$\dot{\sigma}$ is a strategy for ONE in $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$." By Theorem 3,

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{"}\dot{\sigma}(\emptyset) \text{ has a countable subset with union dense in } \check{X}.\text{"}$$

Choose a $\mathbb{C}(\kappa)$ name $\dot{\mathcal{U}}_\emptyset$ such that

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{"}\dot{\mathcal{U}}_\emptyset \subseteq \dot{\sigma}(\emptyset) \text{ is a countable subset with union dense in } \check{X}.\text{"}$$

Thus choose $\mathbb{C}(\kappa)$ names $\dot{U}_n$, $n < \omega$ such that

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{"}\dot{\mathcal{U}}_\emptyset = \{\dot{U}_n : n < \omega\}.\text{"}$$

Then we have

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{``}(\forall n)(\dot\sigma(\dot U_n) \text{ has a countable subset with union dense in } \check X).\text{''}$$

For each n we choose $\mathbb{C}(\kappa)$ names $\dot{\mathcal{U}}_n$ and $\dot U_{n,k}$, $k < \omega$ such that

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{``}\dot{\mathcal{U}}_n \subseteq \dot\sigma(\dot U_n) \text{ is a countable subset with union dense in } \check X\text{''}$$

and

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{``}\dot{\mathcal{U}}_n = \{\dot U_{n,k} : k < \omega\}.\text{''}$$

and so on. In this way we find for each finite sequence in ω $\mathbb{C}(\kappa)$ names $\dot{\mathcal{U}}_{n_1,\cdots,n_k}$ and $\dot U_{n_1,\cdots,n_k}$ such that

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{``}\{\dot U_{n_1,\cdots,n_k,m} : m < \omega\} = \dot{\mathcal{U}}_{n_1,\cdots,n_k}\text{''}$$

and

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{``}\dot{\mathcal{U}}_{n_1,\cdots,n_k} \subseteq \dot\sigma(\dot U_{n_1}, \cdots, \dot U_{n_1,\cdots,n_k})\text{''}$$

and

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{``}\dot{\mathcal{U}}_{n_1,\cdots,n_k} \text{ is a countable subset with union dense in } \check X\text{''}$$

Since $\mathbb{C}(\kappa)$ has countable chain condition and each of the names $\dot{\mathcal{U}}_\tau$ and $\dot U_\tau$ is a name for a open single set or a countable set of open sets, there is an $\alpha < \kappa$ such that each of these is a $\mathbb{C}(\alpha)$ name. Thus, factoring the forcing as $\mathbb{C}(\alpha) * \mathbb{C}([\alpha,\kappa))$ we may assume that all the named objects are in the ground model. Then, in the generic extension by $\mathbb{C}([\alpha,\kappa))$ over this ground model there is a function $f \in {}^\omega\omega$ such that f is not in any first category set definable from parameters in the ground model.

Now for each nonempty open subset V of X in the ground model the set

$$F_V = \{f \in {}^\omega\omega : (\forall k)(V \cap U_{f\restriction k} = \emptyset)\}$$

is first category and is definable from parameters in the ground model only. Thus, in the generic extension by $\mathbb{C}([\alpha,\kappa))$ $\bigcup_{V \in \tilde\tau} F_V \neq {}^\omega\omega$. Choose in this generic extension an f with

$$f \in {}^\omega\omega \setminus \bigcup\{F_V : V \text{ a ground model open set}\}$$

Then in the generic extension the σ-play during which TWO selected the sets $U_{f\restriction n}$, $0 < n < \omega$ is won by TWO. This completes the proof that in the generic extension ONE has no winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O},\mathcal{D})$ on X. $\square$

A weakly Lindelöf space need not be Lindelöf: Consider a separable space which is not Lindelöf. Since every proper forcing preserves being not Lindelöf, Theorem 12 shows that it is consistent that there are non-Menger spaces in which ONE has no winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O},\mathcal{D})$. It is not clear how much of the Rothberger property must be indirectly present when a space has the weak Rothberger property. Is the converse of the following true?

Theorem 13. If X has a dense Rothberger subspace then ONE has no winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O},\mathcal{D})$ on X.

Proof. Let D be a dense Rothberger subspace of X. Let F be a strategy for ONE in $\mathsf{G}_1^\omega(\mathcal{O},\mathcal{D})$ on X. From F define a strategy G for ONE on the space D as follows: $G(\emptyset) = \{U \cap D : U \in F(\emptyset)\}$. For $T \in G(\emptyset)$ choose $U_T \in F(\emptyset)$ with $T = D \cap U_T$. Define $G(T) = \{U \cap D : U \in F(U_T)\}$, and so on.

Now by Theorem 7 G is not a winning strategy for ONE in the game $\mathsf{G}_1^\omega(\mathcal{O},\mathcal{O})$. Thus let $O_1, T_1, \cdots, O_n, T_n, \cdots$ be a G-play of $\mathsf{G}_1^\omega(\mathcal{O},\mathcal{O})$ which is won by TWO on

D. From the definition of G we find a corresponding sequence $B_1, W_1, \cdots, B_n, W_n, \cdots$ where $B_1 = F(\emptyset)$ and $O_1 = \{D \cap U : U \in B_1\}$, $W_1 \in B_1$ is such that $T_1 = D \cap W_1$, and for each n we have $B_{n+1} = F(W_1, \cdots, W_n)$ with $O_{n+1} = \{D \cap U : U \in B_{n+1}\}$ and $T_n = D \cap W_n$. But then we have that

$$D = \bigcup_{n \in \mathbb{N}} T_n \subseteq \bigcup_{n \in \mathbb{N}} W_n.$$

Since D is dense in X, so is $\bigcup_{n \in \mathbb{N}} W_n$. Thus TWO wins this F-play of X. $\square$.

Games and the almost Rothberger property

Theorem 14. *Let (X, τ) be a first countable topological space (but not necessarily T_3). Then the following are equivalent:*

   (1) *X has countable subset D such that for each neighborhood assignment $f : D \to \tau$ with $x \in f(x)$ for each x in D, the family $\{\overline{f(x)} : x \in D\}$ covers X.*
   (2) *TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \overline{\mathcal{O}})$.*

Proof. $(1) \Rightarrow (2)$: Let a countable set D as in (1) be given, and enumerate it as $(d_n : n < \omega)$. TWO's strategy which chooses in inning n an element T_n of ONE's move O_n so that $d_n \in T_n$ is a winning strategy.

Proof of $(2) \Rightarrow (1)$: Let X be a first countable topological space in which TWO has a winning strategy σ in the game $\mathsf{G}_1(\mathcal{O}, \overline{\mathcal{O}})$. Let $\prec$ be a well-order of X. Let θ be an infinite regular cardinal which is so large that X, its topology, $\mathcal{O}$, $^{<\omega}\mathcal{O}$, $\mathcal{D}$, $\prec$, σ, and for each $x \in X$ a neighborhood base $(U_n(x) : n \in \mathbb{N})$ of $\{x\}$, are elements of H_θ. We may assume that $\mathcal{N} = \{(U_n(x) : n \in \mathbb{N}) : x \in X\}$ is a member of H_θ. Let $(\mathcal{M}, \in_\mathcal{M})$ be a countable elementary submodel of $(\mathsf{H}_\theta, \in_\theta)$ such that each of the objects above is an element of $\mathcal{M}$.

Claim 1: For each finite sequence $(\mathcal{U}_1, \cdots, \mathcal{U}_k)$ of open covers of X there is a point $x \in X$ such that for each neighborhood U of x there is an open cover $\mathcal{U}$ such that $U = \sigma(\mathcal{U}_1, \cdots, \mathcal{U}_k, \mathcal{U})$.

This claim holds because a strategy for TWO in game $\mathsf{G}_1^\omega(\mathcal{O}, \overline{\mathcal{O}})$ is also a strategy for TWO in the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$. Now use the argument of Claim 1 of Theorem 9.

The sentence in Claim 1 holds in $(\mathsf{H}_\theta, \in_\theta)$ and all its parameters are in $\mathcal{M}$, so the statement holds in $(\mathcal{M}, \in_\mathcal{M})$. Thus, choose for each finite sequence ν of open covers of X an x_ν the $\prec$-least element of X satisfying Claim 1. Note that x_ν also satisfies Claim 1 in $(\mathsf{H}_\theta, \in_\theta)$, and is in H_θ also the $\prec$-first such element. The set $E = \{x_\nu : \nu \in {}^{<\omega}\mathcal{O}\}$ is definable from $\prec$, σ, X and $^{<\omega}\mathcal{O}$, all parameters in $\mathcal{M}$, and thus is a member of $\mathcal{M}$.

Enumerate $\mathcal{O} \cap \mathcal{M}$ bijectively as $(O_n : n \in \mathbb{N})$. Now $D = E \cap \mathcal{M}$ is the countable set $\{x_\nu : \nu \in {}^{<\omega}\{O_0, O_1, \cdots, O_n, \cdots\}\}$.

Claim 2: D has the property defined in (1).

For suppose the contrary. Choose a neighborhood assignment $f : D \to \tau$ for which $\{\overline{f(x)} : x \in D\}$ does not cover X. Pick a point $y \in X \setminus \bigcup\{\overline{f(x)} : x \in D\}$.

Since $x_\emptyset$ is a member of $\mathcal{M}$, and $\mathcal{N} = \{(U_n(x) : n \in \mathbb{N}) : x \in X\}$ is a member of $\mathcal{M}$, also $(U_n(x_\emptyset) : n \in \mathbb{N})$ (which is definable from $\mathcal{N}$ and $x_\emptyset$, both parameters in $\mathcal{M}$) is in $\mathcal{M}$. But then $\{U_n(x_\emptyset) : n \in \mathbb{N}\}$ is an element, and subset of, $\mathcal{M}$. Fix an m_1 with $U_{m_1}(x_\emptyset) \subset f(x_\emptyset)$. Since y is not in $\overline{f(x_\emptyset)}$, y is also not a member of $\overline{U}_{m_1}(x_\emptyset)$.

Then by Claim 1 $\mathcal{M}$ witnesses that there is an open cover $\mathcal{U}$ of X such that $U_{m_1}(x_\emptyset) = \sigma(\mathcal{U})$. Thus, choose n_1 so that $U_{m_1}(x_\emptyset) = \sigma(O_{n_1})$.

Next, consider $x_{\langle O_{n_1} \rangle}$. By the same considerations as above there is a neighborhood $U_{m_2}(x_{\langle O_{n_1} \rangle})$ with $y \notin \overline{U}_{m_2}(x_{\langle O_{n_1} \rangle})$, and an n_2 such that for the open cover O_{n_2} of X, $U_{m_2}(x_{\langle O_{n_1} \rangle}) = \sigma(O_{n_1}, O_{n_2})$.

Apply these considerations to $x_{\langle O_{n_1}, O_{n_2} \rangle}$ to choose a neighborhood $U_{m_3}(x_{\langle O_{n_1}, O_{n_2} \rangle})$ with y not a member of $\overline{U}_{m_3}(x_{\langle O_{n_1}, O_{n_2} \rangle})$ and an open cover O_{n_3} of X so that $U_{m_3}(x_{\langle O_{n_1}, O_{n_2} \rangle}) = \sigma(O_{n_1}, O_{n_2}, O_{n_3})$, and so on. Proceeding like this we obtain a σ-play

$$O_{n_1}, \sigma(O_{n_1}), O_{n_2}, \sigma(O_{n_1}, O_{n_2}), \cdots, O_{n_k}, \sigma(O_{n_1}, \cdots, O_{n_k}), \cdots$$

for which y is not a member of $\bigcup_{k \in \mathbb{N}} \overline{\sigma(O_{n_1}, \cdots, O_{n_k})}$. This contradicts the fact that σ is a winning strategy of TWO in $\mathsf{G}_1(\mathcal{O}, \overline{\mathcal{O}})$.

This completes the proof of $(2) \Rightarrow (1)$. $\square$

Corollary 15. [3] *If X is a first countable T_2-space such that TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \overline{\mathcal{O}})$, then X is countable.*

Proof. : Let $D \subseteq X$ be the countable subset as in (1) of Theorem 14. We claim that $D = X$. For suppose the contrary and choose $y \in X \setminus D$. Then as X is T_2, choose for each $d \in D$ a neighborhood $f(d)$ with y not in $\overline{f(d)}$. It follows that f is a neighborhood assignment violating the property (1) of D, a contradiction. $\square$

In Theorem 14 some hypothesis like first countability is needed to derive that (2) implies (1): In Example F in the examples section of the paper we present an uncountable T_2 space which is not Lindelöf, not first countable, and TWO has a winning strategy in $\mathsf{G}_1^\omega(\mathcal{O}, \overline{\mathcal{O}})$, and the space does not satisfy property (1) of Theorem 14.

In Theorem 8 of [29] it is shown that there is for each infinite cardinal number κ a T_4 Lindelöf P-space X of cardinality κ for which TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{O})$, and thus also in the games $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$ and $\mathsf{G}_1^\omega(\mathcal{O}, \overline{\mathcal{O}})$. By Galvin's Theorem, Theorem 6, X has some one-element subset which is not a G_δ set. Since X is in fact a topological group it follows that no one-element subset of X is a G_δ set.

Question 16. *Can there be an uncountable T_2-space X such that each one-element subset of X is a G_δ set, and TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \overline{\mathcal{O}})$?*

Question 17. *Can there be an uncountable T_1-space X which is first countable such that TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \overline{\mathcal{O}})$?*

Theorem 18. Let X be a Menger space. The following are equivalent:
  (1) X is almost Rothberger.
  (2) ONE has no winning strategy in $\mathsf{G}_1^\omega(\mathcal{O}, \overline{\mathcal{O}})$.

Proof. The proof of (1) implies (2) proceeds like the proof of the corresponding implication in the proof of Theorem 11. But at the stage of that proof where $\mathsf{S}_1(\mathcal{O}, \mathcal{D})$ is applied to the sequence of $\mathcal{W}_n$'s, apply instead $\mathsf{S}_1(\mathcal{O}, \overline{\mathcal{O}})$. $\square$

Using the same technique as in the proof of Theorem 12 we find:

Theorem 19. *For X an almost Lindelöf space and κ an uncountable cardinal,*

$$\mathbf{1}_{\mathbb{C}(\kappa)} \Vdash \text{"ONE has no winning strategy in the game } \mathsf{G}_1^\omega(\mathcal{O}, \overline{\mathcal{O}}) \text{ on } \check{X}.\text{"}$$

---

[3]A. Bella (personal communication) proved that first countability can be replaced by the weaker hypothesis that each point is the intersection of countably many closed sets.

There are non-Lindelöf, almost Lindelöf spaces. Since Cohen real forcing preserves non-Lindelöf, Theorem 19 implies it is consistent that there are non-Menger spaces for which ONE has no winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \overline{\mathcal{O}})$.

5. The Menger property and weakenings

The following table contains names for properties considered in this section, symbolic definitions, and where available a reference for where the property was introduced:

Property name	Definition	Source
Menger	$\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$	[13]
Almost Menger	$\mathsf{S}_{fin}(\mathcal{O}, \overline{\mathcal{O}})$	[18]
Weakly Menger	$\mathsf{S}_{fin}(\mathcal{O}, \mathcal{D})$	[7], p. 94

The weakly Menger property was investigated in [19]. Figure 1 below depicts the implications among the properties have been introduced thus far:
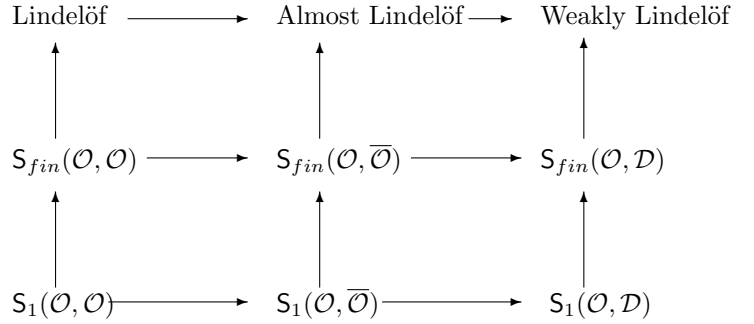

FIGURE 1. Basic Relationships

If a space is T_3, then it is almost Menger if, and only if, it is Menger. However, $\mathsf{S}_{fin}(\mathcal{O}, \overline{\mathcal{O}})$ is not equivalent to $\mathsf{S}_{fin}(\overline{\mathcal{O}}, \overline{\mathcal{O}})$, even in the context of metrizable spaces:

Theorem 20 ([26], Theorem 2). *An uncountable set of real numbers has the property $\mathsf{S}_{fin}(\overline{\mathcal{O}}, \overline{\mathcal{O}})$ if, and only if, it is a Lusin set.*

Games and the Menger property

In Corollary 4 of [34] Telgársky characterized the metrizable spaces for which TWO has a winning strategy as the ones that are σ-compact. A more direct proof of Telgársky's result was given in [27]. There appears to be no satisfactory more general characterizations of topological spaces for which player TWO has a winning strategy in the games $\mathsf{G}_{fin}^\omega(\mathcal{O}, \mathcal{O})$, $\mathsf{G}_{fin}^\omega(\mathcal{O}, \overline{\mathcal{O}})$ or $\mathsf{G}_{fin}^\omega(\mathcal{O}, \mathcal{D})$.

It is not clear how this characterization would generalize to say T_4 spaces. It is false that for T_4-spaces TWO has a winning strategy in the game $\mathsf{G}_{fin}^\omega(\mathcal{O}, \mathcal{O})$ if, and only if, the space is σ-compact. This can be seen by considering the examples in Section 4 of [29]: By Theorem 8 of [29] there is for each infinite cardinal number κ a T_4 Lindelöf P-group of cardinality κ such that TWO has a winning strategy in

the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{O})$ (and thus in $\mathsf{G}_{fin}^\omega(\mathcal{O}, \mathcal{O})$). An uncountable T_4 Lindelöf P-space cannot be a closed subset of a σ-compact space.

It is clear that if a space is σ-compact, then TWO has a winning strategy in the game $\mathsf{G}_{fin}^\omega(\mathcal{O}, \mathcal{O})$. A weaker condition than this permits TWO a winning strategy in the game $\mathsf{G}_{fin}^\omega(\mathcal{O}, \mathcal{D})$:

Definition 21. [21] X is H-*closed* if every open cover $\mathcal{U}$ of X has a finite subfamily $\mathcal{V}$ whose union is dense in X (i.e. $X \subseteq cl_X(\bigcup_{V \in \mathcal{V}} V)$).

It is well-known that a T_2-space is H-closed if, and only if it is a closed subspace of each T_2-space it embeds in. Equally well-known, a T_3-space is H-closed if, and only if, it is compact.

Definition 22. X is σ-H-*closed* if it is a union of countably many subspaces, each of which is H-closed.

Proposition 23. If X contains a dense σ-H-closed subspace then TWO has a winning strategy in the game $\mathsf{G}_{fin}^\omega(\mathcal{O}, \mathcal{D})$.

Proof. Let $Y \subseteq X$ be a dense σ-H-closed subspace of X. Since Y is a σ-H-closed space, write $Y = \bigcup_{n \in \mathbb{N}} Y_n$ where each Y_n is an H-closed subspace of Y. Define a strategy σ for player TWO as follows: When in inning n ONE plays an open cover O_n of X, let $\sigma(O_1, \cdots, O_n) \subseteq O_n$ be a finite set with $\overline{\bigcup \sigma(O_1, \cdots, O_n)} \supseteq Y_n$. Then σ is a winning strategy for TWO. $\square$

Question 24. *(T_2) Is it true that X contains a dense σ-H-closed subspace if, and only if, TWO has a winning strategy in $G_{fin}^\omega(\mathcal{O}, \mathcal{D})$?*[4]

Question 25. *(T_3) Is it true that X has a dense σ-compact subset if, and only if, TWO has a winning strategy in $G_{fin}^\omega(\mathcal{O}, \mathcal{D})$?*

Question 26. *Characterize the topological spaces for which TWO has winning strategy in:*
  (1) *the game $\mathsf{G}_{fin}^\omega(\mathcal{O}, \mathcal{O})$.*
  (2) *the game $\mathsf{G}_{fin}^\omega(\mathcal{O}, \overline{\mathcal{O}})$.*
  (3) *the game $\mathsf{G}_{fin}^\omega(\mathcal{O}, \mathcal{D})$.*

Conditions under which ONE has no winning strategy in these games are somewhat better understood. Hurewicz, who introduced and studied the Menger property in [13], proved there

Theorem 27 (Hurewicz). *For a space X the following are equivalent:*
  (1) *X has the Menger property*
  (2) *ONE has no winning strategy in the game $\mathsf{G}_{fin}^\omega(\mathcal{O}, \mathcal{O})$.*

For Lindelöf spaces Hurewicz's proof of Theorem 27 generalizes to also give the corresponding characterizations for weakly Menger and almost Menger spaces. For the convenience of the reader we now give the proof of the characterization of weakly Menger Lindelöf spaces, and then indicate what modification is needed to also obtain the characterization for Lindelöf almost Menger spaces.

Theorem 28. *Let X be a Lindelöf space. Then the following are equivalent:*

---

[4]A. Bella (personal communication) has shown that the answer to this question is "NO".

(1) X is weakly Menger.

(2) ONE does not have a winning strategy in the game $\mathsf{G}_{fin}^{\omega}(\mathcal{O}, \mathcal{D})$.

Proof. The implication that if a Lindelöf space satisfies $S_{fin}(\mathcal{O}, \mathcal{D})$ then ONE has no winning strategy in the game $\mathsf{G}_{fin}^{\omega}(\mathcal{O}, \mathcal{D})$ requires proof. The argument used here is due to Hurewicz [13] for Menger spaces, and has been used in [25] for a different context. We give some of the details of Hurewicz's argument for the convenience of the reader.

Let X be Lindelöf and let F be a strategy for ONE. Without loss of generality, we may assume that each move of ONE according to the strategy F, is an ascending ω-chain of open sets covering X.

Write $F(\emptyset) = (U(n) : n \in \mathbb{N})$, listed in $\subset$-increasing order. Then, for each n, list $F(U_{(n)})$ in $\subset$-increasing order as $(U_{(n,m)} : m \in \mathbb{N})$, and so on.

Supposing that U_τ, has been defined for each finite sequences τ of length at most k of positive integers, we now define for each $(n_1, ..., n_k)$:

$$F(U_{(n_1)}, ..., U_{(n_1,,,...,n_k)}) = (U_{(n_1,...,n_k,m)} : m \in \mathbb{N}).$$

Then the family $(U_\tau : \tau$ a finite sequence of positive integers) has the following properties for each τ:

(1) If m is less than n, then $U_{\tau \frown (m)}$ is a subset of $U_{\tau \frown (n)}$.

(2) For each n, $U_\tau \subseteq U_{\tau \frown (n)}$.

(3) $\{U_{\tau \frown (n)} : n$ a positive integer$\}$ is an open cover of X.

Define for each n and κ:

$$U_k^n = \begin{cases} U_{(k)}, & \text{if n=1;} \\ (\bigcap\{U_{\tau \frown (k)} : \tau \in^{n-1} \mathbb{N}\}) \cap U_k^{n-1}, & \text{otherwise.} \end{cases}$$

Note that for each n the set $\{U_k^n : k \in \mathbb{N}\}$, denoted $\mathcal{U}_n$, is an open cover of X. To see this, first show (by induction) that for each $(i_1, .., i_n)$ such that $\max\{i_l, ..., i_n\} \geq k$ one has $U_k^n \subseteq U_{(i_1,...,i_n)}$. It then follows that each U_k^n is an intersection of finitely many open sets, and thus is itself open. Now observe that by its definition each $\mathcal{U}_n$ is an increasing chain of open sets such that each $\mathcal{U}_n$ is an open cover of X.

Apply the fact that X is weakly Menger to the sequence $(\mathcal{U}_n : n \in \mathbb{N})$. Since each $\mathcal{U}_n$ is an ascending chain this gives for each n a $U_{k_n}^n \in \mathcal{U}_n$ such that $\{U_{k_n}^n : n \in \mathbb{N}\}$ is in $\mathcal{D}$. Since, for each n, $U_{k_n}^n \subseteq U_{(k_1,...,k_n)}$, the sequence of moves $U_{k_1}, U_{k_1,k_2}, \cdots$ by TWO defeats ONE's strategy F. $\square$

The corresponding theorem for almost Menger is:

Theorem 29. *Let X be a Lindelöf space. Then the following are equivalent:*

(1) X is almost Menger.

(2) ONE does not have a winning strategy in the game $\mathsf{G}_{fin}^{\omega}(\mathcal{O}, \overline{\mathcal{O}})$.

Proof. The proof proceeds like for Theorem 28. In that proof at one point we apply the selection property $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{D})$ to a sequence $(\mathcal{U}_n : n < \omega)$ of special open covers of X. Applying the selection property $\mathsf{S}_{fin}(\mathcal{O}, \overline{\mathcal{O}})$ instead at the same point of the proof produces a proof of Theorem 29. $\square$

The hypothesis that X is Lindelöf in Theorems 28 or 29 is not necessary: By Theorem 12 it is consistent that there is a non-Lindelöf space X for which TWO has a winning strategy in $\mathsf{G}_1^{\omega}(\mathcal{O}, \mathcal{D})$, and thus in $\mathsf{G}_{fin}^{\omega}(\mathcal{O}, \mathcal{D})$. Similar remarks apply to the almost Menger case. We do not know the answers to the following questions:

Question 30. *Is there an almost Menger space for which ONE has a winning strategy in the game $\mathsf{G}^\omega_{fin}(\mathcal{O}, \overline{\mathcal{O}})$?*

Question 31. *Is there a weakly Menger space for which ONE has a winning strategy in the game $\mathsf{G}^\omega_{fin}(\mathcal{O}, \mathcal{D})$?*

6. Examples

The relationships among the covering properties we consider in this paper are indicated in the Figure 1. Also considering the game theoretic versions, the strongest property considered in this paper is the property that TWO has a winning strategy in the game $\mathsf{G}^\omega_1(\mathcal{O}, \mathcal{O})$. The following diagram, Figure 2, is Figure 1 updated to include the classes where TWO has a winning strategy in the corresponding game. We use the symbol $\uparrow \mathsf{G}^\omega_{fin}(\mathcal{A}, \mathcal{B})$ to denote that TWO has a winning strategy in the corresponding game.
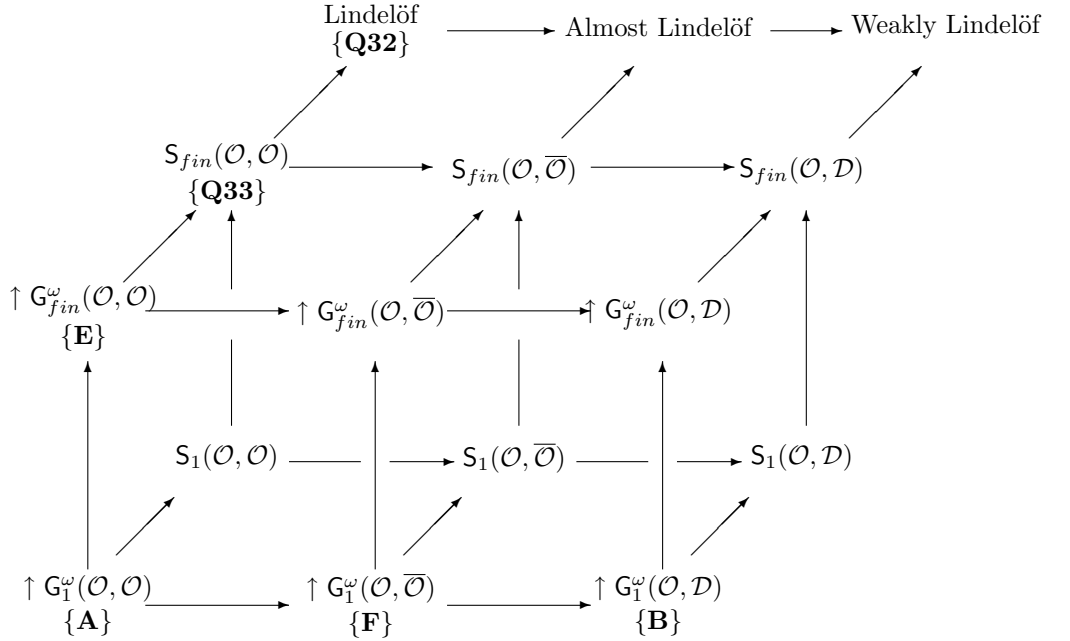

Figure 2. Classes considered in this paper

We now consider examples that distinguish these classes from each other. We are missing two examples, as indicated in the following two questions:

Question 32. *Is there a Lindelöf space which is not weakly Menger?*

Question 33. *Is there a Menger space for which TWO does not have a winning strategy in $\mathsf{G}^\omega_{fin}(\mathcal{O}, \mathcal{D})$?*

These questions are associated with the corresponding vertices in Figure 2. The rest of our examples are as follows:

A: Large compact T_2 spaces where TWO has a winning strategy in $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{O})$.

For an infinite cardinal number κ let D_κ be the one-point compactification of a discrete space of cardinality κ. Then TWO has a winning strategy in $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{O})$ on D_κ.

B: Non-almost Lindelöf spaces where TWO has a winning strategy in $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$.

Note that such an example indicates that none of the implications from the middle panel to the right panel of Figure 2 is reversible.

(a) The space $\mathbb{P}$ of irrational numbers with the topology inherited from the real line is not Menger. As $\mathbb{P}$ is T_3, it follows that this space is also not almost Menger. For any refinement of this topology on $\mathbb{P}$, these statements remain true. As $\mathbb{P}$ is separable, TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$. Define the topology

$$\tau_0 := \{U \setminus C : U \text{ open in the usual topology on } \mathbb{P} \text{ and } C \subseteq \mathbb{P} \text{ is countable}\}$$

Then $(\mathbb{P}, \tau_0)$ is a T_2-space but no longer a T_3-space. It is still Lindelöf and not almost Menger, but it is no longer separable. Yet, TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$ on $(\mathbb{P}, \tau_0)$. This indicates that in Theorem 9 the hypothesis that the space is T_3 cannot be weakened to T_2.

(b) Examination of Examples 6 in [19] shows that also that example is non-separable, T_2 but not T_3, and the space is not almost Lindelöf, but TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$.

(c) Example 11 of [19] is, on the other hand, a separable $\mathsf{T}_{3\frac{1}{2}}$-space which is not almost Lindelöf.

(d) Let $\mathbb{S}$ denote the *Sorgenfrey line*, the topological space obtained from refining the standard topology on the real line by also declaring each interval of the form $[a, b)$ open. Since the set of rational numbers still is a dense subset of $\mathbb{S}$, TWO has a winning strategy in $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$ on $\mathbb{S}$ and all its finite powers. In Lemma 17 of [2] it was shown that $\mathbb{S}$ does not have the property $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$, and since $\mathbb{S}$ is T_3, this means that $\mathbb{S}$ is not almost Menger. Note that finite powers of $\mathbb{S}$ are still separable, but are not Lindelöf and as these powers remain T_3, they also are not almost Lindelöf. Thus, for finite powers of $\mathbb{S}$ TWO has a winning strategy in $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$ while the space is not almost Lindelöf.

(e) The space $\mathbb{Z}^{\omega_1}$ is T_3 and not T_4, it is not Lindelöf and thus also not almost Lindelöf. But it is separable, and so TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$ for this space.

C: Non-Lindelöf (but almost Menger) spaces where TWO has a winning strategy in $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$.

Such an example shows that none of the implications from the top left edge of Figure 2 to the top middle is reversible.

Example 77 of [31], the deleted radius topology in the plane, is a non-Lindelöf space. In [17], Example 1, Kočev points out that the space in this example is almost Menger (and thus not T_3). Indeed, TWO has a winning strategy in the game $\mathsf{G}_{fin}^\omega(\mathcal{O}, \overline{\mathcal{O}})$: When ONE presents TWO with an open cover, TWO may first replace it with basic open sets consisting of the appropriate deleted radius open disks. Ignoring the deleted radii, this would be a move of ONE in the usual topology of $\mathbb{R}^2$, and TWO may, in the n-th inning, choose finitely many of these open disks (including radii) that cover the Euclidean compact set $[-n, n] \times [-n, n]$. Then remove the radii so as to recover sets from the replacement family of basic open sets of the deleted radius topology, and then select finitely many elements of ONE's presented

cover that contain the corresponding finitely many basis elements. This example illustrates that in the proof of Theorem 29 the hypothesis that a space be Lindelöf is not required for the conclusion that TWO has a winning strategy in the almost Menger game.

This space is not almost Rothberger, as can be shown by for each positive integer n letting $\mathcal{U}_n$ be the open cover consisting of all open discs of area less than $\frac{1}{2^n}$ with horizontal radius (excluding the center) removed. If for each n we choose a $U_n \in \mathcal{U}_n$, then the total area covered by the sets $\overline{U}_n$, $n \in \mathbb{N}$, is finite, and so these sets do not cover the plane. But the set D of points in the plane with rational coordinates only is still dense in the deleted radius topology, so that this space is separable. It follows that TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$. Should we refine the deleted radius topology further by declaring all countable sets closed, the resulting space would no longer be separable, but TWO would still have a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$, and in the game $\mathsf{G}_{fin}^\omega(\mathcal{O}, \overline{\mathcal{O}})$.

D: Non-Lindelöf almost Rothberger spaces.
Such an example eliminates one more implication from the middle panel to the left panel of Figure 2.
Consider subspaces of the space in Example 77 of [31], the Euclidean plane with the deleted radius topology. It is a non-Lindelöf space. Now assume X is an uncountable subset of $\mathbb{R}$ such that $X \times X$ has the Rothberger property. Let $\mathsf{R}(X)$ denote the set $X \times X$ with the deleted radius topology inherited. Then one can show that $\mathsf{R}(X)$ still is not Lindelöf, but it is almost Rothberger.

E: Compact T_2 spaces which are not weakly Rothberger.
This example shows that there are no implications from the top level to the bottom level of Figure 2.
Let **I** be the closed unit interval. Let X be a dense subset of **I**. Consider the following subspace $T(X)$ of the *Alexandroff double of* **I**. $T(X) = \mathbf{I} \times \{0\} \cup X \times \{l\}$. For $A \subset \mathbf{I}$ and for $x \in \mathbf{I}$ we write A_i for $A \times \{i\}$ and x_i for (x, i), $i \in \{0, 1\}$. The family $\mathcal{B} = \{U_0 \cup ((U \cap X)_1 \setminus \{x_1\}) : U \text{ open in } \mathbf{I} \text{ and } x \in U \cap X\} \cup \{\{x_1\} : x \in X\}$ is a basis for a topology on $T(X)$. In this topology $T(X)$ is compact and T_2. In particular, TWO has a winning strategy in the game $\mathsf{G}_{fin}^\omega(\mathcal{O}, \mathcal{O})$.

The weakly Rothberger property for $T(X)$ is connected with the classical strong measure zero sets of real numbers. A subset X of the real line $\mathbb{R}$ has *strong measure zero* if there is for every sequence $(\varepsilon_n : n \in \mathbb{N})$ of positive real numbers a sequence $(J_n : n \in \mathbb{N})$ of nonempty open intervals such that each J_n has length at most ε_n, and $X \subseteq \bigcup_{n \in \mathbb{N}} J_n$. This concept was introduced in [5] where Bore1 observed that every countable set of real numbers has this property. Borel conjectured that every strong measure zero set is countable. The truth of tis conjecture is independent of ZFC. In [24] it is shown that for X a dense subset of **I**, the following are equivalent:

(1) X has strong measure zero.
(2) $T(X)$ satisfies the selection hypothesis $S_1(\mathcal{O}, \mathcal{D})$.
(3) ONE has no winning strategy $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$ played on the space $T(X)$.

One can also show that TWO has a winning strategy in $\mathsf{G}_1^\omega(\mathcal{O}, \mathcal{D})$ if, and only if, X is countable. Thus, by choosing $X \subseteq \mathbb{R}$ appropriately we obtain a compact T_2 space which is not weakly Rothberger.

F: Non-Lindelöf T_2 space for which TWO has a winning strategy in the game $\mathsf{G}_1^\omega(\mathcal{O}, \overline{\mathcal{O}})$.

This example demonstrates that none of the implications from the left panel of Figure 2 to its middle panel is reversible.

The following is Example 3.3 of [32]. The space X is a subset of $\mathbb{R} \times \mathbb{R}$ with a special topology: First, choose a subset $\{x_\alpha : \alpha < \omega_1\}$ of $\aleph_1$ distinct elements of the set of nonnegative real numbers.

$$X = \{(x_\alpha, m) : m \text{ an integer larger than } -2 \text{ and } \alpha < \omega_1\} \cup \{(-1, -1)\}.$$

For convenience we also define:

$$A = \{(x_\alpha, -1) : \alpha < \omega_1\}$$

and

$$Y = \{(x_\alpha, n) : 0 \le n \in \mathbb{N} \text{ and } \alpha < \omega_1\}.$$

Topologize X as follows: Declare each element of Y to be an isolated point; for each $\alpha < \omega_1$ and $n < \omega$ the neighborhood $U_{n,\alpha}$ of $(x_\alpha, -1)$ is the set $\{(x_\alpha, -1)\} \cup \{(x_\alpha, m) : m \ge n\}$. Finally, for each α the neighborhood V_α of $(-1, 1)$ is the set $\{(-1, -1)\} \cup \{(x_\beta, n) : \beta > \alpha, -1 < n < \omega\}$.

Since the uncountable subset A of X is a closed and discrete subset of X, X is not Lindelöf. Also, for a fixed α, for neighborhood V_α of $(-1, -1)$ we see that $\overline{V}_\alpha$ contains all but countably many elements of X. Thus, TWO wins $\mathsf{G}_1^\omega(\mathcal{O}, \overline{\mathcal{O}})$ as follows: In the first inning TWO chooses the set T_1 from the open cover O_1 provided by ONE so that T_1 contains a neighborhood of $(-1, -1)$ of the form V_α. Then in the remaining innings TWO makes sure to cover the at most countably many points in the set $X \setminus \overline{T}_1$.

Moreover, the point $(-1, 1)$ does not have a countable neighborhood base, and so this space is not first countable. We now show that this example does not meet condition (1) of Theorem 14: Let D be a countable subset of X. We may assume that $(-1, -1)$ is an element of D. Fix a $\beta < \omega_1$ such that $D \cap \{(x_\alpha, n) : -1 \le n < \omega, \beta < \alpha < \omega_1\} = \emptyset$. Then any neighborhood assignment which uses only neighborhoods from the family

$$\{V_{\beta+\omega}\} \cup \{\{(x_\alpha, n)\} : -1 < n \text{ and } \alpha < \beta + \omega\} \cup \{U_{0,\alpha} : \alpha < \beta + \omega\}$$

witnesses the failure of condition (1) of Theorem 14.

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