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# COMBINATORIAL BOUNDS ON HILBERT FUNCTIONS OF FAT POINTS IN PROJECTIVE SPACE

SUSAN COOPER, BRIAN HARBOURNE, AND ZACH TEITLER

**ABSTRACT.** We study Hilbert functions of certain non-reduced schemes  $A$  supported at finite sets of points in  $\mathbf{P}^N$ , in particular, fat point schemes. We give combinatorially defined upper and lower bounds for the Hilbert function of  $A$  using nothing more than the multiplicities of the points and information about which subsets of the points are linearly dependent. When  $N = 2$ , we give these bounds explicitly and we give a sufficient criterion for the upper and lower bounds to be equal. When this criterion is satisfied, we give both a simple formula for the Hilbert function and combinatorially defined upper and lower bounds on the graded Betti numbers for the ideal  $I_A$  defining  $A$ , generalizing results of Geramita–Migliore–Sabourin [GMS]. We obtain the exact Hilbert functions and graded Betti numbers for many families of examples, interesting combinatorially, geometrically, and algebraically. Our method works in any characteristic.

## 1. INTRODUCTION

The determination of Hilbert functions and graded Betti numbers for ideals of fat point schemes in  $\mathbf{P}^N$  are issues of significant interest in algebraic geometry, even when  $N = 2$  (and related areas; e.g., [AP, Sc, Mi]). Previous work has focused on obtaining exact determinations of Hilbert functions and graded Betti numbers in various circumstances. For example, [H4, G, Hi2, HHH] formulate conjectures for the ideals of fat points in  $\mathbf{P}^2$ . Various special cases have attracted a lot of attention, such as when the number of points is a square [CM3, Ev1, Ev2, HR, R], or when the multiplicities of the points are small [CM1, GMS, I] or the number of points is small [N, FHH, GHM].

Our aim here is to give a *general* iterative method for determining *upper and lower bounds* on the Hilbert function of any fat point scheme in  $\mathbf{P}^N$  (in any characteristic), using only the multiplicities of the points and information about which subsets of the points are linearly dependent (meaning, when  $N = 2$ , which subsets of the points are collinear); i.e., information which can be recovered from the matroid of linear dependencies of the points. The bounds we obtain in case  $N = 2$  are explicit and easy to compute. Although our bounds are tightest when the points are fairly special, the best results for general sets of points depend on understanding specializations (see, for example, [CM2], [Du], [Hi1]), so we expect our work to be useful quite broadly. When  $N = 2$ , we also give a simple criterion for the bounds to coincide, in which case our results determine the Hilbert function uniquely. This condition is satisfied in many cases of interest; see §4 for examples. When the condition is satisfied, we also give upper and lower bounds on the graded Betti numbers of the ideal of the fat point scheme, and we give a condition for these bounds to coincide.

We begin with the very general statement Theorem 1.1.3, then specialize to hyperplanes in  $\mathbf{P}^N$ , then specialize further to lines in  $\mathbf{P}^2$ .

It is an open question to determine the Hilbert functions for specific kinds of fat point schemes in projective space. For example, it is not known which functions occur as Hilbert functions of double point subschemes of  $\mathbf{P}^2$ . (A double point scheme is a fat point scheme consisting of a finite set

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of points all taken with multiplicity 2; i.e., the scheme defined by the symbolic square of the ideal defining the reduced finite set of points.) A start on this problem for points, in the case of double point subschemes of  $\mathbf{P}^2$  in characteristic 0, was made in [GMS] using linkage methods. One of the main results of [GMS] gives a criterion characterizing a class of functions each of which occurs as the Hilbert function of a double point scheme. For each function  $h$  in that class, the approach of [GMS] is to use a sequence of basic double links to construct a double point scheme whose Hilbert function is  $h$ .

We take an approach opposite to that presented in [GMS]. For the case of fat point subschemes of  $\mathbf{P}^2$ , our procedure amounts to tearing down the subscheme as a sequence of residuals with respect to lines containing various subsets of points of the scheme to obtain upper and lower bounds for the Hilbert function of the scheme (for subschemes  $A, B \subseteq \mathbf{P}^N$ , by the residual of  $A$  with respect to  $B$  we mean the subscheme defined by the ideal  $I_A : I_B$ ). Starting with any fat point subscheme  $Z = Z_0$ , we choose a sequence of lines  $L_1, \dots, L_r$  and define  $Z_i$  to be the residual of  $Z_{i-1}$  with respect to  $L_i$  and we define the associated reduction vector  $\mathbf{d} = (d_1, \dots, d_r)$  by taking  $d_i = \deg(L_i \cap Z_{i-1})$ . Given  $\mathbf{d}$  (or indeed any vector  $\mathbf{d} = (d_1, \dots, d_n)$  with non-negative integer entries), we also define functions  $f_{\mathbf{d}}$  and  $F_{\mathbf{d}}$  from the non-negative integers to the non-negative integers (see Definition 2.3.3); these functions are defined completely in terms of  $\mathbf{d}$ . Our procedure works in a similar way for fat point subschemes of  $\mathbf{P}^N$  for  $N > 2$ , except the residuals are taken with respect to hyperplanes, and our bounds are less explicit, since the procedure expresses the bounds for a subscheme  $Z$  of  $\mathbf{P}^N$  in terms of bounds obtained iteratively on the Hilbert functions of  $Z$  and its residuals intersected with various hyperplanes. The iteration terminates with  $\mathbf{P}^2$  by applying Theorem 1.1, which is proved in section §3, ultimately as a corollary of Theorem 1.1.3. AWK scripts implementing the following theorem can be obtained at <http://www.math.unl.edu/~bharbourne1/CHT/Example.html>. (Note: in the original posted version [CHT] of this paper the indexation of the entries of  $\mathbf{d}$  is the reverse of what we use here.)

**Theorem 1.1.** *Let  $Z = Z_0$  be a fat point scheme in  $\mathbf{P}^2$  with reduction vector  $\mathbf{d} = (d_1, \dots, d_r)$  such that  $Z_{r+1} = \emptyset$ . Then the Hilbert function  $h_Z$  is bounded by  $f_{\mathbf{d}} \leq h_Z \leq F_{\mathbf{d}}$ . Furthermore, if  $\mathbf{d}$  is non-increasing and positive, i.e.,  $d_1 \geq d_2 \geq \dots \geq d_r > 0$ , then  $f_{\mathbf{d}} = F_{\mathbf{d}}$  unless  $\mathbf{d}$  contains a subsequence of consecutive entries of the form  $(a, a, a)$ , or of the form  $(a_i, \dots, a_{i+j+1})$  for  $j > 1$  where  $a_i = a_{i+1}$ ,  $a_{i+j} = a_{i+j+1}$ , and  $a_{i+1}, \dots, a_{i+j}$  are consecutive integers. If  $d_1 > d_2 > \dots > d_r > 0$ , then  $\mathbf{d}$  also uniquely determines the graded Betti numbers of the ideal  $I_Z$ .*

We will leave an analysis of the bounds obtained by our iterative procedure when  $N > 2$  to future articles. For the case of  $N = 2$  our examples in §4 show that this theorem gives the Hilbert functions and graded Betti numbers of fat point subschemes of  $\mathbf{P}^2$  in many situations where they were not previously known, in some cases for subschemes defined by arbitrary symbolic powers of ideals of points, substantially extending some of the results of [GMS] which considered only symbolic squares.

Although the results of [GMS] were our main motivation, the methods of our paper were inspired by those of [FL] and [FHL]. In these papers the authors focus their attention on fat point subschemes of  $X \subset \mathbf{P}^N$  whose reduced subscheme  $X_{\text{red}}$  is contained in a hyperplane  $H$ . In this setting, the basic idea of [FL] and [FHL] is to obtain the minimal free resolution of the ideal  $I_X$  (and hence the Hilbert function and graded Betti numbers for  $I_X$ ) in terms of the Hilbert functions and graded

Betti numbers of  $X_i \cap H$ , where  $\emptyset = X_r \subset X_{r-1} \subset \cdots \subset X_0 = X$  are successive residuals of  $X$  with respect to  $H$ .

Not only can our method be used to obtain bounds for Hilbert functions of fat point schemes in  $\mathbf{P}^N$  for  $N > 2$ , in terms of residuation with respect to hyperplanes, but our underlying method can also be applied taking residuals with respect to nonlinear subvarieties, such as curves in  $\mathbf{P}^2$  of degrees greater than 1, or with curves (especially smooth rational curves) on surfaces  $X$  obtained by blowing up points of  $\mathbf{P}^2$ . We focus on the case of lines in  $\mathbf{P}^2$ , partly because the data structures for specifying linear dependencies of points in  $\mathbf{P}^2$  are easier to analyze than those for points in higher dimensions, but also because the work of Geramita, Migliore and Sabourin [GMS] has sparked interest in finding conditions which uniquely determine the Hilbert functions of fat point subschemes of  $\mathbf{P}^2$  (as we do in Theorem 2.2.2, Proposition 2.2.4 and Remark 2.5.1).

**1.1. The key idea.** Our fundamental idea is very simple but it ends up giving remarkably strong results. Given a sheaf  $\mathcal{Q}_0$  on a scheme  $S$ , one often attempts to obtain information about  $h^0(S, \mathcal{Q}_0)$  by finding a short exact sequence

$$(1.1.1) \quad 0 \rightarrow \mathcal{Q}_1 \rightarrow \mathcal{Q}_0 \rightarrow \mathcal{R}_0 \rightarrow 0$$

of sheaves where one knows something about the cohomology of  $\mathcal{Q}_1$  and  $\mathcal{R}_0$ . In particular, the short exact sequence of sheaves gives a long exact sequence of cohomology from which we immediately obtain the following bounds:

$$(1.1.2) \quad \max(h^0(S, \mathcal{Q}_1), h^0(S, \mathcal{Q}_1) + h^0(S, \mathcal{R}_0) - h^1(S, \mathcal{Q}_1)) \leq h^0(S, \mathcal{Q}_0) \leq h^0(S, \mathcal{Q}_1) + h^0(S, \mathcal{R}_0).$$

In our situation we will know everything about  $\mathcal{R}_0$  and we will know all we need to know about  $h^0(S, \mathcal{Q}_1) - h^1(S, \mathcal{Q}_1)$ , but we will not usually know  $h^0(S, \mathcal{Q}_1)$  by itself. However, we will be able to find another short exact sequence in which  $\mathcal{Q}_1$  becomes the middle term. By iteration, we obtain the following theorem (in our typical applications, we will know the quantities  $h^0(\mathcal{Q}_i) - h^1(\mathcal{Q}_i)$ , we will either know the quantities  $h^0(\mathcal{R}_i)$  or have inductively obtained bounds on them, and  $\mathcal{Q}_{n+1}$  will be a sheaf whose cohomology we know):

**Theorem 1.1.3.** *Given short exact sequences of sheaves on a scheme  $S$*

$$(1.1.4) \quad \begin{array}{ccccccc} 0 & \rightarrow & \mathcal{Q}_1 & \rightarrow & \mathcal{Q}_0 & \rightarrow & \mathcal{R}_0 \rightarrow 0 \\ 0 & \rightarrow & \mathcal{Q}_2 & \rightarrow & \mathcal{Q}_1 & \rightarrow & \mathcal{R}_1 \rightarrow 0 \\ & & & & \vdots & & \\ 0 & \rightarrow & \mathcal{Q}_{n+1} & \rightarrow & \mathcal{Q}_n & \rightarrow & \mathcal{R}_n \rightarrow 0, \end{array}$$

*we obtain the following bounds (in which we have suppressed  $S$  in our notation):*

$$h^0(\mathcal{Q}_0) \leq h^0(\mathcal{Q}_{n+1}) + h^0(\mathcal{R}_n) + \cdots + h^0(\mathcal{R}_0)$$

and

$$h^0(\mathcal{Q}_0) \geq \max(h^0(\mathcal{Q}_{n+1}), h^0(\mathcal{R}_n) + h^0(\mathcal{Q}_{n+1}) - h^1(\mathcal{Q}_{n+1}), \dots, h^0(\mathcal{R}_0) + h^0(\mathcal{Q}_1) - h^1(\mathcal{Q}_1)).$$

*Moreover, the upper bound is an equality if and only if each of the short exact sequences of (1.1.4) is exact on global sections, and both the upper and lower bounds are equalities if  $h^0(\mathcal{R}_i)h^1(\mathcal{Q}_{i+1}) = 0$  for  $0 \leq i \leq n$ .*

*Proof.* By applying the bounds in (1.1.2) to successive rows of (1.1.4), we get the upper bound

$$\begin{aligned}
 (1.1.5) \quad h^0(\mathcal{Q}_0) &\leq h^0(\mathcal{Q}_1) + h^0(\mathcal{R}_0) \\
 &\leq h^0(\mathcal{Q}_2) + h^0(\mathcal{R}_1) + h^0(\mathcal{R}_0) \\
 &\leq \dots \\
 &\leq h^0(\mathcal{Q}_{n+1}) + h^0(\mathcal{R}_n) + \dots + h^0(\mathcal{R}_0)
 \end{aligned}$$

and similarly we get the lower bound

$$\begin{aligned}
 (1.1.6) \quad h^0(\mathcal{Q}_0) &\geq \max(h^0(\mathcal{Q}_1), h^0(\mathcal{R}_0) + h^0(\mathcal{Q}_1) - h^1(\mathcal{Q}_1)) \\
 &\geq \max(h^0(\mathcal{Q}_2), h^0(\mathcal{R}_1) + h^0(\mathcal{Q}_2) - h^1(\mathcal{Q}_2), h^0(\mathcal{R}_0) + h^0(\mathcal{Q}_1) - h^1(\mathcal{Q}_1)) \\
 &\geq \dots \\
 &\geq \max(h^0(\mathcal{Q}_{n+1}), h^0(\mathcal{R}_n) + h^0(\mathcal{Q}_{n+1}) - h^1(\mathcal{Q}_{n+1}), \dots, h^0(\mathcal{R}_0) + h^0(\mathcal{Q}_1) - h^1(\mathcal{Q}_1)).
 \end{aligned}$$

Clearly, the upper bound in (1.1.2) is an equality if and only if (1.1.1) is exact on global sections. It follows that the inequalities in (1.1.5) are all equalities if and only if each of the short exact sequences of (1.1.4) is exact on global sections. In particular, if  $h^0(\mathcal{R}_i)h^1(\mathcal{Q}_{i+1}) = 0$  for every  $0 \leq i \leq n$ , then (1.1.5) is an equality. Similarly,  $h^0(\mathcal{Q}_i) = \max(h^0(\mathcal{Q}_{i+1}), h^0(\mathcal{R}_i) + h^0(\mathcal{Q}_{i+1}) - h^1(\mathcal{Q}_{i+1}))$  if  $h^0(\mathcal{R}_i)h^1(\mathcal{Q}_{i+1}) = 0$ , and hence the inequalities of (1.1.6) are all equalities if  $h^0(\mathcal{R}_i)h^1(\mathcal{Q}_{i+1}) = 0$  for  $0 \leq i \leq n$ .  $\square$

**Remark 1.1.7.** If by induction we have upper and lower bounds on  $h^0(\mathcal{R}_i)$  and if  $h^0(\mathcal{Q}_{n+1})$  and the differences  $h^0(\mathcal{Q}_i) - h^1(\mathcal{Q}_i)$  are known, then as a corollary of Theorem 1.1.3 we obtain upper and lower bounds on  $h^0(\mathcal{Q}_0)$ . This is how we obtain upper and lower bounds on the Hilbert function of fat point subschemes of projective space in general, and it is how we obtain  $f_{\mathbf{d}}$  and  $F_{\mathbf{d}}$  in Theorem 1.1 in particular. Moreover, although we will not usually know  $h^1(\mathcal{Q}_{i+1})$ , if we have  $h^0(\mathcal{R}_i)h^1(\mathcal{Q}_{i+1}) = 0$  for all  $i$ , then we have

$$\begin{aligned}
 \max(h^0(\mathcal{Q}_{n+1}), h^0(\mathcal{R}_n) + h^0(\mathcal{Q}_{n+1}) - h^1(\mathcal{Q}_{n+1}), \dots, h^0(\mathcal{R}_0) + h^0(\mathcal{Q}_1) - h^1(\mathcal{Q}_1)) = \\
 h^0(\mathcal{Q}_0) = h^0(\mathcal{Q}_{n+1}) + h^0(\mathcal{R}_n) + \dots + h^0(\mathcal{R}_0)
 \end{aligned}$$

and thus we do know  $h^0(\mathcal{Q}_0)$ . In particular, if  $h^1(\mathcal{Q}_{n+1}) = 0$  and if for some  $i$  we have  $h^1(\mathcal{R}_j) = 0$  for  $j > i$ , then it follows from (1.1.4), working our way up from the bottom of the diagram, that  $h^1(\mathcal{Q}_j) = 0$  for  $j > i$ . Hence if whenever  $h^0(\mathcal{R}_i) > 0$  we have  $h^1(\mathcal{R}_j) = 0$  for  $j > i$ , then we have  $h^0(\mathcal{R}_i)h^1(\mathcal{Q}_{i+1}) = 0$  for all  $i$ , in which case our upper and lower bounds are both equal to  $h^0(\mathcal{Q}_0)$ . This is precisely what happens in those cases where Theorem 1.1 allows us to conclude  $f_{\mathbf{d}} = F_{\mathbf{d}}$ .

**1.2. Background.** We briefly cover the background we will need regarding fat points, Hilbert functions, graded Betti numbers and residuals. For this purpose, let  $R = K[\mathbf{P}^N]$  be the homogeneous coordinate ring for  $\mathbf{P}^N$  over an arbitrary algebraically closed field  $K$ .

**Definition 1.2.1.** Given points  $p_1, \dots, p_r \in \mathbf{P}^N$  and non-negative integers  $m_i$ , the ideal  $I = \bigcap_i I_{p_i}^{m_i} \subset R$ , where  $I_{p_i}$  is the ideal generated by all forms that vanish at  $p_i$ , defines a subscheme  $Z \subset \mathbf{P}^N$  referred to as a **fat point scheme**.

We denote  $Z$  by  $m_1p_1 + \dots + m_r p_r$  and  $I$  by  $I_Z$ . The **support** of  $Z$  is the set of points  $p_i$  with  $m_i > 0$ . If  $X = p_1 + \dots + p_r$  and  $Z = mp_1 + \dots + mp_r$ , we write  $Z = mX$ .

The ideal  $I_Z$  is graded in the usual way by degree. The span of the forms in  $I_Z$  of degree  $t$  is denoted  $(I_Z)_t$ ; thus  $I_Z = \bigoplus_t (I_Z)_t$ . With  $(R/I_Z)_t = R_t/(I_Z)_t$ , we see that the quotient  $R/I_Z$  is also graded. The **Hilbert function**  $h_{I_Z}$  of  $I_Z$  gives the  $K$ -vector space dimension of  $(I_Z)_t$  as a function of  $t \geq 0$ ; i.e.,  $h_{I_Z}(t) = \dim_K((I_Z)_t)$ . Similarly, the **Hilbert function**  $h_Z$  of  $Z$  gives the  $K$ -vector space dimension of  $(R/I_Z)_t$  as a function of  $t \geq 0$ ; i.e.,  $h_Z(t) = \dim_K((R/I_Z)_t)$ . It is convenient to regard  $h_Z$  as the sequence  $(h_Z(0), h_Z(1), \dots)$  and vice versa.

As a graded ideal,  $I_Z$  has a minimal graded free resolution. This is an exact sequence of the form  $0 \rightarrow F_1 \rightarrow F_0 \rightarrow I_Z \rightarrow 0$ , where each  $F_i$  is a graded free  $R$ -module and the map  $F_1 \rightarrow F_0$  is given by a matrix of homogeneous forms of positive degree. As graded free modules, each  $F_i$  is, up to graded isomorphism, either 0 or a direct sum of modules of the form  $R[-j]$ , where  $R[-j]$  is just  $R$  with a shift in the grading; i.e.,  $R[-j]_t = R_{t-j}$ . The **graded Betti numbers** of  $I_Z$  are the numbers of factors of  $R[-j]$  in each sum. In particular, we can write  $F_0$  as  $\bigoplus_{j \geq 0} R[-j]^{\nu_j(Z)}$  and  $F_1$  as  $\bigoplus_{j \geq 0} R[-j]^{\sigma_j(Z)}$ . The graded Betti numbers of  $I_Z$  are just the sequences  $\nu(Z)$  and  $\sigma(Z)$ . (For clarity we suppress the  $Z$  if it is not needed.) The value  $\nu_j$  specifies the number of generators of  $I_Z$  in degree  $j$  in any minimal set of homogeneous generators of  $I_Z$ ; the sequence  $\sigma$  does the same for the syzygies. In cases in which the Hilbert function  $h_Z$  is known, one can recover the sequence  $\sigma$  from  $h_Z$  and  $\nu$  using the formula (see [FHH])  $\sigma_t - \nu_t = -\Delta^3 h_{I_Z}(t)$  (which for  $t > 0$  can be written as  $\sigma_t - \nu_t = \Delta^3 h_Z(t)$ ), where  $\Delta$  is the difference operator. For any sequence  $v = (v_1, v_2, \dots, v_r)$  we define  $\Delta(v)$  to be  $(v_1, v_2 - v_1, \dots, v_r - v_{r-1})$ . For a function  $f$  defined on  $\{0, 1, 2, \dots\}$ , we will define  $\Delta(f)$  to be the function  $\Delta(f)(0) = f(0)$  and  $\Delta(f)(t) = f(t) - f(t-1)$  for  $t > 0$ .

**Definition 1.2.2.** Given  $A = a_1p_1 + \dots + a_r p_r \subset \mathbf{P}^N$ , let  $L$  be a hyperplane and let  $F$  be the linear form defining  $L$ . We do not assume  $L$  passes through any point of  $A$ . The **subscheme  $B$  of  $A$  residual to  $L$**  is defined by the ideal  $I_B = (I_A : F)$ .

Note that the ideal  $(I_A : F)$  is saturated and the subscheme  $B$  it defines is the fat point subscheme  $B = b_1p_1 + \dots + b_r p_r$  where  $b_i = a_i$  if  $p_i \notin L$  and  $b_i = \max(a_i - 1, 0)$  if  $p_i \in L$ . We write  $B = A : L$ , as in the following standard short exact sequence of graded  $K[\mathbf{P}^N]$ -modules:

$$0 \rightarrow I_{A:L}(-1) \xrightarrow{\cdot F} I_A \rightarrow I_A/(I_A \cap (F)) \rightarrow 0.$$

Sheafifying the above short exact sequence and tensoring by  $\mathcal{O}_{\mathbf{P}^N}(t)$  yields the following short exact sequence of  $\mathcal{O}_{\mathbf{P}^N}$ -modules, which will play the role of (1.1.1):

$$(1.2.3) \quad 0 \rightarrow \mathcal{I}_{A:L}(t-1) \rightarrow \mathcal{I}_A(t) \rightarrow \mathcal{I}_{L \cap A, L}(t) \rightarrow 0$$

where  $\mathcal{I}_{L \cap A, L}$  denotes the ideal sheaf of  $L \cap A$  regarded as a subscheme of  $L$  (or, depending on context, its extension by zero to  $\mathbf{P}^N$ ). When  $N = 2$ , then  $L \cong \mathbf{P}^1$  and  $\mathcal{I}_{L \cap A, L}(t) \cong \mathcal{O}_{\mathbf{P}^1}(t-d)$ , where  $d = \sum \{a_i : p_i \in L\}$  is the degree of  $L \cap A$ .

We will use the well-known fact that  $H^0(\mathcal{I}_A(t))$  can be identified with the homogeneous component  $(I_A)_t$  of  $I_A$  in degree  $t$  (see, for example, [H3, Proposition 4.1.1]).

**Remark 1.2.4.** Since (1.2.3) is the critical technical tool of this paper, we provide here a brief justification. It is easy to check that  $I_{A:L} = I_B$  and  $I_A$  sheafify to  $\mathcal{I}_B$  and  $\mathcal{I}_A$  respectively. The inclusion  $FI_B \subset I_A$  sheafifies to  $\mathcal{I}_B \otimes \mathcal{O}_{\mathbf{P}^N}(-L) \subset \mathcal{I}_A$ , which is conventionally written as  $\mathcal{I}_B(-1) \subset$

$\mathcal{I}_A$ . Also,  $I_A/FI_B = I_A/(I_A \cap (F)) = (I_A + (F))/(F)$  is an ideal in the homogeneous coordinate ring  $R/(F) = K[L] \cong K[\mathbf{P}^{N-1}]$  of  $\mathbf{P}^{N-1}$ . In particular  $\mathcal{I}_A/\mathcal{I}_B(-1)$  is the sheafification of  $I_A/FI_B = (I_A + (F))/(F)$ , so it is a subsheaf of the sheafification  $\mathcal{O}_L$  of  $K[L]$  (regarded as a sheaf on  $\mathbf{P}^N$  by extension by 0). Thus we have an exact sequence  $0 \rightarrow \mathcal{I}_B(-1) \rightarrow \mathcal{I}_A \rightarrow \mathcal{I}_{L \cap A, L}$ . By checking stalks, we see that in fact  $\mathcal{I}_A \rightarrow \mathcal{I}_{L \cap A, L}$  is surjective. Thus  $0 \rightarrow \mathcal{I}_B(-1) \rightarrow \mathcal{I}_A \rightarrow \mathcal{I}_{L \cap A, L} \rightarrow 0$  is exact and  $\mathcal{I}_{L \cap A, L}$  is the sheafification of  $I_A/(I_A \cap (F))$ .

By iteration we obtain from (1.2.3) the linked short exact sequences that will play the role of (1.1.4).

**Definition 1.2.5.** Let  $A = a_1p_1 + a_2p_2 + \dots + a_r p_r$  be a fat point scheme in  $\mathbf{P}^N$ . Fix a sequence  $L_1, \dots, L_{n+1}$  of hyperplanes in  $\mathbf{P}^N$ , not necessarily distinct.

- (a) We define fat point schemes  $A_0, \dots, A_{n+1}$  by  $A_0 = A$  and  $A_j = A_{j-1} : L_j$  for  $1 \leq j \leq n+1$ . As a matter of notation, we write  $A : L_1, \dots, L_j = (((A : L_1) : L_2) : \dots : L_j)$ , so we have  $A_j = A : L_1, \dots, L_j$ .
- (b) We will say that the sequence  $L_1, \dots, L_{n+1}$  **totally reduces**  $A$  if  $A_{n+1} = \emptyset$  is the empty scheme, or equivalently, if for each point  $p_i$ , the hyperplane  $L_j$  passes through  $p_i$  for at least  $a_i$  of the indices  $j$ .
- (c) We associate to  $A$  and the sequence  $L_1, \dots, L_{n+1}$  an integer vector  $\mathbf{d} = \mathbf{d}(A; L_1, \dots, L_{n+1}) = (d_1, \dots, d_{n+1})$ , where  $d_j$  is the degree  $\deg(L_j \cap A_{j-1})$  of the scheme theoretic intersection of  $L_j$  with  $A_{j-1}$ . We refer to  $\mathbf{d}$  as the **reduction vector** for  $A$  induced by the sequence  $L_1, \dots, L_{n+1}$ . We will say that  $\mathbf{d}$  is a **full** reduction vector for  $A$  if  $L_1, \dots, L_{n+1}$  totally reduces  $A$ .

**Remark 1.2.6.** If  $N = 2$  and  $L_1, \dots, L_{n+1}$  totally reduce  $A$  with reduction vector  $\mathbf{d} = (d_1, \dots, d_{n+1})$ , we note that  $d_j$  is the sum of the multiplicities  $a_{j-1, i}$  of  $A_{j-1} = a_{j-1, 1}p_1 + \dots + a_{j-1, r}p_r$  at those points  $p_i$  of  $A_{j-1}$  which lie on  $L_j$ . More generally it is not hard to see for any  $N$  that  $\deg(A_i) = d_{i+1} + \dots + d_{n+1}$  for  $0 \leq i \leq n$  and  $\deg(A) = d_1 + \dots + d_{n+1}$ .

## 2. BOUNDS FOR THE HILBERT FUNCTION

In this section we obtain various versions of our bounds on Hilbert functions and give several examples.

**2.1. Bounds for the Hilbert function of the ideal.** In the context of hyperplanes  $L_1, \dots, L_{n+1}$  which totally reduce  $A = a_1p_1 + a_2p_2 + \dots + a_r p_r \subset \mathbf{P}^N$ , let  $\overline{A}_i$  be the scheme theoretic intersection  $L_{i+1} \cap A_i$ . Then (1.1.4) becomes:

$$\begin{aligned}
 (2.1.1) \quad & \begin{array}{ccccccc}
 0 & \rightarrow & \mathcal{I}_{A_1}(t-1) & \rightarrow & \mathcal{I}_{A_0}(t) & \rightarrow & \mathcal{I}_{\overline{A_0, L_1}}(t) & \rightarrow & 0 \\
 0 & \rightarrow & \mathcal{I}_{A_2}(t-2) & \rightarrow & \mathcal{I}_{A_1}(t-1) & \rightarrow & \mathcal{I}_{\overline{A_1, L_2}}(t-1) & \rightarrow & 0 \\
 & & & & \vdots & & & & \\
 0 & \rightarrow & \mathcal{I}_{A_{i+1}}(t-(i+1)) & \rightarrow & \mathcal{I}_{A_i}(t-i) & \rightarrow & \mathcal{I}_{\overline{A_i, L_{i+1}}}(t-i) & \rightarrow & 0 \\
 & & & & \vdots & & & & \\
 0 & \rightarrow & \mathcal{I}_{A_n}(t-n) & \rightarrow & \mathcal{I}_{A_{n-1}}(t-(n-1)) & \rightarrow & \mathcal{I}_{\overline{A_{n-1}, L_n}}(t-(n-1)) & \rightarrow & 0 \\
 0 & \rightarrow & \mathcal{O}_{\mathbf{P}^N}(t-(n+1)) & \rightarrow & \mathcal{I}_{A_n}(t-n) & \rightarrow & \mathcal{I}_{\overline{A_n, L_{n+1}}}(t-n) & \rightarrow & 0
 \end{array}
 \end{aligned}$$

From Theorem 1.1.3 applied to (2.1.1), we almost immediately obtain the following result:



**Corollary 2.1.2.** *Let  $A_0 = \sum_i m_i p_i$  be a fat point subscheme of  $\mathbf{P}^N$  and let  $L_1, \dots, L_{n+1}$  be hyperplanes which totally reduce  $A_0$ . Let  $h_n = h^0(L_{n+1}, \mathcal{I}_{A_n, L_{n+1}}(t-n)) + \binom{t-(n+1)+N}{N}$ , for  $0 \leq i < n$  let*

$$h_i = h^0(L_{i+1}, \mathcal{I}_{A_i, L_{i+1}}(t-i)) + h^0(\mathcal{O}_{\mathbf{P}^N}(t-(i+1))) - \deg(A_{i+1}),$$

let  $h'_{n+1} = h^0(\mathcal{O}_{\mathbf{P}^N}(t-(n+1)))$  and for  $0 \leq i \leq n$  let

$$h'_i = h^0(\mathcal{O}_{\mathbf{P}^N}(t-i)) - \deg(A_i).$$

Then

$$\max_{0 \leq i \leq n+1} (h'_i) \leq \max_{0 \leq i \leq n} (h_i) \leq h^0(\mathbf{P}^N, \mathcal{I}_{A_0}(t)) \leq \binom{t-n+N-1}{N} + \sum_{i=0}^n h^0(L_{i+1}, \mathcal{I}_{A_i, L_{i+1}}(t-i)).$$

*Proof.* The upper bound is immediate from Theorem 1.1.3 using the fact that  $h^0(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(t-(n+1))) = \binom{t-(n+1)+N}{N}$ . The lower bound  $\max_{0 \leq i \leq n} (h_i) \leq h^0(\mathbf{P}^N, \mathcal{I}_{A_0}(t))$  is also precisely the bound given in Theorem 1.1.3. To see this, use the fact that  $h^1(\mathbf{P}^N, \mathcal{O}_{\mathbf{P}^N}(t-(n+1))) = 0$  for all  $t$ , so  $\max(h^0(\mathcal{O}_{\mathbf{P}^N}(t-(n+1))), h^0(L_{n+1}, \mathcal{I}_{A_n, L_{n+1}}(t-n)) + h^0(\mathcal{O}_{\mathbf{P}^N}(t-(n+1))) - h^1(\mathcal{O}_{\mathbf{P}^N}(t-(n+1))))$  is  $h_n = h^0(L_{n+1}, \mathcal{I}_{A_n, L_{n+1}}(t-n)) + \binom{t-(n+1)+N}{N}$ . The remaining terms over which the maximum is taken in (1.1.6) are of the form

$$(2.1.3) \quad h^0(L_{i+1}, \mathcal{I}_{A_i, L_{i+1}}(t-i)) + h^0(\mathcal{I}_{A_{i+1}}(t-(i+1))) - h^1(\mathcal{I}_{A_{i+1}}(t-(i+1)))$$

for  $0 \leq i < n$ . Taking cohomology of the short exact sequence

$$0 \rightarrow \mathcal{I}_{A_{i+1}}(t-(i+1)) \rightarrow \mathcal{O}_{\mathbf{P}^N}(t-(i+1)) \rightarrow \mathcal{O}_{A_{i+1}}(t-(i+1)) \rightarrow 0$$

and using  $h^1(\mathcal{O}_{\mathbf{P}^N}(t-(i+1))) = 0$  gives

$$h^0(\mathcal{I}_{A_{i+1}}(t-(i+1))) - h^1(\mathcal{I}_{A_{i+1}}(t-(i+1))) = h^0(\mathcal{O}_{\mathbf{P}^N}(t-(i+1))) - h^0(\mathcal{O}_{A_{i+1}}(t-(i+1))),$$

but  $h^0(\mathcal{O}_{A_{i+1}}(t-(i+1))) = h^0(\mathcal{O}_{A_{i+1}}) = \deg(A_{i+1})$  since  $\mathcal{O}_{A_{i+1}}(t-(i+1)) \cong \mathcal{O}_{A_{i+1}}$  (because  $\mathcal{O}_{A_{i+1}}$  is a skyscraper sheaf). Thus (2.1.3) becomes

$$h_i = h^0(L_{i+1}, \mathcal{I}_{A_i, L_{i+1}}(t-i)) + h^0(\mathcal{O}_{\mathbf{P}^N}(t-(i+1))) - \deg(A_{i+1})$$

which is exactly what occurs in the lower bound  $\max_{0 \leq i \leq n} (h_i)$ . Moreover, from the long exact sequence of cohomology coming from the top sequence in (2.1.1) we see that (2.1.3) with  $i = 0$  (which we just saw is  $h_0$ ) is at least as big as  $h^0(\mathcal{I}_{A_0}(t)) - h^1(\mathcal{I}_{A_0}(t))$ , which by the same argument as above is  $h'_0$ . Thus  $h_0 \geq h'_0$  and clearly  $h_i \geq h'_{i+1}$  for  $0 \leq i \leq n$ , so we now see that  $\max_{0 \leq i \leq n+1} (h'_i) \leq \max_{0 \leq i \leq n} (h_i)$ .  $\square$

**Remark 2.1.4.** When  $N > 2$  we will not in general know the terms  $h^0(\mathcal{I}_{A_i, L_{i+1}}(t-i))$ , and thus we will not in general know either the upper bound in Corollary 2.1.2 or the lower bound  $\max_{0 \leq i \leq n} (h_i)$ . Nevertheless, by induction on dimension we can obtain upper and lower bounds on  $h^0(\mathcal{I}_{A_i, L_{i+1}}(t-i))$  and thus also on  $h^0(\mathcal{I}_{A_0}(t))$ . On the other hand, we always will know each  $h'_i$  and thus we know the lower bound  $\max_{0 \leq i \leq n+1} (h'_i)$ . Easy examples (such as three collinear points of multiplicity 1 in  $\mathbf{P}^3$ ) show in general that equality in  $\max_{0 \leq i \leq n+1} (h'_i) \leq \max_{0 \leq i \leq n} (h_i)$  fails. When  $N = 2$ , however, equality holds and we have  $h^0(\mathcal{I}_{A_i, L_{i+1}}(t-i)) = \binom{t-i-d_{i+1}+1}{1}$ , so our bounds for  $N = 2$ , which we state explicitly in the next corollary, can be computed exactly. (To see that



equality holds when  $N = 2$ , note that  $h_i = h'_{i+1}$  if  $t - i - d_{i+1} + 1 < 1$ , while if  $t - i - d_{i+1} + 1 \geq 1$ , then  $h^1(\mathcal{I}_{\overline{A_i}, L_{i+1}}(t - i)) = 0$ , so  $h_i = h'_i$ .

**Corollary 2.1.5.** *Let  $A_0 = \sum_i m_i p_i$  be a fat point subscheme of  $\mathbf{P}^2$ , let  $L_1, \dots, L_{n+1}$  be a totally reducing sequence of lines with reduction vector  $\mathbf{d} = (d_1, \dots, d_{n+1})$ , let  $h'_{n+1} = \binom{t - (n+1) + 2}{2}$  and for  $0 \leq i \leq n$  let*

$$h'_i = \binom{t - i + 2}{2} - \sum_{i+1 \leq j \leq n+1} d_j.$$

Then we have

$$\max(h'_0, \dots, h'_{n+1}) \leq h^0(\mathbf{P}^2, \mathcal{I}_{A_0}(t)) \leq \binom{t - n + 1}{2} + \sum_{i=0}^n \binom{t - i - d_{i+1} + 1}{1}.$$

*Proof.* This follows immediately from Corollary 2.1.2 using the facts that  $\deg(A_i) = d_{i+1} + \dots + d_{n+1}$  and  $h^0(L_{i+1}, \mathcal{I}_{\overline{A_i}, L_{i+1}}(t - i)) = h^0(\mathbf{P}^1, \mathcal{O}_{\mathbf{P}^1}(t - i - \deg(\overline{A_i}))) = \binom{t - i - d_{i+1} + 1}{1}$ .  $\square$

**Remark 2.1.6.** In Corollary 2.1.5, the referee notes that the lower bounds  $h'_i \leq h^0(\mathbf{P}^2, \mathcal{I}_{A_0}(t))$  can be seen very simply using the obvious facts that  $h'_i = \binom{t - i + 2}{2} - \deg(A_i) \leq h^0(\mathbf{P}^2, \mathcal{I}_{A_i}(t - i))$  and  $h^0(\mathbf{P}^2, \mathcal{I}_{A_i}(t - i)) \leq h^0(\mathbf{P}^2, \mathcal{I}_{A_0}(t))$ . Moreover, in the case that the lines  $L_i$  are distinct, the referee also shared with us a nice way of obtaining the upper bound. Let  $Z'$  be the fat point scheme resulting from adding  $\binom{t - i - d_{i+1} + 1}{1}$  points (in general position on line  $L_i$ ) to  $Z$ , let  $Z_{n+1}$  and  $Z'_{n+1}$  be the residuals of  $Z$  and  $Z'$  respectively with respect to the sequence of lines  $L_i$ , and note that  $Z'_{n+1} = Z_{n+1}$ . The added points impose at most  $\sum_i \binom{t - i - d_{i+1} + 1}{1}$  additional conditions to forms of degree  $t$  vanishing on  $Z$ , hence  $h_{I_Z}(t) \leq h_{I_{Z'}}(t) + \sum_i \binom{t - i - d_{i+1} + 1}{1}$ . But the number of points added to each line is such that each line is forced by Bézout's Theorem to be a fixed component of  $(I_{Z'})_t$ , so  $h_{I_{Z'}}(t) = h_{I_{Z'_{n+1}}}(t - (n + 1))$ , and since the lines are totally reducing we have that  $Z'_{n+1} = Z_{n+1}$  is empty, so  $h_{I_{Z'_{n+1}}}(t - (n + 1)) = \binom{t - (n+1) + 2}{2}$ .

**2.2. Equality of the bounds.** We now define a property that implies the bounds in Corollary 2.1.5 coincide.

**Definition 2.2.1.** A vector  $\mathbf{v} = (v_1, \dots, v_r)$  will be said to be **non-negative**, respectively **positive**, **non-increasing** or **strictly decreasing** if its entries are. If  $\mathbf{v} = (v_1, \dots, v_r)$  is a non-negative, non-increasing integer vector, we will say  $\mathbf{v}$  is **GMS** if  $v_i - v_j \geq j - i - 1$  for each  $1 \leq i < j \leq r$ .

**Theorem 2.2.2.** *Let  $\mathbf{d} = (d_1, \dots, d_{n+1})$  be the reduction vector of a fat point subscheme of  $\mathbf{P}^2$  with respect to some totally reducing sequence of lines  $L_1, \dots, L_{n+1}$ . If  $\mathbf{d}$  is **GMS**, then the upper and lower bounds given in Corollary 2.1.5 coincide.*

*Proof.* By Remark 1.1.7 it is enough to show that  $h^0(\mathcal{O}_{L_i}(t - (i - 1) - d_i)) > 0$  implies  $h^1(\mathcal{O}_{L_j}(t - (j - 1) - d_j)) = 0$  for all  $j > i$  for all  $t$ . I.e., it suffices to show that  $t - (i - 1) - d_i \geq 0$  implies  $t - (j - 1) - d_j \geq -1$ , but **GMS** means  $d_i - d_j - j + i + 1 \geq 0$ , and adding  $t - (i - 1) - d_i \geq 0$  to  $d_i - d_j - j + i + 1 \geq 0$  gives  $t - (j - 1) - d_j \geq -1$ .  $\square$

The converse of Theorem 2.2.2 usually also holds; see Remark 2.5.1.

**Remark 2.2.3.** Consider  $\mathbf{v} = (v_1, \dots, v_r)$ . The GMS property is weaker than being strictly monotone (each  $v_i - v_j \geq j - i$ ), but stronger than being non-increasing (each  $v_i - v_j \geq 0$ ). The adjective ‘‘GMS’’ is meant to suggest ‘‘generalized monotone sequence’’. In [GMS], a structure analogous to our reduction vector is called a pseudo type vector. When a pseudo type vector satisfies a certain property (specifically Property 3.2 in the statement of Theorem 3.7 of [GMS]), [GMS] shows how to build a double point scheme whose Hilbert function is determined by the pseudo type vector. They build it starting from the empty scheme in a series of steps (called basic double links) related to the entries of the pseudo type vector. We on the other hand start with an arbitrary fat point scheme and in a series of steps corresponding to the entries of a reduction vector reduce the scheme to the empty scheme; when the reduction vector is GMS, it uniquely determines the Hilbert function of the scheme. Because we tear down where [GMS] builds up, given a double point scheme for which we have a reduction vector  $\mathbf{v} = (v_1, \dots, v_r)$ , the corresponding pseudo type vector would be  $(v_r, \dots, v_1)$  (which is the ordering we used in previous versions of our paper also). Property 3.2 of [GMS] applied to  $(v_r, \dots, v_1)$  is precisely that between any two zero entries of  $\Delta(v_r, \dots, v_1)$  there is an entry strictly bigger than 1. We will show in Proposition 2.2.4 that this is equivalent to our condition GMS, which thus, if you like, can be taken to stand for the authors ‘‘Geramita, Migliore, Sabourin’’ of [GMS].

It is also true that when we have a GMS reduction vector for a fat point scheme  $Z$ , the residuation steps taken in reverse are actually basic double links. Thus our results show how to construct fat point schemes other than just double points as a sequence of basic double links. We direct readers interested in further details to section 7.2 of our expository posting [CHT].

**Proposition 2.2.4.** *Let  $\mathbf{v} = (v_1, \dots, v_r)$  be a non-negative, non-increasing integer vector. Then the following are equivalent:*

- (a)  $\mathbf{v}$  is GMS;
- (b) between any two zero entries of  $\Delta(v_r, \dots, v_1)$  there is an entry strictly bigger than 1;
- (c)  $\mathbf{v}$  does not contain a subsequence of consecutive entries of the form  $(a, a, a)$ , or of the form  $(a_i, \dots, a_{i+j+1})$  for  $j > 1$  where  $a_i = a_{i+1}$ ,  $a_{i+j} = a_{i+j+1}$ , and  $a_{i+1}, \dots, a_{i+j}$  are consecutive integers.

*Proof.* Assume (a) holds. Then (b) holds by the pigeon hole principle. To show (b) implies (c), it is enough to check the contrapositive, which is trivial. To show that (c) implies (a), apply the pigeon hole principle to the contrapositive. □

**2.3. Bounds on the Hilbert function of the scheme.** We can reformulate the bounds of Corollary 2.1.5 in terms of the Hilbert function of  $A_0$ , rather than in terms of the ideal of  $A_0$ . The lower bound of the theorem becomes the upper bound

(2.3.1)

$$h_{A_0}(t) \leq \min_{0 \leq i \leq n+1} \left( h^0(\mathbf{P}^2, \mathcal{O}_{\mathbf{P}^2}(t)) - h'_i \right) = \min_{0 \leq i \leq n+1} \left( \binom{t+2}{2} - \binom{t-i+2}{2} + \sum_{i+1 \leq j \leq n+1} d_j \right).$$

From the upper bound  $h^0(\mathbf{P}^2, \mathcal{I}_{A_0}(t)) \leq \binom{t-n+1}{2} + \sum_{i=0}^n \binom{t-i-d_{i+1}+1}{1}$  we obtain

$$\begin{aligned} h_{A_0}(t) &\geq \binom{t+2}{2} - \binom{t-n+1}{2} - \sum_{i=0}^n \binom{t-i-d_{i+1}+1}{1} \\ &= \sum_{i=0}^n \left( \binom{t-i+1}{1} - \binom{t-i-d_{i+1}+1}{1} \right). \end{aligned}$$

We can simplify each term in this last sum with the following identity: if  $a \geq b$  then, by examining the three cases  $0 \leq b \leq a$ ,  $b \leq 0 \leq a$ , and  $b \leq a \leq 0$ , we obtain

$$\binom{a}{1} - \binom{b}{1} = \begin{cases} a-b, & 0 \leq b \leq a \\ a, & b \leq 0 \leq a \\ 0, & b \leq a \leq 0 \end{cases} = \binom{\min(a, a-b)}{1}.$$

We can apply this to each term in our lower bound. This yields

$$(2.3.2) \quad h_{A_0}(t) \geq \sum_{i=0}^n \binom{\min(t-i+1, d_{i+1})}{1}.$$

**Definition 2.3.3.** Given an integer vector  $\mathbf{v} = (v_1, \dots, v_{n+1})$ , we define functions

$$f_{\mathbf{v}}(t) = \sum_{i=0}^n \binom{\min(t-i+1, v_{i+1})}{1}$$

and

$$F_{\mathbf{v}}(t) = \min_{0 \leq i \leq n+1} \left( \binom{t+2}{2} - \binom{t-i+2}{2} + \sum_{i+1 \leq j \leq n+1} v_j \right).$$

**2.4. Optimality issues.** An obvious question is how in Corollary 2.1.5 should one pick the lines  $L_i$  in order to get the tightest bounds for a given fat point subscheme  $A \subset \mathbf{P}^2$ . This is not clear. Experience suggests that one should not deviate too much from a greedy approach in which one tries to pick  $L_{i+1}$  so that  $d_{i+1} = \deg(L_{i+1} \cap A_i)$  is as big as possible. For example, one can always avoid choices such that  $\mathbf{d}$  fails to be non-increasing, which one must do if one hopes to have a GMS reduction vector and thereby determine  $h_A$  exactly by an application of Theorem 2.2.2.

The next example shows, for our lower bound  $f_{\mathbf{d}}$  at least, that making greedy choices to obtain  $\mathbf{d}$  does not always give the best bound. This example also shows that by choosing different sequences, our bounds can sometimes determine the Hilbert function exactly, even when neither sequence is GMS.

**Example 2.4.1.** Consider the scheme  $A$  consisting of three points of multiplicity 2 and twelve points of multiplicity 1 arranged as shown in Figure 2.4.2.

The sequence of lines  $L_1, \dots, L_5$ , where  $L_1 = H_1$ ,  $L_2 = H_2$ ,  $L_3 = H_3$ ,  $L_4 = V_1$ , and  $L_5 = V_2$  is totally reducing for  $A$  and gives  $\mathbf{d} = (6, 6, 6, 2, 1)$  as the reduction vector. The sequence of lines  $L'_1, \dots, L'_7$ , where  $L'_1 = V_1$ ,  $L'_2 = V_2$ ,  $L'_3 = V_3$ ,  $L'_4 = V_4$ ,  $L'_5 = V_5$ ,  $L'_6 = V_1$ , and  $L'_7 = V_2$  is also totally reducing for  $A$  and gives  $\mathbf{d}' = (5, 4, 3, 3, 3, 2, 1)$  as the reduction vector.

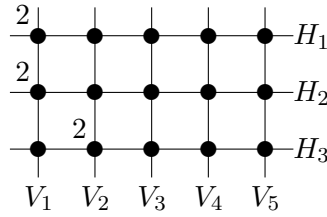


FIGURE 2.4.2. A configuration where greedy choices are not optimal.

Note that  $\mathbf{d}$  is obtained by making only greedy choices, whereas  $\mathbf{d}'$  is not. Neither  $\mathbf{d}$  nor  $\mathbf{d}'$  is GMS, so  $f_{\mathbf{d}} \neq F_{\mathbf{d}}$  and  $f_{\mathbf{d}'} \neq F_{\mathbf{d}'}$ . Hence neither determines  $h_A$ . Nevertheless we compute

$$\begin{aligned} f_{\mathbf{d}} &= (1, 3, 6, 10, 15, 18, 20, 21, 21, \dots) \\ F_{\mathbf{d}} &= (1, 3, 6, 10, 15, 18, 21, 21, \dots) \\ f_{\mathbf{d}'} &= (1, 3, 6, 10, 15, 18, 21, 21, \dots) \\ F_{\mathbf{d}'} &= (1, 3, 6, 10, 15, 21, 21, \dots) \end{aligned}$$

so that  $h_A \leq F_{\mathbf{d}} = f_{\mathbf{d}'} \leq h_A$ , which does determine  $h_A$  uniquely. Observe that  $f_{\mathbf{d}}(6) = 20 < 21 = f_{\mathbf{d}'}(6)$ , so  $\mathbf{d}'$  gives a strictly better lower bound than the reduction vector  $\mathbf{d}$  obtained by making greedy choices.

**Example 2.4.3.** In this example, we have a subscheme  $A = 2p_1 + p_2 + p_3 + 2p_4 + p_5 + p_6$  as shown in Figure 2.4.4 and we are given information about which points are collinear. In particular, for each line  $\ell_i$  we indicate by a 1 each point  $p_j$  which lies on  $\ell_i$ , and by 0 if  $p_j$  does not lie on  $\ell_i$ . We show the bounds we obtain on  $h_Z$  in this case are best possible with the given information.

Points	$p_1$	$p_2$	$p_3$	$p_4$	$p_5$	$p_6$
Multiplicities	2	1	1	2	1	1
$\ell_1$	1	1	0	0	0	0
$\ell_2$	1	0	1	0	0	0
$\ell_3$	0	0	0	1	1	0
$\ell_4$	0	0	0	1	0	1

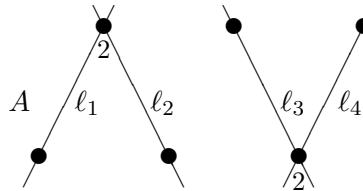


FIGURE 2.4.4. Collinearity data for a given fat point scheme.

One easily checks that  $L_1 = \ell_1, L_2 = \ell_3, L_3 = \ell_2, L_4 = \ell_4$  is a sequence of lines yielding the reduction vector  $\mathbf{d} = (3, 3, 2, 2)$ . We then find

$$\begin{aligned} f_{(3,3,2,2)} &= (1, 3, 6, 9, 10, 10, \dots), \\ F_{(3,3,2,2)} &= (1, 3, 6, 10, 10, \dots). \end{aligned}$$

We now show that the bounds  $f_{(3,3,2,2)}$  and  $F_{(3,3,2,2)}$  are sharp. Consider the schemes  $A'$  and  $A''$  of Figure 2.4.5; these are compatible with the collinearity data given in Figure 2.4.4, but now we are given additional collinearity data, represented by the dotted lines. In particular, there exists a reduction vector  $\mathbf{v}' = (4, 4, 2)$  for  $A'$  obtained using the dotted lines. In this case we have  $f_{\mathbf{v}'} = F_{\mathbf{v}'} = (1, 3, 6, 9, 10, 10, \dots)$  so  $h_{A'}(3) = 9$ . For the fat point scheme  $A''$  in Figure 2.4.5 one can

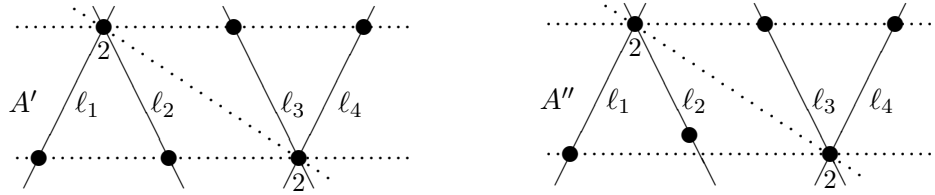


FIGURE 2.4.5. Two fat point schemes compatible with the data in Figure 2.4.4.

choose lines to obtain a reduction vector  $\mathbf{v}'' = (4, 3, 2, 1)$ , giving  $f_{\mathbf{v}''} = F_{\mathbf{v}''} = (1, 3, 6, 10, 10, \dots)$ , hence  $h_{A''}(3) = 10$ . Since both  $f_{(3,3,2,2)}$  and  $F_{(3,3,2,2)}$  are Hilbert functions of fat point schemes having reduction vector  $(3, 3, 2, 2)$ , the bounds  $f_{(3,3,2,2)}$  and  $F_{(3,3,2,2)}$  are optimal.

**2.5. Further remarks.**

**Remark 2.5.1.** When  $Z$  is a fat point subscheme of  $\mathbf{P}^2$  for which  $\mathbf{v}$  occurs as a full reduction vector, then we have  $f_{\mathbf{v}}(t) \leq h_Z(t) \leq F_{\mathbf{v}}(t)$  for all  $t \geq 0$  (by Corollary 2.1.5, Equations (2.3.1) and (2.3.2), and Definition 2.3.3), and if  $\mathbf{v}$  is GMS then by Theorem 2.2.2 we have  $f_{\mathbf{v}} = F_{\mathbf{v}}$ . For any non-negative integer vector  $\mathbf{v}$ , this means that  $f_{\mathbf{v}}(t) \leq F_{\mathbf{v}}(t)$  for all  $t \geq 0$ , with equality if  $\mathbf{v}$  is GMS (because any non-negative integer vector  $\mathbf{v} = (v_1, \dots, v_r)$  is a full reduction vector for some  $Z$ ; just take  $Z$  to be the reduced scheme consisting of  $v_i$  general points on each line  $L_i$  of a general set of lines  $L_1, \dots, L_r$ ). Conversely, if  $\mathbf{v}$  is positive and if  $f_{\mathbf{v}} = F_{\mathbf{v}}$ , then it is possible to show that  $\mathbf{v}$  is GMS (see [CHT, Theorem 8.1.1]). In most cases reduction vectors are positive (this is because a zero entry in a reduction vector means that one has taken a residual with respect to a line that doesn't meet the scheme; this is a null operation, so one would normally avoid such lines). Thus equality of our bounds  $f_{\mathbf{v}}$  and  $F_{\mathbf{v}}$  is essentially equivalent to  $\mathbf{v}$  being GMS.

**Remark 2.5.2.** For those interested in writing code, recursive definitions for the functions  $f_{\mathbf{d}}$  and  $F_{\mathbf{d}}$  may be of interest. The recursions can be justified by checking that the explicit versions of the functions shown above satisfy the recursion. Here we merely exhibit the result; we refer the interested reader to [CHT] for details.

Let  $\mathbf{v} = (v_1, \dots, v_r)$  be a non-negative integer vector. Then  $f_{\mathbf{v}}$  can be defined by setting  $f_{\mathbf{v}}(t) = 0$  if  $t < 0$  or  $n = 0$  (i.e., if  $\mathbf{v}$  is the empty vector), and otherwise we set  $f_{\mathbf{v}}(t) = f_{\mathbf{v}'}(t-1) + \min(t+1, v_1)$ , where  $\mathbf{v}' = (v_2, \dots, v_r)$ . Similarly, if  $t < 0$  or  $\mathbf{v}$  is the empty vector, then  $F_{\mathbf{v}}(t) = 0$ . Otherwise we set  $F_{\mathbf{v}}(t) = \min(t + 1 + F_{\mathbf{v}'}(t - 1), \sum_{i=1}^r v_i)$ .

One additional way to define  $f_{\mathbf{v}}$  may be of interest, since it shows that  $f_{\mathbf{v}}$  is what [GMS] calls the *standard Hilbert function* associated to  $\mathbf{v}$ .

**Definition 2.5.3.** Let  $\mathbf{v} = (v_1, \dots, v_n)$  be a non-negative integer vector. Define the **summation operator**  $\sum$  by

$$\sum \mathbf{v} = (v_1, v_1 + v_2, \dots, v_1 + \dots + v_n).$$

Also define the **standard configuration**  $S_{\mathbf{v}}$  determined by  $\mathbf{v}$  to be the set  $S_{\mathbf{v}} \subset \mathbf{Z} \times \mathbf{Z}$  of all integer lattice points  $(i, j - 1)$  with  $i \geq 0$  and  $1 \leq j \leq n$  such that  $i < v_j$ . Thus  $S_{\mathbf{v}}$  consists of the  $v_j$  leftmost first quadrant lattice points of  $\mathbf{Z} \times \mathbf{Z}$  on each horizontal line with second coordinate  $j - 1$  for  $1 \leq j \leq n$  (we take  $j - 1$  to be the second coordinate so that the bottom row of  $S_{\mathbf{v}}$  is on the horizontal

axis). We also define the **diagonal count operator**  $\text{diag}(\mathbf{v})$  of  $\mathbf{v}$  by  $\text{diag}(\mathbf{v}) = (c_0, c_1, \dots, c_t, \dots)$ , where  $c_t$  is the number of points in  $S_{\mathbf{v}}$  lying on the diagonal line with the equation  $x + y = t$ . Equivalently,

$$c_t = \#\{j : 0 \leq j < n \text{ and } 0 \leq i < v_{j+1} \text{ where } i + j = t\}.$$

**Remark 2.5.4.** Given any non-negative integer vector  $\mathbf{v} = (v_1, \dots, v_n)$ , we have  $f_{\mathbf{v}} = \sum \text{diag}(\mathbf{v})$ . Indeed,  $\binom{\min(t-i+1, v_i)}{1}$  is the number of dots in row  $i$  of  $S_{\mathbf{v}}$  (i.e., with vertical coordinate  $y = i - 1$ ) on or to the left of the line with slope  $-1$  intersecting the horizontal coordinate axis at coordinate  $x = t$ . Thus from Definition 2.3.3 we see that  $f_{\mathbf{v}}(t)$  is the number of points of the standard configuration  $S_{\mathbf{v}}$  in the region  $\{x + y \leq t\}$ ; i.e.,  $f_{\mathbf{v}} = \sum \text{diag}(\mathbf{v})$ , as claimed. Note that this is precisely the standard Hilbert function associated to  $\mathbf{v}$  defined in [GMS]. See [CHT] for further discussion.

**Example 2.5.5.** Let  $\mathbf{v} = (8, 6, 5, 2)$ . Then  $S_{\mathbf{v}}$  is the set of lattice points in  $\mathbf{Z} \times \mathbf{Z}$  shown in Figure 2.5.6. We can regard  $\mathbf{v}$  as giving the row counts for  $S_{\mathbf{v}}$  and  $\text{diag}(\mathbf{v}) = (1, 2, 3, 4, 4, 3, 3, 1, 0, 0, \dots)$  as giving the diagonal counts (along diagonals with slope  $-1$ ).

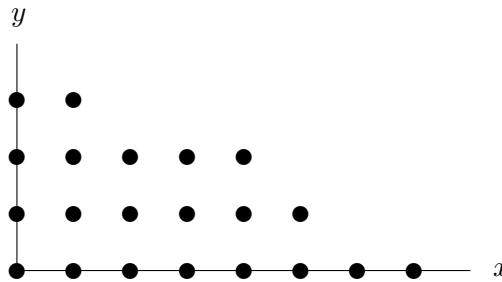


FIGURE 2.5.6.  $S_{(8,6,5,2)}$ .

**Example 2.5.7.** Note that  $\text{diag}(\mathbf{v})$  does not determine  $\mathbf{v}$ . Indeed,  $\text{diag}((2, 2)) = \text{diag}((3, 1)) = (1, 2, 1, 0, 0, \dots)$ .

**Remark 2.5.8.** One may ask whether  $f_{\mathbf{v}}$  and  $F_{\mathbf{v}}$  are differentiable O-sequences, since given a 0-dimensional subscheme  $Z \subset \mathbf{P}^2$ ,  $h_Z$  is a differentiable O-sequence (for details, see [GMR, Ma, St, Mi]). This is equivalent to saying that  $h_Z = \sum \text{diag}(\mathbf{d})$  for an integer vector  $\mathbf{d}$  with positive increasing entries (see [GMR, Corollary 3.4] and [GHS, Theorem 2.6]). In fact, if  $\mathbf{d} = (d_1, \dots, d_r)$  is GMS then  $f_{\mathbf{d}}$  and  $F_{\mathbf{d}}$  are differentiable O-sequences, since by the construction mentioned in Remark 2.5.1 every GMS sequence occurs as a full reduction vector for some fat point scheme  $Z$ , in which case  $f_{\mathbf{d}} = h_Z = F_{\mathbf{d}}$ . More generally, if  $\mathbf{d}$  is merely non-negative, then  $F_{\mathbf{d}} = h_Z$  for some fat point subscheme  $Z$  (but  $Z$  need not have  $\mathbf{d}$  as a reduction vector), and hence  $F_{\mathbf{d}}$  is a differentiable O-sequence. And if  $\mathbf{d}$  is non-negative and non-increasing, then  $f_{\mathbf{d}} = h_{Z'}$  for some fat point subscheme  $Z'$  (again not necessarily with  $\mathbf{d}$  as a reduction vector), and hence  $f_{\mathbf{d}}$  is then also a differentiable O-sequence. See [CHT, Remark 7.5.2] for details.

### 3. GRADED BETTI NUMBERS

We focus in this section on the underlying principles that can be used to give lower and upper bounds for the graded Betti numbers of a fat point scheme  $A \subset \mathbf{P}^2$  in terms of a reduction vector  $\mathbf{d}$ .



The graded Betti numbers determine the modules in a minimal free graded resolution for  $I_A$  up to graded isomorphism. When  $h_A$  is known, the Betti numbers of the syzygy module can be determined from the Betti numbers for the module of generators. Thus we focus on bounding the Betti numbers  $\nu_{t+1}$  for the module of generators, given a GMS reduction vector  $\mathbf{d}$ , where  $\nu_{t+1}(A)$  is the number of generators of  $I_A$  of degree  $t + 1$  in any minimal set of homogeneous generators. Explicitly,  $\nu_{t+1}(A)$  is the dimension of the cokernel of the map  $\mu_t(A) : (I_A)_t \otimes R_1 \rightarrow (I_A)_{t+1}$  given by multiplication of forms in the ideal of degree  $t$  by linear forms.

**3.1. The underlying principles.** Assuming that we have a full GMS reduction vector  $\mathbf{d}$  for a fat point subscheme  $A \subset \mathbf{P}^2$ , then we know  $h_A$  in terms of  $\mathbf{d}$ . In particular, we know  $\alpha(I_A)$ ; i.e., the least degree  $t \geq 0$  such that  $(I_A)_t \neq 0$ . Hence we know that  $\nu_t(A) = 0$  for  $t < \alpha(I_A)$  and that  $\nu_{\alpha(I_A)} = \dim(I_A)_{\alpha(I_A)}$ . We also know the Castelnuovo-Mumford regularity  $\text{reg}(I_A)$ ; i.e., the least degree  $t$  such that  $h_A(t - 1) = \deg A$ . By the well-known fact that one can always choose homogeneous generators for  $I_A$  of degree at most  $\text{reg}(I_A)$ , we see that  $\nu_t(A) = 0$  for  $t > \text{reg}(I_A)$ .

It is easy to give naive bounds on  $\nu_{t+1}(A)$  that depend only on  $\mathbf{d}$  for degrees  $t$  in the range  $\alpha(I_A) \leq t < \text{reg}(I_A)$ . For example, we have

$$(3.1.1) \quad \max(0, h_{I_A}(t + 1) - 3h_{I_A}(t)) \leq \nu_{t+1}(A) \leq h_{I_A}(t + 1) - (2 + h_{I_A}(t)).$$

The lower bound comes from the fact that the dimension of the cokernel of a linear map of vector spaces is at least 0 but also at least the difference in the dimensions of the target and source of the map. Thus the lower bound here is an equality if and only if  $\mu_t$  has maximal rank. The upper bound here is obtained by applying the following lemma with  $V = (I_A)_t \subset R_t$  for  $R = K[\mathbf{P}^2]$ :

**Lemma 3.1.2.** *Let  $t \geq 0$  and let  $V \neq 0$  be a vector subspace of  $R_t$  for  $R = K[\mathbf{P}^N] = K[x_0, \dots, x_N]$ . Then  $R_1V \subseteq R_{t+1}$  has dimension  $\dim_K R_1V \geq N + \dim_K V$ .*

*Proof.* Let  $W_i \subseteq R_1$  be the  $K$ -span of  $x_0, \dots, x_i$ . Then  $x_iV \not\subseteq W_{i-1}V$ , since the highest degree in the  $x_i$  variable among monomial terms of elements is the same for  $V$  as for  $W_{i-1}V$ , but it is higher for  $x_iV$  than it is for  $V$ . Thus we have proper containments  $W_0V \subsetneq W_1V \subsetneq \dots \subsetneq W_NV = R_1V$ , and, since  $\dim_K V = \dim_K W_0V$ , we have  $\dim_K R_1V \geq N + \dim_K V$ .  $\square$

We can improve on the bounds given in (3.1.1) by using more of the information available to us. The key depends on the following lemma.

**Lemma 3.1.3.** *Let  $Z \subset \mathbf{P}^2$  be a fat point subscheme. Assume  $t \geq \alpha(I_Z)$  and  $h^1(\mathbf{P}^2, \mathcal{I}_Z(t)) = 0$  and let  $e = h^1(\mathbf{P}^2, \mathcal{I}_Z(t - 1))$ . Then  $e = \Delta h_Z(t) = h_Z(t) - h_Z(t - 1)$  and  $\max(2e - t, 0) \leq \nu_{t+1}(Z) \leq e$ .*

*Proof.* Let  $L$  be a general line. Taking cohomology of the short exact sequence  $0 \rightarrow \mathcal{I}_Z(t - 1) \rightarrow \mathcal{I}_Z(t) \rightarrow \mathcal{O}_L(t) \rightarrow 0$  gives  $h^0(\mathcal{I}_Z(t - 1)) - h^0(\mathcal{I}_Z(t)) + (t + 1) - e = 0$ , since  $h^1(\mathcal{I}_Z(t)) = 0$ . Rewriting in terms of  $h_Z$  gives  $e = \Delta h_Z(t)$ .

To bound  $\nu_{t+1}(Z) = \dim \text{cok}(\mu_t(Z))$ , we first restrict to  $L$  (defined, say, by a linear form  $F$ ). Twisting  $0 \rightarrow \mathcal{O}_{\mathbf{P}^2}(-L) \rightarrow \mathcal{O}_{\mathbf{P}^2} \rightarrow \mathcal{O}_L \rightarrow 0$  by  $\mathcal{O}_{\mathbf{P}^2}(L) \cong \mathcal{O}_{\mathbf{P}^2}(1)$ , taking global sections and tensoring by  $(I_Z)_t$  gives the short exact sequence  $0 \rightarrow (I_Z)_t \rightarrow (I_Z)_t \otimes R_1 \rightarrow (I_Z)_t \otimes H^0(L, \mathcal{O}_L(1)) \rightarrow 0$ .

We also have a second short exact sequence. Since  $L$  is general, we have  $Z : L = Z$ . Twisting (1.2.3) by  $t + 1$  and taking global sections gives  $0 \rightarrow (I_Z)_t \rightarrow (I_Z)_{t+1} \rightarrow H^0(L, \mathcal{O}_L(t + 1)) \rightarrow 0$ , which is a short exact sequence since  $h^1(\mathbf{P}^2, \mathcal{I}_Z(t)) = 0$ .

The first short exact sequence maps to the second, giving the commutative diagram:

$$\begin{array}{ccccccc}
 0 & \rightarrow & (I_Z)_t & \rightarrow & (I_Z)_t \otimes R_1 & \rightarrow & (I_Z)_t \otimes H^0(L, \mathcal{O}_L(1)) & \rightarrow & 0 \\
 & & \downarrow \mu_1 & & \downarrow \mu_2 & & \downarrow \mu_3 & & \\
 0 & \rightarrow & (I_Z)_t & \rightarrow & (I_Z)_{t+1} & \rightarrow & H^0(L, \mathcal{O}_L(t+1)) & \rightarrow & 0
 \end{array}$$

Since  $\mu_1$  is an isomorphism, we have  $\text{cok } \mu_2 \cong \text{cok } \mu_3$  by the snake lemma. Twisting (1.2.3) by  $\mathcal{O}_{\mathbf{P}^2}(t)$  and taking global sections gives the four term exact sequence  $0 \rightarrow (I_Z)_{t-1} \rightarrow (I_Z)_t \rightarrow H^0(L, \mathcal{O}_L(t)) \rightarrow H^1(\mathbf{P}^2, \mathcal{I}_Z(t-1)) \rightarrow 0$ . Let  $V$  be the image of  $(I_Z)_t$  in  $H^0(L, \mathcal{O}_L(t))$ . Then  $(I_Z)_t \otimes H^0(L, \mathcal{O}_L(1))$  and  $V \otimes H^0(L, \mathcal{O}_L(1))$  have the same image in  $H^0(L, \mathcal{O}_L(t+1))$ . By Lemma 3.1.2, the dimension of the image of  $V \otimes H^0(L, \mathcal{O}_L(1))$  is at least  $\dim V + 1$ , hence  $\dim \text{cok } \mu_2 = \dim \text{cok } \mu_3 \leq h^0(L, \mathcal{O}_L(1)) - (\dim V + 1) = (t+2) - (\dim V + 1)$ . Using the four term exact sequence,  $\dim V = t + 1 - e$ , so we have  $\nu_{t+1}(Z) \leq e$ .

Of course, the maximum possible dimension of the image of  $\mu_3$  is obtained by assuming it has maximal rank. This gives  $\dim \text{Im } \mu_3 \leq \min(2 \dim V, t+2)$ , hence  $\nu_{t+1}(Z) \geq (t+2) - \min(2 \dim V, t+2) = \max(2e - t, 0)$ .  $\square$

**Notation 3.1.4.** Fixing notation, let  $\mathbf{d} = (d_1, \dots, d_{n+1})$  be a GMS reduction vector for a fat point subscheme  $A$  with respect to a totally reducing sequence of lines  $L_1, \dots, L_{n+1}$ , and let the corresponding residual schemes be  $A = A_0, A_1, \dots, A_n, A_{n+1} = \emptyset$ . For  $\alpha(I_A) \leq t < \text{reg}(I_A)$ , let  $j = j(\mathbf{d}, t)$  be the greatest value in the interval  $0 \leq j \leq n + 1$  such that  $h_{I_{A_i}}(t - i) = h_{I_A}(t)$  for  $1 \leq i \leq j$ . Note that each  $A_i$  has a GMS reduction vector given by a truncation of  $\mathbf{d}$ , so each  $h_{I_{A_i}}$  is determined by  $\mathbf{d}$ ; thus  $j$  is indeed determined by  $\mathbf{d}$  and  $t$ . Note also that  $j \leq t$  (either  $j \leq n + 1 \leq t$ ; or if  $t < n + 1$  and  $j > t$ , then  $A_t \neq \emptyset$ , so  $0 = h_{I_{A_t}}(0) = h_{I_A}(t) > 0$ , since  $t \geq \alpha_A$ ).

**Lemma 3.1.5.** *With notation as in 3.1.4, we have:*

(1)  $F = F_1 \cdots F_j$  is a common divisor for  $(I_A)_t$ , where  $F_i$  is a linear form defining  $L_i$ , so that

$$\nu_{t+1}(A) = h_{I_A}(t+1) - h_{I_{A_j}}(t-j+1) + \nu_{t-j+1}(A_j).$$

(2) The fat point scheme  $A_j$  satisfies  $h^1(\mathbf{P}^2, \mathcal{I}_{A_j}(t-j)) = 0$ .

*Proof.* Multiplication by  $F$  gives an injective map  $(I_{A_j})_{t-j} \rightarrow (I_A)_t$ . By assumption these have the same dimension, so also multiplication by  $F$  is surjective, meaning  $F$  is a common divisor for  $(I_A)_t$ . Since multiplication by linear forms commutes with multiplication by  $F$ , the image  $\text{Im}(\mu_t(A))$  of  $\mu_t(A)$  in  $(I_A)_{t+1}$  is  $F \cdot \text{Im}(\mu_{t-j+1}(A_j))$ , which has dimension  $h_{A_j}(t-j+1) - \nu_{t-j+1}(A_j)$ . This gives  $\dim \text{Im}(\mu_t(A))$ , so  $\nu_{t+1}(A) = \text{codim } \text{Im}(\mu_t(A)) = h_A(t+1) - (h_{A_j}(t-j+1) - \nu_{t-j+1}(A_j))$ .

If  $j = n + 1$  then  $\mathcal{I}_{A_j} = \mathcal{O}_{\mathbf{P}^2}$  and the vanishing of  $h^1$  is automatic. Otherwise, we claim  $h^1(\mathbf{P}^2, \mathcal{I}_{A_i}(t-i)) = 0$  for  $j \leq i \leq n + 1$ . This is automatic for  $i = n + 1$ . In the short exact sequence

$$0 \rightarrow \mathcal{I}_{A_{i+1}}(t-i-1) \rightarrow \mathcal{I}_{A_i}(t-i) \rightarrow \mathcal{O}_{L_i}(t-i-d_i) \rightarrow 0,$$

we may by downward induction assume  $h^1(\mathbf{P}^2, \mathcal{I}_{A_{i+1}}(t-i-1)) = 0$ , and we have  $t-i-d_i \geq t-j-d_j-1$  since  $\mathbf{d}$  is GMS. From the short exact sequence

$$0 \rightarrow \mathcal{I}_{A_{j+1}}(t-(j+1)) \rightarrow \mathcal{I}_{A_j}(t-j) \rightarrow \mathcal{O}_{L_j}(t-j-d_j) \rightarrow 0$$

the maximality of  $j$  means  $h_{I_{A_{j+1}}}(t-j-1) < h_{I_{A_j}}(t-j)$ , whence  $h^0(L_j, \mathcal{O}_{L_j}(t-j-d_j)) > 0$ . Thus  $t-i-d_i \geq t-j-d_j-1 \geq -1$ , and  $h^1(\mathcal{O}_{L_i}(t-i-d_i)) = 0$ . This completes the downward induction.  $\square$

We can now give bounds improving on (3.1.1).

**Theorem 3.1.6.** *With notation as in 3.1.4, let  $e = h^1(\mathbf{P}^2, \mathcal{I}_{A_j}(t - j - 1)) = \Delta h_{A_j}(t - j)$ . Then*

$$\begin{aligned} \nu_{t+1}(A) &\geq (h_{I_A}(t + 1) - h_{I_{A_j}}(t - j + 1)) + \max(2e - t + j, 0), \\ \nu_{t+1}(A) &\leq (h_{I_A}(t + 1) - h_{I_{A_j}}(t - j + 1)) + e. \end{aligned}$$

Note that the bounds for  $\nu_{t+1}(A)$  are determined by  $\mathbf{d}$  and  $t$  (via truncations of  $\mathbf{d}$  and  $j = j(\mathbf{d}, t)$ ).

*Proof.* Applying Lemma 3.1.3 with  $Z = A_j$ ,  $t - j$  in place of  $t$  and  $e = h^1(\mathbf{P}^2, \mathcal{I}_{A_j}(t - j - 1))$  to the result of Lemma 3.1.5 gives the result.  $\square$

Since  $t \geq j$ , we see that  $2e - t + j \leq 2e$ , and hence that the bounds in Theorem 3.1.6 agree if  $e = 0$ , and thus in that case we even obtain an exact determination of  $\nu_{t+1}(A)$ . The following theorem gives a criterion for the bounds on  $\nu_t(A)$  to coincide for all  $t$ . A precise determination of exactly those  $\mathbf{d}$  for which the bounds coincide for all  $t$  can also be given; see Remark 3.1.9.

**Theorem 3.1.7.** *Let  $A \subset \mathbf{P}^2$  be a fat point subscheme with full reduction vector  $\mathbf{d} = (d_1, \dots, d_{n+1})$ . If  $\mathbf{d}$  is positive and strictly decreasing, then the bounds on  $\nu_A(t + 1)$  given in Theorem 3.1.6 agree for all  $\alpha(I_A) \leq t < \text{reg}(I_A)$ , and thus  $\mathbf{d}$  determines the graded Betti numbers for  $I_A$ .*

*Proof.* By Proposition 2.2.4,  $\mathbf{d}$  is GMS. Thus by Theorem 2.2.2, Equation (2.3.2) and Definition 2.3.3, we have  $h_A = f_{\mathbf{d}}$ . By Remark 2.5.4,  $f_{\mathbf{d}}(t)$  is the number of dots of  $S_{\mathbf{d}}$  on or to the left of the line  $x + y = t$ . But  $\alpha(I_A)$  is the least  $t$  such that  $f_{\mathbf{d}}(t) = h_A(t) < \binom{t+2}{2}$ ; i.e., it is the least  $t$  such that there is an integer lattice point  $(i, j) \notin S_{\mathbf{d}}$  with  $i \geq 0$ ,  $j \geq 0$  and  $i + j \leq t$ . Since the entries of  $\mathbf{d}$  are strictly decreasing, this is precisely the number of rows of  $S_{\mathbf{d}}$ ; i.e.,  $\alpha(I_A) = n + 1$ . And the regularity  $\text{reg}(I_A)$  is  $t + 1$  for the least degree  $t$  such that  $h_A(t) = \text{deg}(A)$ , which is just the least  $t$  such that every dot of  $S_{\mathbf{d}}$  is on or to the left of the line  $x + y = t$ . Since  $\mathbf{d}$  is strictly decreasing, this  $t$  is  $d_1 - 1$ ; i.e.,  $\text{reg}(I_A) = d_1$ .

Now suppose that  $\alpha(I_A) \leq t < \text{reg}(I_A)$ , and let  $L_1, \dots, L_{n+1}$  be the sequence of lines giving rise to the reduction vector  $\mathbf{d}$ . The divisor  $L_1 + \dots + L_j$  is in the base locus of  $(I_A)_t$  if  $h_{I_{A_i}}(t) = h_{I_{A_i}}(t - i)$  for  $0 < i \leq j$ . Assume we pick the largest  $j$  such that this holds. This is equivalent to saying that the line  $x + y = t$  contains dots of  $S_{\mathbf{d}}$  for each of the rows of  $S_{\mathbf{d}}$  corresponding to  $d_1, \dots, d_j$  but for none of the other rows. In particular,  $t - j \geq \text{reg}(I_{A_j})$  and hence  $e = h^1(\mathbf{P}^2, \mathcal{I}_{A_j}(t - j - 1)) = 0$ , so, as pointed out above following Theorem 3.1.6, our bounds coincide.  $\square$

**Example 3.1.8.** Let  $Z = 3(p_1 + \dots + p_{10})$  where the points  $p_i$  are the points of pairwise intersections of five general lines  $L_1, \dots, L_5$  (as shown in Figure 4.2.4). Using the sequence of lines  $L_1, \dots, L_5, L_1, \dots, L_4$  where  $L_i$  is as shown in the figure, we obtain a full reduction vector  $\mathbf{d} = (12, 11, 10, 9, 8, 4, 3, 2, 1)$  for  $Z$ . This is GMS so we obtain an exact determination  $h_Z = (1, 3, 6, 10, 15, 21, 28, 36, 45, 50, 55, 60, 60, \dots)$  for the Hilbert function of  $Z$ . From this we see that  $\alpha(I_Z) = 9$  and  $\text{reg}(I_Z) = 12$ , hence  $\nu_t(Z) = 0$  for  $t < 9$  and for  $t > 12$ , and  $\nu_9(Z) = \binom{9+2}{2} - h_Z(9) = 5$ . For  $t = 9$ , the bounds given in (3.1.1) are  $0 = \max(0, 11 - 3 \cdot 5) \leq \nu_{10}(Z) \leq 11 - (2 + 5) = 4$ . In this case  $e = 0$ , so Theorem 3.1.6 gives the exact value  $\nu_{10}(Z) = h_{I_Z}(t + 1) - h_{I_{Z_j}}(t - j + 1) = 0$ .

Similarly,  $t = 10$  gives  $\nu_{11}(Z) = 0$ ; now consider  $t = 11$ . Then (3.1.1) gives the bounds  $0 = \max(0, 31 - 3 \cdot 18) \leq \nu_{12}(Z) \leq 31 - (2 + 18) = 11$ , but from Theorem 3.1.6 again  $e = 0$  so we have the exact value,  $\nu_{12}(Z) = h_{I_Z}(t + 1) - h_{I_{Z_j}}(t - j + 1) = 31 - 26 = 5$ .

**Remark 3.1.9.** We have indicated above how to find bounds in terms of a full reduction vector for the graded Betti numbers for fat point subschemes of  $\mathbf{P}^2$  when the reduction vector is GMS. In the preceding example we saw that we obtained exact values for  $\nu_t$  for every  $t$ , as we expect from Theorem 3.1.7 since the reduction vector is positive and strictly decreasing. It is possible to show that our approach gives an exact determination for  $\nu_t$  for all  $t$  for a positive GMS reduction vector  $\mathbf{d}$  whenever (and only when) the reduction vector has one of the following forms (see [CHT, Proposition 5.1.8]):

- (1)  $\mathbf{d}$  is strictly decreasing;
- (2)  $\mathbf{d} = (m, m, m - 1, \dots, 2, 1)$  where  $m \geq 1$ ;
- (3)  $\mathbf{d} = (d_1, \dots, d_k, m, m, m - 1, \dots, 2, 1)$  where  $(d_1, \dots, d_k)$  is strictly decreasing and  $d_k \geq m + 2$ .

Because the proof is somewhat long but does not involve any new ideas, we have omitted the proof here.

**Remark 3.1.10.** Our criterion for when a reduction vector  $\mathbf{d}$  determines the Betti numbers of a fat point scheme is more restrictive than an analogous criterion given in [GMS]. This is because the criterion given in [GMS] is only for the situation of linear configurations (see section §4.1 for the definition), whereas we allow arbitrary fat point schemes. We give an example to illustrate this. Let  $\mathbf{d} = (2, 2)$ . Then  $\mathbf{d}$  satisfies the criterion given in [GMS, Theorem 4.5] for the graded Betti numbers of  $I_A$  for a linear configuration  $A$  of type  $\mathbf{d}$  to be uniquely determined. Indeed,  $A$  being a linear configuration of type  $(2, 2)$  means simply that  $A$  is a reduced set of 4 points in general position. In particular  $\mathbf{d} = (2, 2)$  is a reduction vector for  $A$ , obtained by reducing along two lines through pairs of the points.

Our methods give bounds for the graded Betti numbers of  $I_B$  for any fat point scheme  $B$  having reduction vector  $\mathbf{d} = (2, 2)$ , since  $(2, 2)$  is GMS. However, as indicated by Remark 3.1.9, these bounds do not coincide, so they do not determine the graded Betti numbers of  $I_B$ . Indeed without assuming  $B$  is a linear configuration of type  $(2, 2)$  the graded Betti numbers are not uniquely determined—even assuming  $B$  is reduced is not sufficient. For instance, let  $B$  be the reduced union of 4 points with exactly 3 collinear. Let  $L_2$  be any line determined by the fourth (non-collinear) point and any one of the collinear points, and let  $L_1$  be the line through the other two points. We then obtain the same reduction vector  $\mathbf{d} = (2, 2)$  by taking our reducing sequence of lines to be  $L_2, L_1$ , in that order. But since  $B$  contains 3 collinear points, the graded Betti numbers of  $I_B$  are different from those of  $I_A$ . (Note that  $L_1, L_2$  gives  $(3, 1)$  as a reduction vector for  $B$ , determining the graded Betti numbers of  $I_B$ . Indeed this  $B$  is a linear configuration of type  $(3, 1)$ .)

In conclusion, since the bounds we obtain must be broad enough to accommodate subschemes with different graded Betti numbers but still having the same  $\mathbf{d}$ , there are situations where our bounds will not coincide (such as the case described here of four general points) for which [GMS], as a result of restricting to the case of linear configurations, can give an exact determination of the Betti numbers.

*Proof of Theorem 1.1.* This theorem is an immediate consequence of Corollary 2.1.5 and Theorems 2.2.2 and 3.1.7, in view of Proposition 2.2.4, Equations (2.3.1) and (2.3.2), and Definition 2.3.3.  $\square$

#### 4. APPLICATIONS AND EXAMPLES

In this section we present a collection of examples.

**4.1. Linear and line count configurations.** We first consider some examples which generalize one of the main results of [GMS]. The following notation is needed for these examples.

**Definition 4.1.1.** Given an integer vector  $\mathbf{v} = (v_1, \dots, v_n)$ , we define the **permuting operator**  $\pi(\mathbf{v})$  to be the vector whose entries are the entries of  $\mathbf{v}$  permuted to be non-increasing.

**Definition 4.1.2.** Given positive integer vectors  $\mathbf{a} = (a_1, \dots, a_n)$  and  $\mathbf{m} = (m_1, \dots, m_n)$ , let  $\mathbf{a} \circ \mathbf{m} = (a_1 m_1, (a_1 - 1)m_1, \dots, 2m_1, m_1, a_2 m_2, (a_2 - 1)m_2, \dots, m_2, \dots, a_n m_n, (a_n - 1)m_n, \dots, m_n)$ . The **star operator** is defined by  $\mathbf{a} * \mathbf{m} = \pi(\mathbf{a} \circ \mathbf{m})$ . Note that the number of entries in  $\mathbf{a} \circ \mathbf{m}$  or  $\mathbf{a} * \mathbf{m}$  is  $\sum_{i=1}^n a_i$ .

**Example 4.1.3.** We have

$$(2, 3) * (2, 3) = \pi((4, 2, 9, 6, 3)) = (9, 6, 4, 3, 2)$$

and

$$(3, 2) * (2, 3) = \pi((6, 4, 2, 6, 3)) = (6, 6, 4, 3, 2).$$

**Example 4.1.4.** Note that  $(3, 1) * (2, 3) = (2, 2) * (2, 3) = (6, 4, 3, 2)$ , so we cannot recover  $\mathbf{a}$  from  $\mathbf{a} * \mathbf{m}$  even if we know  $\mathbf{m}$ . Similarly,  $(1, 2) * (2, 4) = (1, 2) * (8, 2) = (8, 4, 2)$ , so we cannot recover  $\mathbf{m}$  from  $\mathbf{a} * \mathbf{m}$  even if we know  $\mathbf{a}$ .

One of the motivating goals of the paper [GMS] was the determination (in characteristic 0) of the Hilbert function for  $2Z$ , whenever  $Z \subset \mathbf{P}^2$  is a **linear configuration of type**  $(m_1, \dots, m_s)$ ; i.e., whenever  $Z = Z_1 + \dots + Z_s$  for some set of distinct lines  $L_1, \dots, L_s$  and finite reduced schemes  $Z_i \subset L_i$ , such that no point of  $Z$  occurs where two of the  $L_i$  meet, and the integers  $m_i = \#(Z_i)$ ,  $i = 1, \dots, s$  are distinct. After re-indexing, we may assume that  $m_1 > \dots > m_s > 0$ . In our terminology, [GMS] showed that  $h_{2Z} = \sum \text{diag}(\mathbf{a} * \mathbf{m})$  where  $\mathbf{m} = (m_1, \dots, m_s)$  and  $\mathbf{a} = (2, \dots, 2)$ , if  $\mathbf{a} * \mathbf{m}$  is GMS.

The following corollary is a generalization of the above result of [GMS] in which we give an exact determination of the Hilbert function in a broader range of cases. If we allow the integers  $m_i$  to be equal in the definition of a linear configuration, then we obtain what we call a **line count configuration**.

**Theorem 4.1.5.** *Given positive integers  $a_i$  and  $m_i$ , let  $\mathbf{a} = (a_1, \dots, a_s)$  and  $\mathbf{m} = (m_1, \dots, m_s)$ . Let  $Z_1 + \dots + Z_s$  be a line count configuration of type  $\mathbf{m}$  corresponding to distinct lines  $L_1, \dots, L_s$  and let  $Z(\mathbf{a}, \mathbf{m}) = a_1 Z_1 + \dots + a_s Z_s$ . Then  $\mathbf{a} * \mathbf{m}$  is a full reduction vector for  $Z(\mathbf{a}, \mathbf{m})$ , hence  $f_{\mathbf{a} * \mathbf{m}} \leq h_{Z(\mathbf{a}, \mathbf{m})} \leq F_{\mathbf{a} * \mathbf{m}}$ . If  $\mathbf{a} * \mathbf{m}$  is GMS, then  $h_{Z(\mathbf{a}, \mathbf{m})} = f_{\mathbf{a} * \mathbf{m}} = F_{\mathbf{a} * \mathbf{m}} = \sum \text{diag}(\mathbf{a} * \mathbf{m})$ .*

*Proof.* Consider the sequence of lines  $\mathbf{L} = \{L_1^{a_1}, \dots, L_s^{a_s}\}$ , where the notation  $L_i^{a_i}$  denotes the sequence  $L_i, \dots, L_i$  with  $a_i$  terms. It is clear that  $\mathbf{L}$  is a totally reducing sequence of lines for  $Z(\mathbf{a}, \mathbf{m})$  with reduction vector  $\mathbf{a} \circ \mathbf{m}$ . By definition,  $\mathbf{a} * \mathbf{m}$  is obtained from  $\mathbf{a} \circ \mathbf{m}$  by appropriately permuting the entries of  $\mathbf{a} \circ \mathbf{m}$ . By applying the same permutation to the entries of  $\mathbf{L}$ , we obtain another totally reducing sequence of lines for  $Z(\mathbf{a}, \mathbf{m})$  but with reduction vector  $\mathbf{a} * \mathbf{m}$ . Thus  $f_{\mathbf{a} * \mathbf{m}} \leq h_{Z(\mathbf{a}, \mathbf{m})} \leq F_{\mathbf{a} * \mathbf{m}}$ , and if  $\mathbf{a} * \mathbf{m}$  is GMS, then  $f_{\mathbf{a} * \mathbf{m}} = F_{\mathbf{a} * \mathbf{m}}$ , and hence  $h_{Z(\mathbf{a}, \mathbf{m})} = f_{\mathbf{a} * \mathbf{m}} = \sum \text{diag}(\mathbf{a} * \mathbf{m})$ .  $\square$

Our methods determine the Hilbert functions of fat point schemes whose support are line count configurations consisting of one and two lines, as the next example shows.



**Example 4.1.6.** Let  $Z$  consist of  $m$  points on a line  $\ell$  in  $\mathbf{P}^2$  and define the fat point scheme  $A = aZ$ . Then the sequence of lines  $\ell^a$  (where again  $\ell^a$  denotes the sequence  $\ell, \dots, \ell$  in which  $\ell$  is repeated  $a$  times) is a totally reducing sequence for  $A$  with GMS reduction vector  $(a) * (m) = (am, \dots, 2m, m)$ . Thus  $h_A = f_{(a)*(m)} = F_{(a)*(m)} = \sum \text{diag}((a) * (m))$ .

Similarly, suppose  $Z = Z_1 + Z_2$  where  $Z_1$  is a set of  $m_1$  points on a line  $\ell_1$  and  $Z_2$  is a set of  $m_2$  points on a line  $\ell_2$  such that no point of  $Z$  lies at the intersection of  $\ell_1$  and  $\ell_2$ . Assume that  $m_1 \geq m_2 > 2$ . Consider the fat point scheme  $A = a_1Z_1 + a_2Z_2 \subset \mathbf{P}^2$ . The sequence of lines  $\ell_1^{a_1}, \ell_2^{a_2}$  is a totally reducing sequence of lines for  $A$  with reduction vector  $\mathbf{a} \circ \mathbf{m}$ , where  $\mathbf{a} = (a_1, a_2)$  and  $\mathbf{m} = (m_1, m_2)$ . Again, after a permutation of the lines, we obtain a totally reducing sequence of lines with reduction vector  $\mathbf{a} * \mathbf{m}$ . But  $\mathbf{a} * \mathbf{m}$  is GMS by Proposition 2.2.4 (since at most two consecutive entries of  $\mathbf{a} * \mathbf{m}$  can be equal and otherwise they differ by at least 2). Thus, again,  $h_A = f_{\mathbf{a}*\mathbf{m}} = F_{\mathbf{a}*\mathbf{m}} = \sum \text{diag}(\mathbf{a} * \mathbf{m})$ .

For fat point schemes supported on line count configurations involving three or more lines, these arguments do not always determine the Hilbert functions. For example, let  $Z = Z_1 + Z_2 + Z_3$ , where each  $Z_i$  is a set of  $m \geq 3$  points on a line  $\ell_i$  such that no point of  $Z$  lies in the intersection of the lines. Consider the fat point scheme  $A = 2Z_1 + 2Z_2 + 2Z_3 \subset \mathbf{P}^2$ . The sequence of lines  $\ell_1, \ell_2, \ell_3, \ell_1, \ell_2, \ell_3$  is a totally reducing sequence of lines for  $A$  with reduction vector  $\mathbf{a} * \mathbf{m}$  where  $\mathbf{a} = (2, 2, 2)$  and  $\mathbf{m} = (m, m, m)$ . However,  $\mathbf{d} = \mathbf{a} * \mathbf{m}$  is not GMS, so  $f_{\mathbf{d}}(t) < F_{\mathbf{d}}(t)$  for at least one value of  $t$ .

**4.2. Intersections of lines.** Two things (at least) make linear configurations interesting. One is that it is easy to write down their Hilbert functions: a (reduced) linear configuration  $Z$  of type  $\mathbf{c} = (c_1, \dots, c_s)$  has Hilbert function  $\sum \text{diag}(\mathbf{c})$ . Another is that according to results of Macaulay [Ma] and Geramita-Harima-Shin [GHS], every Hilbert function for 0-dimensional subschemes of  $\mathbf{P}^2$  already occurs for linear configurations.

However, when considering Hilbert functions of schemes  $aZ$  where  $Z$  is a reduced 0-dimensional subscheme of  $\mathbf{P}^2$ , linear configurations do not behave well. In fact, if  $Z$  is a linear configuration of type  $\mathbf{c} = (c_1, \dots, c_s)$  for  $s > 2$  which we assume for simplicity has  $c_1 > c_2 > \dots > c_s$ , and if  $a \geq c_{s-2}c_{s-1}$ , then  $(a, \dots, a) * \mathbf{c}$  will have an entry equal to  $c_{s-2}c_{s-1}c_s$  which occurs at least three times, and hence is not GMS, and so our upper and lower bounds on the Hilbert function of  $aZ$  will not coincide.

Linear configurations arise by taking finite sets of points on a union of lines  $L_1, \dots, L_s$  but avoiding the points of intersections of the lines. It turns out the opposite extreme, of taking  $Z$  to be the set of all of the points where two or more lines  $L_i \neq L_j$  intersect, behaves better in this respect. In fact, if in this situation the number of lines  $L_i$  through each point of  $Z$  is the same (say  $n$  lines go through each point of  $Z$ ), then our results determine the Hilbert function of  $mnZ$  and the graded Betti numbers of  $I_{mnZ}$  for every  $m \geq 1$ . In particular, Example 4.2.3 handles the case  $n = 2$ , Example 4.2.5 handles the case  $n = q$  where  $q$  is a power of a prime, and Example 4.2.6 handles the case  $n = 3$ . If  $n = 2$ , then we even obtain the Hilbert function of  $mZ$  and the graded Betti numbers of  $I_{mZ}$  for every  $m \geq 1$ .

Here is the general set up. Let  $L_1, \dots, L_s$  be distinct lines in  $\mathbf{P}^2$ , and let  $e_1, \dots, e_s$  be positive integers (these need not be in non-decreasing order). Consider the divisor  $D = \sum_i e_i L_i$ . Let  $S_D$  be the finite set of all points  $p$  which lie simultaneously on two or more of the lines  $L_i$ , and define  $Z(D)$  to be the fat point scheme  $\sum_{p \in S_D} m_p p$ , where  $m_p = \sum_{p \in L_i} e_i$  is the number of lines  $L_i$  which pass through  $p$ , counting with multiplicity. In case  $e_1 = \dots = e_s = 1$  (in which case  $D$  is reduced),



we also define  $Z'(D) = \sum_{p \in S_D} (m_p - 1)p$ . In Example 4.2.1 we consider  $Z'(D)$  for  $D$  reduced. In Example 4.2.2 we consider  $mZ(D)$  for any  $m$  and any  $D$ . In Example 4.2.3, under the special condition that no three lines of  $D$  have a common point, we consider  $mZ'(D)$  for any  $m$  and for  $D$  reduced.

In each case we describe the reduction vectors, which turn out to be not only GMS, but even strictly decreasing. Hence the Hilbert functions are completely determined by the reduction vector and, as pointed out in Remark 3.1.9, so are the graded Betti numbers.

For comparison, Ardila and Postnikov [AP] obtain the Hilbert function both for  $Z'(D)$  and for  $Z(D)$ , not only for  $\mathbf{P}^2$  but for hyperplane arrangements in any dimension, over the complex numbers. They express their solution in terms of the Hilbert series. While on the one hand our results are only for subschemes of  $\mathbf{P}^2$ , on the other hand for us the characteristic is arbitrary, and we obtain the Hilbert functions and Betti numbers, not only for  $Z'(D)$  and for  $Z(D)$  but also as mentioned above for  $mZ(D)$  (in Example 4.2.2) and for  $mZ'(D)$  (in Example 4.2.3).

**Example 4.2.1.** Consider  $Z'_{s-1} = Z'(D)$ , where  $D = L_1 + \cdots + L_s$  for distinct lines  $L_i$ , so  $D$  is reduced. Note that the scheme theoretic intersection  $Z'_{s-1} \cap L_i$  of any of the lines  $L_i$  with  $Z'(D)$  has degree exactly  $\sum_{p \in L_i \cap S_D} (m_p - 1)$ , but this sum is just the number of lines other than  $L_i$ ; i.e., it is  $s - 1$ . The residual  $Z'_{s-2}$  of  $Z'_{s-1}$  with respect to  $L_i$  is  $Z'(D - L_i)$ . Then for any line  $L_j$  with  $j \neq i$ , the intersection  $Z'_{s-2} \cap L_j$  has degree  $s - 2$ . Thus taking residuals with respect to the sequence of lines  $L_2, \dots, L_s$  (or indeed any sequence of  $s - 1$  of the  $s$  lines) results in a full reduction vector  $\mathbf{d} = (s - 1, s - 2, \dots, 3, 2, 1)$ .

**Example 4.2.2.** Now consider  $Z(D)$  for  $D = e_1 L_1 + \cdots + e_s L_s$  for distinct lines  $L_i$  and  $e_i \geq 1$ , so  $D$  need not be reduced. Whereas in Example 4.2.1 we could totally reduce  $Z'(D)$  and obtain a strictly decreasing (and hence GMS) reduction vector without regard to the order in which we chose the lines (as long as we chose  $s - 1$  different lines), now the order will matter, but we can nevertheless obtain a strictly decreasing reduction vector not only for  $Z(D)$  but for  $mZ(D)$  for any  $m > 0$ .

In general, the reduction vector corresponding to an arbitrary choice of a sequence of lines need not be GMS, or even non-increasing; we now show how the sequence of lines can be chosen so that in fact we do get a strictly decreasing reduction vector, by using the following “greedy algorithm”: at any given step, say the residual fat point scheme is  $Z_k = \sum_{p \in S_D} a_p p$ . Choose any one of the lines  $L_i$  maximizing the degree of  $Z_k \cap L_i$  (i.e., maximizing the sum  $\sum_{p \in L_i \cap S_D} a_p$ ), among lines which have not been chosen  $m e_i$  times so far. This is totally reducing, as for each point  $p \in S_D$ , each line  $L_i$  through  $p$  occurs  $m e_i$  times in the reducing sequence, giving a total of  $\sum m e_i = m m_p$  lines through  $p$  (where the sum is over all  $i$  such that  $p \in L_i$ ).

We now explain why the resulting reduction vector in fact contains no repeated values. If the sum of the multiplicities at the  $k$ th step along some line, say  $L_i$ , is the maximum, then in the next step, the sum of the multiplicities along any line, say  $L_j$ , is strictly less, because the multiplicities along line  $L_i$  were reduced and for any  $j \neq i$  the multiplicity of the point  $L_i \cap L_j$  has been reduced by 1, as long as that multiplicity was not already 0. But for the multiplicity to have already been 0, all of the lines through that point must have been chosen already; in particular,  $L_j$  must have been chosen  $m e_j$  times already, and so we do not allow ourselves to choose it again.

**Example 4.2.3.** Here we consider so-called *star configurations*; see Figure 4.2.4. These are of the form  $Z'(D)$  in the special case that  $D = L_1 + \cdots + L_s$  is a sum of distinct lines (so  $D$  is reduced)

under the assumption that at any point  $p = L_i \cap L_j$ ,  $i \neq j$ , where two lines meet, these are the only two lines which contain  $p$ . Such configurations arise in an important way in both [GMS] and [BH] as having extremal behavior. In [GMS],  $h_{mZ'(D)}$  is determined for  $m = 2$ , while [BH] determines  $\alpha(I_{mZ'(D)})$  for every  $m$ . Our results give both  $h_{mZ'}$  and the graded Betti numbers for  $I_{mZ'}$  since the reduction vector is positive and strictly decreasing, for all  $m > 0$ , as we now show. Note that  $Z' = Z'(D)$  is just the reduced union of the  $\binom{s}{2}$  points and  $Z(D) = 2Z'$ . Thus we get the Hilbert function and graded Betti numbers for  $mZ'$  when  $m$  is even by Example 4.2.2, in which case the reduction vector is

$$\begin{aligned} & (m(s-1), m(s-1) - 1, \dots, m(s-1) - (s-1), \\ & (m-2)(s-1), (m-2)(s-1) - 1, \dots, (m-2)(s-1) - (s-1), \\ & \dots, \\ & 2(s-1), 2(s-1) - 1, \dots, 2(s-1) - (s-1)), \end{aligned}$$

where we have used a multi-line display to make the pattern easier to discern. We also get them when  $m$  is odd, as follows. Say  $m = 2a + 1$  is odd. Then  $mZ' = aZ(D) + Z'(D)$ . First as we just showed we can choose a reducing sequence of lines for  $aZ(D)$ , giving a strictly decreasing reduction vector  $\mathbf{d} = (d_1, \dots, d_u)$  as displayed in the even case, then choose a reducing sequence of lines for  $Z'(D)$ , giving the reduction vector  $\mathbf{d}' = (s-1, \dots, 2, 1)$ . Putting these sequences together reduces  $mZ'$  first to  $Z'(D)$ , and then it reduces  $Z'(D)$  to the empty scheme, and the reduction vector is  $(d_1 + s - 1, \dots, d_u + s - 1, s - 1, \dots, 2, 1)$ , which is clearly positive and strictly decreasing.

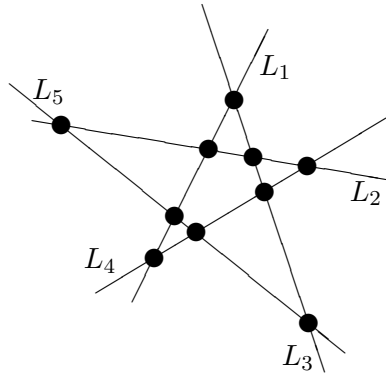


FIGURE 4.2.4. The configuration  $Z(L_1 + L_2 + L_3 + L_4 + L_5)$ .

**Example 4.2.5.** Here we show that in positive characteristic there can be examples where the number of lines through each point is larger than 2. Let  $Z$  be the reduced union of the points of  $\mathbf{P}_K^2$ , where  $q$  is a power of a prime  $p$  and  $K = \mathbf{F}_q$  is the finite field of  $q$  elements. For the set of lines, take all of the lines in  $\mathbf{P}_{\mathbf{F}_q}^2$ . Then there are  $q^2 + q + 1$  points,  $q^2 + q + 1$  lines,  $q + 1$  lines go through every point, and  $q + 1$  points lie on every line. If  $D$  is any divisor with support on all of the lines, then by Example 4.2.2 we obtain  $h_{mZ(D)}$  and the graded Betti numbers for  $I_{mZ(D)}$ , for all  $m > 0$ . In particular, we obtain  $h_{m(q+1)Z}$  and the graded Betti numbers for  $I_{m(q+1)Z}$ , for all  $m > 0$ .

**Example 4.2.6.** Finally we consider a classical example over the complex numbers where the number of lines through each point is 3. In particular, consider the projective dual of the Hesse configuration where  $K$  is the complex numbers (see [To] for another situation in which this has come up recently). The Hesse configuration is the set of nine flex points for a general smooth plane cubic  $C$  and the twelve lines through each pair of flexes. It has the property that any of the twelve lines through two flexes goes through a third, and each of the nine flexes lies on four of the lines. However, it is not true that every pair of the twelve lines meet at a flex, so the Hesse configuration is not of the type we have considered in the previous examples; i.e., it is not of the form  $Z(D)$  for a divisor  $D$  supported on the twelve lines.

Consider now the configuration projectively dual to the Hesse configuration, consisting of twelve points and nine lines, with each point lying on three of the lines and each line passing through four of the points. Now each pair of the lines (corresponding to a pair of flex points on  $C$ ) does intersect at one of the twelve points (corresponding to a line through the two flex points on  $C$ ). For any divisor  $D$  supported on these nine lines we get a reduction vector for the fat point scheme  $Z(D)$ .

In particular, if  $D$  is the reduced union of the nine lines and  $Y$  is the reduced union of the twelve points, then  $Z(D) = 3Y$  and  $Z'(D) = 2Y$ . By the previous examples we get strictly decreasing reduction vectors for  $m3Y$ ,  $2Y$ , and even  $(m3+2)Y$ , for all  $m > 0$ , by an argument similar to that in Example 4.2.3. In fact, in this case one can also check ad hoc that  $(3m+1)Y$  has a GMS (but not strictly decreasing) reduction vector. So we get the Hilbert function of  $mY$  for all  $m > 0$ , and the graded Betti numbers for all  $m$  congruent to 0 or 2 modulo 3.

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