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An Introductory Look at Deterministic Chaos

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SENIOR THESIS

An Introductory Look at Deterministic Chaos

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for the degree of Bachelor of Science*

in

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BOISE STATE UNIVERSITY

Abstract

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Mathematics

Bachelor of Science

An Introductory Look at Deterministic Chaos

by Kenneth COITEUX

This is a brief introduction to deterministic chaos. We will be studying the logistic map and the 'double-humped' logistic map.

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Chapter 1

Introduction

1.1 Where to start?

A few years ago, I was looking for something to watch on the television. While flipping through the channels, I came across a program on Wyoming PBS on fractal geometry. Combined with a distance of time and only seeing the program once, all I can remember is being introduced to the Mandelbrot set. In any case, my curiosity was aroused and I wanted to learn more about fractal geometry.

Casual learning led me to the fact that fractal geometry and chaos related. I also learned that iteration of some functions can create sets of fractal shapes as well as leading to chaos.

Unfortunately, I had no access to formally learning about fractal geometry. However, in a class called 'Communication in the Mathematical Sciences' at Boise State University, I was able to explore a little about iteration and its relation to chaos. That will be the focus on this thesis.

There are volumes of information regarding chaos. This is my attempt at paring down that information into a sort of introduction to iteration and chaos. We will first look at special function called logistic map and then compare that information with what I will call the 'double-humped' logisitc map.

1.2 Definitions

Before beginning this discussion on iteration and chaos, I need to present some definitions.

Definition 1. Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and a fixed point x_0 , *iteration* is the process of evaluating the function at $f(x_0)$ and re-evaluation of the previous results as input.

$$f(x_0) = x_1$$

$$f(x_1) = x_2$$

$$f(x_2) = x_3$$

$$\vdots$$

In regards to notation, f^n represents the n th iteration of f .

The *period* of a point is the minimum number of iterations needed for $f^n(x_0) = x_0$.

Definition 2. Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$, an initial value x_0 is a *periodic point* of f if $f^n(x_0) = x_0$. The *period* of a point is the minimum number of iterations needed for $f^n(x_0) = x_0$.

observe that fixed points are periodic points with a period of 1.

Definition 3. Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and an initial value x_0 , the *orbit* of x_0 is the set of iterations:

$$O(x_0) = f^n(x_0) : n = 1, 2, 3, \dots$$

Other definitions will be given as the thesis progresses.

Chapter 2

The Logistic Map

2.1 What is the logistic map?

The logistic map is defined as $f(x) = ax - ax^2$ where $a > 0$ is a constant parameter. This function is used in a variety of ways such as describing seasonal breeding populations in which generations do not overlap. Other areas that the logistic function provides a good model include epidemiology, economics, and social sciences.

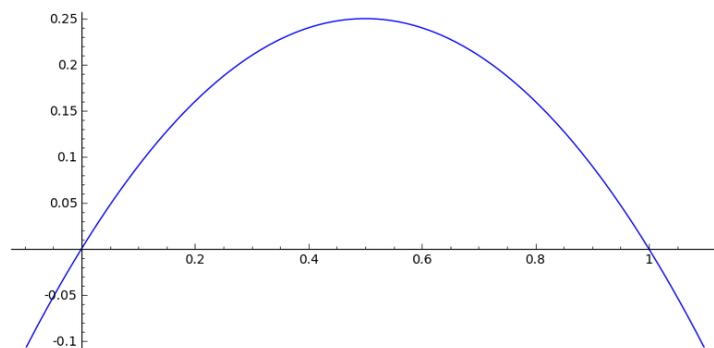


FIGURE 2.1: $f(x) = x - x^2$

Let us look at some basic properties of the logistic function. To find the roots of the function, set it equal to 0 and solve for x .

$$ax - ax^2 = 0$$

$$ax(1 - x) = 0$$

$$x = 0$$

or

$$x = 1$$

To find the maximum or minimum, we need to take the derivative of the logistic function, set it equal to 0 and solve for x .

$$\begin{aligned} f'(x) &= 0 \\ a - 2ax &= 0 \\ x &= 1/2 \end{aligned}$$

Now to determine if $x = 1/2$ is a maximum or minimum, we determine concavity by looking at the sign of $f''(1/2)$.

$$f''(1/2) = -2a$$

Remember that $a > 0$, so the sign of f'' is negative. Because the graph is concave down at the critical point $x = 1/2$, then the point $(1/2, a/4)$ is the maximum for the logistic map.

2.2 Graphical analysis of the logistic map

A reasonably precise graph of the logistic function will allow us to analyze graphically the orbits of elements of the domain. We add the line $y = x$ to the graph of the map function.

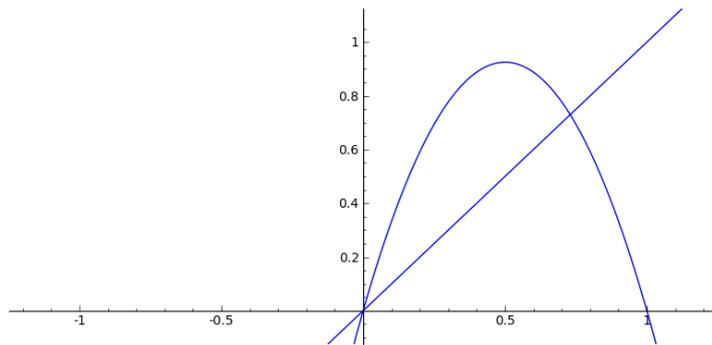


FIGURE 2.2: Adding the line $y = x$ to the graph of $f(x) = 3.7x - 3.7x^2$

Immediately, we see that the line $y = x$ intersects the logistic map. Finding the intersection points:

$$\begin{aligned} x &= ax - ax^2 \\ 0 &= -ax^2 + ax - x \\ 0 &= x(-ax + a - 1) \end{aligned}$$

$$x = 0 \text{ or } x = 1 - 1/a$$

We can visually represent iterations on this graph. Starting at x_0 on the x -axis, draw a vertical line to $f(x)$. For the next iteration, draw a horizontal line to the line $y = x$ and then again vertically to $f(x)$.

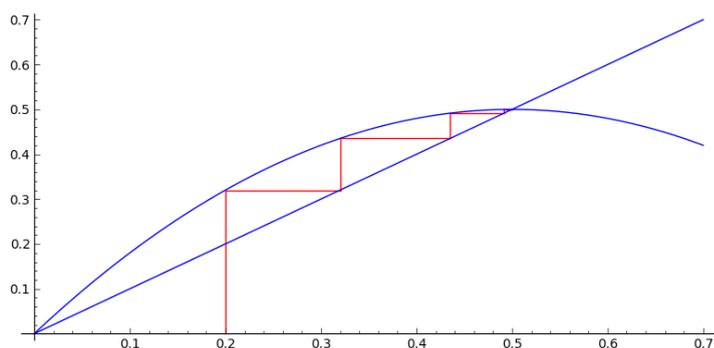


FIGURE 2.3: Graphical representation of the iterations of $f(x) = 2x - 2x^2$ with $x_0 = 0.2$

Earlier, we determined the points of intersection of the logistic map and the line $y = x$. In Figure 2.4, we see just a vertical line. This is a graphical representation of the orbit of a fixed point.

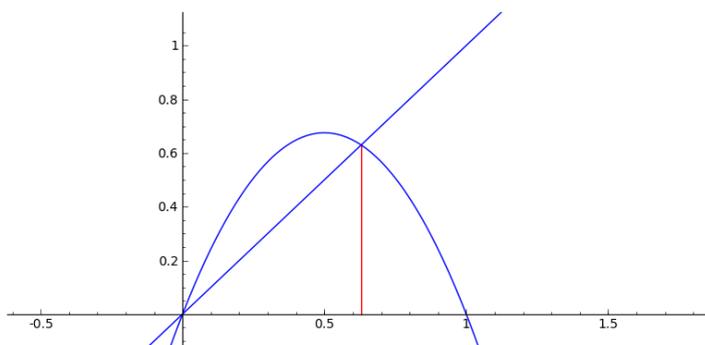


FIGURE 2.4: $f(x) = 2.7x - 2.7x^2$ with $x_0 = 1 - \frac{1}{2.7}$

From this image, we can make a conjecture that the intersections of the line $y = x$ and the logistic map are fixed points for the logistic map.

Evaluating the logistic map at the intersection $x_0 = 0$ gives us

$$\begin{aligned} f(0) &= a(0) - a(0)^2 \\ &= 0. \end{aligned}$$

Then evaluating at the second intersection $x_0 = 1 - \frac{1}{a}$ gives us

$$\begin{aligned} f\left(1 - \frac{1}{a}\right) &= a\left(1 - \frac{1}{a}\right) - a\left(1 - \frac{1}{a}\right)^2 \\ &= (a - 1) - a\left(1 - \frac{2}{a} + \frac{1}{a^2}\right) \\ &= a - 1 - a + 2 - \frac{1}{a} \\ &= 1 - \frac{1}{a}. \end{aligned}$$

Both intersection points are fixed points.

Fixed points can be classified into different types. They can be attracting, repelling, or neither.

Definition 4. Given a function $f : \mathbb{R} \rightarrow \mathbb{R}$ and an $\epsilon > 0$,

1. a fixed point p is said to be *attracting* if there is an $\epsilon > 0$, for all $x \in (p - \epsilon, p + \epsilon)$, $f^n(x) \rightarrow p$ as n increases without bound.
2. a fixed point p is said to be *repelling* if there is an $\epsilon > 0$, for all $x \in (p - \epsilon, p + \epsilon)$, $x \neq p$, $|f(x) - p| > |x - p|$.

Intuitively, points that are close to a fixed point p are either converge to p or diverge from p as the number of iterations increase without bound.

Theorem 1. Suppose that f is differentiable at a fixed point p .

1. If $|f'(p)| < 1$, then p is attracting.
2. If $|f'(p)| > 1$, then p is repelling.

Proof. We will prove both cases by induction

(1) Given any $\epsilon > 0$, we want to show that $\lim_{n \rightarrow \infty} f^n(x) = p$ for all $x \in (p - \epsilon, p + \epsilon)$ when $|f'(p)| < 1$. Suppose that $|f'(x)| < M < 1$ for some $M > 0$. Then as implied by the definition of derivative, there is an open interval $I = (p - \epsilon, p + \epsilon)$ such that $|f'(x)| \leq M < 1$ for all $x \in I$. Then we have

$$\left| \frac{f(x) - f(p)}{x - p} \right| \leq M$$

or $|f(x) - f(p)| \leq M|x - p|$. Since p is a fixed point we have $|f(x) - p| \leq M|x - p|$. Since $M < 1$, this inequality tells us that $M|x - p|$ is small but $|f(x) - p|$ is smaller, therefore $f(x)$ is closer to p than x . Also, since $x \in I$ and since $f(x)$ is closer to p than x , then $f(x)$ is also in I . Assume $|f^n(x) - p| \leq M^n|x - p|$ for all $n > 0$ holds. Then $f^n(x) \in I$

because $0 < M^n < M < 1$. We just showed the case for $n = 1$ yields $f(x) \in I$. Now we want to show that $|f^{n+1}(x) - p| \leq M^{n+1}|x - p|$ holds for $n > 1$.

$$|f^{n+1}(x) - p| = |f(f^n(x)) - p| \leq M|f^n(x) - p| \leq M(M^n|x - p|) = M^{n+1}|x - p|$$

Since $\lim_{n \rightarrow \infty} M^n = 0$, $f^n(x)$ has to be arbitrarily close to p . Thus $\lim_{n \rightarrow \infty} f^n(x) = p$ for all $x \in (p - \epsilon, p + \epsilon)$ when $|f'(p)| < 1$.

(2) Now we want to show that given an $\epsilon > 0$, $f^n(x)$ moves away from p when $x \in (p - \epsilon, p + \epsilon)$ and $x \neq p$. Suppose $|f'(x)| > M > 1$ for some $M > 0$. Then we have

$$\left| \frac{f(x) - f(p)}{x - p} \right| > M$$

or $|f(x) - f(p)| > M|x - p|$. With $M > 1$, this implies that $|f(x) - p| > |x - p|$ and that $f(x)$ is farther away from p than x . Assume $|f^n(x) - p| > M^n|x - p|$ is true for $n > 0$. We just showed that this true for $n = 1$. Now we want to show that $|f^{n+1}(x) - p| > M^{n+1}|x - p|$ holds for $n > 1$.

$$|f^{n+1} - p| = |f(f^n(x)) - p| > M|f^n(x) - p| > M(M^n)|x - p| = M^{n+1}|x - p|.$$

This implies then that as n increases without bound $f^n(x)$, when $x \in (p - \epsilon, p + \epsilon)$ and $x \neq p$, $f^n(x)$ moves farther away from p , thus p is repelling. \square

What happens if $|f'(x)| = 1$? It turns out that the fixed point may be attracting or repelling or both. This will need to be evaluated on a case by case basis.

Earlier, we found the fixed points of the logistic map to be 0 and $1 - 1/a$. We can now use Theorem 1 to determine whether these fixed points are attracting or repelling. First find the derivative of the logistic function, $f'(x) = a - 2ax$ and evaluate the absolute value of $f'(x)$ at the fixed points.

$x = 0$

1.

$$|f'(0)| < 1$$

$$-1 < a - 2a(0) < 1$$

$$-1 < a < 1$$

2.

$$|f'(0)| > 1$$

$$a - 2a(0) > 1 \text{ or } a - 2a(0) < -1$$

$$a > 1 \text{ or } a < -1$$

Now that $a \in [0, 4]$ we can say $x = 0$ is attracting when $a \in$ the open interval $(0, 1)$ and repelling when $a \in$ the semi-open interval $(1, 4]$.

$$x = 1 - 1/a$$

1.

$$|f'(1 - 1/a)| < 1$$

$$-1 < -a + 2 < 1$$

$$3 > a > 1$$

2.

$$|f'(1 - 1/a)| > 1$$

$$-a + 2 > 1 \text{ or } -a + 2 < -1$$

$$a < 1 \text{ or } a > 3$$

Since $a \in [0, 4]$ we can say $x = 1 - 1/a$ is attracting when $a \in (1, 3)$ (Figure 2.5) and repelling when $a \in (0, 1) \cup (3, 4)$ (Figure 2.6).

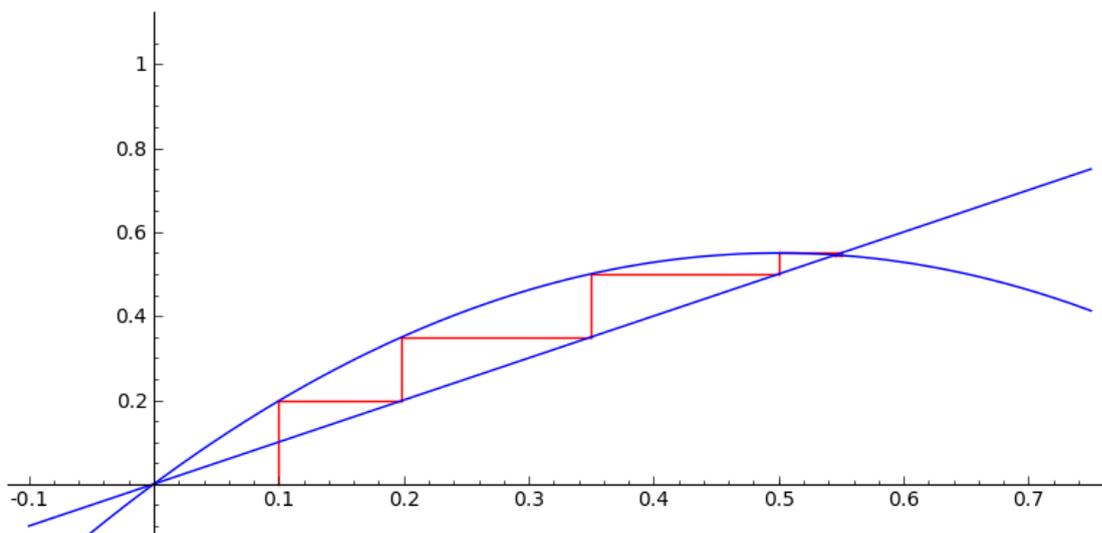
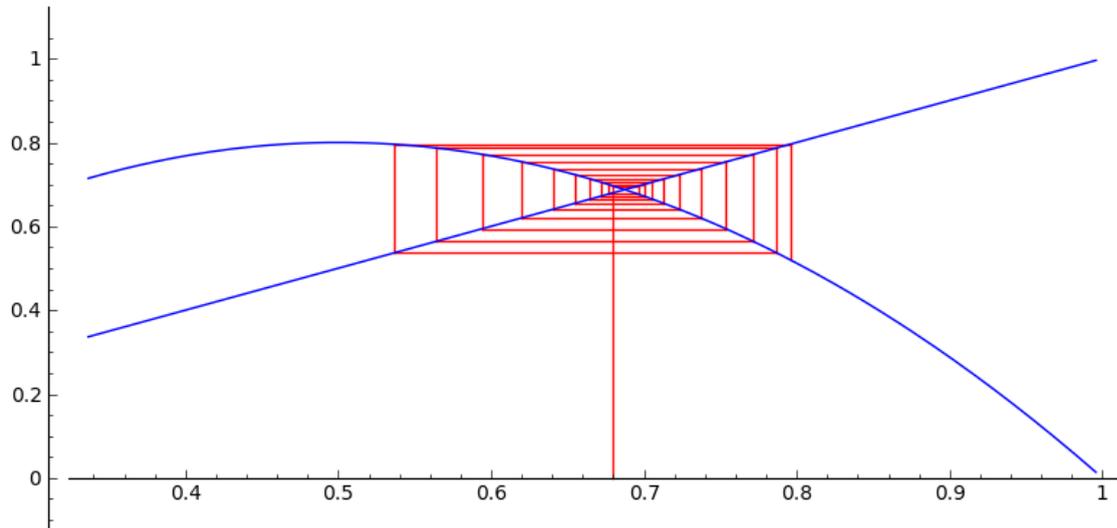
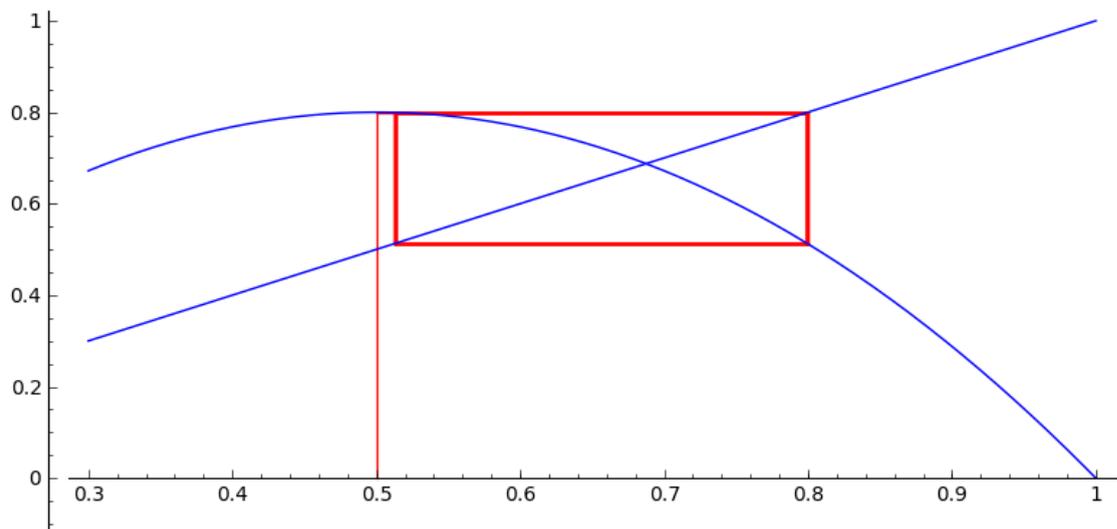


FIGURE 2.5: attracting fixed point $a = 2.2, x_0 = 0.1$

FIGURE 2.6: repelling fixed point $a = 3.2, x_0 = 0.68$

2.3 Bifurcation

Consider the map $f(x) = 3.2x - 3.2x^2$ with an initial value $x_0 = 0.5$. The fixed points are 0 and 0.6875, both of which are repelling since $|f'(x)| > 0$ at both fixed points. Notice what happens to $x_0 = 0.5$ as $f(x)$ is iterated, (Figure 2.7). This image represents a 2-period graph. For example, with an initial value $x_0 = 0.5$, repeated iterations of $f(x) = 3.2x - 3.2x^2$ alternates between two values. We can also say that 0.5 is a period-2 point for the map $f(x) = 3.2x - 3.2x^2$ with an orbit $\mathcal{O} = \{0.513, 0.799\}$.

FIGURE 2.7: $f(x) = 3.2x - 3.2x^2$

A series of questions can arise. Are there other initial values that have period-2? Are there initial values having period-3, period-4, etc.? Then what happens when we change the parameter a ?

These questions can be answered by creating a bifurcation diagram for the logistic map. First is to fix x_0 . Then we are going to vary the parameter a from 0 to 4. Using a computer program, we will compute 1000 iterations of the logistic map evaluated at x_0 while a varies between 0 and 4. On a Cartesian plane, we will plot a along the x -axis and the last 50 values of $f^{1000}(x_0)$ along the y -axis. Figure 2.8 represents this graph.

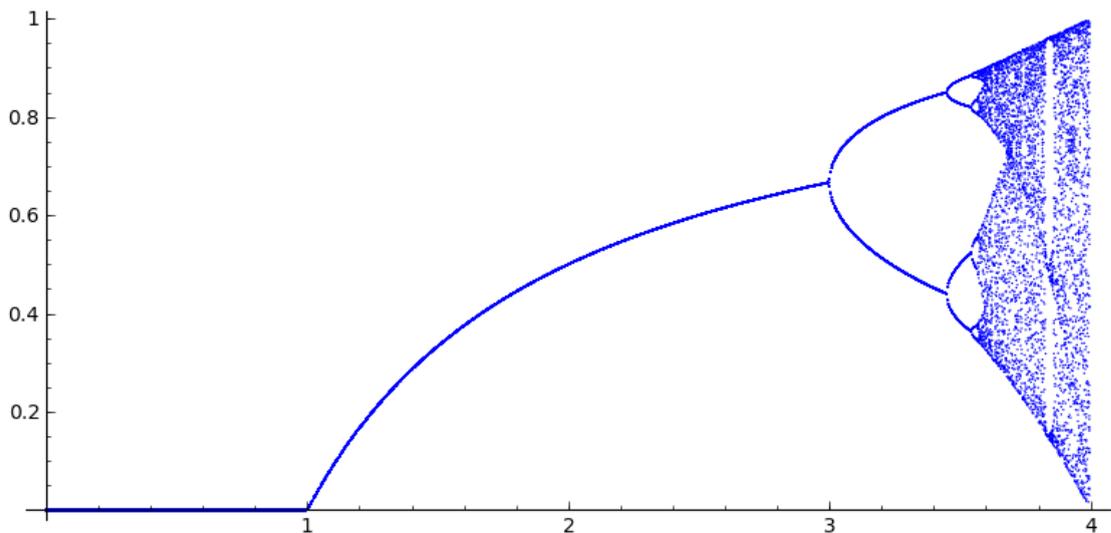


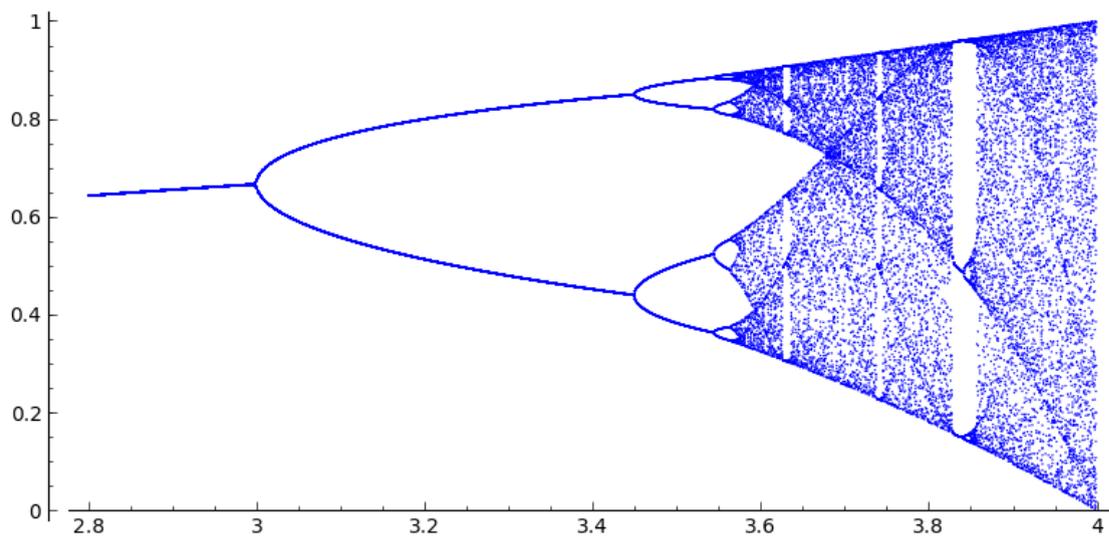
FIGURE 2.8: a on x -axis, $f^n(x_0)$ on y -axis

This is called the bifurcation diagram for the logistic map. Here's the information we can read from this diagram:

1. When $0 \leq a \leq 1$, $f^n(x_0) \rightarrow 0$.
2. When $1 < a < 3$, $f^n(x_0)$ converges to a single value ranging from 0 to approximately 0.625.

Remember Figure 2.7 demonstrated a point having period-2. The parameter a in this case was 3.2. In Figure 2.8, notice there is a split (or bifurcation) when $a > 3$. this bifurcation represents the number of periods an initial value has when a is a certain value.

A zoom on the bifurcation diagram is seen in Figure 2.9. Notice that some more bifurcations have become visible. In fact, the more we zoom in on the graph, the more bifurcations become visible. At each value of a where there is a bifurcation, the number of periods for an initial value doubles.

FIGURE 2.9: a on x -axis, $f^n(x_0)$ on y -axis

A question arises as to the rate of bifurcation as a increases. A possible way to determine the rate is to compare the ratios of distances between the bifurcations.

The first bifurcation occurs appears to occur at $a = 3$. Zooming in on the diagram will tell us the value for a at the next bifurcation. For ease, we will be just looking at the bifurcations on the lower part of the graph.

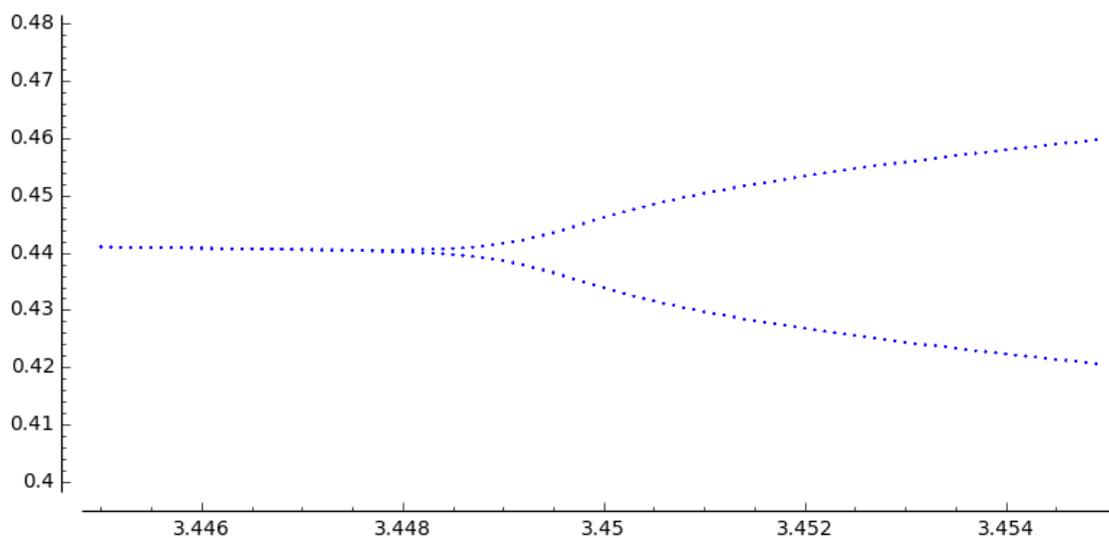


FIGURE 2.10: Second bifurcation at 3.4485

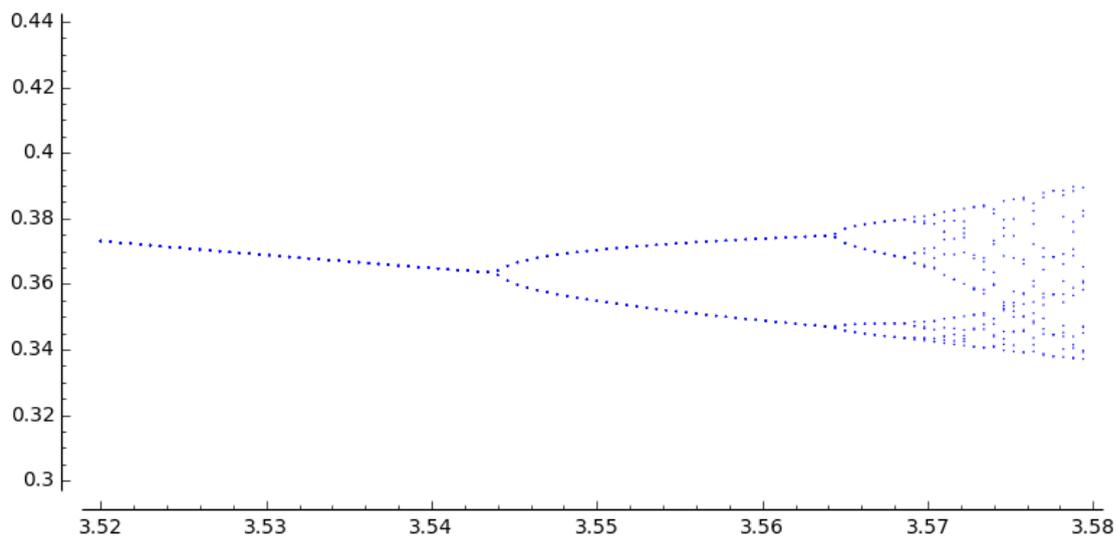


FIGURE 2.11: third bifurcation at 3.5438

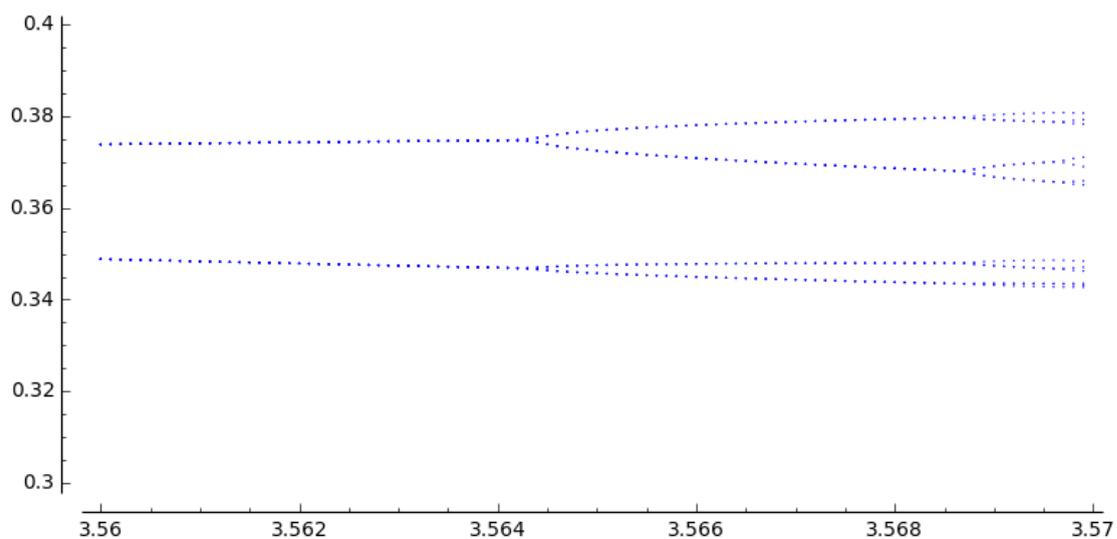


FIGURE 2.12: fourth bifurcation at 3.5645

Admittedly, this is not the most precise method. Perhaps an algebraic version may be better.

In Section 2.2, we found the fixed points for the logistic function. Suppose we try to find the fixed points for $f^2(x)$. First, let's evaluate $f^2(x)$.

$$\begin{aligned}
 f^2(x) &= a(ax(1-x))(1-(ax(1-x))) \\
 &= a^2x(1-x)(ax^2-ax+1) \\
 &= a^2x - (a^3+a^2)x^2 + 2a^3x^3 - a^3x^4
 \end{aligned}$$

n	bifurcations	a_n	$a_{n-1} - a_{n-2}$	$\frac{a_{n-1}-a_{n-2}}{a_n-a_{n-1}}$
2	2	3	NA	NA
3	4	3.4485	NA	NA
4	8	3.5438	0.4485	4.706
5	16	3.5645	0.0953	4.603

TABLE 2.1: Table of bifurcations where a_n is the value of a where we see a bifurcation.

Now we'll set this equation equal to x and solve.

$$\begin{aligned}
 a^2x - (a^3 + a^2)x^2 + 2a^3x^3 - a^3x^4 &= x \\
 a^2x - (a^3 + a^2)x^2 + 2a^3x^3 - a^3x^4 - x &= 0 \\
 x(a^2 - 1 - a^2x(1 + a) + 2a^3x^2 - a^3x^3) &= 0 \\
 -x(x - 1 + 1/a)(a^2x^2 - (a^2 + a)x + a + 1) &= 0
 \end{aligned}$$

$$x = 0, 1 - 1/a, \frac{\pm\sqrt{a^2 - 2a - 3} + a + 1}{2a}$$

We want x to be real, so we need the discriminant to be bigger than or equal to 0. So

$$\begin{aligned}
 a^2 - 2a - 3 &\geq 0 \\
 (a - 3)(a + 1) &\geq 0 \\
 a &\geq 3 \text{ or } a \leq -1
 \end{aligned}$$

However, we need $a > 0$. Notice in the bifurcation graph, the first bifurcation begins at $a = 3$.

When the next bifurcation occurs, we get initial values with period-4 cycles. Hence, we need to solve $f^4(x) = 0$. This process continues at each bifurcation as a gets bigger. Mitchell Feigenbaum worked on this process and came up with the following table. What

n	bifurcations	a_n	$a_{n-1} - a_{n-2}$	$\frac{a_{n-1}-a_{n-2}}{a_n-a_{n-1}}$
1	2	3	NA	NA
2	4	3.449490	NA	NA
3	8	3.544090	0.44949	4.7515
4	16	3.564407	0.09460	4.6562
5	32	3.568759	0.020317	4.6684
6	64	3.56989	0.004352	4.6692
7	128	3.56993	0.001131	4.6694

TABLE 2.2: Table of bifurcations where a_n is the value of a calculated algebraically.

he discovered is now called the Feigenbaum constant and is defined by

$$\lim_{n \rightarrow \infty} \frac{a_{n-1}a_{n-2}}{a_n - a_{n-1}} \approx 4.669\dots$$

This number can be used to predict subsequent values of a where there is a split on the bifurcation diagram. In other words, we can predict the values of a where we can find a period doubling.

Returning to the questions at the beginning of this section, we can deduce answers from the bifurcation diagram. Period-2 points of f appear when $3 < a < 3.44$. Period-4 points appear when $3.44 < a < 3.54$. Period-3 points appear in the small strip of open space that appears before $a = 4$. There will be more on this in the next section.

2.4 Chaos

We have looked at fixed points, periodic points and the bifurcation graph of the logistic function. Now there is one more characteristic of the logistic map to consider: chaos.

Consider the natural numbers. Alexander Sharkovsky re-ordered the natural numbers in the following way:

1. Start listing the odd numbers, starting with 3
2. Then list the odd numbers again, this time multiplied by 2
3. Again, list the odd numbers, this time multiplied by 2^2
4. continue until all odd numbers and powers of 2 are exhausted
5. list in descending order the powers of 2
6. end the list with the number 1

$$3 \triangleright 5 \triangleright 7 \triangleright \dots \triangleright 2 \cdot 3 \triangleright 2 \cdot 5 \triangleright 2 \cdot 7 \triangleright \dots \triangleright 2^2 \cdot 3 \triangleright 2^2 \cdot 5 \triangleright 2^2 \cdot 7 \triangleright \dots \triangleright 2^3 \triangleright 2^2 \triangleright 2 \triangleright 1$$

where $m \triangleright l$ is read as m is to the left of l .

Theorem 2 (Sharkovsky). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous. If f has a periodic point with period- m and $m \triangleright l$, then f also has a periodic point with period- l .

Sharkovsky published his theorem in the Ukrainian Mathematics Journal in 1964. Unfortunately, it was not widely known until Li and Yorke published their theorem in 1975.

Theorem 3 (Li and Yorke). Let $f : \mathbb{R} \rightarrow \mathbb{R}$ be continuous. If there exists a periodic point with period-3 in f , then all periods exist in f .

The obvious question is whether or not the logistic map has period 3. Referring to Figure 2.8, there is an relatively open gap shortly after $a = 3.8$. Zooming into this section we see that there is a period-3 in this gap.

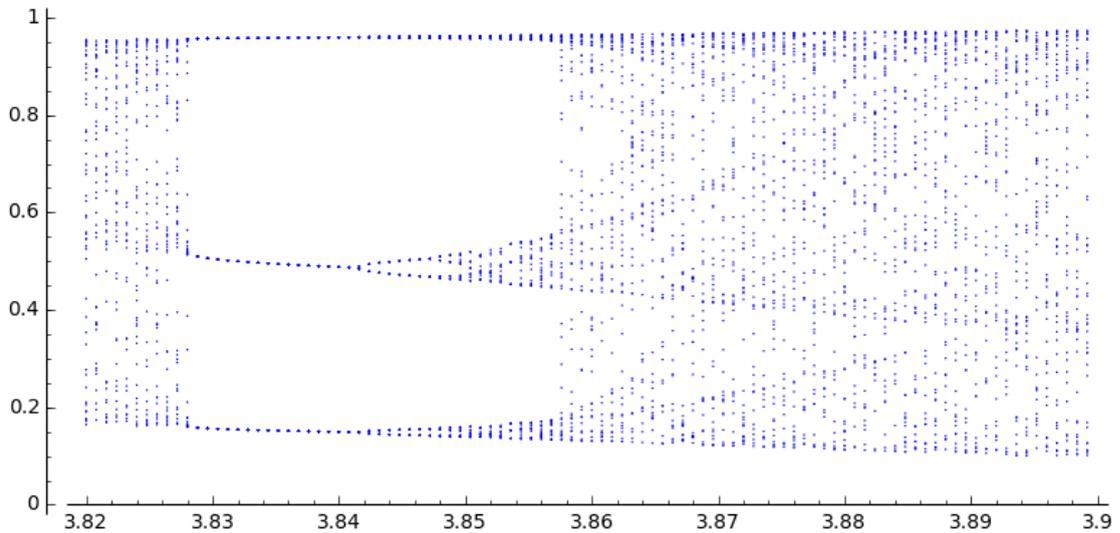


FIGURE 2.13: Zoom to $3.82 < a < 4.0$

In Figure 2.13, it appears that period-3 points exist when a is approximately between 3.83 and 3.84.

n^{th} iteration	value
f^{992}	0.156149315683605
f^{993}	0.504666487408413
f^{994}	0.957416597518873
f^{995}	0.156149315683605
f^{996}	0.504666487408413
f^{997}	0.957416597518873
f^{998}	0.1561493156836054
f^{999}	0.504666487408413
f^{1000}	0.957416597518873

TABLE 2.3: The last 9 iterations of a period-3 point for $3.83x - 3.83x^2$.

As we can see from Figure 2.13 and from Table 2.3, we safely say that the logistic map has a periodic point with period 3.

Referring to Li and Yorke, since there is a periodic point for the d-h logistic function, then one can find periodic points with all periods. Sharkovsky's Theorem reinforces that idea.

Visually, this can be seen in Figure 2.8 at $a = 4$. Points appear in a solid stripe above $a = 4$. In a graphical analysis map, chaos fills in nearly the whole image.

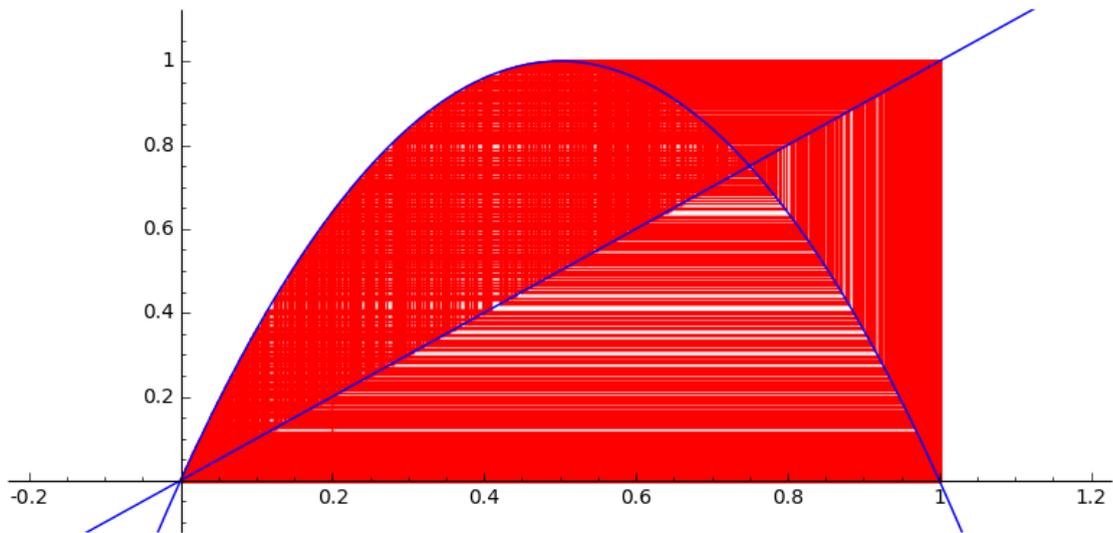


FIGURE 2.14: $4x - 4x^2$ with $x_0 = .2$

Let's review what we have so far. When $3 \leq a \leq 3.4485$, x is a period-2 point for the logistic map. The period doubles at a regular rate as determined by the Feigenbaum number. When a is approximately between 3.83 and 3.84, x is a period-3 point and begins period doubling again after $a > 3.84$. When $a = 4$, we have chaos. What this means is that x has an infinite period for the logistic map.

Different literature define chaos in different ways. One common feature, however, is the idea of being sensitive initial conditions.

x_0	$\frac{1}{3}$	0.333
$f(x_0)$	$\frac{8}{9} \approx 0.888$	0.888
$f^2(x_0)$	$\frac{32}{81} \approx 0.395$	0.396
$f^3(x_0)$	$\frac{6272}{6561} \approx 0.956$	0.957
$f^4(x_0)$	$\frac{7250432}{43046721} \approx 0.168$	0.164
$f^5(x_0)$	≈ 0.560	0.549

TABLE 2.4: Five iterations of $4x - 4x^2$ with different representations of x_0 .

Notice what is happening in Table 2.4. Though 0.333 is an approximation of $1/3$, we start to see a difference in values in f^2 . From the table, we can infer that as the number of iterations increase, the difference between the resulting values increases.

Definition 5. Let J be an interval and suppose that $f : J \rightarrow J$. Then f has sensitive dependence on initial conditions at x , or just sensitive dependence at x if there is an

$\epsilon > 0$ such that for each $\delta > 0$ there is a y in J and a positive integer n such that

$$|x - y| < \delta$$

and

$$|f^n(x) - f^n(y)| > \epsilon.$$

Another way to state this is that if we fix ϵ , no matter how close we choose 2 initial values, there will be an n where the difference of f^n of the initial values will be greater than ϵ .

In regards to Table 2.4, the difference between x and y is $\delta = 0.000333$. If we choose $\epsilon > 0.001$, we see that the difference between the two values at f^4 exceeds ϵ . Now say we want x and y to be closer, say $y = 0.3333$. Then $\delta = 0.0000333$. Remember that we fixed $\epsilon = 0.001$. At f^7 , the resulting difference between the values is 0.002, which exceeds our ϵ . We can say that the logistic map is sensitive to initial conditions.

Considering that we have sensitivity to initial conditions for the logistic map, we can assume that the logistic map is a chaotic system.

Chapter 3

The Double-Humped Logistic Map

3.1 What is the double-humped logistic map?

Previously, we have looked at the logistic map. For further investigation into iteration and chaos, I was curious as to what happens when the logistic map has an extra hump added to it. What we get is what I will call the 'double-humped' logistic map. It is defined as

$$f(x) = a(x - 1)^2(1 - (x - 1)^2)$$

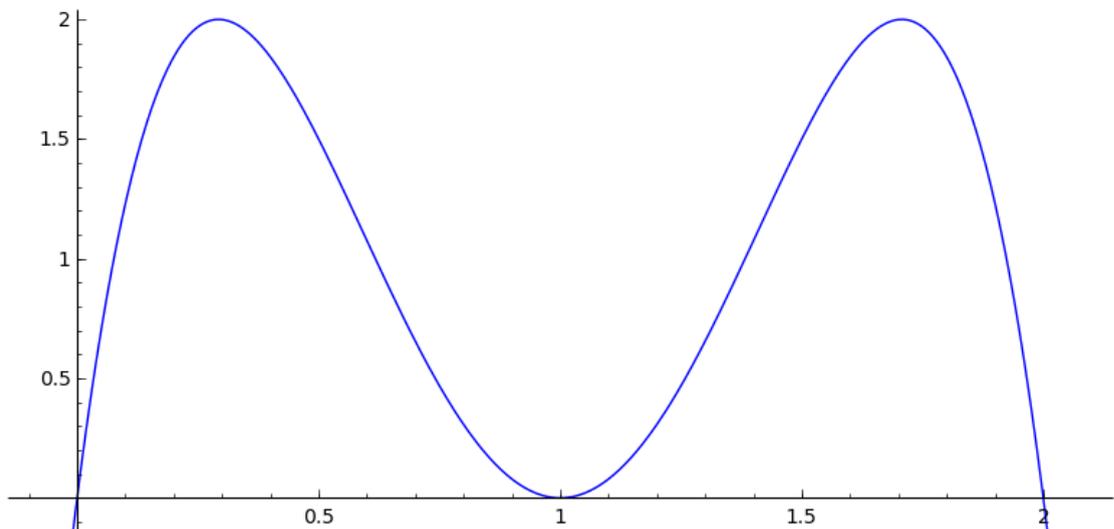


FIGURE 3.1: $f(x) = (x - 1)^2(1 - (x - 1)^2)$

3.2 Properties

Visually, Figure 3.1 tells us the points of intersection are at $x = 0$, $x = 1$ and $x = 2$. To verify algebraically:

$$\begin{aligned} a(x-1)^2(1-(x-1)^2) &= 0 \\ a(x-1)^2 &= 0 \text{ or } (1-(x-1)^2) = 0 \\ x &= 1 \text{ or } 2x-x^2 = 0 \\ x &= 1 \text{ or } x = 0 \text{ or } x = 2 \end{aligned}$$

In finding the maximums/minimums we take the derivative and set it equal to 0.

$$\begin{aligned} f'(x) &= 0 \\ a(-4x^3 + 12x^2 - 10x + 2) &= 0 \\ x &= 1 \text{ or } x = 1 - \frac{1}{\sqrt{2}} \text{ or } x = 1 + \frac{1}{\sqrt{2}} \end{aligned}$$

Now to check concavity, evaluate f'' at the three points.

$$\begin{aligned} f''(1) &= -2a(6(1)^2 - 12(1) + 5) = 2a \\ f''\left(1 - \frac{1}{\sqrt{2}}\right) &= -2a\left(6\left(1 - \frac{1}{\sqrt{2}}\right)^2 - 12\left(1 - \frac{1}{\sqrt{2}}\right) + 5\right) = -4a \\ f''\left(1 + \frac{1}{\sqrt{2}}\right) &= -2a\left(6\left(1 + \frac{1}{\sqrt{2}}\right)^2 - 12\left(1 + \frac{1}{\sqrt{2}}\right) + 5\right) = -4a \end{aligned}$$

Keeping $0 < a < 8$, the signs do not change. Thus f is concave up at $x = 1$ and is a minimum, while f is concave down at $x = 1 \pm \frac{1}{\sqrt{2}}$ signifies those points are maximums.

3.2.1 Graphical Analysis

As with the logistic function, we are going to add the line $y = x$ to the graph of the d-h-logistic function. Then we are going to determine the points of intersection which tells us the fixed points.

This will be tricky to do for depending on the parameter a , there may be 1 or 2 or 3 or 4 points of intersection. However in all cases, we know that one point of intersection is $x = 0$.

It is possible to determine the points of intersection using the quartic formula. It is even possible to factor out the root $x = 0$ and use the cubic formula on the result. In this case, if there are three points of intersection, the cubic formula will return 3 real numbers, two points of intersection will return 2 real roots and 1 complex root, and one point of intersection will return one real root and 2 complex roots. It may be easier to break $a \in (0, 8]$ into intervals.

1. $0 < a \leq 0.5$

There is one point of intersection: $x = 0$

2. $.5 < a < 6.75$

There are two points of intersection: $x = 0$ and a very ugly solution to a cubic equation

3. $a = 6.75$

There are three points of intersection: $x = 0, x = 2/3, x = 5/3$

4. $6.75 < a \leq 8$

There are four points of intersection: $x = 0$ and very ugly solutions to a quartic equation.

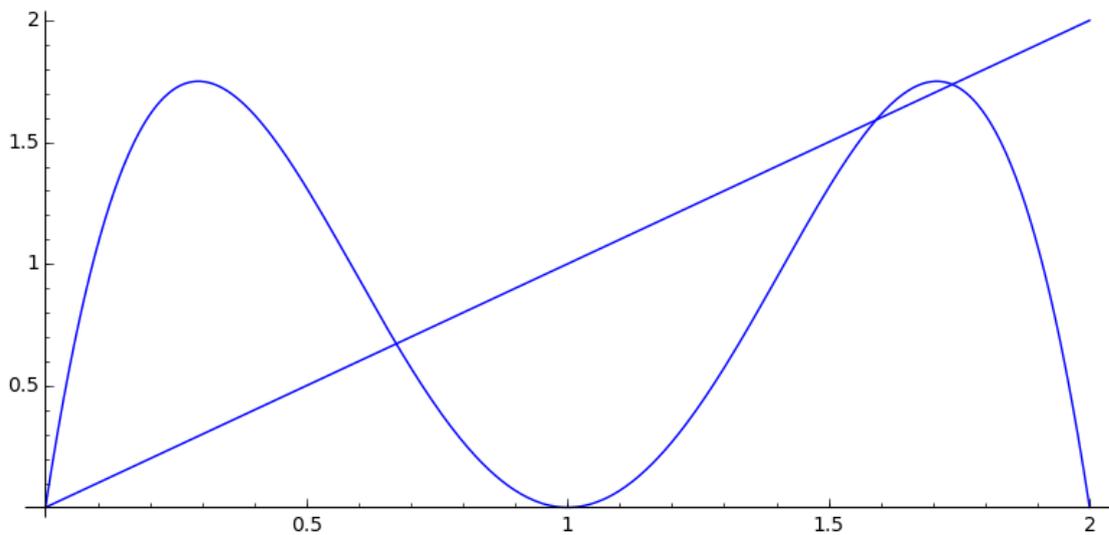


FIGURE 3.2: $f(x) = 7(x - 1)^2(1 - (x - 1)^2)$

3.2.2 Bifurcation diagram

Once again, we repeat the process we did with the logistic map and the resulting bifurcation diagram can be seen in Figure 3.3

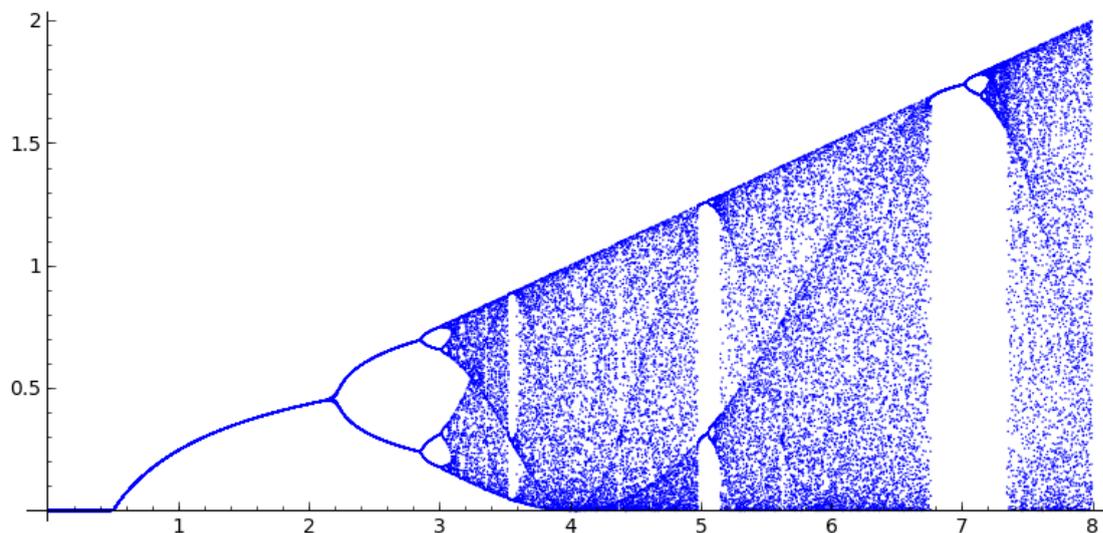
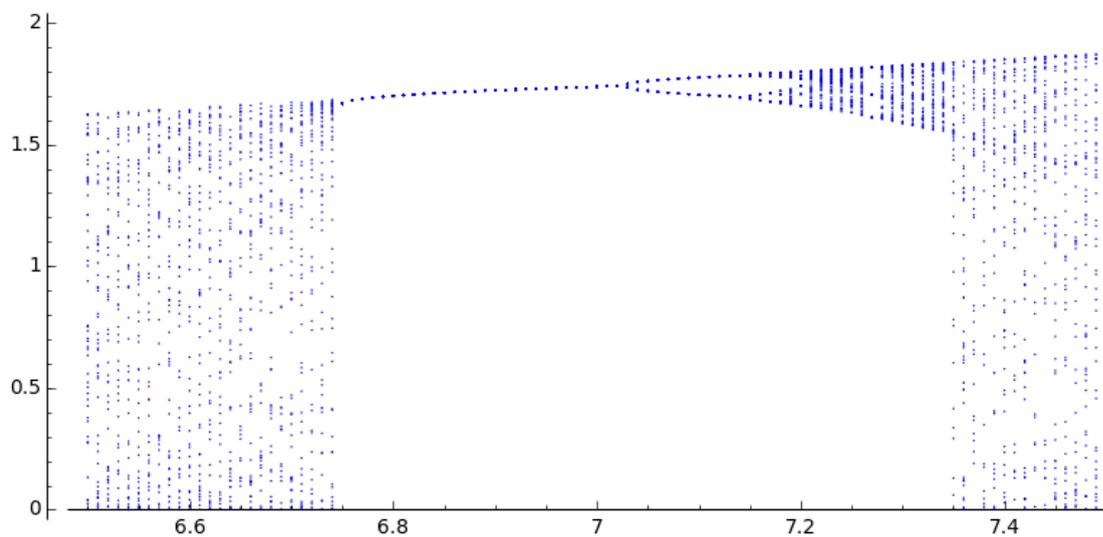


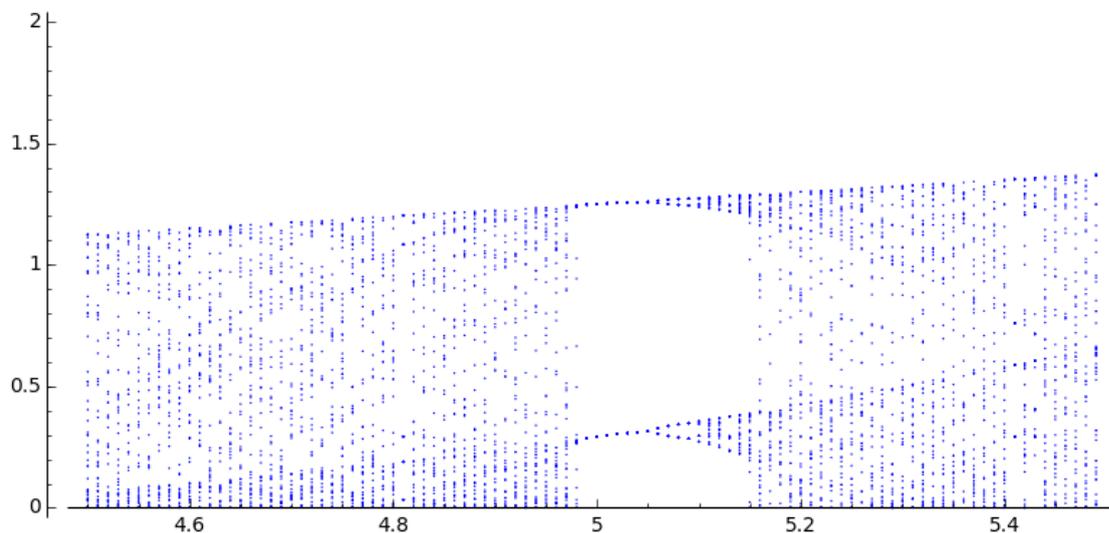
FIGURE 3.3: Bifurcation diagram for the double-humped logistic map

The first thing noticed is that this is very similar to the bifurcation diagram for the logistic function. We see that there is a possible repeated bifurcation and that there is a large gap around $a = 7$. Let's look at that first.

FIGURE 3.4: zoom to $6.5 < a < 7.5$

It appears that when a is approximately between 6.75 and 7.0, f^n converges to a single point.

The next large gap appears to be around $a = 5$. If we zoom into that region (Figure 3.6), we see that f^n produces a 2-cycle.

FIGURE 3.5: zoom to $4.5 < a < 5.5$

Yet another gap to look at is around $a = 3.6$. Another zoom to this region appears to show a 3-cycle when $3.4 < a < 3.6$. If this is a true 3-cycle, then we can say that the double-humped logistic map is a chaotic system.

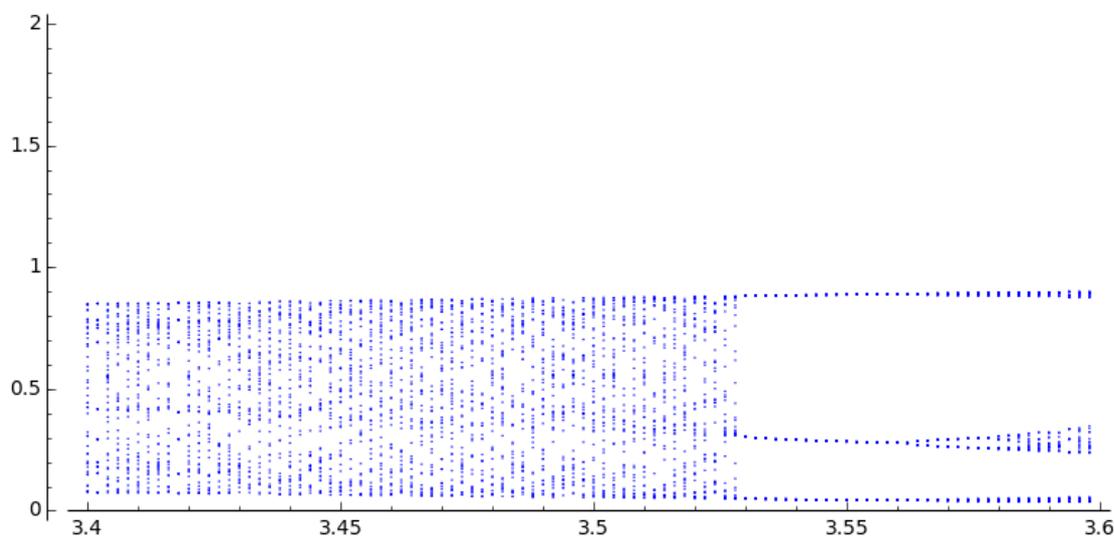
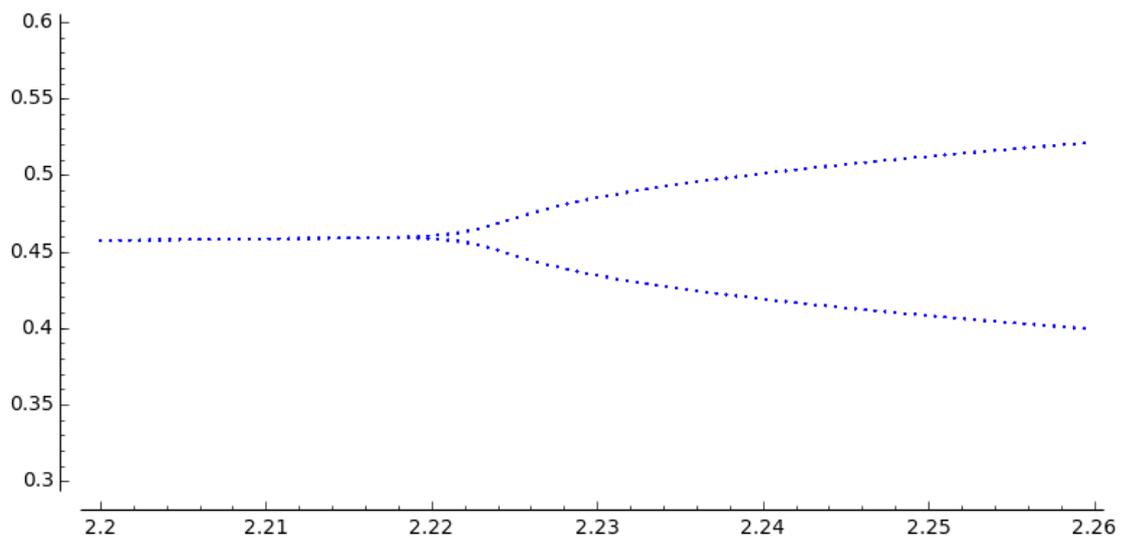
FIGURE 3.6: zoom to $3.4 < a < 3.6$

Table 3.1 shows that after multiple iterations, a period-3 point becomes evident, so we can truly say that the d-h logistic map is chaotic.

Now let's check the points of bifurcation

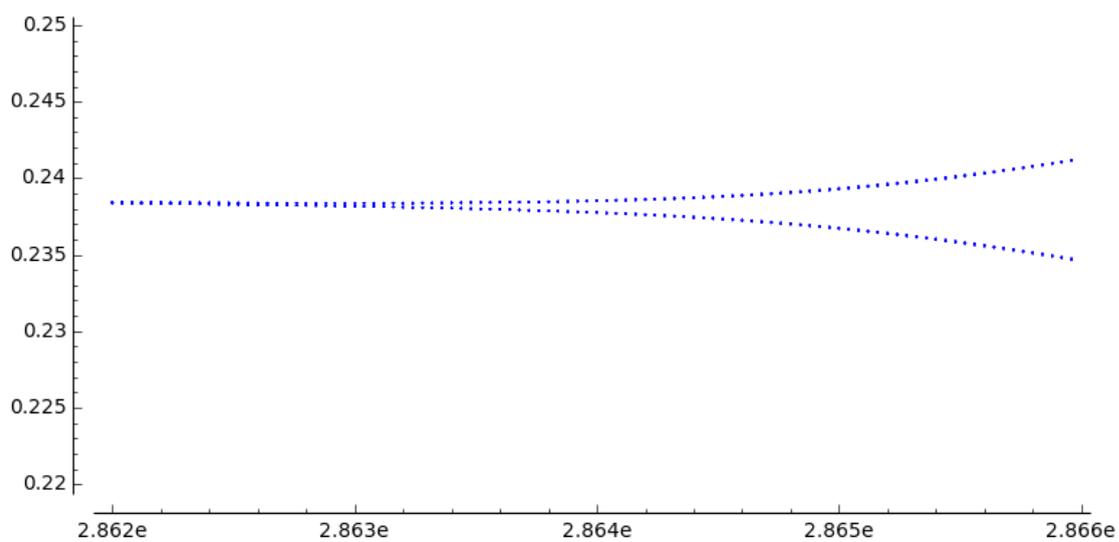
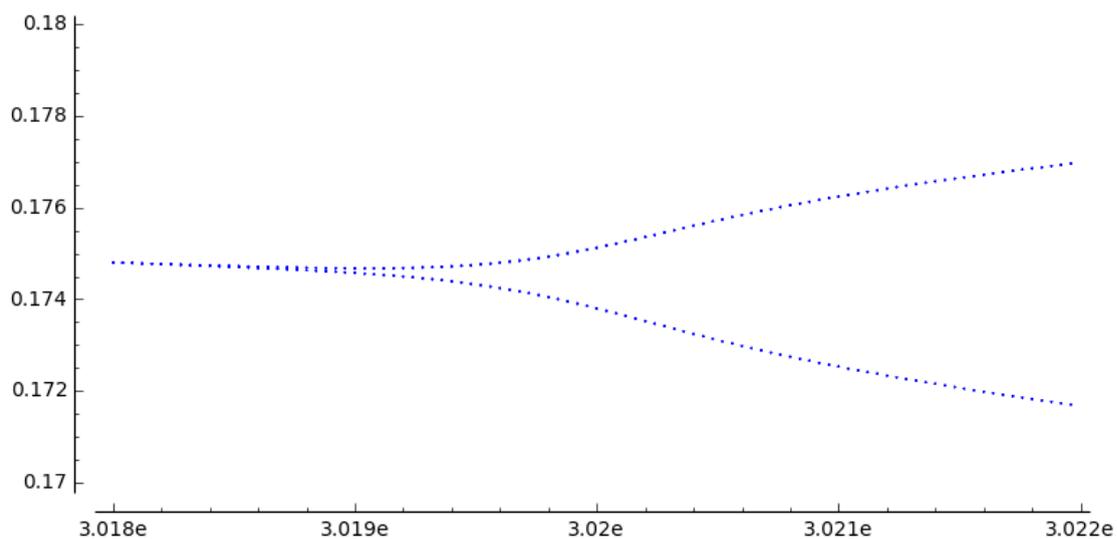
n^{th} iteration	value
f^{992}	0.0448213566327823
f^{993}	0.283836996403593
f^{994}	0.886910210588010
f^{995}	0.0448213566327823
f^{996}	0.283836996403593
f^{997}	0.886910210588010
f^{998}	0.0448213566327823
f^{999}	0.283836996403593
f^{1000}	0.886910210588010

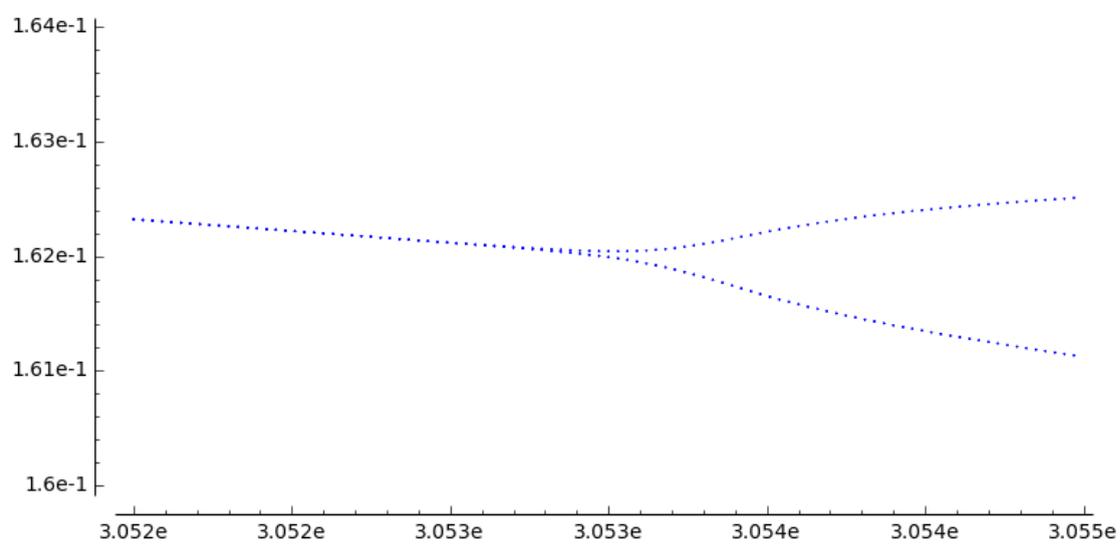
TABLE 3.1: The last 9 iterations of a period-3 point for $3.55x^2(1-x^2)$.FIGURE 3.7: First bifurcation: zoom to $2.20 < a < 2.26$

n	bifurcations	a_n	$a_{n-1} - a_{n-2}$	$\frac{a_{n-1} - a_{n-2}}{a_n - a_{n-1}}$
2	2	2.219	NA	NA
3	4	2.863	NA	NA
4	8	3.019	0.644	4.117
5	16	3.053	0.156	4.588
6	32	3.061	0.034	4.250

TABLE 3.2: Table of bifurcations where a_n is the value of a where we see a bifurcation.

In Table 3.2, I visually determined a by zooming in on successive bifurcations. Unfortunately, there may be some human error in determining these values. These numbers could also be found algebraically, but we will need advanced math analysis to determine these ratios. I suspect we will see Feigenbaum's number again. For my visual numbers are close to what we have seen earlier in the logistic map. With better computers, I may be able to get more precise images to better determine bifurcations.

FIGURE 3.8: Second bifurcation: zoom to $2.862 < a < 2.866$ FIGURE 3.9: Third bifurcation: zoom to $3.018 < a < 3.022$

FIGURE 3.10: Fourth bifurcation: zoom to $3.052 < a < 3.055$

Chapter 4

Conclusion

4.1 Conclusion

What I have just presented is a brief overview of deterministic chaos. We have learned that the logistic map is a good model for studying chaos. A slightly different map, the double humped logistic map, can also be used to study chaos. There was also attracting and repelling fixed points as well as periodic finding points with a period of 3 indicate that both maps are chaotic.

Then of course there is the sensitivity to initial conditions. This is a topic that arises with computer calculations. Table 2.4 was calculated with the use of a computer. But notice the error that arises after a few calculations. Using a decimal representation for a fraction may lead to computing errors.

In my research, I found the Feigenbaum number appears in the quadratic class of functions. With the double-humped logistic function being a quadratic, this is why I suspected we would see the Feigenbaum number appear in Table 3.2

I found that with the logistic map, the Feigenbaum number is easier to calculate. This is mainly because of the exponents involved do not increase as fast as with the d-h logistic function. Graphical analyzation works to an extent, but numerical analysis would have to be applied when exponents are larger than 4.

One question I would like to research more has to do with the period-3 window in the bifurcation diagrams. I am curious as to why all of the sudden period doubling stops and changes to a period-3 before and then restarts the period doubling. I am also curious as to why the period-1 window appears in the d-h logistic bifurcation diagram and then explodes back into a number of bifurcations.

Appendix A

Coding

Below is a list of coding developed in SAGE for research in this thesis.

```
def iterquad(a,x0,N=100):
    iterates = [x0]
    for i in range(N):
        iterates.append( a*iterates[-1]*(1-iterates[-1]) )
    return iterates

def itergraph ( a, x0, n=1000 ):\\
    iterates = iterquad( a,x0,n )\\
    G = Graphics()\\
    G += line( [ (iterates[0],0), (iterates[0],iterates[1]) ], color="red" )\\
    for i in range(n-1):\\
        G += line( [ (iterates[i],iterates[i+1]), (iterates[i+1],iterates[i+1]) ], color="red" )\\
        G += line( [ (iterates[i+1],iterates[i+1]), (iterates[i+1],iterates[i+2]) ], color="red" )\\
    G += plot ( x, G.xmin()-.2, G.xmax()+.2,xmin=-.1,ymin=-.1,xmax=1.1,ymax=1.1 )\\
    G += plot ( a*x*(1-x), G.xmin(), G.xmax(),xmin=-.1,ymin=-.1,xmax=1.1,ymax=1.1 )\\
    G.show()$

$def feigenbaum( x0 ):\\
    output = []\\
    for i in range(0,1000):\\
        a = 4*i/1000\\
        iterates = iterquad( a, x0, 1000 )\\
```

```
        for i in range(951,1000):\\
            output.append( [a,iterates[i]] )\\
    return point(output,size=1)$

$def feigenbaum_zoom( x0, l, r, y1, y2 ):\\
    output = []\\
    for i in range(0,100):\\
        a = l + (r-l)*i/100\\
        iterates = iterquad( a, x0, 1000 )\\
        for i in range(950,1000):\\
            output.append( [a,iterates[i]] )\\
    return point(output,ymin=y1,ymax=y2,size=1)
```

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