1-1-2009

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2009 American Control Conference
Hyatt Regency Riverfront, St. Louis, MO, USA
June 10-12, 2009

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***Abstract*— The induction machine is widely utilized in the industry and exists in a plethora of applications. Although it is characterized by its inherent stability over a wide range of operating conditions, this characterization is based on steady-state arguments. This work develops a rigorous approach to the open-loop stability of the induction machine. In particular, a condition for the global asymptotic stability of the induction machine in the sense of Lyapunov is presented. These conditions are met if the machine is lightly loaded. Hence, meeting these conditions guarantees that the motor will reach (or return to) the desired equilibrium point regardless of how far it has been perturbed from it. The analysis is based on the standard nonlinear differential equation model of the induction machine taking into account transient responses.**

***Index Terms*— Induction Machine, Lyapunov Stability, Open-Loop Stability**

I. INTRODUCTION

The classical method that depicts the range of stable operation for the induction machine is a torque versus (normalized) slip curve as shown below (see [1])

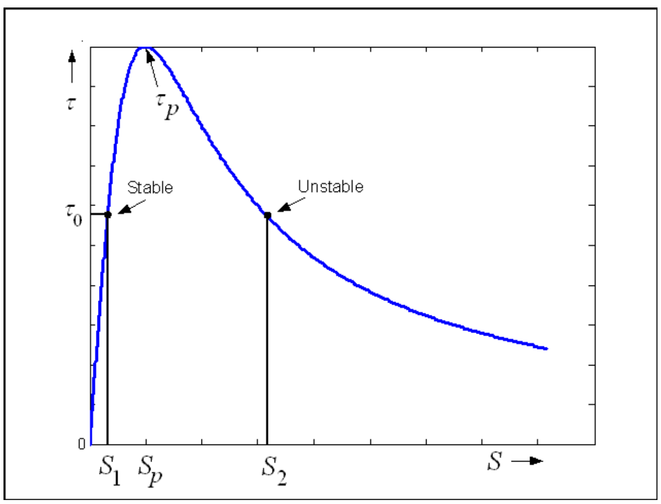

Fig. 1. Torque versus normalized slip curve

τ is the steady state output torque, τ_p is the peak load torque, S is the normalized slip defined as

$$S \triangleq \frac{\omega_S - n_p\omega_R}{\omega_S} \tag{1}$$

(i.e. the normalized difference between the electrical frequency ω_S, and the angular speed $n_p\omega_R$), and S_p is the pull-out slip which corresponds to the peak torque τ_p. The curve indicates the stability of the induction machine about steady-state operating points. The stable steady-state operating points for motoring must satisfy $0 < S < S_p$. For example, if the motor is operating at slip S_1 producing the torque τ_0 as shown in Figure 1. Then any increase in the load torque (but the total load torque not exceeding τ_p) would result in a *decrease* in the steady-state speed ω_R [see (1)] with a consequent *increase* in the steady-state slip S (i.e. a shift to the right from the operating point S_1 in Figure 1). The increased slip gives an *increase* in the steady-state output torque to accommodate the increase in the load torque. On the other hand, consider the motor operating at the slip $S_2 > S_p$ in Figure 1. Any *increase* in the load torque (even a minimal one) would again result in a *decrease* in ω_R [see (1)] and thus an increased slip to the right of the original steady-state slip S_2 in Figure 1. But now a lower output torque is produced which cannot meet the increased load demand. Hence the motor will stall. Note that this argument is based on steady-state conditions and does not account for transients. In fact though the operating points for $S > S_p$ are always unstable, operating points with $S < S_p$ can also be unstable.

At rest ($\omega_R = 0$), $S = 1$ and typically $S_p << 1$. Thus at startup of the motor the (instantaneous) slip $S >> S_p$ and, as Figure 1 shows, the torque produced by the motor is low. As a result, the machine must be lightly loaded so that it can come up to full (near synchronous) speed under open-loop conditions. After getting up to full speed, the motor can then be loaded and run stably.

In this work, we give a rigorous treatment of the stability issue by accounting for transients. Specifically, a sufficient condition for the global stability of an open-loop induction machine is derived using Lyapunov theory based on the well-known nonlinear differential equation model of the induction machine. It is shown that the conditions for global stability hold if the machine is lightly loaded. We begin in Section II by deriving an error-dynamics model of the induction motor in the stator field coordinate system. In Section III a power balance equation of the motor is developed that is then transformed into the error state variables. The results are then utilized in Section IV to develop a Lyapunov function that gives sufficient conditions for global stability of the induction machine. Section V provides a numerical example that is used to demonstrate the application of the theorem. Finally, concluding remarks are presented in Section VI.

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II. Stator Field Model of the Induction Motor

The starting point for the analysis is the two-phase equivalent model of the machine (see [1] and [2]). The parameters of the two phase induction motor are the stator-side inductance L_S and resistance R_S, the rotor-side inductance L_R and resistance R_R, the mutual inductance M, the number of rotor pole pairs n_p, the moment of inertia of the rotor J, and the rotational friction f.

The variables consist of the angular position of the rotor θ_R, the angular speed ω_R, the load torque τ_L, the stator currents i_{Sa} and i_{Sb}, the stator voltages u_{Sa} and u_{Sb}, and the rotor currents i_{Ra} and i_{Rb} where a and b denote the equivalent two phases of the motor.

A. Space Vector Model

A space vector model of the induction machine is ([1] and [2])

$$\begin{aligned} R_S\underline{i}_S + L_S\frac{d}{dt}\underline{i}_S + M\frac{d}{dt}\left(\underline{i}_R e^{jn_p\theta_R}\right) &= \underline{u}_S \\ R_R\underline{i}_R + L_R\frac{d}{dt}\underline{i}_R + M\frac{d}{dt}\left(\underline{i}_S e^{-jn_p\theta_R}\right) &= 0 \\ n_pM\,\mathrm{Im}\left\{\underline{i}_S\left(\underline{i}_R e^{jn_p\theta_R}\right)^*\right\} - \tau_L &= J\frac{d\omega_R}{dt} \end{aligned} \quad (2)$$

where the state vector's (complex) stator current, rotor current and stator voltage are defined as

$$\begin{aligned} \underline{i}_S &\triangleq i_{Sa} + ji_{Sb} \\ \underline{i}_R &\triangleq i_{Ra} + ji_{Rb} \\ \underline{u}_S &\triangleq u_{Sa} + ju_{Sb} \end{aligned}$$

The total load torque on the motor τ_L is defined as

$$\tau_L \triangleq f\omega_R + \tau_{L0}$$

where τ_{L0} denotes the external load torque exerted on the rotor, and is henceforth assumed to be constant.

B. Stator Field Coordinate System Model

Next, the model (2) is transformed into a stator field coordinate system. The transformation is defined as

$$\begin{aligned} \underline{i}_{Sdq} &\triangleq i_{Sd} + ji_{Sq} \triangleq \underline{i}_S e^{-j\omega_S t} \\ \underline{i}_{Rdq} &\triangleq i_{Rd} + ji_{Rq} \triangleq \underline{i}_R e^{jn_p\theta_R} e^{-j\omega_S t} \\ \underline{u}_{Sdq} &\triangleq u_{Sd} + ju_{Sq} \triangleq \underline{u}_S e^{-j\omega_S t} \end{aligned} \quad (3)$$

or

$$\begin{aligned} \underline{i}_S &= \underline{i}_{Sdq} e^{j\omega_S t} \\ \underline{i}_R &= \underline{i}_{Rdq} e^{-jn_p\theta_R} e^{j\omega_S t} \\ \underline{u}_S &= \underline{u}_{Sdq} e^{j\omega_S t} \end{aligned} \quad (4)$$

where ω_S is the electrical frequency of the voltage source applied to the stator and is assumed to be constant.

Substituting (4) into the space vector model (2) and simplifying results in

$$R_S\underline{i}_{Sdq} + L_S\frac{d\underline{i}_{Sdq}}{dt} + j\omega_S L_S\underline{i}_{Sdq} + M\frac{d\underline{i}_{Rdq}}{dt} + j\omega_S M\underline{i}_{Rdq} = \underline{u}_{Sdq}$$

$$R_R\underline{i}_{Rdq} + L_R\frac{d\underline{i}_{Rdq}}{dt} + j\left(\omega_S - n_p\omega_R\right)L_R\underline{i}_{Rdq} + M\frac{d\underline{i}_{Sdq}}{dt} + j\left(\omega_S - n_p\omega_R\right)M\underline{i}_{Sdq} = 0 \quad (5)$$

$$n_pM\,\mathrm{Im}\left\{\underline{i}_{Sdq}\left(\underline{i}_{Rdq}\right)^*\right\} - \left(f\omega_R + \tau_{L0}\right) = J\frac{d\omega_R}{dt}$$

Expanding into real and imaginary parts, we obtain the state space representation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + \mathbf{B}\mathbf{u} \quad (6)$$

where

$$\begin{aligned} \mathbf{x} &\triangleq \left[\begin{array}{ccccc} \omega_R & i_{Sd} & i_{Sq} & i_{Rd} & i_{Rq} \end{array}\right]^T \\ \mathbf{u} &\triangleq \left[\begin{array}{ccc} u_{Sd} & u_{Sq} & \tau_{L0} \end{array}\right]^T \end{aligned}$$

$$\mathbf{f}(\mathbf{x}) \triangleq \begin{bmatrix} \frac{n_pM}{J}\left(i_{Sq}i_{Rd} - i_{Sd}i_{Rq}\right) - \frac{f}{J}\omega_R \\ \frac{R_RM}{\sigma L_S L_R}i_{Rd} + \frac{n_pM}{\sigma L_S}\omega_R i_{Rq} - \frac{R_S}{\sigma L_S}i_{Sd} + \frac{n_pM^2}{\sigma L_S L_R}\omega_R i_{Sq} + \omega_S i_{Sq} \\ \frac{R_RM}{\sigma L_S L_R}i_{Rq} - \frac{n_pM}{\sigma L_S}\omega_R i_{Rd} - \frac{R_S}{\sigma L_S}i_{Sq} - \frac{n_pM^2}{\sigma L_S L_R}\omega_R i_{Sd} - \omega_S i_{Sd} \\ \frac{R_S\left(1-\sigma\right)}{\sigma M}i_{Sd} - \frac{n_pM}{\sigma L_R}\omega_R i_{Sq} - \frac{R_R}{\sigma L_R}i_{Rd} - \frac{n_p}{\sigma}\omega_R i_{Rq} + \omega_S i_{Rq} \\ \frac{R_S\left(1-\sigma\right)}{\sigma M}i_{Sq} + \frac{n_pM}{\sigma L_R}\omega_R i_{Sd} - \frac{R_R}{\sigma L_R}i_{Rq} + \frac{n_p}{\sigma}\omega_R i_{Rd} - \omega_S i_{Rd} \end{bmatrix}$$

$$B \triangleq \begin{bmatrix} 0 & 0 & -\frac{1}{J} \\ \frac{1}{\sigma L_S} & 0 & 0 \\ 0 & \frac{1}{\sigma L_S} & 0 \\ \frac{\sigma-1}{\sigma M} & 0 & 0 \\ 0 & \frac{\sigma-1}{\sigma M} & 0 \end{bmatrix}$$

and σ is the leakage factor defined as

$$\sigma \triangleq 1 - \frac{M^2}{L_S L_R} .$$

The equilibrium conditions are obtained by setting the derivatives in the stator field model (5) to zero and then equating the real and imaginary parts to obtain

$$R_S i_{Sd0} - \omega_S L_S i_{Sq0} - \omega_S M i_{Rq0} = u_{Sd0}$$

$$R_S i_{Sq0} + \omega_S L_S i_{Sd0} + \omega_S M i_{Rd0} = u_{Sq0}$$

$$\begin{aligned} R_R i_{Rd0} - \omega_S L_R i_{Rq0} + n_p \omega_{R0} L_R i_{Rq0} - \omega_S M i_{Sq0} \\ + n_p \omega_{R0} M i_{Sq0} = 0 \end{aligned} \tag{7}$$

$$\begin{aligned} R_R i_{Rq0} + \omega_S L_R i_{Rd0} - n_p \omega_{R0} L_R i_{Rd0} + \omega_S M i_{Sd0} \\ - n_p \omega_{R0} M i_{Sd0} = 0 \end{aligned}$$

$$n_p M \left(i_{Sq0} i_{Rd0} - i_{Sd0} i_{Rq0} \right) - \left(f \omega_{R0} + \tau_{L0} \right) = 0$$

C. Error Model

Next, to facilitate the Lyapunov analysis of the induction machine we derive an error model. This is achieved by translating the origin of the system (6) to an arbitrary equilibrium point $\mathbf{x}_0$ as defined by (7). Specifically, a set of error state variables about an equilibrium point is defined as

$$\begin{aligned} e_1 &\triangleq \omega_R - \omega_{R0} \\ e_2 &\triangleq i_{Sd} - i_{Sd0} \\ e_3 &\triangleq i_{Sq} - i_{Sq0} \\ e_4 &\triangleq i_{Rd} - i_{Rd0} \\ e_5 &\triangleq i_{Rq} - i_{Rq0} \end{aligned} \tag{8}$$

or

$$\begin{aligned} \omega_R &= e_1 + \omega_{R0} \\ i_{Sd} &= e_2 + i_{Sd0} \\ i_{Sq} &= e_3 + i_{Sq0} \\ i_{Rd} &= e_4 + i_{Rd0} \\ i_{Rq} &= e_5 + i_{Rq0}. \end{aligned} \tag{9}$$

Then, substituting these expressions for the state variables of the stator field into the model (6), we obtain the error model of the induction machine

$$\dot{\mathbf{e}} = A(\mathbf{x}_0)\mathbf{e} + \mathbf{g}(\mathbf{e}) \tag{10}$$

where

$$\mathbf{e} = \begin{bmatrix} e_1 & e_2 & e_3 & e_4 & e_5 \end{bmatrix}^T$$

$A(\mathbf{x}_0) \triangleq$

$$\begin{bmatrix} -\frac{f}{J} & -\frac{n_p M}{J} i_{Rq0} & \frac{n_p M}{J} i_{Rd0} & \frac{n_p M}{J} i_{Sq0} & -\frac{n_p M}{J} i_{Sd0} \\ -\frac{n_p(\sigma-1)}{\sigma}\left(\frac{L_R}{M} i_{Rq0} + i_{Sq0}\right) & -\frac{R_S}{\sigma L_S} & -\frac{n_p(\sigma-1)}{\sigma}\omega_{R0} + \omega_S & \frac{R_R M}{\sigma L_S L_R} & \frac{n_p M}{\sigma L_S}\omega_{R0} \\ \frac{n_p(\sigma-1)}{\sigma}\left(\frac{L_R}{M} i_{Rd0} + i_{Sd0}\right) & \frac{n_p(\sigma-1)}{\sigma}\omega_{R0} - \omega_S & -\frac{R_S}{\sigma L_S} & -\frac{n_p M}{\sigma L_S}\omega_{R0} & \frac{R_R M}{\sigma L_S L_R} \\ -\frac{n_p}{\sigma}\left(\frac{M}{L_R} i_{Sq0} + i_{Rq0}\right) & -\frac{R_S(\sigma-1)}{\sigma M} & -\frac{n_p M}{\sigma L_R}\omega_{R0} & -\frac{R_R}{\sigma L_R} & \omega_S - \frac{n_p}{\sigma}\omega_{R0} \\ \frac{n_p}{\sigma}\left(\frac{M}{L_R} i_{Sd0} + i_{Rd0}\right) & \frac{n_p M}{\sigma L_R}\omega_{R0} & -\frac{R_S(\sigma-1)}{\sigma M} & \frac{n_p}{\sigma}\omega_{R0} - \omega_S & -\frac{R_R}{\sigma L_R} \end{bmatrix} \tag{11}$$

and

$$\mathbf{g}(\mathbf{e}) \triangleq \begin{bmatrix} \frac{n_p M}{J} e_3 e_4 - \frac{n_p M}{J} e_2 e_5 \\ \frac{n_p M}{\sigma L_S} e_1 e_5 + \frac{n_p M^2}{\sigma L_S L_R} e_1 e_3 \\ -\frac{n_p M}{\sigma L_S} e_1 e_4 - \frac{n_p M^2}{\sigma L_S L_R} e_1 e_2 \\ -\frac{n_p M}{\sigma L_R} e_1 e_3 - \frac{n_p}{\sigma} e_1 e_5 \\ \frac{n_p M}{\sigma L_R} e_1 e_2 + \frac{n_p}{\sigma} e_1 e_4 \end{bmatrix} . \tag{12}$$

The error model consists of quadratic terms which vanish near the equilibrium point (where the matrix $A(\mathbf{x}_0)$ dominates), and the stability of the linearized system is dependent on the choice of the equilibrium point as $A(\mathbf{x}_0)$ depends on the equilibrium point.

The system (7) which determines the equilibrium points may be rewritten as

$$n_p M \left(i_{Sq0} i_{Rd0} - i_{Sd0} i_{Rq0} \right) - \left(f \omega_{R0} + \tau_{L0} \right) = 0 \tag{13}$$

$$K \begin{bmatrix} i_{Sd0} \\ i_{Sq0} \\ i_{Rd0} \\ i_{Rq0} \end{bmatrix} = \begin{bmatrix} u_{Sd0} \\ u_{Sq0} \\ 0 \\ 0 \end{bmatrix} \tag{14}$$

where $K \triangleq$

$$\begin{bmatrix} R_S & -\omega_S L_S & 0 & -\omega_S M \\ \omega_S L_S & R_S & \omega_S M & 0 \\ 0 & M\left(n_p \omega_{R0} - \omega_S\right) & R_R & L_R\left(n_p \omega_{R0} - \omega_S\right) \\ -M\left(n_p \omega_{R0} - \omega_S\right) & 0 & -L_R\left(n_p \omega_{R0} - \omega_S\right) & R_R \end{bmatrix} .$$

Therefore, one possible scenario is to select the set-points for the speed ω_{R0}, and voltages u_{Sd0} and u_{Sq0}, with the currents i_{Sd0}, i_{Sq0}, i_{Rd0} and i_{Rq0} then specified by equation (14). The resulting load torque τ_{L0} is determined by equation (13). In other words, one specifies ω_{R0}, u_{Sd0} and u_{Sq0}, and then uses

$$\begin{bmatrix} i_{Sd0} \\ i_{Sq0} \\ i_{Rd0} \\ i_{Rq0} \end{bmatrix} = K^{-1} \begin{bmatrix} u_{Sd0} \\ u_{Sq0} \\ 0 \\ 0 \end{bmatrix} \tag{15}$$

and

$$\tau_{L0} = n_p M \left(i_{Sq0} i_{Rd0} - i_{Sd0} i_{Rq0} \right) - f\omega_{R0} \tag{16}$$

to obtain the currents and load torque.

III. Power Balance Equation

The Lyapunov candidate function will be derived from a power balance equation that characterizes the power transfer between the input and output of the motor.

A. Power Balance Equation

First we define the magnetic field energy of the motor W_f and the mechanical energy W_J as (see [1])

$$\begin{aligned} W_f \;\triangleq\; & \frac{1}{2}L_S\left(i_{Sd}^2 + i_{Sq}^2\right) + \frac{1}{2}L_R\left(i_{Rd}^2 + i_{Rq}^2\right) \\ & + M \begin{bmatrix} i_{Sd} & i_{Sq} \end{bmatrix} \begin{bmatrix} i_{Rd} \\ i_{Rq} \end{bmatrix} \end{aligned} \tag{17}$$

and

$$W_J \triangleq \frac{1}{2}J\omega_R^2. \tag{18}$$

The power balance equation in terms of the stator field coordinate variables is given by

$$\begin{aligned} \frac{d}{dt}\left(W_f + W_J\right) = & \begin{bmatrix} u_{Sd} & u_{Sq} & -\tau_L \end{bmatrix} \begin{bmatrix} i_{Sd} \\ i_{Sq} \\ \omega_R \end{bmatrix} \\ & - R_S i_{Sd}^2 - R_S i_{Sq}^2 - R_R i_{Rd}^2 - R_R i_{Rq}^2. \end{aligned} \tag{19}$$

B. Error State Variables

Next, substituting for the state variables (7) of the stator field into the power balance equation (19), and simplifying using the equilibrium conditions (7) results in the power balance equation given in terms of the error state variables as

$$\begin{aligned} \frac{d}{dt}\left(W_f + W_J\right) = & \; u_{Sd}e_2 + u_{Sq}e_3 \\ & - \left(fe_1^2 + 2fe_1\omega_{R0} + \tau_{L0}e_1\right) \\ & - R_S\left(e_2^2 + 2e_2 i_{Sd0}\right) - R_S\left(e_3^2 + 2e_3 i_{Sq0}\right) \\ & - R_R\left(e_4^2 + 2e_4 i_{Rd0}\right) - R_R\left(e_5^2 + 2e_5 i_{Rq0}\right) \end{aligned} \tag{20}$$

where

$$\begin{aligned} W_f \;=\; & \frac{1}{2}L_S\left(e_2^2 + e_3^2 + 2e_2 i_{Sd0} + 2e_3 i_{Sq0}\right) \\ & + \frac{1}{2}L_R\left(e_4^2 + e_5^2 + 2e_4 i_{Rd0} + 2e_5 i_{Rq0}\right) + Me_2e_4 \\ & + M\left(i_{Sd0}e_4 + e_2 i_{Rd0} + e_3e_5 + i_{Sq0}e_5 + e_3 i_{Rq0}\right) \\ & + \frac{1}{2}L_S\left(i_{Sd0}^2 + i_{Sq0}^2\right) + \frac{1}{2}L_R\left(i_{Rd0}^2 + i_{Rq0}^2\right) \\ & + M\left(i_{Sd0}i_{Rd0} + i_{Sq0}i_{Rq0}\right) \end{aligned}$$

and

$$W_J = \frac{1}{2}J\left(e_1^2 + 2e_1\omega_{R0}\right) + \frac{1}{2}J\omega_{R0}^2.$$

IV. Lyapunov Stability of the Induction Machine

In this section, the power balance equation (20) is used to obtain a Lyapunov candidate function V. Define the function $W(\mathbf{e})$ by

$$W(\mathbf{e}) \triangleq W_f + W_J - \left(W_f(\mathbf{0}) + W_J(\mathbf{0})\right)$$

where

$$\begin{aligned} W_f(\mathbf{0}) \;=\; & \frac{1}{2}L_S\left(i_{Sd0}^2 + i_{Sq0}^2\right) + \frac{1}{2}L_R\left(i_{Rd0}^2 + i_{Rq0}^2\right) \\ & + M\left(i_{Sd0}i_{Rd0} + i_{Sq0}i_{Rq0}\right) \end{aligned}$$

and

$$W_J(\mathbf{0}) = \frac{1}{2}J\omega_{R0}^2.$$

This ensures $W(\mathbf{0}) = 0$, however W is not assured to be positive definite. Next rewrite $W(\mathbf{e})$ as

$$W(\mathbf{e}) = \mathbf{e}^T P \mathbf{e} + \mathbf{d}^T \mathbf{e} \tag{21}$$

where

$$P \triangleq \frac{1}{2}\begin{bmatrix} J & 0 & 0 & 0 & 0 \\ 0 & L_S & 0 & M & 0 \\ 0 & 0 & L_S & 0 & M \\ 0 & M & 0 & L_R & 0 \\ 0 & 0 & M & 0 & L_R \end{bmatrix} \tag{22}$$

and

$$\mathbf{d} \triangleq \begin{bmatrix} J\omega_{R0} \\ L_S i_{Sd0} + M i_{Rd0} \\ L_S i_{Sq0} + M i_{Rq0} \\ L_R i_{Rd0} + M i_{Sd0} \\ L_R i_{Rq0} + M i_{Sq0} \end{bmatrix}. \tag{23}$$

The derivative of $W(\mathbf{e})$ is of course equal to the right-hand side of the power balance equation (20), which is now rewritten as

$$\frac{dW}{dt} = -\mathbf{e}^T Q_W \mathbf{e} - \mathbf{c}_W^T \mathbf{e} \tag{24}$$

where

$$Q_W \triangleq \begin{bmatrix} f & 0 & 0 & 0 & 0 \\ 0 & R_S & 0 & 0 & 0 \\ 0 & 0 & R_S & 0 & 0 \\ 0 & 0 & 0 & R_R & 0 \\ 0 & 0 & 0 & 0 & R_R \end{bmatrix}$$

and

$$\mathbf{c}_W \triangleq \begin{bmatrix} 2f\omega_{R0}+\tau_{L0} \\ 2R_S i_{Sd0} - u_{Sd} \\ 2R_S i_{Sq0} - u_{Sq} \\ 2R_R i_{Rd0} \\ 2R_R i_{Rq0} \end{bmatrix}.$$

A. Lyapunov Candidate Function and its Derivative

Next, using P as defined in (22) above, a candidate Lyapunov function V is constructed by defining

$$V \triangleq \mathbf{e}^T P \mathbf{e}. \tag{25}$$

The derivative of this Lyapunov candidate function is thus

$$\frac{dV}{dt} = -\mathbf{e}^T Q_W \mathbf{e} - \mathbf{c}_W^T \mathbf{e} - \mathbf{d}^T \dot{\mathbf{e}}.$$

Using (10) this becomes

$$\frac{dV}{dt} = -\mathbf{e}^T Q_W \mathbf{e} - \mathbf{c}_W^T \mathbf{e} - \mathbf{d}^T (\mathbf{g}(\mathbf{e}) + A(\mathbf{x}_0)\mathbf{e})$$

which can be rewritten as

$$\frac{dV}{dt} = -\mathbf{e}^T Q \mathbf{e} - \mathbf{c}^T \mathbf{e} \tag{26}$$

where $Q \triangleq$

$$\begin{bmatrix} f & \frac{1}{2}n_p M i_{Rq0} & -\frac{1}{2}n_p M i_{Rd0} & \frac{1}{2}n_p L_R i_{Rq0} & -\frac{1}{2}n_p L_R i_{Rd0} \\ \frac{1}{2}n_p M i_{Rq0} & R_S & 0 & 0 & -\frac{1}{2}M n_p \omega_{R0} \\ -\frac{1}{2}n_p M i_{Rd0} & 0 & R_S & \frac{1}{2}M n_p \omega_{R0} & 0 \\ \frac{1}{2}n_p L_R i_{Rq0} & 0 & \frac{1}{2}M n_p \omega_{R0} & R_R & 0 \\ -\frac{1}{2}n_p L_R i_{Rd0} & -\frac{1}{2}M n_p \omega_{R0} & 0 & 0 & R_R \end{bmatrix} \tag{27}$$

and $\mathbf{c}^T = \mathbf{c}_W^T + \mathbf{d}^T A(\mathbf{x}_0)$ or explicitly

$$\mathbf{c} = \begin{bmatrix} f\omega_{R0} - n_p M\left(i_{Sq0} i_{Rd0} - i_{Sd0} i_{Rq0}\right) + \tau_{L0} \\ R_S i_{Sd0} - \left(L_S i_{Sq0} + M i_{Rq0}\right)\omega_S - u_{Sd0} \\ R_S i_{Sq0} + \left(L_S i_{Sd0} + M i_{Rd0}\right)\omega_S - u_{Sq0} \\ R_R i_{Rd0} + \left(M i_{Sq0} + L_R i_{Rq0}\right)\left(n_p\omega_{R0} - \omega_S\right) \\ R_R i_{Rq0} - \left(M i_{Sd0} + L_R i_{Rd0}\right)\left(n_p\omega_{R0} - \omega_S\right) \end{bmatrix}.$$

However, with reference to the equilibrium conditions (7) one sees that $\mathbf{c} \equiv \mathbf{0}$ regardless of the equilibrium point. Therefore, the Lyapunov candidate function and its derivative are

$$V \triangleq \mathbf{e}^T P \mathbf{e} \tag{28}$$

and

$$\frac{dV}{dt} = -\mathbf{e}^T Q \mathbf{e}. \tag{29}$$

B. Sufficient Conditions for Global Stability

The induction machine is globally asymptotically stable in the sense of Lyapunov if (see [3])

(a) $V(\mathbf{e}) > 0 \ \ \forall\ \mathbf{e} \neq \mathbf{0}$, and $V(\mathbf{0}) = 0$

(b) $dV(\mathbf{e})/dt < 0 \ \ \forall\ \mathbf{e} \neq \mathbf{0}$

(c) $V(\mathbf{e}) \to \infty$ as $\|\mathbf{e}\| \to \infty$

The leading principal minors of the matrix P are

$$\pi_1 = \frac{1}{2}J > 0,\ \pi_2 = \frac{1}{4}JL_S > 0,\ \pi_3 = \frac{1}{8}JL_S^2 > 0$$

$$\pi_4 = \frac{1}{16}J\sigma L_S^2 L_R > 0,\ \pi_5 = \frac{1}{32}J\sigma^2 L_S^2 L_R^2 > 0$$

As all of the leading principal minors of P are positive, P is positive definite. Moreover, $V(\mathbf{0}) = 0$ so that condition (a) is always satisfied. Furthermore, $V = \mathbf{e}^T P \mathbf{e} \geq \lambda_{\min}(P)\,\mathbf{e}^T\mathbf{e}$ and as $\lambda_{\min}(P) > 0$ we have $V(\mathbf{e}) \to \infty$ as $\|\mathbf{e}\| \to \infty$ thus fulfilling condition (c).

The matrix Q in (27) can be written as a function of just (S, ω_S), i.e., $Q = Q(S, \omega_S)$ by using (15) to eliminate the currents and $n_p\omega_{R0} = \omega_S(1-S)$ to eliminate $n_p\omega_{R0}$. Doing so, the leading principal minors of $Q(S, \omega_S)$ are computed and letting $S \to 0$ results in

$$\begin{array}{lcl} \Pi_1 & = & f \\ \Pi_2 & \to & fR_S \\ \Pi_3 & \to & fR_S^2 \\ \Pi_4 & \to & fR_S\left(R_S R_R - \frac{1}{4}M^2\omega_S^2\right) \\ \Pi_5 & \to & f\left(R_S R_R - \frac{1}{4}M^2\omega_S^2\right)^2 \end{array} \tag{30}$$

so that for

$$4R_S R_R - M^2\omega_S^2 > 0$$

and small enough S, the system is globally asymptotically stable. Summarizing, the main result is that for sufficiently small normalized slip S (i.e. the motor is lightly loaded), the system is globally asymptotically stable.

V. Numerical Example

Consider an induction machine with the following parameter values (see [1]): $M = 0.0117$ H, $L_R = 0.014$ H, $L_S = 0.014$ H, $R_S = 1.7\ \Omega$, $R_R = 3.9\ \Omega$, $f = 0.00014$ N-m/rad/sec, $J = 0.00011$ Kg-m^2, $n_p = 3$, $\omega_S = 2\pi \times 60$ rad/sec. The condition for globally asymptotically stable under light loads is

$$\frac{4R_S R_R}{M^2\omega_S^2} = 1.363 > 1.$$

For example, with the following set points: $u_{Sd0} = 50$ V, $u_{Sq0} = 0$ V, and $\omega_{R0} = 124$ rad/sec the normalized slip is

$$S = \frac{377 - 3 \times 124}{377} = \frac{377 - 372}{377} = 0.0132$$

and the corresponding equilibrium currents and load torque set point are computed from equations (15) and (16) as

$$\begin{aligned} i_{Sd0} &= +2.852 \text{ A} \\ i_{Sq0} &= -8.521 \text{ A} \\ i_{Rd0} &= -0.128 \text{ A} \\ i_{Rq0} &= -0.040 \text{ A} \\ \tau_{L0} &= +0.025 \text{ N-m.} \end{aligned}$$

Substituting these into the expression (27) for the matrix Q and numerically computing the five eigenvalues gives

$$\begin{bmatrix} 0.000121 \\ 0.361589 \\ 0.361607 \\ 5.238411 \\ 5.238412 \end{bmatrix}$$

which are all positive showing the system is globally asymptotically stable under these operating conditions.

VI. Conclusions and Future Work

Sufficient conditions for the global asymptotic stability of an open-loop induction machine have been derived in this work. Under lightly loaded conditions, global asymptotic stability holds meaning the motor will eventually converge to its equilibrium point no matter how far away it starts from the equilibrium point.

Future work is intended to focus on obtaining local stability results that set bounds on the error variables. These are expected to apply to larger set of operating conditions, but not result in global stability.

References

[1] John Chiasson, *Modeling and High-Performance Control of Electric Machines*, John Wiley & Sons, 2005.

[2] Werner Leonhard, *Control of Electrical Drives*, 3rd Edition, Springer-Verlag, Berlin, 2001.

[3] Hassan K. Khalil, *Nonlinear Systems, Third Edition*, Prentice-Hall, 2002.