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An Exploration of the Chromatic Polynomial

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1 Introduction

In 1912, George Birkhoff was studying the Four Color Problem, and in doing so introduced the concept of the chromatic polynomial [2]. While this did not end up directly contributing to proving that every map could be colored with four colors such that no region shares a border with another region of the same color, the chromatic polynomial has been found to have some very interesting properties. In this paper, it will be our goal to examine some of these properties and use them to determine information about their corresponding graphs.

1.1 Definitions

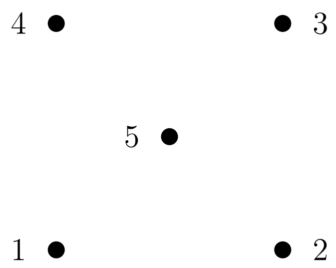
We begin by noting some preliminary definitions that we will use throughout this paper.

Definition 1.1. A **graph** G is a set of vertices and edges, where each edge is connected to two vertices. We say two vertices are **adjacent** if they are connected by an edge.

Definition 1.2. We define the **order** of the graph n as the number of vertices and the **size** of the graph m as the number of edges.

Definition 1.3. A **connected component** of a graph G is a connected subgraph of G that is not connected to any other vertex in G.

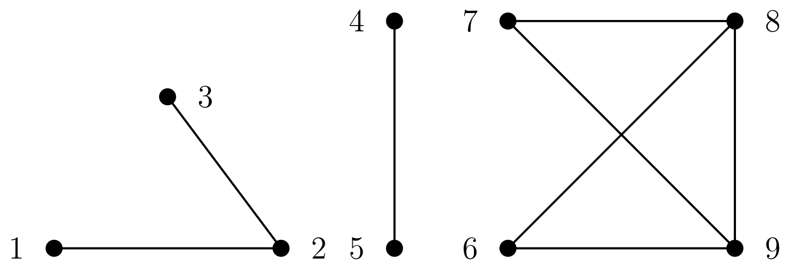
Definition 1.4. The **empty graph** on n vertices is a graph with n vertices and no edges.



The empty graph on 5 vertices

Definition 1.5. A **proper coloring** of a graph is a coloring in which no adjacent vertices share the same color.

To illustrate these definitions, consider the following graph G.



Note the following:

- G has order $n = 9$ and size $m = 8$.
- G has 3 connected components, defined by the sets of vertices {1,2,3}, {4,5}, and {6,7,8,9}.
- An example of a proper coloring of G is coloring the vertex set {1,3,4,6,7} with one color, {2,5,9} with another color, and {8} with a third color. In this way, no vertex shares an edge with any other vertex with the same color.

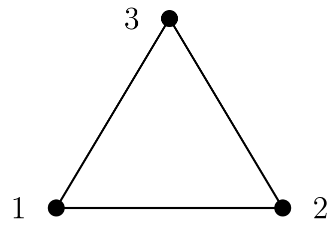
2 The Chromatic Polynomial

2.1 Deletion-Contraction

Consider the following definition:

Definition 2.1. The **chromatic polynomial** of a graph, denoted $P(G, x)$ is a function which gives the number of proper colorings of a graph G using x colors. We will see in Theorem 2.6 that this function is, in fact, a polynomial in x.

For a simple example of how this works, consider the following graph:



Suppose we are given a set of x colors with which to color this graph. We can systematically create a proper coloring in the following way:

1. First color vertex 1. At this point, we have x options of colors we can use to do this.

2. When we go to color vertex 2, we only have $x - 1$ options since it is adjacent to vertex 1.

3. Since vertex 3 is adjacent to both vertex 1 and vertex 2, we cannot use either of the colors we have already used, so we only have $x - 2$ options.

Because of the method in which this proper coloring is produced, we can easily see that the number of ways to color this graph using x colors, or the chromatic polynomial, is

$$P(G, x) = x(x - 1)(x - 2).$$

However, calculating the chromatic polynomial of a graph is usually not this straightforward. The choice of how to color a graph so as to minimize the number of colors used can actually be quite complicated. Thus, we need a better method by which we can consistently obtain the chromatic polynomial of a graph. To do this, we will use the Deletion-Contraction theorem.

Definition 2.2. Consider a graph G with an edge e and its associated vertices u and v. Let $G - e$ be the graph G without e, and let G/e be the graph G where e is removed and u and v are combined into a single vertex. We call $G - e$ the **deletion** of e and G/e the **contraction** of e.

Theorem 2.3 (Deletion-Contraction)**.** *For a graph G and one of its edges e, the chromatic polynomial of G is:*

$$P(G, x) = P(G - e, x) - P(G/e, x).$$

Proof. Consider a graph G and one of its edges e, and let u and v be the two vertices connected to e. To be a proper coloring, it must be the case that u and v are different colors. In the graph G/e, u and v are represented by a single vertex, and thus, they have the same color. In the graph $G - e$, u and v are no longer adjacent, and thus, they can either be colored with the same color or with different ones. Thus, the number of colorings of $G - e$ is the same as the total number of colorings of G and G/e. Thus,

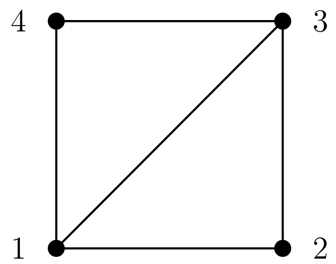
$$P(G - e, x) = P(G, x) + P(G/e, x).$$

Simply rearranging the terms of this equality gives:

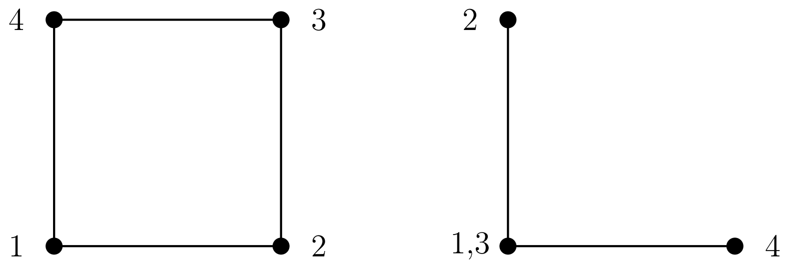
$$P(G, x) = P(G - e, x) - P(G/e, x).$$

□

In order to see how Deletion-Contraction works, consider the following graph G:



Now suppose that the edge e which we want to perform Deletion-Contraction on is the one that connects vertices 1 and 3. Then for the deletion, we simply remove e from G to get the graph $G - e$. For the contraction G/e, we once more remove e, but we must also combine 1 and 3 into the same vertex while maintaining all connections that both vertices originally had with 3 and 4. When we perform this contraction, we also remove the multiple edges that would have been created. So the deletion and contraction look like this:



$G - e$ $\qquad$ G/e

Now, using Theorem 2.1, we know that our chromatic polynomial for G is given by the difference of the chromatic polynomials for $G-e$ and G/e. As we will observe in Section 3.2, $P(G - e, x) = (x - 1)^4 + (x - 1)$ and $P(G/e, x) = x(x - 1)^2$. Thus, we have that

$$P(G, x) = (x - 1)^4 + (x - 1) - x(x - 1)^2.$$

Now there are clearly much more complicated examples where it takes more than one Deletion-Contraction step to obtain graphs for which we know the chromatic polynomial. In this case, we continue to recursively apply Theorem 3.2 as long as necessary.

It is important to note that when we perform Deletion-Contraction, if G is not a simple graph, it is possible that two vertices u and v could be connected by more than one edge. However, the only reason u and v must be different colors in order for G to have a proper coloring is that they are connected. Thus, having multiple edges between any two vertices simply provides redundant information, so we can handle this by beginning any Deletion-Contraction process on a graph by first deleting any multiple edges so that at exactly one edge connects any set of adjacent vertices.

Also note that when we perform a contraction of u and v, as seen in our example, it is entirely possible that the process will create either loops or multiple edges. But if we consider loops, we find that any vertex containing a loop is adjacent to itself, and thus, that the graph cannot have a proper coloring. However, in the context of graph coloring, we want to be able to color every vertex in any given graph. For this reason, we will make the decision to simply delete any loops that arise. We can also once again delete any multiple edges that result from the contraction step.

Throughout this paper, we will assume that we are dealing exclusively with simple graphs and that the deletion of unnecessary edges is completed after each Deletion-Contraction step. However, it is easy to see that we can also find the chromatic polynomial of a multigraph by deleting extra edges at the beginning and treating it as a simple graph.

2.2 Calculating the Chromatic Polynomial

As we have seen, we can always use Deletion-Contraction to find the chromatic polynomial of any given graph. By doing this, we are reducing the graph in each step to one with fewer edges and, in the case of a contraction, fewer vertices. As shown in the example in Section 2.1, once we have a graph for which we already know the chromatic polynomial, there is no need to continue the process of Deletion-Contraction on it. We can simply simply use the chromatic polynomials that we already know.

However, if we don't know the chromatic polynomial of any intermediate graph, the Deletion-Contraction process would produce a series of empty graphs. So if we know the chromatic polynomial of an empty graph, we will always be able to determine the chromatic polynomial of a given graph using Deletion-Contraction.

In order to find this chromatic polynomial, we notice that the empty graph on more than one vertex is not connected. As it turns out, it is simple to find the chromatic polynomial of a disconnected graph in terms of its connected components, and this concept will extend directly to finding the chromatic polynomial of an empty graph.

Theorem 2.4. *Let G be a graph and let $G_1, G_2, \dots, G_k$ be connected components of G. Then*

$$P(G,x) = P(G_1,x) \cdot P(G_2,x) \cdots P(G_k,x).$$

Proof. Since the connected components of a graph are disjoint, given a set of x colors, each component can be colored with the same x colors independently of the colorings of the other components. So we can find the number of ways to color G by multiplying the number of colorings of each component together. □

Using this tool, we can now find the chromatic polynomial of the empty graph.

Theorem 2.5. *The chromatic polynomial of the empty graph is $P(G,x) = x^n$.*

Proof. Let G be the empty graph on n vertices and a set of x colors. Since there are no adjacent vertices, it follows that G consists of n connected components. Since each of these components consists of a single vertex with no edges, each component can be colored in x ways. So we have

$$P(G, x) = x \cdot x \cdots x = x^n.$$

□

We will see that being able to find the chromatic polynomial of a graph in terms of its components is extremely useful, and it often simplifies the Deletion-Contraction process a great deal. While it is definitely useful to be able to reduce a graph down to a set of empty graphs, which have very simple chromatic polynomials, this is not always very efficient. Notice that every time we use a Deletion-Contraction step on our current set of graphs, we double the amount of graphs we are working with because each has both a deletion and a contraction. For graphs with a large number of edges, this process would take a very long time to complete. So it is incredibly useful to be able to simplify the process as much as possible and end it as early as we can. Besides simply reducing a graph into smaller graphs with disjoint components, many families of graphs have well-known chromatic polynomials, and we can use these to simplify the process as well. We will see some examples of these families of graphs in Section 3.2.

At this point, it is also important to note the following:

Theorem 2.6. *The chromatic polynomial $P(G, x)$ of a graph G is always a polynomial in x.*

Proof. Consider a graph with 0 edges and some number of vertices n. Then, as we already showed, the chromatic polynomial is x^n, which is a polynomial in x.

Now, using strong induction, assume that all graphs with fewer than m edges have chromatic polynomials in x, and let G be a graph with m edges. Then, by Deletion-Contraction, using some arbitrary edge e, the chromatic polynomial is

$$P(G, x) = P(G - e, x) - P(G/e, x).$$

Since $G - e$ has exactly $m - 1$ edges and G/e has strictly fewer than m edges, the chromatic polynomials of both are polynomials in x. Since the chromatic polynomial of G is formed by simply subtracting $P(G - e, x)$ and $P(G/e, x)$, this means that $P(G, x)$ is also a polynomial in x. Thus, our hypothesis holds for any given graph. □

This property is essential because it establishes the fact that, no matter how large or complicated our graph gets, we will never need more than one variable to express its chromatic polynomial. This fact, especially with assistance from the Deletion-Contraction method, is what makes it so simple to guarantee that we can find a chromatic polynomial of any possible graph.

3 Properties of the Chromatic Polynomial

3.1 Properties of the Coefficients

Even though graphs can be vastly different, and thus have very different chromatic polynomials, it turns out that they all have some similar characteristics. In fact, we will see that there is quite a bit of information about a graph that can be determined by its chromatic polynomial. Many of these properties were introduced by Read in 1968 [7], and have become very well-known since then. We will now discuss some of the more interesting patterns that appear in chromatic polynomials in general.

Given a graph, one of the first things that one is likely to take note of is the number of vertices and edges that it has. In fact, these are both very accessible pieces of information from the chromatic polynomial. The next three theorems explain why this is true, as well as introducing some other interesting properties of the coefficients of chromatic polynomials.

Theorem 3.1. *For any graph G, the degree of its chromatic polynomial $P(G,x)$ is the number of vertices in G.*

Proof. Consider an empty graph on n vertices. We have already shown that the chromatic polynomial of this graph is $P(G,x) = x^n$, and since this polynomial has degree n, our hypothesis holds when $m = 0$.

We proceed by strong induction. Assume the hypothesis holds for all graphs with m or fewer edges, and let G be a graph with n vertices and $m+1$ edges. We know that the chromatic polynomial of G is $P(G,x) = P(G-e,x) - P(G/e,x)$. As $P(G-e,x)$ is the chromatic polynomial of G with an edge deleted, $G-e$ still has n vertices and m edges. Then it must be that the degree of $P(G-e,x)$ is n. Also, since $P(G/e,x)$ is the chromatic polynomial of G with an edge contracted, G/e has $n-1$ vertices and fewer than $m+1$ edges. Because this is the case, we know that $P(G/e,x)$ has degree $n-1$. Then $P(G,x)$ is a degree $n-1$ polynomial subtracted from a degree n polynomial. Since this subtraction has no way to cancel out the degree n term in $P(G-e,x)$ and no term of a higher degree than n can appear, it is necessarily the case that $P(G,x)$ is also a degree n polynomial. So our hypothesis is true. □

Theorem 3.2. *Let G be a graph with chromatic polynomial $P(G,x)$. Then the following are true:*

- *The leading coefficient of $P(G,x)$ of any graph is 1.*
- *The absolute value of the coefficient of the x^{n-1} term in $P(G,x)$ is the number of edges.*
- *The first coefficient of $P(G,x)$ is positive, and all terms alternate in sign.*
- *All coefficients are integers.*
- *If the coefficient of x^k is 0, then so is the coefficient of x^{k-1}.*

Proof. Let G be the empty graph on n vertices. Then the chromatic polynomial is $P(G,x) = x^n$. We can easily see that the leading coefficient is 1, which is an integer. Also, all other coefficients are 0, so the coefficient of x^{n-1} is indeed equal to the number of edges in G. Because there is only one term, it has the property that terms alternate in sign. Finally, note that our first coefficient equal to 0 is the x^{n-1} term, and all subsequent terms also have a coefficient of 0. Then our conclusion holds for $m = 0$.

Now, using strong induction on the number of edges, assume that each of these properties hold for all graphs with m or fewer edges, and let G be a graph on n vertices with $m+1$ edges. We know $P(G,x) = P(G-e,x) - P(G/e,x)$. Because $G-e$ has one fewer edge than G and the same number of vertices, by our inductive hypothesis, the chromatic polynomial is of the form

$$P(G-e,x) = x^n - mx^{n-1} + c_1x^{n-2} - c_2x^{n-3} + \cdots$$

where each c_i is a nonnegative integer. Similarly, since G/e has at least one fewer edge than G and exactly one fewer vertex than G, it follows that the chromatic polynomial has the form

$$P(G/e,x) = x^{n-1} - m_2x^{n-2} + d_1x^{n-3} - d_2x^{n-4} + \cdots$$

where m_2 is the number of edges in G/e and each d_i is a nonnegative integer. Then we have:

$$\begin{aligned} P(G,x) &= P(G-e,x) - P(G/e,x) \\ &= (x^n - mx^{n-1} + c_1x^{n-2} - c_2x^{n-3} + \cdots) - (x^{n-1} - m_2x^{n-2} + d_1x^{n-3} - d_2x^{n-4} + \cdots) \\ &= x^n - (m+1)x^{n-1} + (c_1 + m_2)x^{n-2} - (c_2 + d_1)x^{n-3} + \cdots \end{aligned}$$

Consider the first coefficient in $P(G,x)$ which is equal to 0. Then we have that the corresponding $c_i + d_j = 0$. Since both c_i and d_j are nonnegative, this means that both c_i and d_j are equal to 0. Then since both $P(G-e,x)$ and $P(G/e,x)$ have fewer than $m+1$ edges, all subsequent c_k's and d_k's are 0 as well. Thus, the coefficients of each subsequent term of $P(G,x)$ are 0 also because they are simply combinations of these 0 coefficients.

Now note these other properties of $P(G,x)$.

- The coefficient of x^n is 1.
- The absolute value of the coefficient of the x^{n-1} term is $m+1$, the number of edges in G.
- The coefficients alternate in sign.
- Each coefficient is an integer.

Thus, each of these properties hold for the chromatic polynomial of any graph. □

Theorem 3.3. *The constant term of the chromatic polynomial of any graph is* 0.

Proof. Suppose on the contrary that the constant term of the chromatic polynomial of some graph is equal to some $c \neq 0$. Then $P(G, 0) = c$. But since there is no way to color a graph with 0 colors, this cannot be true. Thus, the constant term of every graph's chromatic polynomial must be 0. □

Theorem 3.4. *For a nonempty graph G with n vertices, the coefficient of x in $P(G, x)$ is greater than or equal to 0 if the number of vertices is odd and less than or equal to 0 if it is even.*

Proof. We know that, since the chromatic polynomial of a graph G with n vertices has degree n and no constant term, it has terms corresponding to $x, x^2, \ldots, x^n$. Since the x^n term is positive and the coefficients alternate signs, it follows that the coefficients of $x^n, x^{n-2}, x^{n-4}, \ldots$ will all be positive until they become 0, and, similarly, the coefficients of $x^{n-1}, x^{n-3}, x^{n-5}, \ldots$ will all be negative until they become 0. Since we are considering exactly n terms, we can see that if n is odd, then the x term will be in the first list, and if n is even, then it will be in the second list. Thus, an odd number of vertices gives a coefficient for x that is either positive or 0 and an even number gives a coefficient that is either negative or 0. □

The next result was conjectured by Read in his 1968 paper [7], but nobody was able to prove it in full until 2012, though it had been successfully shown for certain types of graphs prior to this.

Theorem 3.5 (Huh, [4])**.** *For a given chromatic polynomial $P(G, x)$ with coefficients $a_0, \ldots, a_n$, $P(G, x)$ is unimodal, i.e., there is some k such that*

$$|a_n| \leq |a_{n-1}| \leq \cdots |a_{k+1}| \leq |a_k| \geq |a_{k-1}| \geq \cdots \geq |a_0|$$

This seemingly simple property turned out to be quite difficult to prove. In fact, success in general wasn't met until Huh showed that chromatic polynomials are all log-concave, meaning that for all coefficients $a_0, \ldots, a_n$ of the polynomial, $a_{i-1}a_{i+1} \leq a_i^2$ for each $0 < i < n$. By proving this, he also proved that chromatic polynomials are, in fact, unimodal.

Note that this also tells us that if a coefficient a_k is 0, then all a_i with $i > k$ are 0 as well. If this were not the case, then there would be a point in the sequence of coefficients where it decreased to 0 and then increased again, meaning the sequence was not unimodal in the first place.

Based on the last few theorems, it is worth noting that, not only can we determine many properties of a graph's chromatic polynomial just by knowing properties of the graph, but we can also use these facts as a way to tell if any given polynomial is be a chromatic polynomial. While we may not be able to know the answer to this for sure, we can at least rule out a large number of polynomials. For example, if a given polynomial is not unimodal, doesn't alternate in sign, or if it has non-integer coefficients, it is impossible for it to be the chromatic polynomial of any graph.

3.2 Graph Families and their Chromatic Polynomials

As was mentioned earlier, there are certain types of graphs that have a very distinct form of chromatic polynomial that they share with other graphs in the same family.

The first of these that we will discuss is the tree graph. Recall that a tree is a connected, acyclic graph with n vertices and $n-1$ edges.

Theorem 3.6. *The chromatic polynomial of a tree T_n is*

$$P(T_n, x) = x(x-1)^{n-1}.$$

Proof. Consider a tree with exactly one vertex. Then, given x colors, it can be colored in x different ways. So $P(G,x) = x = x(x-1)^{1-1}$. So our hypothesis holds when $n = 1$. Now assume it holds for trees with n vertices. Let T_{n+1} be an arbitrary tree on $n+1$ vertices, and pick an edge e that is connected to a leaf of the tree. When we delete e, we get the disjoint union of a tree on n vertices and a one-vertex tree. When we contract e, we obtain a tree on n vertices. So by the deletion-contraction theorem, we have:

$$\begin{aligned} P(T_{n+1}, x) &= P(T_{n+1} - e, x) - P(T_{n+1}/e, x) \\ &= x[x(x-1)^{n-1}] - x(x-1)^{n-1} \\ &= x(x-1)^{n-1}(x-1) \\ &= x(x-1)^n \end{aligned}$$

Thus, the hypothesis holds for trees on $n+1$ vertices, and thus, by induction, is true for all n. □

There is also a very distinct pattern for the chromatic polynomial of a complete graph on n vertices, meaning that each vertex is connected to each of the other $n-1$ vertices.

Theorem 3.7. *The chromatic polynomial of the complete graph K_n is*

$$P(K_n, x) = x(x-1)\cdots(x-n+1).$$

Proof. Consider the complete graph K_n and a set of x colors. The first vertex colored, say v_1, can be colored in x ways. Because each of the other $n-1$ vertices is adjacent to v_1, the next vertex colored, say v_2, can be colored in $x-1$ ways. Let v_k be the kth vertex to be colored. Because v_k is adjacent to each of the other $k-1$ vertices that have already been colored, it can now be colored in $x-(k-1) = x-k+1$ ways. Since this is true for all n vertices, we have:

$$P(K_n, x) = x(x-1)\cdots(x-n+1).$$

□

Given an n-cycle, we can once again write down the chromatic polynomial in the following way:

Theorem 3.8. *The chromatic polynomial of an n-cycle C_n with $n \geq 3$ is:*

$$(x-1)^n + (-1)^n(x-1).$$

Proof. Consider a 3-cycle, and pick an edge e. Using deletion-contraction on e, when we delete e, we get a path on 3 vertices, and when we contract it, we get a path on 2 vertices. So our chromatic polynomial is:

$$\begin{aligned} P(C_3, x) &= P(C_3 - e, x) - P(C_3/e, x) \\ &= x(x-1)^2 - x(x-1) \\ &= x(x-1)[(x-1)-1] \\ &= x(x-1)(x-2) \\ &= (x-1)(x^2-2x) \\ &= (x-1)[(x-1)^2-1] \\ &= (x-1)^3 - (x-1) \end{aligned}$$

So our conclusion holds for $n = 3$. Now assume it holds for an n-cycle, and consider an $(n+1)-$cycle. We pick an edge e and delete the edge to obtain a tree on $n+1$ vertices. We can also contract the same edge to get an $n-$cycle. So by the deletion-contraction theorem,

$$\begin{aligned} P(C_{n+1}, x) &= P(C_{n+1} - e, x) - P(C_{n+1}/e, x) \\ &= x(x-1)^n - [(x-1)^n + (-1)^n(x-1)] \\ &= x(x-1)^n - (x-1)^n + (-1)^{n+1}(x-1) \\ &= (x-1)^n(x-1) + (-1)^{n+1}(x-1) \\ &= (x-1)^{n+1} + (-1)^{n+1}(x-1) \end{aligned}$$

So our conclusion holds for all $n-$cycles. □

Knowing the form of the chromatic polynomial of these types of graphs, as well as those of other common graphs, can often save a substantial amount of time and effort when calculating the chromatic polynomial of a graph. When we are given an arbitrary graph G for which we want to know the chromatic polynomial, we almost always begin performing Deletion-Contraction, but as we have discussed this can often take quite a bit of time and effort if we go about it blindly. However, with this new tool, if one of our Deletion-Contraction steps creates a new graph from one of these families, we can simply write down that chromatic polynomial and only worry about the rest of the graphs we have created up to that point.

4 Properties of Graphs From Their Chromatic Polynomial

We have seen that much about the chromatic polynomial can be predicted by examining a given graph, but it is also interesting to ask about the converse. When we are given only a chromatic polynomial, how much can we determine about the graph it describes? Unfortunately, the answer is that the quantity of information is rarely very substantial.

It is not always easy to find a graph that has the exact chromatic polynomial we are looking at, and even if we can find one, there may be other graphs that fit the polynomial as well. A simple example of this lies with trees. If we are given the polynomial $P(G,x) = x(x-1)^{n-1}$, our first instinct may be that it describes a tree on n vertices, but how do we know which one? Unfortunately, there really isn't a way to tell.

Another small complication comes from the fact that we always treat graphs as though they have no loops nor multiple edges. Because any graph can be modified to contain these characteristics while keeping the same chromatic polynomial, in this sense, it is technically always impossible to decide which graph we are describing. That being said, we can at least determine some of the properties properties of the simple analog of any given graph. We will now examine some of these properties.

First, recall from Section 3.1 that the degree of the chromatic polynomial is the number of vertices in a graph and the absolute value of the coefficient of x^{n-1} is the number of edges. It turns out that it is also easy to tell whether or not a chromatic polynomial is describing a connected graph.

Theorem 4.1. *Let G be a graph with chromatic polynomial $P(G,x)$. Then G is connected if and only if the coefficient of x in $P(G,x)$ is nonzero.*

Proof. We will first prove that if a graph is not connected, the x term in its chromatic polynomial is 0. Consider a disconnected graph G with components $G_1, \ldots, G_k$. Then we know that

$$P(G,x) = P(G_1,x)\cdots P(G_k,x).$$

Now, since the constant term of each $P(G_i,x)$ is 0, the term of lowest degree in each chromatic polynomial is no less than $c_i x$, where c_i is a nonzero integer. Then, if each $P(G_i,x)$ has a nonzero x term, the lowest possible degree term in $P(G,x)$ is

$$c_1 x \cdot c_2 x \cdots c_k x = (c_1 \cdot c_2 \cdots c_k)x^k,$$

which has a coefficient of 0 with the x term. If there are some number of the components that have a coefficient of 0 with x, this clearly means that the term of lowest degree will have degree greater than k. Then in any case, the chromatic polynomial of a disconnected graph has a coefficient of 0 with the x term. Then by contraposition, if the coefficient of x in a chromatic polynomial is nonzero, then the associated graph is connected.

We will now check that the chromatic polynomial of a connected graph always has a non-zero x term. We proceed by induction on the number of edges in G. When we consider the connected graph with $m = 0$ edges, we must have the empty graph on 1 vertex, so we simply have$P(G,x) = x$, which indeed has a non-zero coefficient with the x term.

Now assume that every connected graph with fewer than m edges has a chromatic polynomial with a non-zero coefficient of x. Let G be a connected graph with m edges with n vertices. We know that $P(G,x) = P(G-e,x) - P(G/e,x)$. Because $G-e$ is G with an edge deleted, it still has n vertices, but the number of edges is now $m_1 = m-1$. Note that $G-e$ could be either connected or disconnected. However, since G/e is just G with an edge contracted, we know that it is still connected and that it has exactly $n-1$ vertices and some number m_2 of edges, where $m_2 < m$. Because of this and the fact that G/e is still connected, we know that the x term has a nonzero coefficient. Then we have

$$P(G-e,x) = x^n - m_1x^{n-1} + \cdots + (-1)^{n-1}c_1x$$

and

$$P(G/e,x) = x^{n-1} - m_2x^{n-2} + \cdots + (-1)^{n-2}c_2x,$$

where c_1 and c_2 are nonnegative integers, and c_2 must be nonzero. Because $(-1)^{n-1}$ and $(-1)^{n-2}$ must have opposite signs, we have that

$$\begin{aligned} P(G,x) &= x^n - m_1x^{n-1} + \cdots + (-1)^{n-1}c_1x - [x^n - m_2x^{n-2} + \cdots + (-1)^{n-2}c_2x] \\ &= x^n - (m_1+1)x^{n-1} + \cdots \pm (c_1+c_2)x \end{aligned}$$

So the coefficient of x is nonzero, and our hypothesis holds for all connected graphs. □

Corollary 4.2. *For a graph G and the associated chromatic polynomial $P(G,x)$, the smallest number k such that x^k has a nonzero coefficient in $P(G,x)$ is the number of connected components of G.*

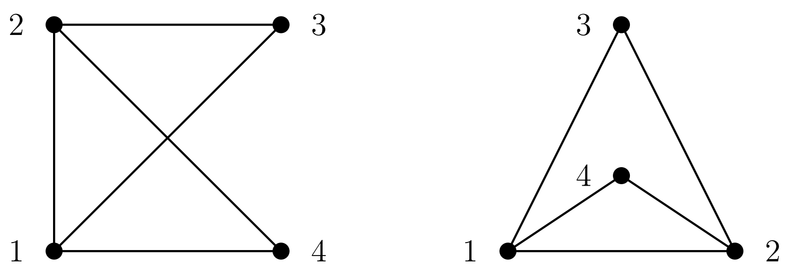
Proof. As shown in Theorem 2.4, since the chromatic polynomial of a disconnected graph is found by multiplying the x terms of the chromatic polynomials of each component, the lowest degree monomial is x^k, where k is the number of connected components. □

It is also possible to determine more properties of a graph from the coefficients of a chromatic polynomial, including the number of K_3 and K_4 in the graph. Some of these properties are outlined in [5].

4.1 Chromatic Uniqueness

We have already seen that, in many cases, a chromatic polynomial describes more than one graph. However, there are, in fact, some cases where this is not true.

Definition 4.3. Two graphs are said to be **isomorphic** if they have the same vertex and edge sets, i.e., if they have the same number of vertices and their edges connect the same vertices.



Two isomorphic graphs

Definition 4.4. A graph G is said to be **chromatically unique** if for any graph H that has the same chromatic polynomial as G, G is isomorphic to H. If G and H are not isomorphic, then they are **chromatically equivalent**.

Because it is not usually the case, it is interesting to find cases where a graph is chromatically unique. In fact, all of the families of graphs discussed in Section 3.2 fit this definition, at least to an extent.

We saw that each tree on n vertices is chromatically equivalent to each other tree on n vertices. However, even though the graph can't be completely determined, we can tell when a chromatic polynomial is describing a tree.

Theorem 4.5. *If the chromatic polynomial of a graph G is $P(G, x) = x(x-1)^{n-1}$, then G is a tree on n vertices.*

Proof. Consider a graph G with a chromatic polynomial such that $P(G, x) = x(x-1)^{n-1}$. We can expand this polynomial in the following way:

$$\begin{aligned}
P(G,x) &= x(x-1)^{n-1} \\
&= x\left[\binom{n-1}{0}x^{n-1} - \binom{n-1}{1}x^{n-2} + \cdots + (-1)^{n-1}\binom{n-1}{n-1}x^0\right] \\
&= \binom{n-1}{0}x^n - \binom{n-1}{1}x^{n-1} + \cdots + (-1)^{n-1}\binom{n-1}{n-1}x \\
&= x^n - (n-1)x^{n-1} + \cdots + (-1)^{n-1}x
\end{aligned}$$

From the expanded form of $P(G, x)$, we see that the x term has a nonzero coefficient, so G is connected. Also, because the degree of the polynomial is n and the coefficient of the second term is $n-1$, we know that G has n vertices and $n-1$ edges. Thus, G must be a tree. □

Unlike the tree, complete graphs and $n-$cycles are chromatically unique. When a chromatic polynomial describes one of these graphs, we know exactly which one it is.

Theorem 4.6. *If the chromatic polynomial of a graph G is $P(G, x) = x \cdot (x-1)\cdots(x-n+1)$, then G is the complete graph on n vertices.*

Proof. Consider a graph G such that the chromatic polynomial is $P(G, x) = x \cdot (x-1)... \cdot (x-n+1)$. When we expand this, we get a polynomial of the form

$$\begin{aligned}
P(G,x) &= x \cdot (x-1)\cdots(x-n+1) \\
&= x^n - x^{n-1} - 2x^{n-1} - \cdots - (n-1)x^{n-1} + \cdots + (-1)^{n-1}(n-1)!x \\
&= x^n - \left(\sum_{i=1}^{n} i\right)x^{n-1} + \cdots + (-1)^{n-1}(n-1)!x \\
&= x^n - \frac{(n-1)n}{2}x^{n-1} + \cdots + (-1)^{n-1}(n-1)!x \\
&= x^n - \frac{n!}{2!(n-2)!}x^{n-1} + \cdots + (-1)^{n-1}(n-1)!x \\
&= x^n - \binom{n}{2}x^{n-1} + \cdots + (-1)^{n-1}(n-1)!x
\end{aligned}$$

From this polynomial, we immediately see that G is connected since the coefficient of x is nonzero. We also know that it has n vertices and $\binom{n}{2}$ edges. Since a simple graph on n vertices can have at most $\binom{n}{2}$ edges, G fits the definition of a complete graph. □

Theorem 4.7. *If the chromatic polynomial of a graph G is $P(G,x) = (x-1)^n + (-1)^n(x-1)$, then G is an $n-$cycle.*

Proof. Consider a graph G with the chromatic polynomial $P(G,x) = (x-1)^n + (-1)^n(x-1)$. When this is expanded, we get

$$\begin{aligned}P(G,x) &= (x-1)^n + (-1)^n(x-1)\\ &= \binom{n}{0}x^n - \binom{n}{1}x^{n-1} + \cdots + \binom{n}{n-1}(-1)^{n-1}x + \binom{n}{n}(-1)^n + (-1)^n(x-1)\\ &= x^n - nx^{n-1} + \cdots + (-1)^{n-1}nx + (-1)^n + (-1)^n x - (-1)^n\\ &= x^n - nx^{n-1} + \cdots + (-1)^{n-1}(n-1)x\end{aligned}$$

From this expansion, we know that G has n vertices, n edges, and is connected. Now we know that a tree on n vertices necessarily has exactly $n-1$ edges, and adding an additional edge to the same vertex set would create a cycle. Then since G is a connected graph on n vertices with n edges, it must contain a cycle.

To see that G contains no more than one cycle, assume on the contrary that it has $l \geq 2$ cycles. Since the graph is connected, this means that there is a path between each pair of vertices in G. Since there are cycles in G, this property will be maintained if we remove exactly one edge from each cycle. However, when we remove these edges, we still have n vertices but only $n-l$ edges. Since a minimum of $n-1$ edges is needed to make a graph on n vertices connected, and $l > 1$, this means that G is disconnected, which is a contradiction.

So we now know that G contains exactly one cycle. We will say that the number of vertices in this cycle is k, where $k \leq n$. Then there are only $n-k$ vertices not contained in the cycle. We can reconstruct the chromatic polynomial of this graph as follows:

Begin by considering the k-cycle. By itself, we know that this has the chromatic polynomial $(x-1)^k + (-1)^k(x-1)$. Now since the other $n-k$ vertices are connected but don't create any additional cycles, we can color them by working away from the cycle one vertex at a time. Because of this, each new vertex that we color only has one adjacent vertex that has already been colored, and because of this, we are only limited to $x-1$ colors for each new vertex. So each vertex contributes a factor of $x-1$ to the chromatic polynomial of the graph. Then our chromatic polynomial is simply

$$P(G,x) = [(x-1)^k + (-1)^k(x-1)](x-1)^{n-k}.$$

But we also know that $P(G,x) = (x-1)^n + (-1)^n(x-1)$. So we now see that

$$\begin{aligned}[(x-1)^k + (-1)^k(x-1)](x-1)^{n-k} &= (x-1)^n + (-1)^n(x-1)\\ (x-1)^n + (-1)^k(x-1)^{n-k+1} &= (x-1)^n + (-1)^n(x-1)\\ (-1)^k(x-1)^{n-k+1} &= (-1)^n(x-1)\end{aligned}$$

So it must be that $(-1)^k = (-1)^n$ and $(x-1)^{n-k+1} = (x-1)$, and thus, $n = k$. Thus, the length of the cycle in G is exactly n, and G is an n-cycle. □

These are by no means all of the known chromatically unique graphs. Much work has been done in recent years to determine when a graph is chromatically unique and when it is chromatically equivalent to some other graph. Many of these results can be found in [5] and [6]. More information on the properties of chromatic polynomials can be found in [3] and [7] as well as in their references.

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