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Regressive functions on pairs

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Abstract

We compute an explicit upper bound for the regressive Ramsey numbers by a combinatorial argument, the corresponding function being of Ackermannian growth. For this, we look at the more general problem of bounding $g(n, m)$, the least l such that any regressive function $f : [m, l]^{[2]} \rightarrow \mathbb{N}$ admits a min-homogeneous set of size n . Analysis of this function also leads to the simplest known proof that the regressive Ramsey numbers have rate of growth at least Ackermannian. Together, these results give a purely combinatorial proof that, for each m , $g(\cdot, m)$ has rate of growth precisely Ackermannian, considerably improve the previously known bounds on the size of regressive Ramsey numbers, and provide the right rate of growth of the levels of g . For small numbers we also find bounds on their value under g improving the ones provided by our general argument.

Key words: Kanamori-McAloon theorem, regressive Ramsey numbers, Ackermann's function.

2000 MSC: 05D10, 03D20.

1. Introduction

Throughout this paper, $\mathbb{N} = \{0, 1, \dots\}$. For $1 \leq n, k \leq m$, let $m \rightarrow (n)_{reg}^k$ be the following assertion:

Whenever $f : [1, m]^{[k]} \rightarrow [0, m - k]$ is regressive, there is $H \in [1, m]^{[n]}$ min-homogeneous for f .

Similarly, for $X \subseteq \mathbb{N}$ infinite, let $X \rightarrow (\mathbb{N})_{reg}^k$ mean that for every regressive $f : X^{[k]} \rightarrow \mathbb{N}$ there is $H \subseteq X$ infinite and min-homogeneous for f . Here,

- $X^{[k]}$ is the collection of k -sized subsets of X .
- $f : X^{[k]} \rightarrow \mathbb{N}$ is regressive iff $f(s) < \min(s)$ whenever $s \in X^{[k]}$ and $\min(s) > 0$ (where $\min(s)$ is the least element of s).

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- For such an f , $H \subseteq X$ is min-homogeneous for f iff $0 \notin H$ and, whenever $s, t \in H^{[k]}$ and $\min(s) = \min(t)$, then $f(s) = f(t)$.
- $[n, m] = \{n, n + 1, \dots, m\}$. Similarly for other interval notation.

The following is the main result of Kanamori-McAloon [5]:

Theorem 1.1. 1. For any $k, n \in \mathbb{N}$, there is m such that $m \rightarrow (n)_{reg}^k$.
 2. Item 1 is not a theorem of Peano Arithmetic PA.

In fact, in Kanamori-McAloon [5] a level-by-level correspondence is established between the values of k and the amount of induction required to prove the existence of the function that to n assigns the least m as in Theorem 1.1.1; see Carlucci-Lee-Weiermann [2] for more on this.

In this paper, I only deal with $k = 2$ although, in Section 3, I present a short proof of Theorem 1.1.1. In Section 4, I show that

$$g(n) = \text{least } l \text{ such that } l \rightarrow (n)_{reg}^2$$

is provably total in PA. In fact, I provide an explicit (recursive) upper bound for $g(n)$, thus showing by purely elementary means that its rate of growth is at most Ackermannian.

To state the result, let $g(n, m)$ be the least l such that for any regressive

$$f : [m, l]^{[2]} \rightarrow [0, l - 2],$$

there is a min-homogeneous set for f of size n . (From now on, all mentions of g refer to this two-variable function.) Clearly $g(n, m) \leq g(n, m + 1)$, $g(2, m) = m + 1$ and, by the pigeonhole principle, $g(3, m) = 2m + 1$.

Let $G(n, m)$ be the least l such that for any regressive $f : [m, l]^{[2]} \rightarrow [0, l - 2]$, there is a min-homogeneous set for f of size n whose minimum element is m . It may not be immediate that G is well-defined, but this is addressed by Remark 3.3 and the proof of Theorem 4.1.

We have $G(2, m) = g(2, m)$, $G(3, m) = g(3, m)$, $G(n + 1, 1) = g(n + 1, 1) = g(n, 2)$ and, in general, $g(n, m) \leq G(n, m)$. Finally, set $g^0(n, m) = m$ and $g^{k+1}(n, m) = g(n, g^k(n, m))$. We then have:

Theorem 1.2. 1. $G(4, m) = 2^m(m + 2) - 1$.
 2. Let $\alpha_{-1} = 0$ and, for $0 \leq i < m$, let $d_i = g^i(4, m + 1)$ and

$$\alpha_i = (\alpha_{i-1} + m + 3 + i)(2^{d_i} - 1).$$

Then $g(5, m) \leq (2m + 1) + \sum_{i=0}^{m-1} \alpha_i$.

3. For all n , there is a constant c_n such that $G(n, m) < A_{n-1}(c_n m)$ for almost all m .

Here, $A_n = A(n, \cdot)$ where A is Ackermann's function, see Section 2. Theorem 1.2.2 is proven by adapting the argument of Blanchard [1, Lemma 3.1] (that bounds $g(5, 2)$) to the more general problem of bounding $g(5, m)$. In Kojman-Shelah [7], explicit lower bounds for g are computed, showing that g is at least of Ackermannian growth (our notion of "Ackermannian growth" is more restrictive than that of Kojman-Shelah [7] or Kojman-Lee-Omri-Weiermann [6], and is discussed in Section 2). In Section 5, I find lower bounds for $G(n, m)$ and $g(n, m)$ in terms of iterates of $g(n - 1, \cdot)$, and conclude:

Theorem 1.3. $g(n, m) \geq A_{n-1}(m - 1)$ for all $n \geq 2$.

The proof of Theorem 1.3 is simpler and shorter than the proofs of lower bounds in Kojman-Shelah [7] and Kojman *et al.* [6], and increases these bounds significantly. Thus the results of Sections 4 and 5 combine to give a very accessible and purely combinatorial proof of the result obtained in Kanamori-McAloon [5] by model theoretic methods, that g is not provably total in Primitive Recursive Arithmetic PRA, but is “just shy” of it; in fact, the argument gives that, for each m , the function $g(\cdot, m)$ has Ackermannian rate of growth. These results also establish the rate of growth of the function $g(n, \cdot)$ as being precisely that of the $(n - 1)^{\text{st}}$ level of the Ackermann hierarchy of fast growing functions.

In the literature, the values of g (more precisely, the values of $g(\cdot, 2)$) are referred to as “regressive Ramsey numbers.” In Section 6, I improve the upper bound for $g(4, m)$ and show:

Theorem 1.4. $g(4, 3) = 37$.

I also improve the upper bound for $g(4, 4)$ provided by the general argument of Section 6. The figures so obtained improve the previously known bounds for small regressive Ramsey numbers obtained in Blanchard [1] and Kojman *et al.* [6].

I occasionally abuse notation by writing $f(t_1, t_2)$ for $f(t)$ where $t_1 < t_2$ and $t = \{t_1, t_2\}$.

2. Preliminaries on Ackermannian functions

In this section I collect several standard results about Ackermannian growth; notice that the notion I use is more restrictive than the version used in Kojman-Shelah [7] or Kojman *et al.* [6], where a function is called Ackermannian simply if it eventually dominates each primitive recursive function.

Definition 2.1. Given functions $g, h : \mathbb{N} \rightarrow \mathbb{N}$, say that h *eventually dominates* g , in symbols $g <_* h$, iff $g(m) < h(m)$ for all but finitely many values of m .

Definition 2.2. Ackermann’s function $A : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{N}$ is defined by double recursion as follows:

- $A(0, m) = m + 1$.
- $A(n, 0) = A(n - 1, 1)$ for $n > 0$.
- $A(n, m) = A(n - 1, A(n, m - 1))$ for $n, m > 0$.

Let $\text{Ack}(n) = A(n, n)$ and $A_n = A(n, \cdot)$. Sometimes, in the literature, it is Ack that is referred to as Ackermann’s function. This is the standard example of a recursive but not primitive recursive function. The version presented above is due to Rafael Robinson and Rózsa Péter, see Robinson [8]. Notice that $A_1(m) = m + 2$, $A_2(m) = 2m + 3$, A_3 has exponential rate of growth and A_4 grows like a tower of exponentials.

Definition 2.3. Let $f_0(m) = m + 1$ and $f_{n+1}(m) = f_n^m(m)$ where the superindex indicates that f_n is iterated m times. Continue this hierarchy by letting $f_\omega(m) = f_m(m)$ and $f_{\omega+1}(m) = f_\omega^m(m)$.

Notice that what in Kojman *et al.* [6] is called Ackermann's function is the map $A'(n, m) = f_{n-1}(m)$.

Definition 2.4. A function $f : \mathbb{N} \rightarrow \mathbb{N}$ is (precisely) of Ackermannian growth if and only if there are constants $c, C > 0$ such that for all but finitely many m , $f_\omega(cm) \leq f(m) \leq f_\omega(Cm)$.

Similarly, say that a function's rate of growth is like that of the n^{th} level of the Ackermann hierarchy if there are constants $c, C > 0$ such that for all but finitely many m , $A_n(cm) \leq f(m) \leq A_n(Cm)$.

(Compare with Graham-Rothschild-Spencer [4, Section 2.7], where the relevant notion is called *Ackermannic*.)

The following two lemmas are standard and collect together several folklore results; see for example Graham-Rothschild-Spencer [4] and Cori-Lascar [3].

Lemma 2.5. 1. For all n , $A_n < A_{n+1}$ and $f_n <_* f_{n+1}$. In fact, for any $C > 0$ and almost all m , $A_n(Cm) < A_{n+1}(m)$ for $n > 0$, and $f_n(Cm) < f_{n+1}(m)$ for all n .
 2. For all $n > 0$, $A_{n+1} <_* f_n$ and $f_n(m) < A_{n+1}(cm)$ for some constant $c = c_n$ and all m .
 3. f_ω and Ack are of Ackermannian growth. □

More precise quantitative versions of the above are possible, but Lemma 2.5 as stated suffices for our purposes.

Lemma 2.6. 1. If f is of Ackermannian growth, it eventually dominates each primitive recursive function. In particular, it eventually dominates each f_n .
 2. If f is of Ackermannian growth then it is eventually dominated by $f_{\omega+1}$.
 3. There is a function f that eventually dominates each f_n and is eventually dominated by $f_{\omega+1}$ but is not of Ackermannian growth.
 4. If g, h are strictly increasing primitive recursive functions and f is of Ackermannian growth, then so is $g \circ f \circ h$. □

3. Regressive functions

I start by proving the infinite version of Theorem 1.1.1. This is also done in Kanamori-McAloon [5], but the argument to follow is easier (in Kanamori-McAloon [5] this is accomplished using the Erdős-Rado canonization theorem). The proof of Theorem 1.2 in Section 4 was obtained by trying to produce a finitary and effective version of this argument for $k = 2$.

Lemma 3.1. If $X \subseteq \mathbb{N}$ is infinite, then for any k , $X \rightarrow (\mathbb{N})_{reg}^k$.

Proof. Let $f : X^{[k]} \rightarrow \mathbb{N}$ be regressive. Without loss, $k > 1$. Define a decreasing sequence of infinite subsets of X , $X \setminus \{0\} = H_0 \supset H_1 \supset H_2 \supset \dots$ such that, letting $m_n = \min H_n$, then $(m_n)_{n \geq 0}$ is strictly increasing, as follows: Given H_n , let

$$\varphi : (H_n \setminus \{m_n\})^{[k-1]} \rightarrow [0, m_n - 1]$$

be the function $\varphi(s) = f(\{m_n\} \cup s)$. By Ramsey's theorem, there is H_{n+1} infinite and homogeneous for φ .

Then $\{m_n : n \in \mathbb{N}\}$ is min-homogeneous for f . □

Theorem 1.1.1 follows now from a standard compactness argument:

Corollary 3.2. $\forall n \forall k \exists l (l \rightarrow (n)_{reg}^k)$.

Proof. Fix n and k counterexamples to the corollary. For each $m \geq n, k$, it follows that there are regressive functions $f : [1, m]^{[k]} \rightarrow [0, m - k]$ without min-homogeneous sets of size n . Consider the collection \mathcal{T} of all these functions, ordered by extension: Given $f_1, f_2 \in \mathcal{T}$, $f_1 : [1, m_1]^{[k]} \rightarrow [0, m_1 - k]$, $f_2 : [1, m_2]^{[k]} \rightarrow [0, m_2 - k]$, set $f_1 < f_2$ iff $m_1 < m_2$, and $f_2 \upharpoonright [1, m_1]^{[k]} = f_1$. Then $(\mathcal{T}, <)$ is an infinite finitely branching tree so, by König's lemma, it has an infinite branch. The functions along this branch fit together into a regressive function $f : \mathbb{N}^{[k]} \rightarrow \mathbb{N}$ which contradicts Lemma 3.1 since it does not even admit min-homogeneous sets of size n . \square

Remark 3.3. Notice that using this argument one can easily show that $G(n, m)$ is well defined. Our argument next section will also show this.

4. An Ackermannian upper bound for G

Here I prove Theorem 1.2.3; the argument resembles the “color focusing” technique from Ramsey theory.

Theorem 4.1. *For each fixed m , $G(n, m)$ is bounded by a function of Ackermannian growth. In particular, so is $g(n, 2) \leq G(n, 2)$.*

Proof. I find an upper bound for the function $G(n, \cdot)$ by induction on n . In order to do this, I introduce numbers $s_i = s(i, n, m)$ for all $n \geq 4$, $m \geq 2$, and $1 \leq i \leq m$, and argue that $G(n, m) \leq s(m, n, m)$.

Fix $n \geq 4$. The numbers s_i are computed in terms of the function $G(n - 1, \cdot)$. Fix m , which we may assume is at least 2.

Define $s(1, n, m), \dots, s(m, n, m)$ and t_0, t_1, \dots, t_{m-1} recursively as follows.

- Let $t_0 = m + 1$.
- Let $s_1 = g(n - 1, t_0)$ and, for $1 \leq i < m$, let $s_{i+1} = G(n - 1, t_i)$.
- For $1 \leq j \leq m$, let $B_j^{n,m} = B_j = \bigcup_{i=1}^j [t_{i-1}, s_i]$, and denote by $\prod B_j$ the Cartesian product $\prod_{i \in B_j} [0, i - 1]$.
- For $1 \leq j < m$, let $t_j = (j + 1) \times |\prod B_j|$.

We claim that $G(n, m) \leq s(m, n, m)$. To see this, suppose a regressive function $f : [m, s_m]^{[2]} \rightarrow [0, s_m - 2]$ is given.

Fix j , $1 < j \leq m$. Suppose $f(m, \cdot) \upharpoonright B_j$ takes at most j values. (This holds trivially for $j = m$.) We claim that either there is a min-homogeneous set for f of size n contained in $\{m\} \cup B_j$ whose minimum element is m , or else $f(m, \cdot) \upharpoonright B_{j-1}$ takes at most $j - 1$ values.

Consider the regressive function

$$\psi : [t_{j-1}, s_j]^{[2]} \rightarrow [0, s_j - 2]$$

given by

$$\psi(u) = \begin{cases} f(u) & \text{if } u_1 > t_{j-1}, \\ \langle f(l, u_2) : l \in \{m\} \cup B_{j-1} \rangle & \text{if } u_1 = t_{j-1}, \end{cases}$$

where $\langle \dots \rangle$ is a bijection from the Cartesian product $C_j \times \prod B_{j-1}$ onto $[0, t_{j-1})$, where $C_j \subset [0, m-1]$ has size j and contains the possible values that $f(m, \cdot) \upharpoonright B_j$ can take.

Then (by definition of s_j) there is a set $\{a_1, \dots, a_{n-2}\} \subseteq [t_{j-1} + 1, s_j]$ that is min-homogeneous for f and such that for all $k \in \{m\} \cup B_{j-1}$, $\{k, a_1, \dots, a_{n-2}\}$ is also min-homogeneous for f . Let $f(m, a_1) = c$. If $f(m, k) = c$ for any $k \in B_{j-1}$, then $\{m, k, a_1, \dots, a_{n-2}\}$ is the min-homogeneous set we are looking for. Otherwise, $f(m, \cdot) \upharpoonright B_{j-1}$ takes at most $j-1$ values, as claimed.

There is therefore no loss in assuming that $f(m, \cdot) \upharpoonright B_1$ is constant. But then, by definition of s_1 , there is $\{a_1, \dots, a_{n-1}\} \subseteq B_1$ min-homogeneous for f . Then $\{m\} \cup \{a_1, \dots, a_{n-1}\}$ is also min-homogeneous, and we are done.

Define a function $H(n, m)$ as follows: $H(n, \cdot) = G(n, \cdot)$ for $n \leq 4$ (see also Fact 5.3 below); in the argument above, let s'_i be the function resulting from replacing $G(n-1, \cdot)$ with $H(n-1, \cdot)$ in the definition of s_i , and let $H(n, m) = s'(m, n, m)$, so clearly $G \leq H$. It is easy to see, using standard arguments (or consider the proof of Theorem 1.2.3 below) that $n \mapsto H(n, m)$ (for any fixed m) is of Ackermannian growth. This completes the proof. \square

Remark 4.2. Since the argument above only requires f to be defined on

$$(\{m\} \cup B_m^{n,m})^{[2]},$$

it follows (by “translation”) that $g(n, m) \leq m + |B_m^{n,m}|$.

That $G(4, m) = 2^m(m+2) - 1$ is shown in Fact 5.3, and the upper bound on $g(5, \cdot)$ is shown in Theorem 7.1. Using this (all I need is that $G(4, m)$ has exponential rate of growth) and the argument of Theorem 4.1, Theorem 1.2.3 follows easily:

Proof. Use the notation of the proof above, and argue by induction on $n \geq 5$ since the result is clear for $n \leq 4$ from the explicit formulas for $G(n, \cdot)$. Notice the easy estimate $l! < 2^{l(l-1)/2}$ and the obvious inequality $s(i+1, n, m) = s_{i+1} \leq G(n-1, s_i!)$ for $i < m$. From this and Fact 5.3 we have that for $n = 5$ there is a constant c_5 such that s_i is bounded by a tower of two’s of length $c_5 i$ applied at m ,

$$s_i \leq 2^{2^{\dots^{2^m}}}$$

In fact any c_5 slightly larger than 3 suffices (with room to spare). This proves the result for $n = 5$; for $n > 5$ use Lemma 2.5 and proceed by a straightforward induction to show that $c_{n-1} = n-1$ suffices (and therefore for each m , $g(\cdot, m)$ has rate of growth precisely Ackermannian). \square

Question 4.3. *Can the value of the constants c_n be significantly improved? This seems to require a more careful analysis than the one above, perhaps combined with fine detail considerations, as in the proof of Theorem 7.1.*

5. Lower bounds for g and G

Here I prove Theorem 1.3.

Theorem 5.1. 1. $G(n+1, m) \geq g^m(n, m+1)$.
 2. $g(n+1, m+1) \geq g(n, g(n+1, m) + 1)$. In particular, for $n \geq 2$ and $m \geq 1$, $g(n, m) \geq A_{n-1}(m-1)$, the inequality being strict for $n > 2$ and, for example, $g(4, m) > 2^{m+2}$ for $m > 1$.

Proof. I exhibit a regressive function $f : [m, g^m(n, m+1) - 1]^{[2]} \rightarrow \mathbb{N}$ without min-homogeneous sets of size $n+1$ whose minimum element is m . Start by choosing regressive functions

$$F_k : [g^k(n, m+1), g^{k+1}(n, m+1) - 1]^{[2]} \rightarrow \mathbb{N}$$

without min-homogeneous sets of size n , for $k < m$; this is possible by definition of $g(n, \cdot)$. Now set, for $m < a \leq g^m(n, m+1) - 1$,

$$f(m, a) = k \iff g^k(n, m+1) \leq a < g^{k+1}(n, m+1),$$

and, for such a , and $b \in (a, g^{k+1}(n, m+1) - 1]$,

$$f(a, b) = F_k(a, b).$$

Define $f(a, b)$ for other values of a and b arbitrarily (below a). This function works, for if $\min(H) > m$ and $\{m\} \cup H$ is min-homogeneous for f , then H is completely contained in some interval $[g^k(n, m+1), g^{k+1}(n, m+1))$ for some $k < m$, but then H is min-homogeneous for F_k , so $|H| < n$.

I now prove item 2. Let $F_m : [m, g(n+1, m)]^{[2]} \rightarrow \mathbb{N}$ be a regressive function without min-homogeneous sets of size $n+1$, and let

$$h_m : [g(n+1, m) + 1, g(n, g(n+1, m) + 1)]^{[2]} \rightarrow \mathbb{N}$$

be a regressive function without min-homogeneous sets of size n . Define

$$F_{m+1} : [m+1, g(n, g(n+1, m) + 1)]^{[2]} \rightarrow \mathbb{N}$$

by

$$F_{m+1}(a, b) = \begin{cases} F_m(a-1, b-1) & \text{if } b \leq g(n+1, m), \\ a-1 & \text{if } a \leq g(n+1, m) < b, \\ h_m(a, b) & \text{if } g(n+1, m) < a. \end{cases}$$

Then F_{m+1} is regressive. If H is min-homogeneous for F_{m+1} and $|H| \geq 2$, let $a = \min(H)$ and $b = \min(H \setminus \{a\})$. If $b \leq g(n+1, m)$ then $F_{m+1}(a, b) = F_m(a-1, b-1) < a-1$ so $H \subseteq [m+1, g(n+1, m)]$ and $\{h-1 : h \in H\}$ is min-homogeneous for F_m , so $|H| \leq n$.

If $g(n+1, m) < b$ then $H \setminus \{a\}$ is min-homogeneous for h_m , so $|H \setminus \{a\}| < n$ and $|H| < n+1$ in this case as well. \square

Remark 5.2. Notice that for $n = 3$, the argument of Theorem 5.1.1 describes (up to trivial renamings) all the examples of regressive functions $f : [m, g^m(3, m+1) - 1]^{[2]} \rightarrow \mathbb{N}$

not admitting min-homogeneous sets of size 4 with minimum element m . It is easy now to give an example of a regressive $f : [2, 14]^{[2]} \rightarrow \mathbb{N}$ witnessing $14 \not\prec (5)_{reg}^2$:

$$f(i, j) = \begin{cases} j - i - 1 \pmod{i} & \text{if } i \geq 6, \\ 0 & \text{if } i = 2 \text{ and } j \leq 6, \\ & \text{if } i \in [3, 5] \text{ and } j = i + 1, \\ 1 & \text{if } i = 2 \text{ and } 7 \leq j, \\ & \text{if } i = 3 \text{ and } j \in \{5, 7, 8\}, \\ & \text{if } i \in \{4, 5\} \text{ and } j = i + 1, \\ 2 & \text{if } i = 3 \text{ and } j \in \{6\} \cup [9, 14], \\ & \text{if } i = 4 \text{ and } j = 7, \\ & \text{if } i = 5 \text{ and } 8 \leq j, \\ 3 & \text{if } i = 4 \text{ and } 8 \leq j. \end{cases}$$

I leave to the reader the easy verification that this example works; in Theorem 6.1.2, I analyze a more difficult example witnessing $g(4, 3) \geq 37$. See Blanchard [1] for an analysis of a different example also witnessing $g(4, 2) \geq 15$; the function I have presented is closer in spirit to the other constructions in this paper.

Now I prove Theorem 1.2.1:

Fact 5.3. $G(4, m) = 2^m(m + 2) - 1$.

Proof. Notice that $2^m(m + 2) - 1 = g^m(3, m + 1) \leq G(4, m)$ by Theorem 5.1.1. Suppose $f : [m, 2^m(m + 2) - 1]^{[2]} \rightarrow \mathbb{N}$ is regressive. A straightforward induction on $k \leq m$ shows that either $f(m, \cdot) \upharpoonright [m + 1, 2^k(m + 1) + 2^k - 1]$ takes at least $k + 1$ values, or else f admits a min-homogeneous set $A \in [m, 2^k(m + 1) + 2^k - 1]^{[4]}$ with $m \in A$ (see also the proof of Theorem 6.1.1 for a more detailed presentation of a similar approach). When $k = m$, this shows that $G(4, m) \leq 2^m(m + 2) - 1$. \square

Remark 5.4. Thus, $g(4, 2) = G(4, 2) = 15$. In the next section, I improve the upper bound for $g(4, m)$, $m > 2$.

Corollary 5.5. $g(5, 2) > 2^{18}$.

This significantly improves the bound $g(5, 2) \geq 195$ claimed in Blanchard [1].

Proof. $g(5, 2) \geq g(4, g(5, 1) + 1) = g(4, 16) > 2^{18}$. \square

Remark 5.6. In fact, by Theorem 6.1.2, $g(4, 3) = 37$, so $g(4, m) \geq 5 \times 2^m - 3$ for $m \geq 3$, and $g(5, 2) \geq 5 \times 2^{16} - 3$.

Theorem 5.1.2 also improves significantly the bound $g(81, 2) > f_{51}(2^{2^{274}})$ obtained in Kojman *et al.* [6, Claim 2.32] (here, f_{51} is as in Section 2; to see that the new bound is an improvement, a slightly more precise version of Lemma 2.5 is necessary).

6. Bounds for $g(4, \cdot)$

From Section 5 it follows that $g(4, m) \leq 2^m(m+2) - 1$. Here I improve this bound and prove Theorem 1.4.

Theorem 6.1. 1. For $m \geq 2$, $g(4, m) \leq 2^m(m+2) - 2^{m-1} + 1$.
 2. $g(4, 3) = 37$.
 3. $g(4, 4) \leq 85$.

Proof. I have already shown that $g(4, 2) = 15$. Assume $m \geq 3$, let

$$n = 2^m(m+2) - 2^{m-1} + 1,$$

and suppose a regressive $f : [m, n]^{[2]} \rightarrow \mathbb{N}$ is given. I need to argue that there is $H \in [m, n]^{[4]}$ min-homogeneous for f . For $i < m$, let $a_i = \min\{j : f(m, j) = i\}$ and $C_i = \{j > a_i : f(m, j) = i\}$. One may assume that, as long as the a_i are defined, they occur in order, so $m+1 = a_0 < a_1 < \dots$

If $f(m+1, a) = f(m+1, b)$ for $a \neq b$ in C_0 , then $H = \{m, m+1, a, b\}$ is as required. Assume now that $f(m+1, \cdot) \upharpoonright C_0$ is injective and, in particular, $|C_0| \leq m+1$.

For $i \in C_0$ let $B_i = \{j > i : f(m+1, j) = f(m+1, i)\}$. I claim that for all $k \in [1, m-2]$, either $a_k \leq 2^k(m+2) - 2^{k-1} - 1$, or else there is an H as required and either of the form $\{m, a_i, a, b\}$ for some $i < k$ and some $a, b \in C_i$, or of the form $\{m+1, i, a, b\}$ for some $i \in C_0$ and some $a, b \in B_i$.

The proof is by induction on k . Fix a least counterexample. Then

$$a_t \leq 2^t(m+2) - 2^{t-1} - 1$$

for all $t \in [1, k)$ and $1 \leq k < m-1$. Then $a_k \leq 2^k(m+2) - 2^{k-1}$. Otherwise, for some $i < k$, $|C_i| > a_i$. If $a_k = 2^k(m+2) - 2^{k-1}$, then $a_t = 2^t(m+2) - 2^{t-1} - 1$ for all $t \in [1, k)$ (or else, again, some C_i for $i < k$ has size larger than a_i). Also, there is some $j \in (2m+1, a_k)$ in C_0 . But then $|B_i| > i$ for some $i \in C_0$, and the claim follows: Otherwise,

$$\begin{aligned} \sum_{i \in C_0} |B_i| &\leq \sum_{i \in [m+2, 2m+1] \cup \{j\}} i \leq \sum_{i=m+2}^{2m+1} i + 2^k(m+2) - 2^{k-1} - 1 \\ &= \frac{3}{2}m(m+1) + 2^k(m+2) - 2^{k-1} - 1 \\ &< n - 2(m+1) = |[2m+2, n] \setminus \{j\}| \end{aligned}$$

because $(3+2m)(2^m - 2^k) \geq 3(3+2m)2^{m-2} > 3m^2 + 7m$ for $m \geq 3$.

It follows that one may assume $a_{m-1} \leq 2^{m-1}(m+2) - 2^{m-2}$, but then, since $n \geq 2a_{m-1} + 1$, some C_i must have size larger than a_i , and the proof is complete.

Now I show that $g(4, 3) = 37$. The upper bound follows from the argument above. To see that $g(4, 3) \geq 37$, I exhibit a regressive $f : [3, 36]^{[2]} \rightarrow \mathbb{N}$ without min-homogeneous

sets of size 4. Consider the function f shown below: For $3 \leq i < j \leq 36$, set

$$f(i, j) = \left\{ \begin{array}{ll} j - i - 1 \pmod{i} & \text{if } \begin{array}{l} i \geq 16, \\ 8 \leq i \leq 15 \text{ and } j \leq 16, \\ 12 \leq i \leq 15 \text{ and } j \leq 19, \\ 4 \leq i \leq 6 \text{ and } j \leq 7, \\ i = 6 \text{ and } j \leq 11, \end{array} \\ 0 & \text{if } \begin{array}{l} i = 3 \text{ and } (j \leq 7 \text{ or } j = 17), \\ i = 5 \text{ and } 8 \leq j \leq 11, \\ i = 6 \text{ and } 12 \leq j \leq 16, \\ i = 7 \text{ and } j \leq 12, \end{array} \\ 1 & \text{if } \begin{array}{l} i = 3 \text{ and } 8 \leq j \leq 16, \\ i = 4 \text{ and } 8 \leq j \leq 11, \\ i = 5 \text{ and } 12 \leq j \leq 16, \\ i = 6 \text{ and } j = 18, \\ i = 7 \text{ and } j = 13, \end{array} \\ 2 & \text{if } \begin{array}{l} i = 3 \text{ and } 18 \leq j, \\ i = 4 \text{ and } j \in [12, 19] \setminus \{17\}, \\ i = 5 \text{ and } j = 17, \\ i = 6 \text{ and } j = 19, \\ i = 7 \text{ and } j = 14, \\ i = 15 \text{ and } 21 \leq j, \end{array} \\ 3 & \text{if } \begin{array}{l} i = 4 \text{ and } (j = 17 \text{ or } 20 \leq j), \\ i = 5 \text{ and } 18 \leq j, \\ i = 7 \text{ and } j = 15, \\ i = 11 \text{ and } 17 \leq j \leq 20, \\ i = 14 \text{ and } 20 \leq j, \end{array} \\ 4 & \text{if } \begin{array}{l} i = 7 \text{ and } j = 16, \\ i = 10 \text{ and } 17 \leq j \leq 20, \\ i = 11 \text{ and } 21 \leq j, \\ i = 13 \text{ and } 20 \leq j, \\ i = 15 \text{ and } j = 20, \end{array} \\ 5 & \text{if } \begin{array}{l} i = 6 \text{ and } (j = 17 \text{ or } 20 \leq j), \\ i = 7 \text{ and } (j = 17 \text{ or } j = 19), \\ i = 9 \text{ and } 17 \leq j \leq 20, \\ i = 10 \text{ and } 21 \leq j, \\ i = 12 \text{ and } 20 \leq j, \end{array} \\ 6 & \text{if } \begin{array}{l} i = 7 \text{ and } (j = 18 \text{ or } 20 \leq j), \\ i = 8 \text{ and } 17 \leq j \leq 20, \\ i = 9 \text{ and } 21 \leq j, \end{array} \\ 7 & \text{if } i = 8 \text{ and } 21 \leq j. \end{array}$$

To help understand the example somewhat, notice that the argument above shows that one must have $a_1 = 8$ and $a_2 = 18$, $f(i, \cdot)$ must be injective for $i \geq 18$ and similarly

$f(i, \cdot) \upharpoonright C_i$ must be injective for $i \in [4, 7]$ and $C_i = \{j > i : f(3, j) = f(3, 4)\}$, or $i \in [8, 16] \cap \{j : f(3, j) = f(3, 8)\}$ and $C_i = [i + 1, 17] \cap \{j : f(3, j) = f(3, 8)\}$. If f is any function satisfying these conditions, $a < b < c < d$, and $A = \{a, b, c, d\}$ is min-homogeneous for f , then $a > 3$ and $b < 18$.

The function f displayed above satisfies the conditions just described. Let A as above be a putative min-homogeneous set. Then $a < 16$ since otherwise $f(a, \cdot)$ does not take any value more than twice.

In fact, $a < 12$, since $12 \leq a \leq 15$ would imply (for the same reason) that $b \geq 18$. If $8 \leq a \leq 11$, then $b \geq 15$. Since $f(i, \cdot) \upharpoonright D_i$ is injective for $i \in \{15\} \cup [17, 20]$ and $D_i = (i, 20]$, or $i = 16$ and $D_i = [21, 36]$, this is not possible.

If $a = 7$ then $b \notin [8, 12]$ as $f(i, \cdot) \upharpoonright (i, 12]$ is injective for $i \in [8, 12]$. This forces $b \geq 18$.

If $a = 6$ then $b \notin \{7\} \cup [12, 16]$ as $f(b, \cdot) \upharpoonright [\max(b + 1, 12), 16]$ is then injective. This forces $b = 17$ but $f(17, \cdot) \upharpoonright [20, 36]$ is injective, so this cannot be the case.

The analysis above already rules out $a = 5$ since $f(6, \cdot) \upharpoonright [8, 11]$ is injective. Since $f(7, \cdot) \upharpoonright [12, 16] \cup \{18, 19\}$ is also injective, it also rules out $a = 4$, completing the argument.

Finally, I argue that $g(4, 4) \leq 85$. Let a regressive $f : [4, 85]^{[2]} \rightarrow \mathbb{N}$ be given. Use notation as before. Then one can assume (from the argument for item 1) that $a_1 \leq 10$. If $a_1 = 10$, since $6 + 7 + 8 + 9 = 30$, one can assume that there is $b \leq 40$ such that $f(5, b) = 4$ (while $f(5, j) = j - 6$ for $j \in [6, 9]$). But then there is a min-homogeneous set for f of size 4 with minimum element 5 and maximum at most 81.

If $a_1 \leq 9$ then $a_2 \leq 21$. If $a_2 = 21$ then one can assume $f(5, j) = j - 6$ for $j \in [6, 8]$ and there are b_1, b_2 with $f(5, b_1) = 3$, $f(5, b_2) = 4$, $b_1 \leq 19$ and $b_2 \leq 20$. Since $6 + 7 + 8 + 19 + 20 = 60$, there is again a min-homogeneous set of size 4 in this case. If $a_2 \leq 20$, then $a_3 \leq 42$ and $|A_i| > a_i$ for some $i < 4$. This shows $g(4, 4) \leq 85$. \square

7. Bounds for $g(5, \cdot)$

In this section I briefly sketch how to adapt the proof of Blanchard [1, Lemma 3.1] to prove the more general statement below, which concludes the proof of Theorem 1.2. The bound for $g(5, 2)$ is smaller than the one in Blanchard [1] because I take advantage of the fact that $g(4, 3) = 37$, as established in Theorem 6.1.2.

Theorem 7.1. *Let m be given. For $i < m$, set $d_i = g^i(4, m + 1)$. Let $\alpha_{-1} = 0$ and $\alpha_i = (\alpha_{i-1} + m + 3 + i)(2^{d_i} - 1)$ for $0 \leq i < m$. Then*

$$g(5, m) \leq (2m + 1) + \sum_{i=0}^{m-1} \alpha_i.$$

In particular, $g(5, 2) \leq 41 \times 2^{37} - 1$.

Proof. Let n be the purported upper bound displayed above and consider a regressive function $f : [m, n]^{[2]} \rightarrow \mathbb{N}$. For $i < m$, let

$$B_i = \{x \in [m + 1, n] : f(m, x) = i\}$$

and, if $B_i \neq \emptyset$, set $a_i = \min(B_i)$. Without loss, $a_0 = m + 1 < a_1 < \dots$. Clearly, we may assume that $a_i \leq g^i(4, m + 1) = d_i$ for all those $i < m$ for which a_i is defined. In

particular, since n is sufficiently large, we may assume that the a_i are defined for all $i < m$.

Consider $B_{ij} = \{x \in [a_i + 1, n] : f(m, x) = i, f(a_i, x) = j\}$ for $i < m$ and $j < a_i$ and, if $B_{ij} \neq \emptyset$, set $a_{ij} = \min(B_{ij})$. Let $D = \{B_{ij} : B_{ij} \neq \emptyset\}$ and $q = |D|$, so $q \leq \sum_{i=0}^{m-1} d_i$. Let $\{C_s : s < q\}$ be the enumeration of D such that, setting $c_s = \min(C_s)$, then the sequence $(c_s : s < q)$ is strictly increasing.

Notice that $a_i \notin C_l$ for any i, l , and $a_i < a_{ij}$ for all i, j such that a_{ij} is defined. For $i < m$, define k_i as the least $k < q$ such that $a_i < c_k$. Then

$$k_i \leq \sum_{j=0}^{i-1} a_j \leq \sum_{j=0}^{i-1} d_j.$$

I now proceed to find an upper bound l_s on the size of C_s beyond which one is guaranteed to find a min-homogeneous set of size 5. The value of n displayed above is obtained by first observing that

$$[m, n] = \{m\} \cup \{a_i : i < m\} \cup \bigcup_{s=0}^{q-1} C_s,$$

so $n - m + 1 = m + 1 + \sum_{s=0}^{q-1} |C_s|$, and then setting $n \geq 2m + \sum_s l_s + 1$.

To find l_s , notice that

$$[m, c_s] \subseteq \{m\} \cup \{a_i : a_i < c_s\} \cup \bigcup_0^{s-1} C_j \cup \{c_s\},$$

so $c_s - m + 1 \leq 2 + (i + 1) + \sum_0^{s-1} |C_j|$, where $s \in [k_{i-1}, k_i)$, or

$$c_s \leq m + 1 + (i + 1) + \sum_0^{s-1} |C_j|.$$

Let $C'_s = C_s \setminus \{c_s\}$. If

$$|C'_s| \geq (m + 2) + (i + 1) + \sum_0^{s-1} |C_j|,$$

then $f(c_s, \cdot) \upharpoonright C'_s$ is not injective, so there are $d < e$ in C'_s such that $f(c_s, d) = f(c_s, e)$ and $\{m, a_j, c_s, d, e\}$ is min-homogeneous, where $j \leq i$ is chosen so that $C_s = B_{jk}$ for some k .

This gives the upper bound $l_s \leq (m + i + 3) + \sum_0^{s-1} l_j$ so, by a straightforward induction,

- $l_s \leq 2^s(m + 3)$ for $s < d_0$,
- $l_s \leq 2^{s-d_0}((m + 3)(2^{d_0} - 1) + (m + 4))$ for $d_0 \leq s < d_0 + d_1$,
- and, in general, for $i < m$, and $\sum_{j=0}^{i-1} d_j \leq s < \sum_{j=0}^i d_j$, we have

$$l_s \leq 2^{s-d_{i-1}}((\dots((m + 3)(2^{d_0} - 1) + (m + 4))(2^{d_1} - 1) + \dots)(2^{d_{i-1}} - 1) + (m + 3 + i)).$$

These upper bounds give the value of n that I started with, and the claimed inequality $g(5, m) \leq n$ follows. In the case $m = 2$, it implies

$$\begin{aligned} g(5, 2) &\leq (2 \times 2 + 1) + (2 + 3)(2^{2+1} - 1) + (5(2^3 - 1) + 6)(2^{g(4,3)} - 1) \\ &= 40 + 41(2^{37} - 1) = 41 \times 2^{37} - 1. \end{aligned}$$

This completes the proof. □

I conclude with some questions:

Question 7.2. *Is $G(n + 1, m) > g^m(n, m + 1)$ for $n > 4$?*

Question 7.3. *Is $2^m(m + 1) \leq g(4, m)$ for all m ?*

The proofs of Theorems 6.1 and 7.1 suggest that to fully understand g requires to solve the following question:

For any n, m and regressive $f : [m, g(n, m)]^{[2]} \rightarrow \mathbb{N}$, set

$$k_f = \min\{\min(H) : H \in [m, g(n, m)]^{[m]} \text{ is min-homogeneous for } f\},$$

and let

$$k(n, m) = \max\{k_f : f : [m, g(n, m)]^{[2]} \rightarrow \mathbb{N} \text{ is regressive}\}.$$

Question 7.4. *What is the rate of growth of the function $k(n, m)$?*

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