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Pythagorean Theorem Area Proofs

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This composition is intended to walk the reader through four proofs of the pythagorean theorem that are based on area. It could be used in a classroom to solidify the pythagorean theorem after studying Neutral and Euclidean Geometries.

In order to use area to prove anything it must be defined. First, remember the two dimensional plane is the set of all possible points. Any line divides the plane into two distinct halves, or half planes. Given any triangle ABC, choose the half plane of AB that contains C, the half plane of BC that contains A, and the half plane of AC that contains B. The set of all points in the intersection of these half planes is considered the **interior of a triangle**. The **triangular region** is the union of the interior points and the points that make up the triangle itself. A **polygonal region** is a subset of the plane that can be written as a finite set of triangular regions, where intersections of any two triangular regions are contained in an edge of each one, with no overlap of the interiors. The triangle is one dimensional and has no area while the triangular region is two dimensional and has a specific area.

Next the proofs will require the following postulates and theorems. Theorems 1-5 are area theorems, 6-8 come from Neutral Geometry.

Neutral Area Postulate (NAP):
1. Congruence: two congruent triangles have equal triangular region areas.
2. Additivity: If *R* is the area of two non overlapping triangular regions *a* and *b*, then the area of *a* plus the area of *b* equals *R*.

Necessary Theorems:
1. If ABC is a triangle and D lies on AB, then the triangular region of ABC is equal to the sum of the triangular regions of ADC and BDC. ADC and BDC are non overlapping.
2. If ABCD is a convex quadrilateral, then the area of ABCD is equal to the sum of the triangular regions of ACD and ABC or ABD and BCD. ACD and ABC as well as ABD and BCD are non overlapping.
3. The area of a rectangle ABCD = AB*BC
4. The area of a triangle ABD = 1/2 *base*height,* where base is the length of any side and height is the length of a perpendicular from the base to the vertex.
5. If two triangles are similar then the ratio of their areas is equal to the square of the ratio of the lengths of any two corresponding sides.
6. SAS, if two angles share corresponding side, angle and side, then the two angles are congruent.
7. The sum of the angles of any given triangle is 180°
8. If an altitude is drawn from the right angle of any right triangle, then the two triangles formed are similar to the original triangle and all three triangles are similar to each other.

Overview of Proofs:

First Proof: Euclid

The first proof uses Theorems 2, 3, 4, 5 and NAP 1. It begins with the construction of squares with side lengths corresponding to the sides of a right triangle. It then cleverly divides the largest square, c^2, into two parts and proves that each part has equivalent area to one of the smaller two squares, a^2 and b^2.

Second Proof: Bhaskara

This proof uses Theorems 2, 3 and NAP 2. A polygon is constructed by stacking two squares with side lengths a and b on top of each other. Then it divides the polygon into rectangles with side lengths a and b. Finally, the rectangles are dissected into triangular regions that are rearranged to form a square with side lengths c. This proof would make an excellent hands on activity for students as they can draw, cut and rearrange the the polygonal regions from start to finish.

Third Proof: Pythagoras

This proof uses NAP 2 and Theorems 6 and 7. A square is strategically dissected into triangles then rearranged to prove the pythagorean theorem. This proof could also be structured as a hands on activity where students are given a square and asked to cut and rearrange it to fit into a given template of the same size.

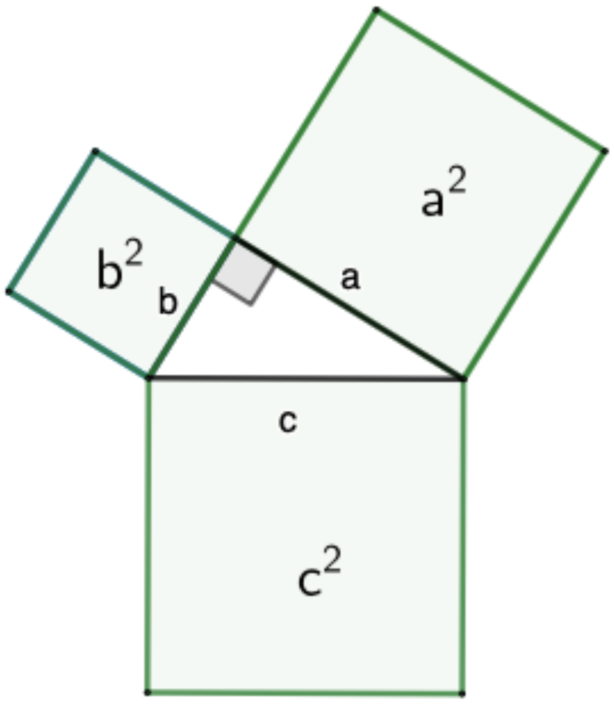
Fourth Proof: Euclid

The final proof shown relies on Theorems 1,5 and 8. It divides any given right triangle into similar triangles, then uses equivalent ratios to prove $a^2 + b^2 = c^2$. This last proof is the only of the four that requires setting up and solving an algebraic equation and would be an excellent exercise for students ready for applied algebra practice problems.

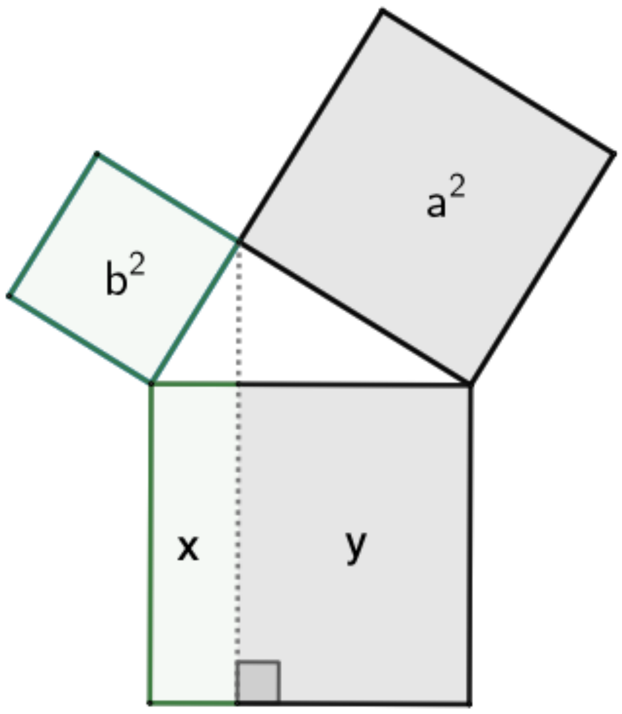
References

Venema, G. (2012). *Foundations of geometry*. Boston, Massachusetts: Pearson. Print.

First Proof: Euclid



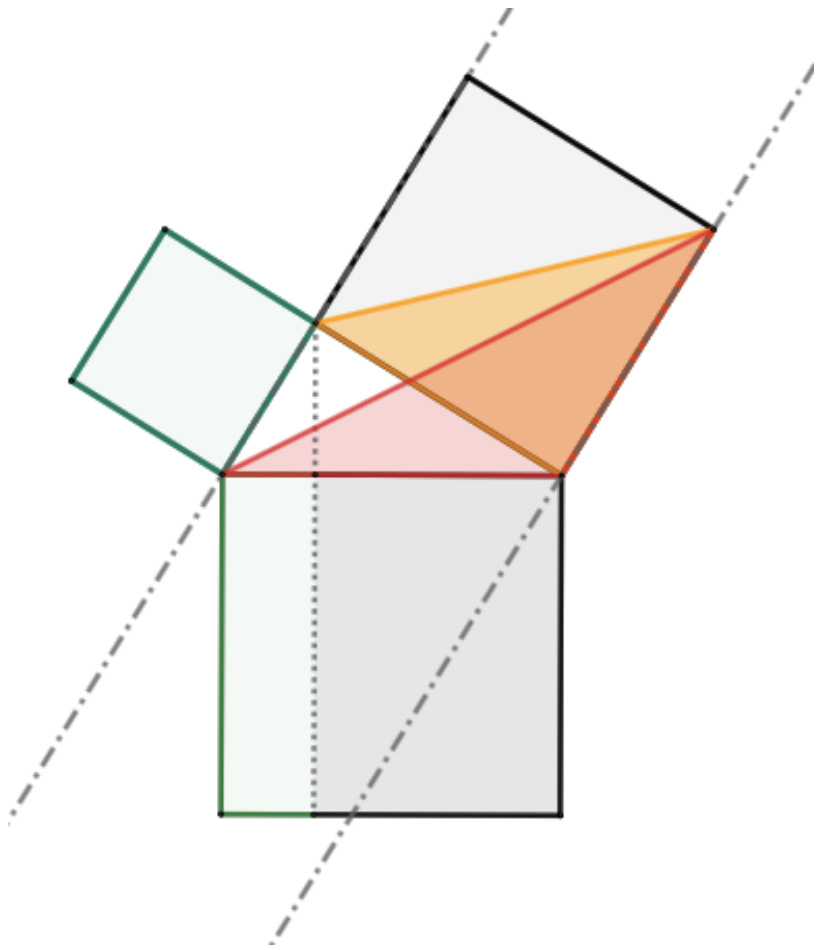
First, from a right triangle, squares are constructed from each side length to represent the area's a^2, b^2 and c^2. (Theorem 3)



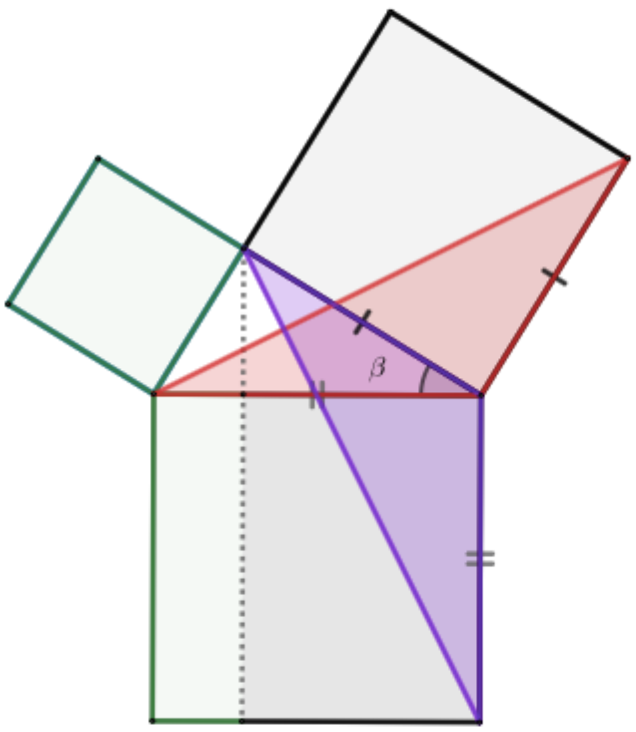
Then a perpendicular line is dropped from the vertex of the right angle to the farthest side of the c^2 square.

This splits c^2 into two rectangles, x and y.

The claim is that the area of a^2 equals the area of y and the area of b^2 equals the area of x, thus $a^2 + b^2 = c^2$.

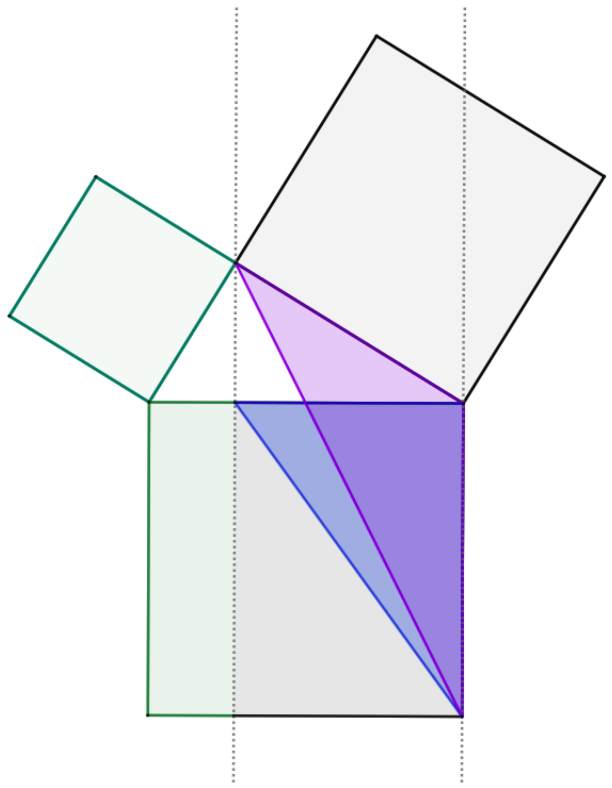


The orange triangle and the red triangle share a base and have the same height, so have the same area. (Theorem 4)

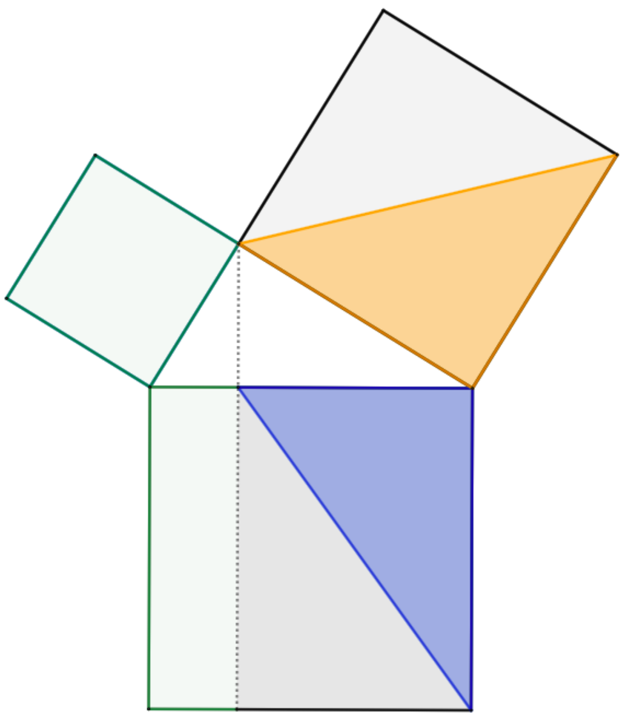


The red triangle and the purple triangle have sides of corresponding length as indicated, and both obtuse angles are 90° plus β.

So the red and blue triangles are congruent and have the same area. (Theorem 5 and NAP 1)



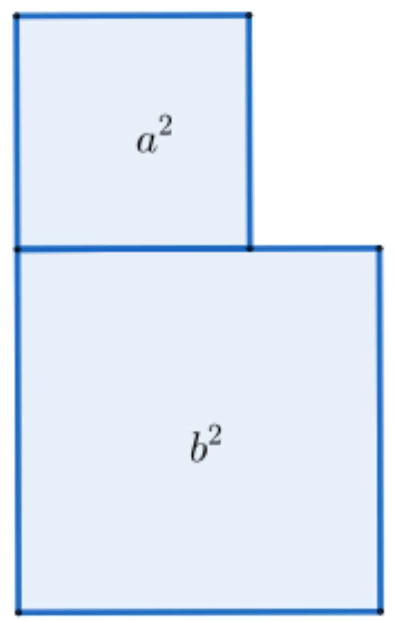
The purple and blue triangles share a base and height so have the same area. (Theorem 4)



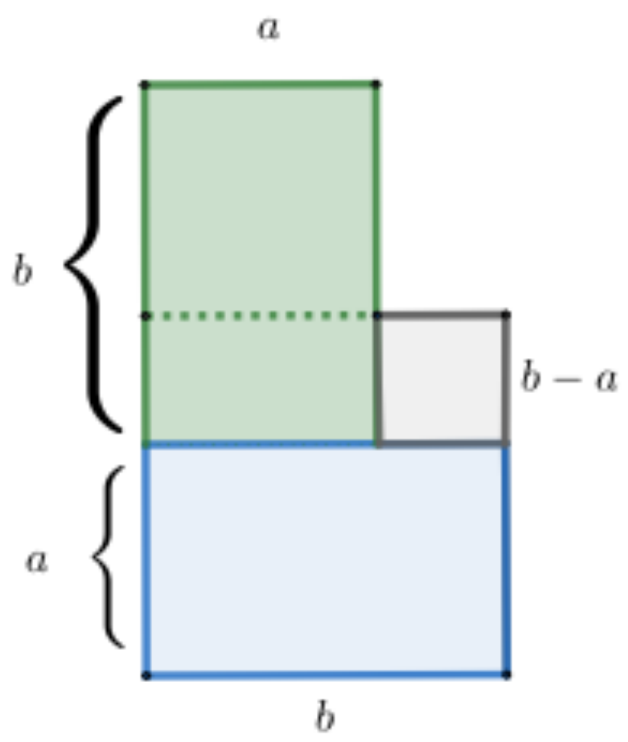
It follows that the orange and blue triangles have the same area and so the area of a^2 is equal to y. (Theorem 2)

By a parallel argument, the area of b^2 is equal to x.

Second Proof: Bhaskara



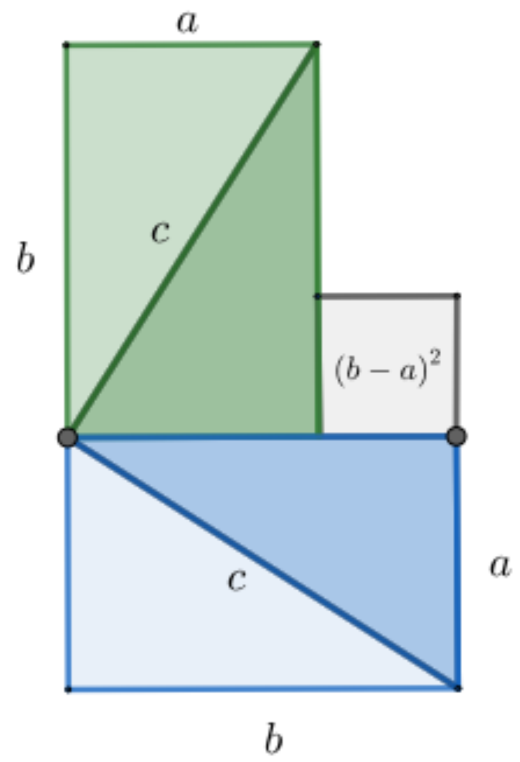
Begin with squares a^2 and b^2 stacked.
(Theorem 3)



From the bottom of the stack measure up length a and construct the blue rectangle which will be length a and width b.

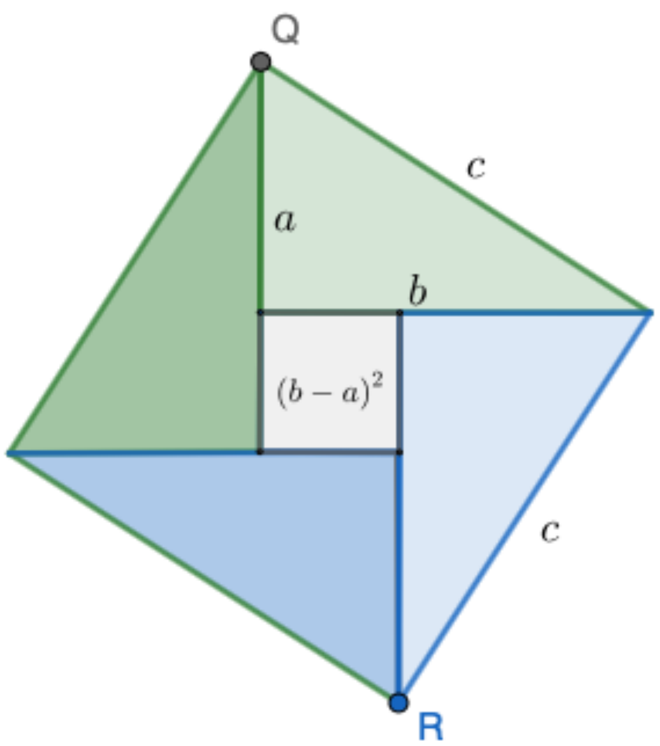
Then construct the green rectangle which is length b and width a.

The remaining area is the grey square with area $(b - a)^2$.



Construct diagonals in the green and blue rectangles creating congruent triangles with hypotenuse $\overline{c}$.

(Theorem 2)



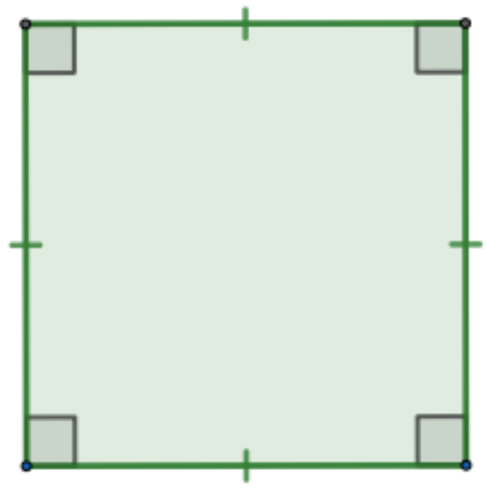
Rotate the light green square clockwise about point Q.

Rotate the light blue square counterclockwise about point R.

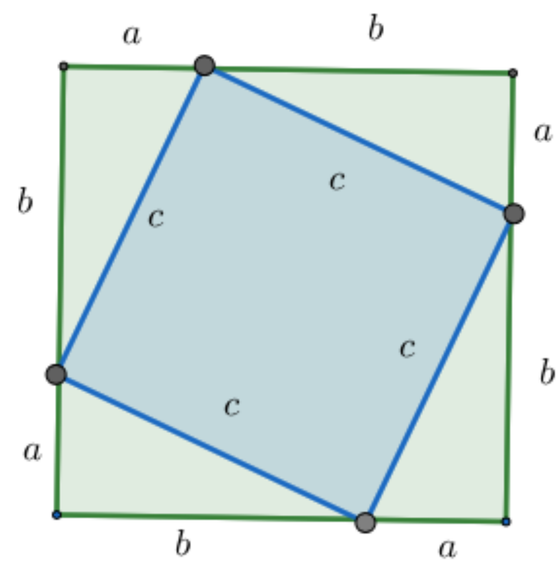
The square with area c^2 is formed from the 4 triangles and the square $(b-a)^2$.

By rearrangement, $a^2 + b^2 = c^2$. (NAP 2)

Third Proof: Pythagoras



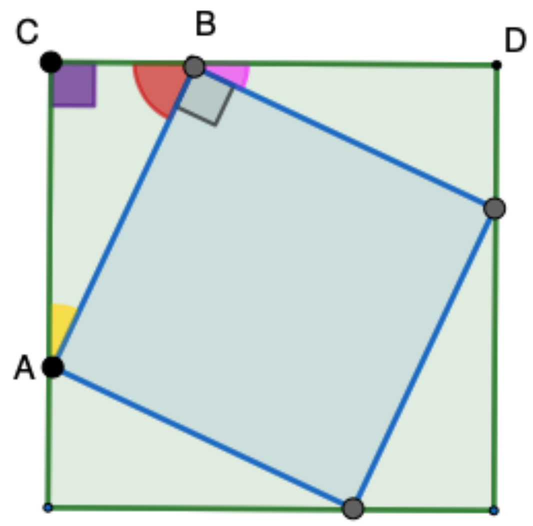
Begin with a square with sides of any given length.



Divide each side of the square into two lengths, a and b.

Connect the points to form the four green triangles.

By Theorem 6 the green triangles are congruent.

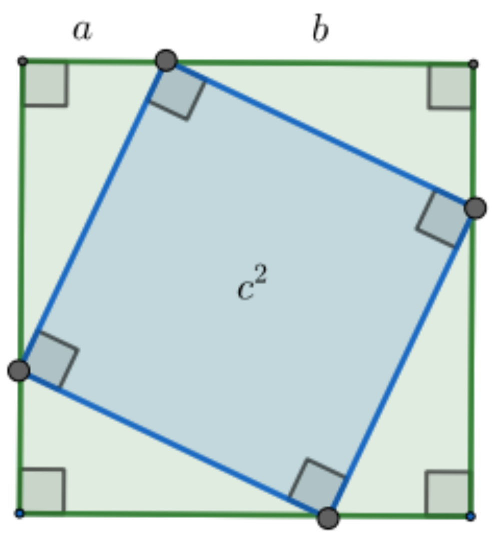


By theorem 7 the sum of the yellow, red, and purple angles sum to 180°.

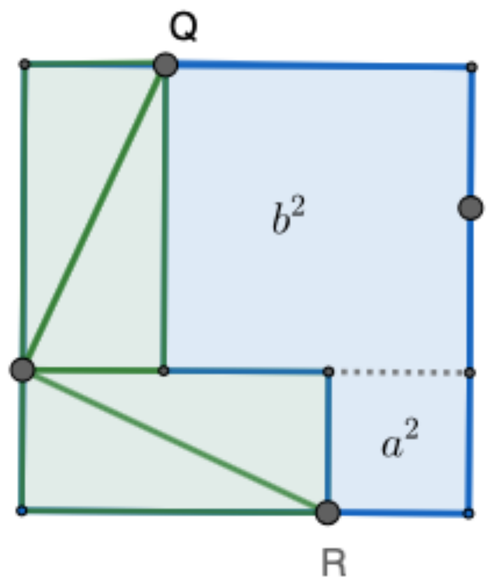
It follows that the green and red angles sum to 90°, as the purple angle is right.

Because the green triangles are congruent, the yellow and pink angles are congruent, so the red and pink angles sum to 90° also.

The red pink and gray angles form a straight line, 180°, so the gray angle equals 90°.

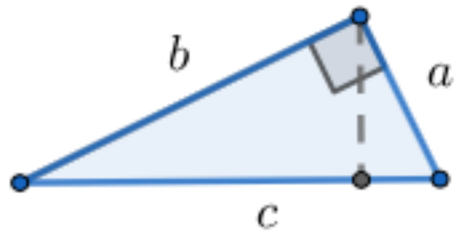


By a symmetric argument, all four angles of the blue polygon are 90°, thus the blue polygon is a square with side lengths c and area c^2 .



Rotate the green triangles on the right side about points Q and R forming two green rectangles. The blue area is rearranged into two squares of area a^2 and b^2 , providing the desired result $a^2 + b^2 = c^2$. (NAP 2)

Fourth Proof: Euclid

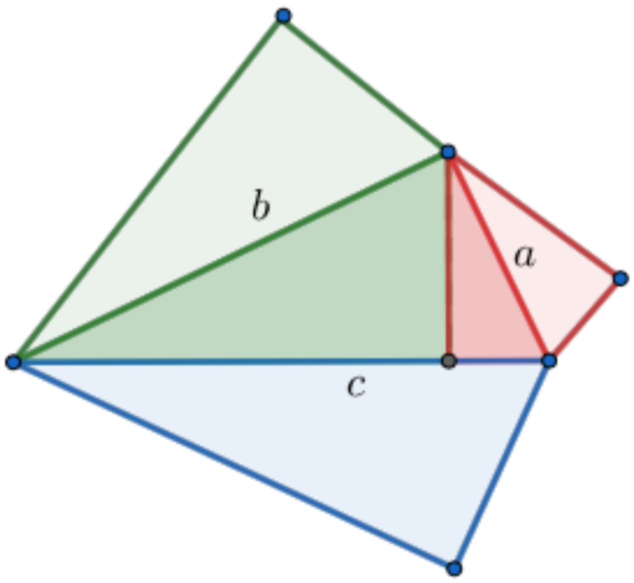


Begin with a right triangle with hypotenuse c and legs a and b.

Drop a perpendicular from the vertex of the right angle to the hypotenuse, which creates the altitude of the triangle.

This divides the blue triangle into two smaller similar triangles, shown in green and red.

By Theorem 8 the green and red triangles are similar to each other and the original.



Reflect the green and red triangles across the legs of the original, and reflect the original triangle across the hypotenuse.

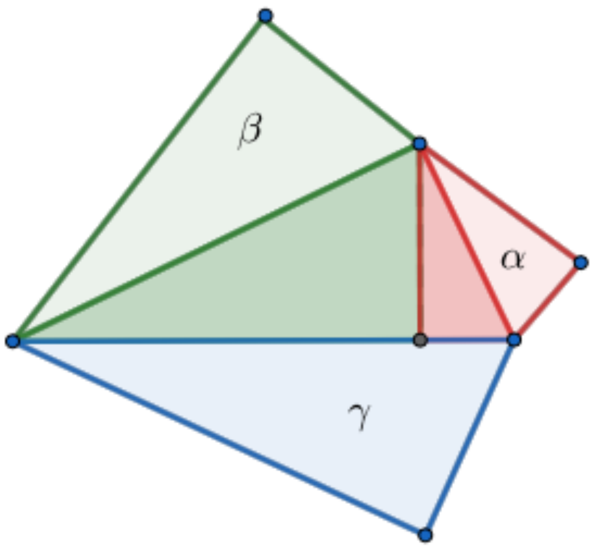
Name the areas of the red, green and blue triangles α, β and γ respectively.

By theorem 1, $\alpha + \beta = \gamma$.

From Theorem 5 we can make the following conclusion:

$\overline{a}$ and $\overline{b}$ are corresponding sides of the red and green triangles, so $\frac{\alpha}{\beta} = (\frac{a}{b})^2$ and $\alpha = (\frac{a}{b})^2\beta$.

$\overline{c}$ and $\overline{b}$ are corresponding sides of the Blue and Green triangles so $\frac{\gamma}{\beta} = (\frac{c}{b})^2$ so $\gamma = (\frac{c}{b})^2\beta$.



Rewrite $\alpha + \beta = \gamma$:

$(\frac{a}{b})^2\beta + \beta = (\frac{c}{b})^2\beta$ (divide by β)

$(\frac{a}{b})^2 + 1 = (\frac{c}{b})^2$ (multiply by b^2)

$a^2 + b^2 = c^2$