### Stably Free Modules over the Klein Bottle

### Andrew Misseldine

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### Polynomials

Let R be a ring.

Then the collection of all polynomials with coefficients from R is a ring.

Denote this ring as

R[x].

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# Polynomials

Let R be a ring.

A **Laurent polynomial** is a polynomial which allows negative exponents.

Then the collection of all Laurent polynomials with coefficients from R is a ring.

Denote this ring as

 $R[x, x^{-1}].$ 

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# Polynomials

Let R be a ring and let  $\sigma : R \rightarrow R$  be a ring isomorphism.

A skew Laurent polynomial is a Laurent polynomial but with the extra condition that  $rx = xr^{\sigma}$ .

Then the collection of all skew Laurent polynomials with coefficients from R is a ring.

Denote this ring as

$$R[x, x^{-1}; \sigma].$$

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Constructing a Stably Free Module Applications to the Klein Bottle

### **Projective Modules**

### Definition - Free Module

### An *R*-module *M* is **free** iff $M \cong \bigoplus_n R$ for some cardinal *n*.

### {Free Modules}

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### **Projective Modules**

### Definition - Projective Module

An *R*-module *P* is **projective** iff *P* is a direct summand of a free module, that is,  $\exists Q \ R$ -module such that  $P \oplus Q \cong R^n$ .

 ${Free Modules} \subseteq {Projective Modules}$ 

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### **Projective Modules**

### Definition - Stably Free Module

An *R*-module *P* is **stably free** iff there exists natural numbers *m*, *n* such that  $P \oplus R^m \cong R^n$ .

 $\{Free Modules\} \subseteq \{Stably Free Modules\} \subseteq \{Projective Modules\}$ 

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### Quillen-Suslin Theorem

Projective modules over group algebras play a key role in many aspects of geometry and topology.

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## Quillen-Suslin Theorem

Projective modules over group algebras play a key role in many aspects of geometry and topology.

Theorem (The Quillen-Suslin Theorem (1976))

Let k be a commutative ring. Then all projective modules over  $k[x_1, \ldots, x_n]$  are free.

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### Quillen-Suslin Theorem

Projective modules over group algebras play a key role in many aspects of geometry and topology.

Theorem (Generalized Quillen-Suslin Theorem (Swan 1978))

Let k be a commutative ring. Then all projective modules over  $k[x_1, \ldots, x_n, x_1^{-1}, \ldots, x_n^{-1}]$  are free.

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Constructing a Stably Free Module Applications to the Klein Bottle

### This is a Klein Bottle



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Constructing a Stably Free Module Applications to the Klein Bottle

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We see from the fundamental square, that

$$\pi(X, v_0) = \langle x, y | x^{-1}yx = y^{-1} \rangle$$
$$= \langle x, y | yx = xy^{-1} \rangle$$

Call this group G.

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Call this group G.

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In light of the presentation of G,

$$\mathbb{Z}G = R[x, x^{-1}; \sigma]$$

where

$$R = \mathbb{Z}[y, y^{-1}]$$

and  $\boldsymbol{\sigma}$  is the isomorphism induced by

$$\sigma: y \longmapsto y^{-1}.$$

In particular,

$$yx = xy^{-1}$$

as polynomials in  $\mathbb{Z}G$ .

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$$(G:\langle y,x^2\rangle)=2.$$

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We note that

$$\langle y, x^2 \rangle \leqslant G,$$

$$\langle y, x^2 \rangle \cong \mathbb{Z} \times \mathbb{Z},$$

$$(G:\langle y,x^2\rangle)=2.$$

This tiny bit of noncommutativity guarantees the existence of nonfree projective modules over the Klein bottle.

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## The $\mathbb{Z}G$ -module K

Let

$$r=1+y+y^3\in\mathbb{Z}G,$$

and let

 $s = r^{\sigma^{-1}} \in \mathbb{Z}G.$ 

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Let

$$K = \{f \in \mathbb{Z}G \mid rf = (x+s)g, g \in kG\}$$
$$\cong \langle r \rangle \cap \langle x+s \rangle$$
$$\cong \langle 1+y+y^3 \rangle \cap \langle x+1+y^{-1}+y^{-3} \rangle$$

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### Some Important Maps

Let  $f \in K$ . Then  $\exists g \in \mathbb{Z}G$  such that

$$rf = (x+s)g.$$

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Let  $i: K \to \mathbb{Z}G \oplus \mathbb{Z}G$  as

$$i: f \mapsto \left( \begin{array}{c} f \\ -\theta(f) \end{array} 
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Let  $\pi : \mathbb{Z}G \oplus \mathbb{Z}G \to \mathbb{Z}G$  as

$$\pi = \left( \begin{array}{cc} r & x+s \end{array} \right).$$

### K is Stably Free

#### Theorem

# The sequence $0 \longrightarrow K \xrightarrow{i} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\pi} \mathbb{Z}G \longrightarrow 0$ is exact.

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#### Theorem

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(Pf)  $\pi$  is surjective.

$$\pi \begin{pmatrix} (sx^{-2}) \\ (x^{-1} - rx^{-2}) \end{pmatrix} = r(sx^{-2}) + (x + s)(x^{-1} - rx^{-2})$$
  
=  $r(r^{\sigma^{-1}}x^{-2}) + (x + r^{\sigma^{-1}})(x^{-1} - rx^{-2})$   
= 1.

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(Pf) im  $i \subseteq \ker \pi$ .

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$$\pi i(f) = \pi \begin{pmatrix} f \\ -g \end{pmatrix}$$
$$= rf - (x+s)g$$
$$= 0.$$

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$$\pi \begin{pmatrix} f \\ g \end{pmatrix} = 0$$
  

$$\Rightarrow rf + (x + s)g = 0$$
  

$$\Rightarrow rf = -(x + s)g = (x + s)(-g)$$
  

$$\Rightarrow f \in K$$
  

$$\Rightarrow \begin{pmatrix} f \\ g \end{pmatrix} = \begin{pmatrix} f \\ -(-g) \end{pmatrix} = i(f) \in \operatorname{im} i.$$

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#### Theorem

The sequence 
$$0 \longrightarrow K \xrightarrow{i} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\pi} \mathbb{Z}G \longrightarrow 0$$
 is exact.

Now, the above sequence is exact. But  $\mathbb{Z}G$  is projective. Thus,  $\exists s : \mathbb{Z}G \to \mathbb{Z}G^2$  such that

 $\pi \mathfrak{L} = \mathbf{1}_{\mathbb{Z}G}.$ 

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 is exact.

Define  $\mathfrak{z}: \mathbb{Z}G \to \mathbb{Z}G \oplus \mathbb{Z}G$  as

$$\mathfrak{L} = \left(\begin{array}{c} \mathfrak{s} \mathfrak{x}^{-2} \\ \mathfrak{x}^{-1} - \mathfrak{r} \mathfrak{x}^{-2} \end{array}\right)$$

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$$\delta = \left(\begin{array}{c} sx^{-2} \\ x^{-1} - rx^{-2} \end{array}\right)$$

Corollary

K is a stably free  $\mathbb{Z}G$ -module, that is,  $K \oplus \mathbb{Z}G \cong \mathbb{Z}G \oplus \mathbb{Z}G$ .

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Define  $p : \mathbb{Z}G \oplus \mathbb{Z}G \to K$  as

$$p: \vec{m} \mapsto i^{-1}(\vec{m} - \lambda \pi \vec{m}).$$

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The sequence 
$$0 \longrightarrow \mathbb{Z}G \xrightarrow{s} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{p} K \longrightarrow 0$$
 is exact.

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Define  $p : \mathbb{Z}G \oplus \mathbb{Z}G \to K$  as

$$p = (1 - (sx^{-2})r - (sx^{-2})(x+s)).$$

### Theorem

The sequence 
$$0 \longrightarrow \mathbb{Z}G \xrightarrow{s} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{p} K \longrightarrow 0$$
 is exact.

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Define  $p : \mathbb{Z}G \oplus \mathbb{Z}G \to K$  as

$$p = (1 - (sx^{-2})r - (sx^{-2})(x + s)).$$

### Theorem

The sequence 
$$0 \longrightarrow \mathbb{Z}G \xrightarrow{s} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{p} K \longrightarrow 0$$
 is exact.

Hence, we see the following diagram with exactness in both directions.



Presentation of K

### $0 \longrightarrow \mathbb{Z}G \xrightarrow{\scriptscriptstyle \wedge} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{p} K \longrightarrow 0$

Andrew Misseldine Stably Free Modules over the Klein Bottle

Presentation of K

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{s} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{p} K \longrightarrow 0$$

$$K \cong \mathbb{Z}G^2 / \operatorname{im} \mathfrak{Z}$$

Andrew Misseldine Stably Free Modules over the Klein Bottle

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Presentation of K

$$0 \longrightarrow \mathbb{Z}G \xrightarrow{\delta} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{p} K \longrightarrow 0$$

 $K \cong \mathbb{Z}G^2 / \operatorname{im} \mathfrak{Z}$ 

 $\mathcal{K} = \langle e_1, e_2 \mid \mathfrak{L}(1) \rangle$ 

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$$\mathcal{K} = \langle e_1, \ e_2 \mid \mathfrak{L}(1) \rangle$$

$$\mathcal{K} = \langle e_1, e_2 \mid e_1 s + e_2 (x - r) \rangle$$

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$$0 \longrightarrow \mathbb{Z}G \xrightarrow{s} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{p} K \longrightarrow 0$$
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$$K = \langle e_1, e_2 | e_1 s + e_2 (x - r) \rangle$$

$$\mathcal{K} = \left\langle e_1, \ e_2 \ \middle| \ e_1(1+y^{-1}+y^{-3}) + e_2(x-1-y-y^3) \right\rangle.$$

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### Generators of K

 $p: \mathbb{Z}G^2 \longrightarrow K$  is a surjective map.

 $K = \langle p(e_1), p(e_2) \rangle$ 

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### Generators of K

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$$p: \mathbb{Z}G^2 \longrightarrow K$$
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$$\begin{split} \mathcal{K} &= \left\langle x^2 - (1 + y^{-1} + y^{-3})(1 + y + y^3), \\ &\quad x(1 + y + y^3) + (1 + y^{-1} + y^{-3})^2 \right\rangle \\ \mathcal{K} &= \left\langle x^2 - y^3 - y^2 - y - 3 - y^{-1} - y^{-2} - y^{-3}, \\ &\quad xy^3 + xy + x + 1 + 2y^{-1} + y^{-2} + 2y^{-3} + 2y^{-4} + y^{-6} \right\rangle \end{split}$$

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#### Theorem (Stafford (1985))

Let R be a commutative Noetherian domain. Suppose  $S = R[x, x^{-1}; \sigma]$  is a skew Laurent extension of R with elements  $r, s \in R$  such that

- r is not a unit in S,
- **2** rS + (x + s)S = S,

$$\bullet$$
 sr <sup>$\sigma$</sup>   $\notin$  rS.

Then the S-module  $K = \{f \in S \mid rf \in (x + s)S\}$  is a non-free, stably free right ideal of S, satisfying  $K \oplus S \cong S \oplus S$ .

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(Pf)

 $r = 1 + y + y^3$  is not a unit since r is not a monomial.

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$$\{r, x + s\}$$
 generates  $\mathbb{Z}G$  since  $\pi$  is surjective.

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$$sr^{\sigma} = \left(1 + y^{-1} + y^{-3}\right)^2$$

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$$\mathit{sr}^{\sigma} = 1 + 2y^{-1} + y^{-2} + 2y^{-3} + 2y^{-4} + y^{-6}$$

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$$sr^{\sigma} = (y^6 + 2y^5 + y^4 + 2y^3 + 2y^2 + 1)y^6$$

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$$(1+y+y^3) \not| (y^6+2y^5+y^4+2y^3+2y^2+1)$$

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Hence, K is a non-free, stably free module over  $\mathbb{Z}G_{.\Box}$ 

# (G, 2)-Complexes

#### Definition

Let G be a group. A (G, 2)-complex is a 2-dimensional CW-complex with fundamental group G.

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# (G, 2)-Complexes

#### Definition

Let G be a group. A (G, 2)-complex is a 2-dimensional CW-complex with fundamental group G.

Let X be the CW-complex of the Klein Bottle. Thus, X is a (G, 2)-complex with  $G = \pi_1(X, v_0)$ .



### **Euler Characteristic**

#### Definition

Let Y be a finite 2-dimensional CW-complex. Then the **Euler** Characteristic of Y, denoted  $\chi(Y)$ , is the alternating sum  $\sum_{k=0}^{2} (-1)^{k} c_{i}$  where  $c_{k}$  is the number of k-cells in Y.

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#### Theorem (Harlander and Jensen (2006))

The Klein bottle is the only complex with  $\chi(X) = 0$ , which is the minimum Euler characteristic for any (G, 2)-complex, up to homotopy.

Algebraic (G, 2)-complexes

#### Definition

An algebraic (G, 2)-complex is an exact sequence

$$\mathcal{C}_* \hspace{.1in} : \hspace{.1in} F_2 \rightarrow F_1 \rightarrow F_0 \rightarrow \mathbb{Z} \rightarrow 0$$

where the  $F_i$  are finitely generated, free  $\mathbb{Z}G$ -modules. The **Euler Characteristic** of  $C_*$ , denoted  $\chi(C_*)$ , is the alternating sum  $\sum_{k=0}^{2} (-1)^k c_i$  where  $c_k$  is the rank of  $F_k$  in  $C_*$ .

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where the  $F_i$  are finitely generated, free  $\mathbb{Z}G$ -modules. The **Euler Characteristic** of  $\mathcal{C}_*$ , denoted  $\chi(\mathcal{C}_*)$ , is the alternating sum  $\sum_{k=0}^{2} (-1)^k c_i$  where  $c_k$  is the rank of  $F_k$  in  $\mathcal{C}_*$ .

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# Algebraic Complexes of G

Since

$$X = \left| \langle x, y \mid x^{-1}yx = y^{-1} \rangle \right|,$$

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$$X = \left| \langle x, y \mid x^{-1} y x = y^{-1} \rangle \right|,$$

X has the celluar chain complex

$$\mathcal{C}_*(X)$$
 :  $\mathbb{Z}\langle A \rangle \longrightarrow \mathbb{Z}\langle x, y \rangle \longrightarrow \mathbb{Z}\langle v_0 \rangle \rightarrow 0.$ 

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$$\mathcal{C}_*(X) \quad : \quad \mathbb{Z}\langle A \rangle \longrightarrow \mathbb{Z}\langle x, \ y \rangle \longrightarrow \mathbb{Z}\langle v_0 \rangle \to 0.$$

Then,

$$\mathcal{C}_*(\widetilde{X})$$
 :  $\mathbb{Z}G \xrightarrow{\partial_2} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0$ 

is an algebraic (G, 2)-complex.

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# Algebraic Complexes of G

#### Since

$$X_1 = \left| \langle x, y \mid x^{-1}yx = y^{-1}, 1 \rangle \right| = X \vee S^2,$$

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and

$$\mathcal{C}_*(\widetilde{X_1}) \hspace{2mm} : \hspace{2mm} \mathbb{Z} G \oplus \mathbb{Z} G \xrightarrow{\partial_2 \oplus 0} \mathbb{Z} G \oplus \mathbb{Z} G \xrightarrow{\partial_1} \mathbb{Z} G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

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Notice that

$$\pi_2(X_1) \cong H_2(\widetilde{X_1}) = \ker(\partial_2 \oplus 0) = \mathbb{Z}G.$$

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## The Algebraic Complex $\mathcal{K}_*$

We now will construct an algebraic (G, 2)-complex with no obvious geometric interpretation. Let  $\mathcal{K}_*$  to be the resolution

$$\mathcal{K}_* : \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\partial_2 \circ \pi} \mathbb{Z}G \oplus \mathbb{Z}G \xrightarrow{\partial_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \longrightarrow 0.$$

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But,  $H_2(\mathcal{K}_*) = K \cong \mathbb{Z}G = H_2(\mathcal{C}_*(X_2)).$ 

Therefore,

 $\mathcal{K}_* \not\simeq \mathcal{C}_*(X_2).$ 

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But, 
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Therefore,

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### Geometric Realization

#### Theorem

There exist chain-homotopically distinct, algebraic (G, 2)-complexes with Euler characteristic 1, where G is the fundamental group of the Klein bottle.

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### Geometric Realization

#### Theorem

There exist chain-homotopically distinct, algebraic (G,2)-complexes with Euler characteristic 1, where G is the fundamental group of the Klein bottle.

#### Question

Does  $\mathcal{K}_*$  arise as an algebraic complex of some geometric (G, 2)-complex?

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