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The Hyperreals: Do You Prefer Non-Standard Analysis Over Standard Analysis?

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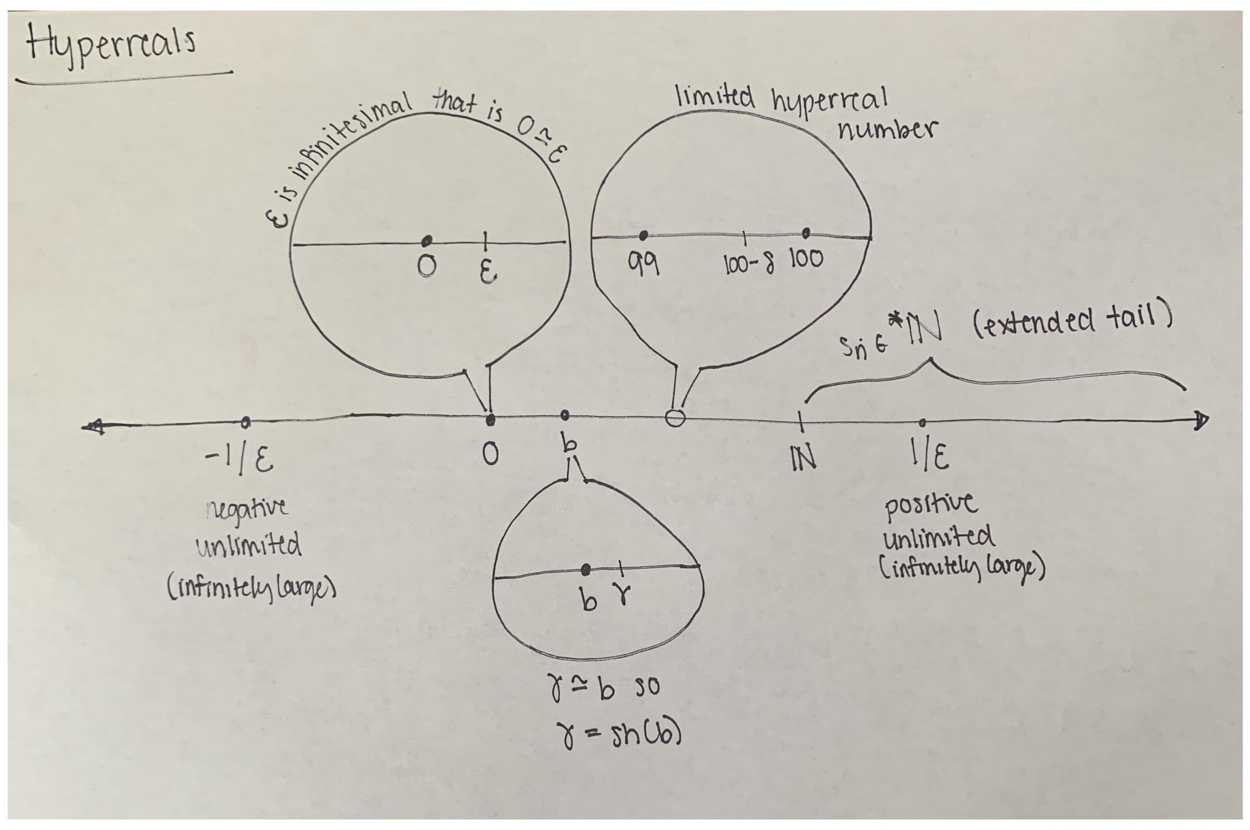
Overview

The hyperreal number system $^*\mathbb{R}$ forms an ordered field that contains $\mathbb{R}$ as a subfield as well as infinitely many large and small numbers. A number is defined to be infinitely large if $|\omega| > n$ for all $n = 1, 2, 3, \ldots$ and infinitely small if $|\epsilon| < \frac{1}{n}$ for all $n = 1, 2, 3\ldots$ This number system is built out of the real number system analogous to Cantor's construction of $\mathbb{R}$ out of $\mathbb{Q}$. The new entities in $^*\mathbb{R}$ and the relationship between the reals and hyperreals provides an appealing alternate approach to real (standard) analysis referred to as nonstandard analysis.

This approach is based around that principle that if a property holds for all real numbers then it holds for all hypereal numbers, known as the transfer principle. By only using the fact that $^*\mathbb{R}$ is an ordered field that has $\mathbb{R}$ as a subfield, includes unlimited numbers $N \in^* \mathbb{N} - \mathbb{N}$ and satisfies the transfer principle the topics of analysis can be explored.

Definitions

- A hyperreal number b is:
  - *limited* if $r < b < s$ for $r, s \in \mathbb{R}$
  - *unlimited* if $r < b$ (positive) or $b < r$ (negative) for $r \in \mathbb{R}$
  - *infinitesimal* if $0 < b < r$ (positive), $r < b < 0$ (negative) or 0 for $r \in \mathbb{R}$
- *Transfer Principle*: A defined $\mathcal{L}_{\mathbb{R}}$-sentence ϕ is true iff $^*\phi$ is true
- Hyperreal b is *infinitely close* to hyperreal c, $b \simeq c$, if $b - c$ is infinitesimal
- *Halo*$(b) = c \in^* \mathbb{R} : b \simeq c$
- Hyperreals b, c are of *limited distance apart*, $b \sim c$, if $b - c$ is limited
- *Galaxy*$(b) = c \in^* \mathbb{R} : b \sim c$
- *Extended tail* of s is the collection $s_n : n \in^* \mathbb{N}_\infty$ such that $^*\mathbb{N}_\infty$ is the set of unlimited hyper-naturals



Theorems

- *Shadow Theorem*: Every limited hyperreal b is infinitely close to exactly one real number called the **shadow** of b, $sh(b)$
- *Infinitely Close Theorem*: f is continuous at the real point c if and only if $f(x) \simeq f(c)$ for all $x \in^* \mathbb{R}$ such that $x \simeq c$, i.e. $f(hal(c)) \subseteq hal(f(c))$
- *Intermediate Value Theorem (IVT)*: If the real function f is continuous on the closed interval $[a, b]$ in $\mathbb{R}$, then for every real number d strictly between $f(a)$ and $f(b)$ there exists a real number $c \in (a, b)$ such that $f(c) = d$
  -Take a look at the two proofs of the IVT below. Which do you prefer, the non-standard or standard proof?

IVT Non-Standard Proof

For each limited $n \in \mathbb{N}$, partition $[a, b]$ into n equal subintervals of width $(b - a)/n$. Thus these intervals have endpoints $p_k = a + k(b - a)/n$ for $0 \le k \le n$. Then let s_n be the greatest partition point whose f-value is less than d. The set

$$\{p_k : f(p_k) < d\}$$

is finite and nonempty. Hence s_n exists as the maximum of this set and is given by some p_k with $k < n$.

Now for all $n \in \mathbb{N}$ we have

$$a \le s_n \le b \text{ and } f(s_n) \le d \le f(s_n + (b - a)/n)$$

so by the transfer principle, these condition hold for all $n \in^* \mathbb{N}$

To obtain an infinitesimal width partition, choose an unlimited hypernatural N. Then s_N is limited as $a \le s_N \le b$, so has a shadow $c = sh(s_N) \in \mathbb{R}$. Note that by transfer principle, s_N is a number of the form $a + K(b - a)/N$ for some $K \in^* \mathbb{N}$. But $(b - a)/N$ is infinitesimal so s_N and $s_N + (b - a)/N$ are both infinitely close to c. Since f is continuous at c and c is real it follows by theorem 7.1.1 that $f(s_N)$ and $f(s_N + (b - a)/N$ are both infinitely close to $f(c)$. But

$$f(s_N) < d < f(s_N + (b - a)/N)$$

so d is also infinitely close to $f(c)$. Since $f(c)$ and d are both real, they must then be equal.

IVT Standard Proof

Let

$$S = \{x \in [a, b] : f(x) < 0\}$$

Then $a \in S$ since $f(a) < d$ thus S is nonempty. Clearly b is an upper bound for S. Therefore, by Least Upper Bound Property, S has a least upper bound L. We claim that $f(L) = d$. If not, set $r = f(L)$ and assume $r > d$.

Since f is continuous there exists a number $\delta > d$ sch that

$$|x - L| < \delta \Rightarrow |f(x) - f(L)| = |f(x) - r| < \tfrac{1}{2}r$$

Equivalently,

$$|x - L| < \delta \Rightarrow \tfrac{1}{2}r < f(x) < \tfrac{3}{2}r$$

The number $\frac{1}{2}r$ is positive so we conclude that

$$L - \delta < x < L + \delta \Rightarrow f(x) > d$$

By definition of L, $f(x) \le d$ for all $x \in [a, b]$ such that $x > L$, and thus $f(x) \le d$ for all $x \in [a, b]$ such that $x > L - \delta$. Thus $L - \delta$ is an upper bound for S. This is a contradiction since L is the least upper bound of S and it follows that $r = f(L)$ cannot satisfy $r > d$. Similarly, r cannot satisfy $r < d$. We concluded that $f(L) = d$ as desired.