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Nonlinear Integral Equations and Their Solutions

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SENIOR THESIS

Nonlinear Integral Equations and Their Solutions

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Declaration of Authorship

I, Caleb RICHARDS, declare that this thesis titled, "Nonlinear Integral Equations and Their Solutions" and the work presented in it are my own. I confirm that:

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BOISE STATE UNIVERSITY

Abstract

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Bachelor of Science in Applied Mathematics

Nonlinear Integral Equations and Their Solutions

by Caleb RICHARDS

We shall investigate nonlinear integral equations and their properties and solutions. Proofs and examples for the existence of unique solutions to nonlinear integral equations are provided. Some other areas explored are properties of solutions to systems of integral equations, integral inequalities, and multiple solutions to such equations.

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Contents

List of Symbols

$A \times B$	The set $\{(a,b) : a \in A,\ b \in B\}$, where A and B are nonempty sets
$\mathbb{R}$	The set of all real numbers
$\mathbb{R}_+$	The set of all nonnegative real numbers, $\mathbb{R}_+ = [0,\infty)$
$\mathbb{R}^n$	The space of n dimensional vectors whose components are real numbers, $\mathbb{R}^n = \underbrace{\mathbb{R} \times \cdots \times \mathbb{R}}_{\text{n-times}}$
$\mathbb{N}$	The set of nonnegative integers
$f : A \to B$	A mapping from A to B, where A and B are arbitrary sets
$C(A,B)$	The space of continuous functions from A to B
$\|f\|$	$\max\left\{\lvert f(x)\rvert : x \in [a,b]\right\} = \max\limits_{x\in[a,b]} \left\{\lvert f(x)\rvert\right\}$, where $f \in C([a,b],\mathbb{R})$

Dedicated to my wife.

Chapter 1

Introduction

The aim of this thesis is to investigate nonlinear integral equations and their solutions. We start with definitions and theorems needed to address questions about their properties.

1.1 Definitions

The Contraction Mapping Theorem is a fundamental tool when addressing the question of existence and uniqueness of solutions to integral as well as differential equations. To introduce the theorem we will need the following definitions that can be found, for example, in [2].

Definition 1. *Let X be an arbitrary linear space (vector space) with a metric $d : X \times X \to \mathbb{R}_+$ that satisfies the following conditions:*

(i) $d(x, y) = 0$ *if and only if* $x = y$*,*

(ii) $d(x, y) = d(y, x)$*,*

(iii) $d(x, y) \leq d(x, z) + d(z, y)$*,*

for arbitrary vectors $x, y, z \in X$. Then, X equipped by the metric d is called a metric space. The last condition is referred to as the triangle inequality.

Definition 2. *Let X be a metric space. Then, a sequence $\{x_n\}_{n=0}^{\infty} \subseteq X$ is convergent in X if and only if there exists $x \in X$ such that for any $\epsilon > 0$ there exists $N \in \mathbb{N}$ such* *that for all* $n \geq N$ *the inequality* $d(x_n, x) < \epsilon$ *holds. We use the notation*

$$\lim_{n\to\infty} d(x_n, x) = 0.$$

Definition 3. *Let* X *be a metric space. Then, a sequence* $\{x_n\}_{n=0}^{\infty} \subseteq X$ *is called a Cauchy sequence if and only if for any* $\epsilon > 0$ *there exists* $N \in \mathbb{N}$ *such that for all* $n, m \geq N$ *the inequality* $d(x_n, x_m) < \epsilon$ *holds. We use the notation*

$$\lim_{n,m\to\infty} d(x_n, x_m) = 0.$$

Definition 4. *Let* X *be a metric space with a metric function* d. *If every Cauchy sequence* $\{x_n\}_{n=0}^{\infty} \subseteq X$ *converges to a vector* $x \in X$, *then* X *is called a complete metric space.*

Definition 5. *Let* X *be a metric space. A mapping* $F : X \to X$ *is called a contraction if and only if there exists a constant* $k \in (0, 1)$ *such that the inequality*

$$d(F(x_1), F(x_2)) \leq kd(x_1, x_2), \tag{1.1}$$

holds for all $x_1, x_2 \in X$.

Definition 6. *A point* $\xi \in X$ *is called a fixed point of a mapping* $F : X \to X$ *if and only if* $F(\xi) = \xi$.

A mapping $F : X \to X$ may have many fixed points, exactly one fixed point, or may have no fixed points at all. In order to address the question of whether or not a fixed point is unique and in order to find it, we consider a sequence of successive iterates

$$\begin{aligned} &x_0, \quad x_1 := F(x_0), \quad x_2 := F(x_1) = F(F(x_0)) = F^2(x_0), \ldots, \\ &x_n := F(F^{n-1}(x_0)) = F^n(x_0), \end{aligned}$$

where $n = 1, 2, \ldots,$ and x_0 (called a starting iterate) is an arbitrary vector in X.

Using the metric d from Definition 1 and the concept of convergence introduced in Definition 2, we investigate whether or not there exists a limit $\tilde{x}$ in X of the sequence of iterates x_n (note that $x_n \in X$). It may happen that the sequence $\{x_n\}_{n=0}^{\infty}$ is not convergent. Then, the method of successive iterates fails. However, for the class of mappings F that are contractions, the method of successive iterates gives a fixed point $\tilde{x} \in X$ of F, provided X is a complete metric space. Moreover, under these conditions, $\tilde{x}$ is unique. In the next section, we present the Contraction Mapping Theorem which states all these conditions.

1.2 Contraction Mapping Theorem

The following theorem can be found, for example, in [2], (Chapter 4, Theorem 1).

Contraction Mapping Theorem. *Suppose*

(i) X is a nonempty complete metric space,

(ii) $F : X \to X$ is a contraction,

(iii) the sequence $\{x_n\}_{n=0}^{\infty}$ is defined by

$$x_{n+1} = F(x_n),$$

$n = 0, 1, \ldots,$ where $x_0 \in X$ is arbitrary.

Then, F has a unique fixed point $\xi \in X$ and

$$\lim_{n \to \infty} d(x_n, \xi) = 0.$$

The proof of the Contraction Mapping Theorem can be found, for example, in [2] or [6]. The following corollary is an illustration of the Contraction Mapping Theorem in the specific case in which X is a closed interval $[a, b]$ equipped by the metric $d(x, y) = |x - y|$, for $x, y \in [a, b]$ and can be found in [6], (Theorem 1.3, Section 1.2).

Corollary 1. *Let the real function $f : [a, b] \to [a, b]$ be a contraction. Then there exists a unique fixed point x of f, where $x \in [a, b]$. Moreover, x is the limit of any sequence $\{x_n\}_{n=0}^{\infty}$ such that $x_{n+1} = f(x_n)$, where the starting point $x_0 \in [a, b]$ is arbitrary.*

In the next chapter, we apply the Contraction Mapping Theorem in the case in which X is the space of all continuous functions defined on a closed interval and F is relevant to nonlinear integral equations.

Chapter 2

Integral Equations

In this chapter, we follow the ideas of [2] and [4] and address the question of whether or not a unique solution exists as well as outline sufficient conditions for existence and uniqueness.

We will start with a general form of an integral equation:

$$x(s) = h(s) + \lambda \int_0^1 g(s, t, x(t))dt, \tag{2.1}$$

where $s \in [0, 1]$. In this equation, λ is a given real parameter and $h \in C([0, 1], \mathbb{R})$ and $g \in C(D, \mathbb{R})$ are given functions, where the domain of the function g is defined by $D = [0, 1] \times [0, 1] \times \mathbb{R}$. The function $x : [0, 1] \to \mathbb{R}$ is an unknown solution to (2.1).

2.1 Uniqueness of Solutions to Nonlinear Integral Equations

In order to investigate the uniqueness of solutions to equation (2.1), we introduce the following definition needed for the integrand g.

Definition 7. *If $f : \mathbb{R} \to \mathbb{R}$ satisfies the inequality*

$$|f(x_1) - f(x_2)| \leq k|x_1 - x_2|,$$

for all $x_1, x_2 \in \mathbb{R}$, then f is called Lipschitz continuous and k is called a Lipschitz constant.

Note that if f is a contraction then it is Lipschitz continuous. However, not all Lipschitz continuous functions are contractions.

We assume that the function g used to define the integral equation (2.1) is Lipschitz continuous with respect to the third argument, that is,

$$|g(s,t,x_1) - g(s,t,x_2)| \leq k|x_1 - x_2|, \tag{2.2}$$

for all $(s,t,x_1), (s,t,x_2) \in D$, where k is a Lipschitz constant which will satisfy a certain condition imposed in Theorem 1. The reason for this assumption is made clear in the last step of the proof of Theorem 1.

We need an explicit statement on the uniqueness of solutions to our general integral equation. We pose this statement as a closely related result to the Contraction Mapping Theorem. We will need to make use of the following theorem, which can be found, for example, in [2] (Theorem 5).

Theorem 1. *Suppose*

(i) $g \in C(D, \mathbb{R})$ satisfies condition (2.2),

(ii) the parameter λ and the Lipschitz constant k satisfy the following strict inequality:

$$|\lambda|k < 1.$$

Then, there exists a unique solution $x \in C([0,1], \mathbb{R})$ to integral equation (2.1). Moreover, for any starting function $x_0 \in C([0,1], \mathbb{R})$, the sequence $\{x_n\}_{n=0}^{\infty}$ such that

$$x_{n+1}(s) = h(s) + \lambda \int_0^1 g(s,t,x_n(t))dt, \tag{2.3}$$

$n = 0, 1, 2, \ldots,$ *satisfies*

$$\lim_{n \to \infty} \max_{s \in [0,1]} |x_n(s) - x(s)| = 0.$$

Proof. We first define $X = C([0,1],\mathbb{R})$ and $d : X \times X \to \mathbb{R}_+$ such that

$$d(y_1, y_2) = \|y_1 - y_2\|,$$

for all $y_1, y_2 \in X$, where the norm $\|\cdot\|$ is defined for $y \in C([0,1],\mathbb{R})$ by

$$\|y\| = \max\{|y(s)| : s \in [0,1]\}.$$

Then, X is a complete metric space (see e.g. the monograph by Cheney [2], Section 1.2) and the first condition of the Contraction Mapping Theorem is satisfied. We now define $F : X \to X$ by

$$(Fx)(s) = h(s) + \lambda \int_0^1 g(s,t,x(t))dt, \tag{2.4}$$

for $x \in X$, $s \in [0,1]$, and verify whether the second condition of the Contraction Mapping Theorem is satisfied. From the definition of the metric d and the norm $\|\cdot\|$, we may see the following

$$\begin{aligned} d\big((Fx_1),(Fx_2)\big) &= \|(Fx_1) - (Fx_2)\| = \max_{s\in[0,1]} \big|\big((Fx_1) - (Fx_2)\big)(s)\big| = \\ &= \max_{s\in[0,1]} |(Fx_1)(s) - (Fx_2)(s)| \end{aligned}$$

From this and (2.4), we get

$$\begin{aligned} d\big((Fx_1),(Fx_2)\big) &= \max_{s\in[0,1]} \left| h(s) + \lambda \int_0^1 g(s,t,x_1(t))dt - h(s) - \lambda \int_0^1 g(s,t,x_2(t))dt \right| \\ &= \max_{s\in[0,1]} \left| \lambda \int_0^1 g(s,t,x_1(t))dt - \lambda \int_0^1 g(s,t,x_2(t))dt \right| \\ &= |\lambda| \max_{s\in[0,1]} \left| \int_0^1 g(s,t,x_1(t)) - g(s,t,x_2(t))dt \right| \\ &\le |\lambda| \max_{s\in[0,1]} \int_0^1 \Big| g(s,t,x_1(t)) - g(s,t,x_2(t)) \Big| dt. \end{aligned}$$

Then, by the assumptions given by (2.2),

$$\begin{aligned} d\big((Fx_1),(Fx_2)\big) &\le |\lambda| \max_{s\in[0,1]} \int_0^1 k\Big|x_1(t)-x_2(t)\Big|dt \\ &= |\lambda| \int_0^1 k\Big|x_1(t)-x_2(t)\Big|dt \\ &\le |\lambda| k \int_0^1 \max_{s\in[0,1]} |x_1(s)-x_2(s)|dt \\ &= |\lambda|\ k\ \|x_1-x_2\| \int_0^1 dt = |\lambda| k d(x_1,x_2). \end{aligned}$$

So we clearly see that if $|\lambda|k < 1$, then F satisfies condition (1.1) with the constant $|\lambda|k$ and assumption *(ii)* of the Contraction Mapping Theorem is satisfied. By this theorem, the sequence $\{x_n\}_{n=0}^{\infty}$ of functions $x_n \in C([0,1],\mathbb{R})$ defined by $x_{n+1} = F(x_n)$ (note that this is equivalent to (2.3)) with any starting function $x_0 \in X$ converges to some $x \in C([0,1],\mathbb{R})$, that is, $\lim_{n\to\infty} d(x_n,x) = \lim_{n\to\infty} \|x_n - x\| = \lim_{n\to\infty} \max_{s\in[0,1]} |x_n(s) - x(s)| = 0$. Here, x is a unique fixed point of F, that is, $Fx = x$ and $(Fx)(s) = x(s)$, for all $s \in [0,1]$. Therefore, from (2.4), we conclude that x is the unique solution of the nonlinear integral equation (2.1). This finishes the proof of the theorem. □

2.2 Examples of Integral Equations

We now present two examples of (2.1) to which we apply Theorem 1 to conclude that their solutions exist and are unique.

Example 1. Consider the following integral equation

$$x(s) = 2(1-2s^2) - \lambda \int_0^1 stx(t)dt, \tag{2.5}$$

for $s \in [0,1]$ and with $|\lambda| < 1$. Here, $h(s) = 2(1-2s^2)$, $g(s,t,y) = sty$ for $s,t \in [0,1]$ and $y \in \mathbb{R}$. Then,

$$|g(s,t,y_1) - g(s,t,y_2)| = |sty_1 - sty_2| = st|y_1 - y_2| \leq |y_1 - y_2|,$$

for all $s,t \in [0,1]$ and $y_1, y_2 \in \mathbb{R}$. Here, $k = 1$. Therefore, $k|\lambda| < 1$ and by Theorem 1, equation (2.5) has a unique solution $x \in C([0,1],\mathbb{R})$.

By the Contraction Mapping Theorem, the unique solution x to (2.5) is a limit of the sequence defined recursively as follows

$$\begin{cases} x_0(s) = 1, \\ x_{n+1}(s) = 2(1-2s^2) + \lambda \displaystyle\int_0^1 stx_n(t)dt, \end{cases} \tag{2.6}$$

for $n = 0, 1, 2, \ldots,$ and $s \in [0,1]$.

For $n = 0$

$$\begin{aligned} x_1(s) &= 2(1-2s^2) + \lambda \int_0^1 stx_0(t)dt \\ &= 2(1-2s^2) + \lambda s \int_0^1 tdt \\ &= h(s) + \frac{1}{2}\lambda s. \end{aligned}$$

We now assume

$$x_{n-1}(s) = h(s) + \frac{1}{2}\lambda s \left(\frac{\lambda}{3}\right)^{n-2} \tag{2.7}$$

for a certain $n \in \mathbb{N}, n > 1$, and prove

$$x_n(s) = h(s) + \frac{1}{2}\lambda s \left(\frac{\lambda}{3}\right)^{n-1}. \tag{2.8}$$

From (2.6) and (2.7), we get

$$\begin{aligned} x_n(s) &= h(s) + \lambda \int_0^1 st x_{n-1}(t)dt \\ &= h(s) + \lambda \int_0^1 st\left(h(t) + \frac{1}{2}\lambda t\left(\frac{\lambda}{3}\right)^{n-2}\right)dt \\ &= h(s) + \lambda s \int_0^1 2t - 4t^3 + \frac{1}{2}\lambda\left(\frac{\lambda}{3}\right)^{n-2} t^2 dt \\ &= h(s) + \lambda s\Big(t^2 - t^4 + \frac{1}{2}\lambda\left(\frac{\lambda}{3}\right)^{n-2} \cdot \frac{1}{3}t^3\Big)\Big|_{t=0}^{t=1} \\ &= h(s) + \lambda s \cdot \frac{1}{2}\left(\frac{\lambda}{3}\right)^{n-1}, \end{aligned}$$

which proves the hypothesis given by equation (2.8). Therefore,

$$\max_{s\in[0,1]} |x_n(s) - h(s)| = \max_{s\in[0,1]} \left|\frac{1}{2}\lambda s\left(\frac{\lambda}{3}\right)^{n-1}\right| = \frac{|\lambda|}{2}\left(\frac{|\lambda|}{3}\right)^{n-1}.$$

Thus, the unique solution defined in terms of the sequence can be found by taking the limit as $n \to \infty$,

$$\lim_{n\to\infty} \max_{s\in[0,1]} |x_n(s) - h(s)| = \lim_{n\to\infty} \frac{|\lambda|}{2}\left(\frac{|\lambda|}{3}\right)^{n-1}.$$

Note that λ is an arbitrary parameter of the integral equation such that $|\lambda| < 1$. Therefore, $\frac{1}{3}|\lambda| < 1$ and the right limit goes to zero, giving the final result $x(s) = \lim_{n\to\infty} x_n(s) = h(s)$, that is, the unique solution to (2.5) can be written in the form

$$x(s) = 2(1 - 2s^2).$$

Example 2. Consider the following nonlinear integral equation

$$x(s) = s(\pi s - \frac{1}{5}) + \frac{\pi}{5}\int_0^1 st \sin x(t)dt, \tag{2.9}$$

for $s \in [0,1]$. Here, $\lambda = \dfrac{\pi}{5}$, $h(s) = s(\pi s - \frac{1}{5})$, $g(s,t,y) = st \sin y$, for $s,t \in [0,1]$ and $y \in \mathbb{R}$. Then, $g \in C(D,\mathbb{R})$ and

$$\begin{aligned} &|g(s,t,y_1) - g(s,t,y_2)| = |st \sin y_1 - st \sin y_2| = st|\sin y_1 - \sin y_2| \\ &= st|\cos \eta| \cdot |y_1 - y_2| \leq |y_1 - y_2|, \end{aligned}$$

for all $s,t \in [0,1]$ and $y_1, y_2 \in \mathbb{R}$, where η is between y_1 and y_2. Here, $k = 1$ and $k|\lambda| = \frac{\pi}{5} < 1$. Therefore, by Theorem 1, equation (2.9) has a unique solution $x \in C([0,1],\mathbb{R})$.

More examples of nonlinear integral equations are provided by Davis [3]. Some equations given in [3] model predator and prey populations. Other applications are also included.

Theorem 1 shows that the sequences given by (2.3) generated by taking any starting function $x_0 \in C([0,1],\mathbb{R})$ can be used as successive approximations to the unique solution x of (2.1).

Chapter 3

Volterra Integral Equations

In this chapter, we investigate nonlinear Volterra integral equations and corresponding inequalities. Examples of such equations are presented, e.g. by Linz [5] and Tricomi [7] .

3.1 A Pair of Integral Inequalities

We will now follow the ideas from the monograph [4] and look to explore integral inequalities. Like in the previous section, we look for properties of solutions to given nonlinear integral equations. We start with a theorem on integral inequalities related to the integral equation

$$x(s) = h(s) + \int_{s_0}^{s} K(s, t, x(t))dt, \tag{3.1}$$

defined for $s \in [s_0, \alpha]$, where $K \in C([s_0, \alpha] \times [s_0, \alpha] \times \mathbb{R})$, $h \in C([s_0, \alpha], \mathbb{R})$ are given functions and $x \in C([s_0, \alpha], \mathbb{R})$ is an unknown solution.

The goal is to establish inequalities between functions that satisfy corresponding integral inequalities. To realize this goal we need to introduce the following definition for the given function K.

Definition 8. *We say that a given* $K \in C([s_0, \alpha] \times [s_0, \alpha] \times \mathbb{R}, \mathbb{R})$ *is nondecreasing with respect to the third argument if and only if*

$$K(s, t, x_1) \leq K(s, t, x_2),$$

for all $s, t \in [s_0, \alpha]$ *and* $x_1, x_2 \in \mathbb{R}$ *such that* $x_1 \leq x_2$*.*

The following theorem is presented in [4] and we follow the ideas from [4] to prove it.

Theorem 2. *Assume* $s_0 < \alpha$ *and* $h \in C([s_0, \alpha], \mathbb{R})$ *are arbitrary. Moreover, suppose that the following conditions are satisfied.*

(i) $K \in C([s_0, \alpha] \times [s_0, \alpha] \times \mathbb{R}, \mathbb{R})$ *is nondecreasing with respect to the third argument,*

(ii) $x, y \in C([s_0, \alpha], \mathbb{R})$ *are any two functions that satisfy the inequalities*

$$\begin{cases} x(s) < h(s) + \displaystyle\int_{s_0}^{s} K(s, t, x(t))dt, \\ y(s) \geq h(s) + \displaystyle\int_{s_0}^{s} K(s, t, y(t))dt, \end{cases} \tag{3.2}$$

for all $s \in [s_0, \alpha]$*,*

(iii) x *and* y *satisfy the strict initial inequality*

$$x(s_0) < y(s_0).$$

Then,

$$x(s) < y(s), \tag{3.3}$$

for all $s \in [s_0, \alpha]$*.*

Proof. Let us first assume by contradiction that (3.3) is false.

Then, since the functions x and y are continuous, there must exist some $s_1 \in (s_0, \alpha]$ such that

$$x(s_1) = y(s_1). \tag{3.4}$$

Note that there may be more than one s_1 that satisfies (3.4). We now apply *(iii)* and choose s_1 such that the strict inequality

$$x(s) < y(s) \tag{3.5}$$

is satisfied for all $s \in [s_0, s_1)$. Next, we combine (3.5) with the assumption that K is nondecreasing with respect to the third argument and conclude that the following inequality is satisfied:

$$K(s_1, t, x(t)) \leq K(s_1, t, y(t)),$$

for all $s_0 \leq t \leq s_1$. Using this fact along with the inequalities given by (3.2), we derive the inequality

$$\begin{aligned} x(s_1) &< h(s_1) + \int_{s_0}^{s_1} K(s_1, t, x(t))dt \\ &\leq h(s_1) + \int_{s_0}^{s_1} K(s_1, t, y(t))dt \\ &\leq y(s_1). \end{aligned} \tag{3.6}$$

We get a final statement $x(s_1) < y(s_1)$ which contradicts the assumption that $x(s_1) = y(s_1)$ so the inequality (3.3) is proven true. □

Remark. Note that the first and last inequalities in (3.6) can be swapped so that if the first inequality in (3.6) is weak, then since the last will be a strict inequality, this will lead to the same conclusion that $x(s_1) < y(s_1)$, contradicting $x(s_1) = y(s_1)$, which proves (3.3). Therefore, we have the following corollary.

Corollary 2. *Theorem 2 remains valid if the first inequality in (3.2) is weak and the second inequality in (3.2) is strict.*

In the next section, we consider integral inequalities written in terms of integral operators.

3.2 Integral Operators and Inequalities

Now let us look at the integral operator $\mathcal{K} : C([s_0, \alpha], \mathbb{R}) \to C([s_0, \alpha], \mathbb{R})$ defined by

$$(\mathcal{K}\phi)(s) = \int_{s_0}^{s} K(s, t, \phi(t))dt, \tag{3.7}$$

for $\phi \in C([s_0, \alpha], \mathbb{R})$, where $s_0 \leq s \leq \alpha$ and $K \in C([s_0, \alpha] \times [s_0, \alpha] \times \mathbb{R}, \mathbb{R})$, as introduced in [4]. Note that the integral equation (3.1) can be written in the form

$$x(s) = h(s) + (\mathcal{K}x)(s), \tag{3.8}$$

where $s \in [s_0, \alpha]$. We define the integral operator in this particular way so that we may use it to formulate a theorem similar to Theorem 1 but such that the integral inequalities are expressed in terms of the operator (3.7). To realize this goal, we follow the ideas from [4] and introduce the following definition imposed on the operator $\mathcal{K}$.

Definition 9. *We say that the integral operator $\mathcal{K}$ is nondecreasing if for any $\phi_1, \phi_2 \in C([s_0, \alpha], \mathbb{R})$ the inequalities*

$$\phi_1(s) \leq \phi_2(s)$$

satisfied for all $s \in [s_0, s_1)$ imply that

$$(\mathcal{K}\phi_1)(s_1) \leq (\mathcal{K}\phi_2)(s_1),$$

where s_1 is any point such that $s_1 > s_0$.

Now we proceed with our integral operator in a similar manner to that of part (ii) of Theorem 2. We follow the ideas from [4] and prove the following theorem on integral inequalities formulated in terms of the nondecreasing integral operator $\mathcal{K}$. The theorem can be found in [4] (Theorem 5.1.2).

Theorem 3. *Suppose $x, y \in C([s_0, \alpha], \mathbb{R})$ satisfy the inequality*

$$x(s) - (\mathcal{K}x)(s) < y(s) - (\mathcal{K}y)(s), \tag{3.9}$$

for all $s > s_0$, where $\mathcal{K}$, defined by (3.7), is nondecreasing. Moreover, assume the initial inequality

$$x(s_0) < y(s_0).$$

Then,

$$x(s) < y(s),$$

for all $s \geq s_0$.

Proof. The proof is similar to the proof of Theorem 2. We assume by contradiction that there exists an s_1 such that the conditions for (3.4) and (3.5) are still true. The operator $\mathcal{K}$ is nondecreasing, so from the suppositions (3.4) and (3.5), we conclude that

$$(\mathcal{K}x)(s_1) \leq (\mathcal{K}y)(s_1). \tag{3.10}$$

On the other hand, from (3.9), for $s = s_1$, we have that

$$\begin{aligned} x(s_1) = &\quad x(s_1) - (\mathcal{K}x)(s_1) + (\mathcal{K}x)(s_1) \\ < &\quad y(s_1) - (\mathcal{K}y)(s_1) + (\mathcal{K}x)(s_1). \end{aligned} \tag{3.11}$$

Now, since we have from (3.10) that $(\mathcal{K}x)(s_1) \leq (\mathcal{K}y)(s_1)$, this means

$$-(\mathcal{K}y)(s_1) + (\mathcal{K}x)(s_1) \leq 0$$

and therefore,

$$y(s_1) - (\mathcal{K}y)(s_1) + (\mathcal{K}x)(s_1) \leq y(s_1). \tag{3.12}$$

We now combine (3.11) with (3.12) and get

$$x(s_1) < y(s_1).$$

This contradicts the assumption that $x(s_1) = y(s_1)$. Therefore, Theorem 3 is proved. □

We now introduce **under** and **over functions** of (3.8) as outlined in Definition 5.1.2 in [4].

Definition 10. *Let $h \in C([s_0, \alpha], \mathbb{R})$ and $\mathcal{K} : C([s_0, \alpha], \mathbb{R}) \to C([s_0, \alpha], \mathbb{R})$ be defined by (3.7). If $u \in C([s_0, \alpha], \mathbb{R})$ satisfies the inequality*

$$u(s) < h(s) + (\mathcal{K}u)(s), \tag{3.13}$$

for all $s \in [s_0, \alpha]$, then it is said to be an **under function** *of the integral equation (3.8).*

On the other hand, if $v \in C([s_0, \alpha], \mathbb{R})$ satisfies the opposite inequality

$$v(s) > h(s) + (\mathcal{K}v)(s), \tag{3.14}$$

for all $s \in [s_0, \alpha]$, then it is called an **over function** *of (3.8).*

If $x \in C([s_0, \alpha], \mathbb{R})$ satisfies (3.8), then we call it a solution of (3.8).

We now apply Theorem 3 and follow the ideas presented in [4] to demonstrate a relation between $u, x,$ and v introduced in Definition 10. The relation is formulated in the following theorem found also in [4] but without proof. We prove the assertion, below.

Theorem 4. *Let u, x, $v \in C([s_0, \alpha], \mathbb{R})$ be an under function, solution, and over function of (3.8) respectively, on $[s_0, \alpha]$, where the integral operator $\mathcal{K}$ defined by (3.7)* *is nondecreasing. Moreover, assume the initial inequalities*

$$u(s_0) < x(s_0) < v(s_0).$$

Then,

$$u(s) < x(s) < v(s), \tag{3.15}$$

holds for all $s \in [s_0, \alpha]$.

Proof. We first verify the assumptions of Theorem 3 so that we may apply it to our proof. We will need to verify the assumptions twice; once for the first application involving u and x and again for the second application involving x and v. Proceeding with the assumptions of Theorem 3, we note that the operator $\mathcal{K}$ is nondecreasing. We now want to show that (3.9) holds for all $s > s_0$. Since u is an under function and x is a solution of (3.8), we obtain the following strict inequality

$$\begin{aligned} u(s) - (\mathcal{K}u)(s) &< h(s) \\ &= x(s) - (\mathcal{K}x)(s), \end{aligned}$$

which shows that (3.9) is satisfied. Therefore, from the initial inequality $u(s_0) < x(s_0)$, by Theorem 3, we conclude that $u(s) < x(s)$ for all $s \in [s_0, \alpha]$. To show the second inequality in (3.15), we now combine the facts that v is an over function and x is a solution of (3.8) to obtain

$$\begin{aligned} x(s) - (\mathcal{K}x)(s) &= h(s) \\ &< v(s) - (\mathcal{K}v)(s). \end{aligned}$$

This shows that (3.9) is satisfied for x and v. Moreover, $x(s_0) < v(s_0)$ and all assumptions of Theorem 3 are satisfied. Therefore,

$$x(s) < v(s),$$

for all $s \in [s_0, \alpha]$, and the assertion of Theorem 4 is proved. □

Examples of systems of nonlinear Volterra integral equations can be found e.g. in [3] and [5]. In the next Chapter, we investigate systems of integral inequalities.

Chapter 4

Integral Systems

The result of Theorem 3 can be generalized to systems of integral inequalities. We will start this chapter by exploring a closely related result to Theorem 2. With this aim, we define the bold inequalities " $\leq$ " and " $<$ " for systems of integral inequalities.

Definition 11. *Let $u = (u_1, u_2, \dots, u_n) \in \mathbb{R}^n$ and $v = (v_1, v_2, \dots, v_n) \in \mathbb{R}^n$. Then,*

$$v \leq u \text{ if and only if } v_i \leq u_i \text{ for all } i = 1, 2, \dots, n.$$

Also,

$$v < u \text{ if and only if } v_i < u_i \text{ for all } i = 1, 2, \dots, n.$$

The opposite strong and weak inequalities " $>$ " and " $\geq$ " are defined in a similar manner.

We now use Definition 11 to define monotonic vector functions.

Definition 12. *We say that a given $\mathbf{K} \in C([s_0, \alpha] \times [s_0, \alpha] \times \mathbb{R}^n, \mathbb{R}^n)$ is nondecreasing with respect to the third argument if and only if*

$$\mathbf{K}(s, t, x_1) \leq \mathbf{K}(s, t, x_2),$$

for all $s, t \in [s_0, \alpha]$ and $x_1, x_2 \in \mathbb{R}^n$ such that $x_1 \leq x_2$.

We proceed by proving the following theorem found in [4].

Theorem 5. *Suppose* $x, y \in C([s_0, \alpha], \mathbb{R}^n)$ *satisfy the following conditions*

(i) *the component-wise inequalities*

$$\begin{cases} x(s) \leq h(s) + \displaystyle\int_{s_0}^{s} \mathbf{K}(s, t, x(t))dt, \\ y(s) > h(s) + \displaystyle\int_{s_0}^{s} \mathbf{K}(s, t, y(t))dt, \end{cases} \tag{4.1}$$

are satisfied for all $s \in [s_0, \alpha]$*, where* $h \in C([s_0, \alpha], \mathbb{R}^n)$*, and* $\mathbf{K} \in C\big([s_0, \alpha] \times [s_0, \alpha] \times \mathbb{R}^n, \mathbb{R}^n\big)$ *is nondecreasing with respect to the third argument,*

(ii) x *and* y *satisfy the strict initial inequalities* $x(s_0) < y(s_0)$*.*

Then,

$$x(s) < y(s), \tag{4.2}$$

for all $s \in [s_0, \alpha]$*.*

Proof. Let us assume by contradiction that the assumption of the theorem is not true. Then, by (ii) and by the continuity of x and y, there exist $s_1 \in (s_0, \alpha]$ and $j \in \{1, 2, \ldots, n\}$ such that $x(s) < y(s)$, for all $s \in [s_0, s_1)$, but $x(s_1) \leq y(s_1)$ and $x_j(s_1) = y_j(s_1)$. From this and *(i)*, we get

$$K_j(s_1, t, x(t)) \leq K_j(s_1, t, y(t)), \tag{4.3}$$

for all $t \in [s_0, s_1]$. From the first inequality in (4.1) and (4.3), we get the following

$$\begin{aligned} x_j(s_1) &\leq h_j(s_1) + \int_{s_0}^{s_1} K_j(s_1, t, x(t))dt \\ &\leq h_j(s_1) + \int_{s_0}^{s_1} K_j(s_1, t, y(t))dt, \end{aligned}$$

and from the second inequality in (4.1), we get

$$h_j(s_1) + \int_{s_0}^{s_1} K_j(s_1, t, y(t))dt < y_j(s_1).$$

Therefore,

$$x_j(s_1) < y_j(s_1).$$

Herein lies a contradiction from our false assumption that $x_j(s_1) = y_j(s_1)$. Therefore Theorem 5 is proved. □

Remark. *Note that we can prove a related theorem where in (4.1) the first inequality is strict and the second is weak. The proof of this is similar to the proof of Theorem 5.*

Applications of systems of Volterra integral equations are presented by Davis [3] (for example, the mutual growth of two conflicting populations).

A collection of further theorems, results, and applications of systems of integral equations are given in [5]. In particular, in Chapter 3, Section 4 of [5], it has also been concluded that many results that are applicable to single integral equations may also pertain to systems. The steps to proving such results are often quite similar. Tricomi [7] shortly mentions results on systems of Volterra integral equations, where a brief account of such a correspondence can be found.

Chapter 5

Maximal and Minimal Solutions

Integral equations may, in general, have more than one solution. In this chapter, we consider maximal and minimal solutions to the nonlinear integral equation (3.1) and by following the ideas from the monograph [4], we introduce the following definition.

Definition 13. *Suppose* $K \in C([s_0, \alpha] \times [s_0, \alpha] \times \mathbb{R}, \mathbb{R})$, $h \in C([s_0, \alpha], \mathbb{R})$, *and* $\tilde{x} \in C([s_0, \alpha], \mathbb{R})$ *is such that*

$$\tilde{x}(s) = h(s) + \int_{s_0}^{s} K(s, t, \tilde{x}(t))dt, \tag{5.1}$$

for all $s \in [s_0, \alpha]$. *If* $x \in C([s_0, \alpha], \mathbb{R})$ *is any other solution to (3.1) on* $[s_0, \alpha]$ *and the inequality*

$$x(s) \leq \tilde{x}(s),$$

is satisfied for all $s \in [s_0, \alpha]$, *then we refer to* $\tilde{x}$ *as the maximal solution of (3.1). By reversing this inequality, we similarly define the minimal solution of (3.1).*

We want to be able to compare a function that satisfies an integral inequality to the maximal solution of the corresponding integral equation. Theorem 6, found in [4], gives a result that does exactly that. The proof of Theorem 6 is based on the ideas from [4] and presents an application of Theorem 2 (proved in Chapter 3) and the remark provided directly afterwards.

Theorem 6. *Suppose that*

(i) $\tilde{x} \in C([s_0, \alpha], \mathbb{R}_+)$ *is the maximal solution of (3.1) on* $[s_0, \alpha]$*,*

(ii) $x \in C([s_0, \alpha], \mathbb{R}_+)$ *is such that*

$$x(s) \leq h(s) + \int_{s_0}^{s} K(s, t, x(t))dt, \tag{5.2}$$

for all $s \in [s_0, \alpha]$*, where* $h \in C([s_0, \alpha], \mathbb{R}_+)$ *and* $K \in C\big([s_0, \alpha] \times [s_0, \alpha] \times \mathbb{R}_+, \mathbb{R}\big)$*,*

(iii) K *is nondecreasing in the third argument,*

(iv) the initial inequality

$$x(s_0) \leq h(s_0)$$

holds.

Then,

$$x(s) \leq \tilde{x}(s), \tag{5.3}$$

for all $s \in [s_0, \alpha]$*.*

Proof. The goal is to show that

$$x(s) < x_\epsilon(s), \tag{5.4}$$

for all $s \in [s_0, \alpha]$, where x_ϵ is any solution of the integral equation

$$x_\epsilon(s) = h(s) + \int_{s_0}^{s} K(s, t, x_\epsilon(t))dt + \epsilon$$

and $\epsilon > 0$ is taken to be sufficiently small. The fact that

$$x(s_0) \leq h(s_0) < h(s_0) + \epsilon = x_\epsilon(s_0)$$

is evident because of *(iv)* and the additional positive component ϵ. Also,

$$x_\epsilon(s) > h(s) + \int_{s_0}^{s} K(s,t,x_\epsilon(t))dt. \tag{5.5}$$

From (5.2) and (5.5), by Theorem 2 (and the remark provided directly afterwards), we get the strict inequality $x(s) < x_\epsilon(s)$, for all $s \in [s_0, \alpha]$. Combining this inequality with the fact that $\lim_{\epsilon\to 0} x_\epsilon(s) = \tilde{x}(s)$ we have the result that $x(s) \leq \tilde{x}(s)$ desired in (5.3), for all $s \in [s_0, \alpha]$, which finishes the proof. □

Results of this type are useful, for example, in proving uniqueness properties for integral equations as well as in deriving error bounds for approximate solutions. More comparison theorems useful for this purpose can be found in e.g. [4] and [5]. Comparison theorems that explore multiple solutions, including maximal and minimal solutions, are given in [4]. These solutions are also important in finding error bounds for integral equations. Numerical solutions to Volterra integral as well as integro-differential equations are investigated in [1].

In this thesis, we have explored integral equations, inequalities, systems, and their solutions. We also explored Volterra integral equations and their properties. More applications and explorations of such equations may be found in the references provided in [1]-[7]. This collection of studies also extends beyond the scope of this thesis, which has thoroughly explained and elaborated on various theorems presented in [2] and [4].

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