5-2017

Complementary Coffee Cups

Brandon Sams
Boise State University, brandonkeithsams@gmail.com

Complementary Coffee Cups

Brandon Sams
Boise State University
Advisement from Dr. Zach Teitler

April 2017

1 Introduction

In a 2006 paper [1], mathematician Thomas Banchoff posed a new calculus problem, when he filled two cups with coffee. One was convex, while the other was concave, but in such a way so as to fit together perfectly. These cups were thus, complementary. The next natural question he asked was "Which cup has more volume?", so that he could be gracious enough to give the larger of the two to his wife. Banchoff decided to explore this topic, but he left some unanswered questions along the way. Under what conditions would two Complementary Coffee Cups have the same volume? Does the answer depend on the curve between the two cups? Can a coffee cup be complementary to itself? If so, under what conditions does this occur? Does the volume of a pair of solids have anything to do with lateral surface area. This paper will explore the mathematical relationship between pairs of complementary coffee cups by answering these questions.

Acknowledgements

Dr. Zach Teitler has been a tremendous source of guidance during this process, and I am very thankful for his assistance. Also, my parents, Walter and Janelle Sams have been incredibly supportive for my education since the beginning, so thank you to them as well.

2 Background: Modelling the Problem

This paper will model one coffee cup as a volume of revolution about the y-axis, defined by the radius $x = f(y)$. The lower bound for the integral will be $y = a$, and the upper bound will be $y = b$.

The complementary cup will be modeled as a volume of revolution about another axis, $x = k$, where the radius of the complementary volume will be $k - f(y)$. Again, the lower bound will be $y = a$, and the upper bound will be $y = b$, so as to have the two solids of revolution have the same height.

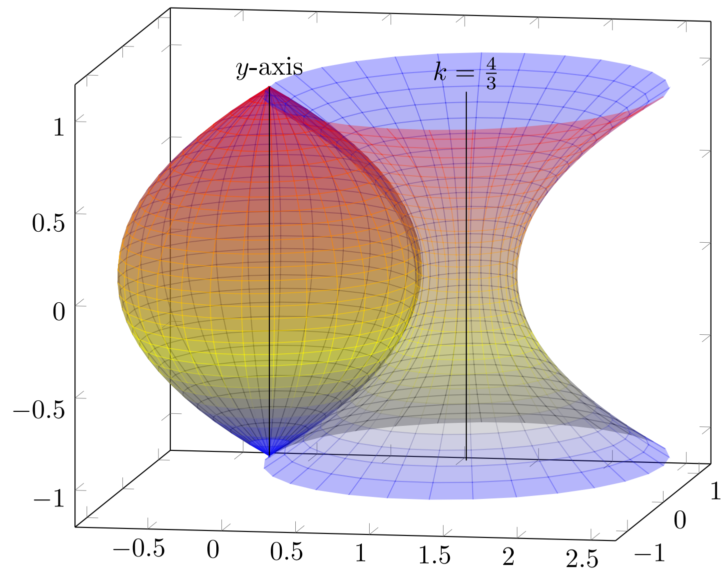

Figure 1: Complementary Coffee Cups arising from a parabola.

The cups will be assumed to be handle-free. They also have zero thickness. Both these assumptions are done to make the coffee cup be approximated succinctly by a volume of revolution.

Setting up the problem in this way requires a few restrictions.

$$a < b$$
$$0 \le f(y) \le k \text{ for } y \in [a, b]$$

3 A Simple Example

Before looking at more complex problems, it will be useful to examine a case which is simple, but non-trivial. See Figure 1. We will begin by taking the curve between the two cups to be a specific parabola.

Let $f(y) = 1 - y^2$. Therefore, since $f(y)$ crosses the y-axis at $y = \pm 1$, $-1 \le a < b \le 1$. In this case, for the sake of demonstrating an example, let $a = -1$, and $b = 1$.

The two volumes, V_1 and V_2, will be calculated using the Disk Method [2, Section 6.3]. This method will be the best to use in further examples, as it allows each volume to be calculated using a single integral.

We begin by assuming the two volumes are equivalent, with the goal of finding a specific value of k.

$$V_1 = V_2$$

Using the Disk Method, we only need to know a function for the radius of each solid of revolution to calculate the volume. For V_1, the radius is $R_1 = f(y)$. For V_2, the radius is $R_2 = (k - f(y))$.

$$\int_a^b \pi(f(y))^2 dy = \int_a^b \pi(k - f(y))^2 dy$$

Now we substitute in the relevant information. That is, $f(y) = 1 - y^2$, $a = -1$, and $b = 1$.

$$\int_{-1}^{1} \pi(1 - y^2)^2 dy = \int_{-1}^{1} \pi(k - (1 - y^2))^2 dy$$

We expand both the integrands on the left and right hand sides.

$$\int_{-1}^{1} \pi(1 - 2y^2 + y^4) dy = \int_{-1}^{1} \pi(k^2 + 2ky^2 - 2k + y^4 - 2y^2 + 1) dy$$

Cancel the π from both sides.

$$\int_{-1}^{1} (1 - 2y^2 + y^4) dy = \int_{-1}^{1} (k^2 + 2ky^2 - 2k + y^4 - 2y^2 + 1) dy$$

Group terms on the right hand side, and split into two separate integrals.

$$\int_{-1}^{1} (1 - 2y^2 + y^4) dy = \int_{-1}^{1} (1 - 2y^2 + y^4) dy + \int_{-1}^{1} (k^2 + 2ky^2 - 2k) dy$$

Subtract $\int_{-1}^{1} (1 - 2y^2 + y^4) dy$ from both sides.

$$0 = \int_{-1}^{1} (k^2 + 2ky^2 - 2k) dy$$

Integrate.

$$0 = k^2 y + \frac{2}{3} k y^3 - 2ky \Big|_{-1}^{1}$$

Substitute the limits of integration.

$$0 = (k^2 + \tfrac{2}{3}k - 2k) - (-k^2 - \tfrac{2}{3}k + 2k)$$

Combine like terms.

$$0 = 2k^2 + \tfrac{4}{3}k - 4k$$

Factor out k, and cancel. This is valid beacuse $k \neq 0$. Combine like terms, and solve for k.

$$\begin{aligned} 0 &= 2k - \tfrac{8}{3} \\ \tfrac{8}{3} &= 2k \\ \tfrac{4}{3} &= k \end{aligned}$$

Therefore, for the simple example of a parabola $f(y) = 1 - y^2$ rotated about the y-axis and evaluated over the interval $[-1, 1]$, the complementary "coffee cup" will be rotated about the axis $x = \frac{4}{3}$. See Figure 1. With k determined, we can calculate V_1 and V_2 exactly.

$$\begin{aligned} V_1 &= V_2 \\ \int_a^b \pi(f(y))^2 dy &= \int_a^b \pi(k - f(y))^2 dy \\ \int_{-1}^1 \pi(1 - y^2)^2 dy &= \int_{-1}^1 \pi(\tfrac{4}{3} - (1 - y^2))^2 dy \\ V_1 = V_2 &= \frac{16\pi}{15} \end{aligned}$$

4 The Arbitrary Case: Disk Method

The previous section was good for demonstration purposes, but this section will show all the work that can take place before the substitution step. This work results in the following theorem. See Figure 2.

Theorem 1. *Two complementary solids of revolution have equal volumes if and only if the cross sections of those complementary solids have equal areas.*

Proof. We begin in a similar fashion, by assuming the two solids of revolution have the same volume.

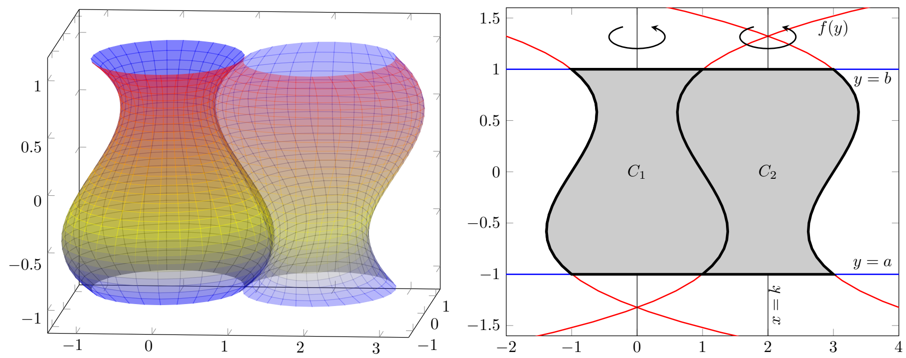

Figure 2: $V_1 = V_2 \iff C_1 = C_2$

$$V_1 = V_2$$

Using the Disk Method, we only need to know a function for the radius of each solid of revolution to calculate the volume. For V_1, the radius is $R_1 = f(y)$. For V_2, the radius is $R_2 = (k - f(y))$.

$$\int_a^b \pi(f(y))^2 dy = \int_a^b \pi(k - f(y))^2 dy$$

Cancel the π from both sides.

$$\int_a^b (f(y))^2 dy = \int_a^b (k - f(y))^2 dy$$

Expand the integrand on the right hand side.

$$\int_a^b f(y)^2 dy = \int_a^b (k^2 - 2kf(y) + f(y)^2) dy$$

Split the right hand side into two separate integrals.

$$\int_a^b f(y)^2 dy = \int_a^b (k^2 - 2kf(y)) dy + \int_a^b f(y)^2 dy$$

Subtract $\int_a^b f(y)^2 dy$ from both sides of the equation.

$$0 = \int_a^b (k^2 - 2kf(y))dy$$

Factor out the constant k.

$$0 = k\int_a^b (k - 2f(y))dy$$

Divide both sides by the constant k. Note that this only works for $k > 0$. If $k = 0$, then since $0 \leq f(y) \leq k$ that means $f(y) = 0$ by the Squeeze Theorem [2, page 96]. This forces the cross sections, C_1 and C_2 to be equivalent, as they are both zero. This completes the proof in this case. The rest of the proof will proceed assuming $k > 0$.

$$0 = \int_a^b (k - 2f(y))dy$$

Add $\int_a^b 2f(y)dy$ to both sides.

$$\int_a^b 2f(y)dy = \int_a^b kdy$$

Factor out the constant, 2, from the integral on the left hand side.

$$2\int_a^b f(y)dy = \int_a^b kdy$$

Integrate the right hand side.

$$2\int_a^b f(y)dy = ky\Big|_a^b$$

Substitute in the limits of integration on the right hand side.

$$2\int_a^b f(y)dy = k(b - a)$$

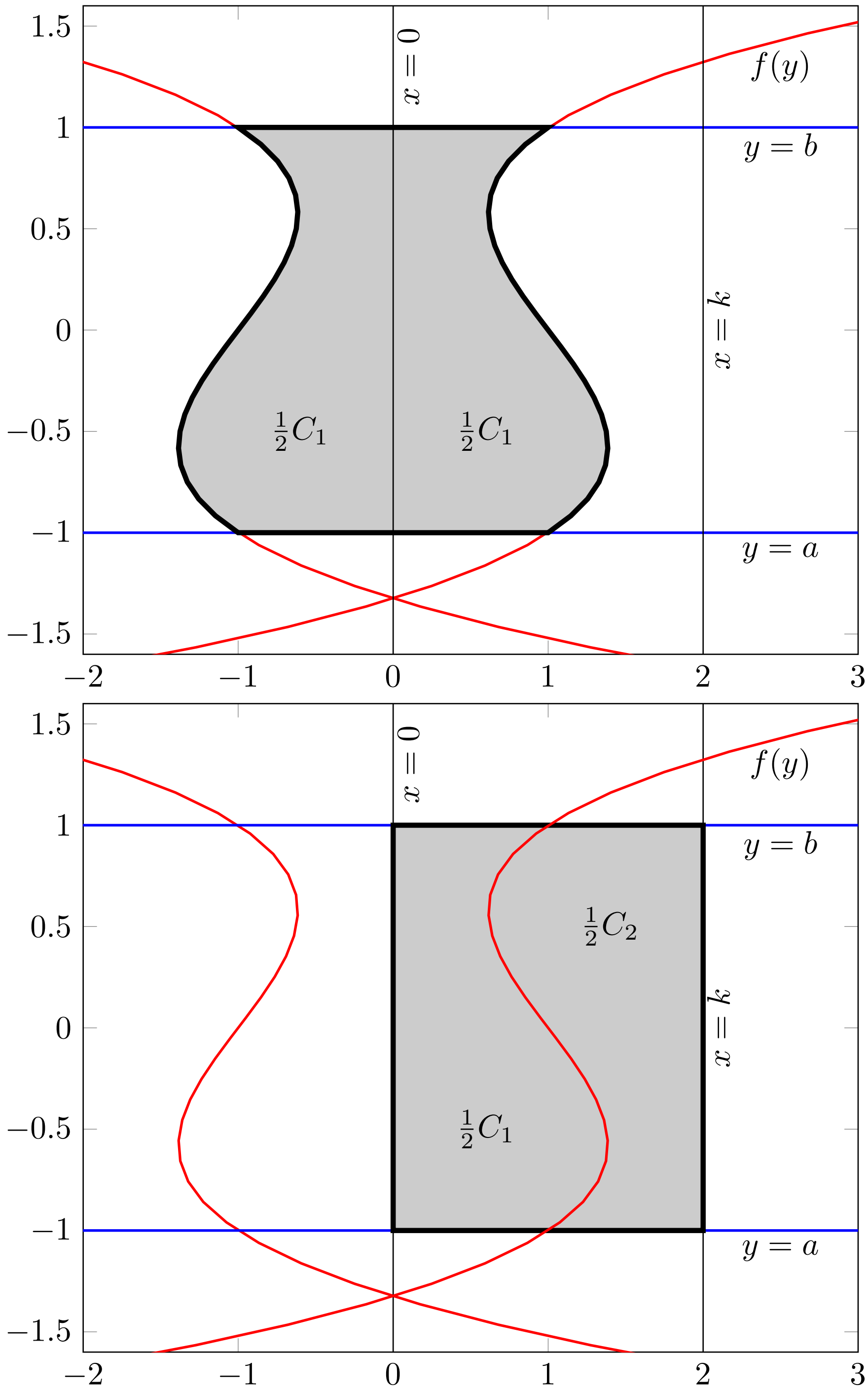

Figure 3: Decomposition of the C_1 and the rectangle between the axes of revolution.

Here, we can take a step back, and examine what this has produced. The left hand side is the area of the cross section of V_1, denoted C_1 in figure 2, and the right hand side is the area of a rectangle with side lengths equal to k and $(b-a)$. Since the Left Hand Side is:

$$2\int_a^b f(y)dy = \frac{1}{2}C_1 + \frac{1}{2}C_1$$

and the Right Hand Side is:

$$k(b-a) = \frac{1}{2}C_1 + \frac{1}{2}C_2$$

and we know these are equivalent, we know that:

$$\frac{1}{2}C_1 + \frac{1}{2}C_1 = \frac{1}{2}C_1 + \frac{1}{2}C_2$$

This is illustrated in figure 3. We can multiply all parts by 2.

$$C_1 + C_1 = C_1 + C_2$$

Subtract C_1 from both sides.

$$C_1 = C_2$$

By reversing the above steps, this grants the converse. □

5 Arbitrary Case: via Pappus's Theorem

The Disk Method is not the only way to calculate the volume of a solid of revolution. A common and very powerful theorem, called Pappus's Theorem [2, page 919], is also sufficient. It runs as follows:

Theorem 2 (Pappus's Theorem). *Let D be a region in the plane and let L be a line in the plane of D. If L does not meet D, then the volume of the solid generated when D is rotated around L is given by*

$$V = AP$$

where A is the area of D, and P is the perimeter (or circumference) of the circle described by the centroid of D as it revolves around L.

Using Pappus's Theorem, we can provide an alternative proof of Theorem 1.

Proof. We begin by assuming

$$V_1 = V_2$$

By Pappus's Theorem,

$$V_1 = A_1 P_1$$

Where

$$A_1 = \int_a^b f(y)dy$$

and

$$P_1 = 2\pi X_C$$

where X_C is the x-coordinate of the centroid, calculated by [2, Section 8.3]

$$X_C = \frac{\frac{1}{2}\int_a^b f(y)^2 dy}{\int_a^b f(y)dy}$$

Therefore,

$$V_1 = \left(\int_a^b f(y)dy\right)\left(2\pi\frac{\frac{1}{2}\int_a^b f(y)^2 dy}{\int_a^b f(y)dy}\right)$$

The $\frac{1}{2}$ cancels with the factor of 2, and $\int_a^b f(y)dy$ cancels from both the numerator and denominator as well, assuming it is nonzero. (If it is zero, then the cross section $C_1 = 0$. Also, $C_2 = 0$ and the proof in this case is completed.) Thus,

$$V_1 = \pi\int_a^b (f(y))^2 dy$$

Repeating a similar process with V_2 yields

$$V_2 = \pi\int_a^b (k - f(y))^2 dy$$

These equations should look familiar, because these are the formulas for calculating the volume of a solid of revolution using the Disk Method. To complete the proof from this point, see section 4 of this paper. □

6 Self-Complementary Coffee Cups

In this paper, we have examined a coffee cup and its complement, where the complement is not necessarily the same as the original cup. However, it may be the case that the complementary cup is exactly the same shape as the original cup! A simple example would be a cylinder of height $b - a$, and radius of $\frac{k}{2}$, but this is certainly not the only case where this is true. Under what conditions would a solid of revolution be complementary to itself?

Theorem 3. *A solid of revolution is its own complement if and only if the function, $f(y)$, that defines the boundary between the two solids has the following property:*

$$\frac{f(y + \frac{a+b}{2}) + f(-y - \frac{a+b}{2})}{2} = \frac{k}{2}$$

for all $y \in [a, b]$.

Proof. All we need is for the function to have 180 degree rotational symmetry about a certain point, $P = (\frac{k}{2}, \frac{a+b}{2})$. We can translate the original function $f(y)$ in the following way:

$$g(y) = f(y + \tfrac{a+b}{2}) - \frac{k}{2}$$

Now, for the function $g(y)$, the point of interest is $P' = (0,0)$. This new function $g(y)$ has rotational symmetry about P' if $g(y)$ is odd, and therefore has the following property [2, page 6]:

$$g(y) = -g(-y)$$

Substituting,

$$f(y + \tfrac{a+b}{2}) - \frac{k}{2} = -(f(-(y + \tfrac{a+b}{2})) - \frac{k}{2})$$

Simplifying the Right Hand Side,

$$f(y + \tfrac{a+b}{2}) - \frac{k}{2} = -f(-y - \tfrac{a+b}{2}) + \frac{k}{2}$$

Add $f(-y - \frac{a+b}{2}) + \frac{k}{2}$ to both sides.

$$f(y + \tfrac{a+b}{2}) + f(-y - \tfrac{a+b}{2}) = k$$

Divide both sides by 2.

$$\frac{f(y + \frac{a+b}{2}) + f(-y - \frac{a+b}{2})}{2} = \frac{k}{2}$$

Reversing the above steps grants the converse. □

This gives us a not particularly elegant formula, but a formula nonetheless. One could have stopped at the second to last line, but dividing both sides by 2 frames the theorem as the average of two separate x values, which must be equal to $\frac{k}{2}$.

7 Lateral Surface Area

It is tempting to consider that the volumes may be related to lateral surface area. However,

Theorem 4. *For the respective volumes V_1 and V_2, and the surface areas S_1 and S_2 for two complementary solids of revolution,*

$$V_1 = V_2 \not\Rightarrow S_1 = S_2$$

Proof. The most direct way to prove the above statement is the provide a counterexample. This is where the simple example from section 3 will be useful. Recall that in that example, we know several key pieces of information:

$$f(y) = 1 - y^2$$
$$\frac{dx}{dy} = -2y$$
$$a = -1$$
$$b = 1$$
$$k = \frac{4}{3}$$
$$V_1 = V_2 = \frac{16\pi}{15}$$

The formula for lateral surface area of a volume of revolution about a vertical axis is [2, page 623]:

$$\int_a^b 2\pi f(y)\sqrt{1 + (\frac{dx}{dy})^2}dy$$

Using this formula, we can plug in the known information and determine the surface area for this specific pair of complementary solids of revolution.

$$S_1 = \int_{-1}^{1} 2\pi(1 - y^2)(\sqrt{1 + (-2y)^2}dy$$
$$\approx 10.965$$
$$S_2 = \int_{-1}^{1} 2\pi(\frac{4}{3} - (1 - y^2))(\sqrt{1 + (-2y)^2}dy$$
$$\approx 13.814$$

Therefore, $S_1 \neq S_2$, and $V_1 = V_2 \not\Rightarrow S_1 = S_2$. □

8 Further Research Questions

If anybody is looking for further rabbit holes to fall down related to this topic, one could consider filling the "coffee cups" with a nonuniform density fluid, and defining the complementary object to be one with the same mass, but not necessarily the same volume.

One could also consider the space filling efficiency of packing n pairs of "complementary cups" into boxes. If anybody can figure out how to do this without resorting to using the dreaded $max()$ function, then they should be very proud of themselves. From my research, one needed to use the $max(f(y), k - f(y))$ to determine the size of the box that holds a set of cups. While one can solve for the maximum value for a specific function that defines the boundary between the solids of revolution, it does not appear to be possible to do it in general.

Also, the converse of Theorem 4 is possible to prove as well.

References

[1] Thomas Banchoff, *Complementary Coffee Cups*, The College Mathematics Journal **37** (2006), no. 3, 170–175.

[2] Jon Rogawski, *Calculus: Early Transcendentals*, 2nd Edition (2012). W.H. Freeman and Company, NY.