THE DENSITY TOPOLOGY ON THE REALS

WITH ANALOGUES ON OTHER SPACES

by

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ABSTRACT

A point x is a *density point* of a set A if all of the points except a measure zero set near to x are contained in A. In the usual topology on $\mathbb{R}$, a set is open if shrinking intervals around each point are eventually contained in the set. The density topology relaxes this requirement. A set is open in the density topology if for each point,

$$\lim_{h \to 0} \frac{\mu\big(A \cap (x-h, x+h)\big)}{\mu\big((x-h, x+h)\big)} = 1. \tag{1}$$

That is, for any point x and a small enough interval I_x, I_x has measure in A arbitrarily close to the measure of I_x. If x has property (1), it is a density point of A.

The density topology is a refinement of the usual topology. As such, it inherits many topological properties from the usual topology. The topology is both Hausdorff and completely regular. This paper will define the density topology starting from Lebesgue measure. After defining the topology, we will demonstrate topological properties including separation and connectedness properties. The density function is related to the topological operations of interior and closure. In addition, the Lebesgue measurable sets are precisely the Borel sets in the density topology.

The density topology can be defined on any space that has a Lebesgue measure and for which the Lebesgue Density Theorem holds. The topology is easily defined on the Cantor space, but is more difficult to define on the space of continuous functions $C[0, 1]$. We explore these results in the final chapters, including a cursory introduction to prevalent and shy sets, an infinite-dimensional analogue of the density topology.

TABLE OF CONTENTS

ABSTRACT v

1 Introduction 1

2 Preliminaries 4

2.1 Topology 4

2.2 Borel Sets 7

2.3 Measure Theory 10

2.3.1 Metric Spaces 10

3 Density 18

3.1 Definition and First Examples 19

3.2 Lebesgue Density Theorem 22

3.3 Properties of the Density Function 25

4 Density Topology 30

4.1 Defining a Topology using Density 30

4.2 Approximate Continuity 36

4.3 Topological Properties of the Density Topology 41

5 Density Topology on the Cantor Space 47

5.1 A Metric on the Cantor Space 47

5.2 A Measure on the Cantor Space 50

5.3 Density in the Cantor Space . . . 53

5.4 Properties of the Density Topology on C . . . 55

6 General Properties of the Density Topology . . . 61

6.1 Interior and Closure Properties . . . 61

6.2 A Naive Approach to the Density Topology . . . 67

6.3 A Different Approach: Prevalence . . . 72

REFERENCES . . . 75

CHAPTER 1

INTRODUCTION

Students of mathematics often encounter the paradox that sets which are "large" in one sense may be very "small" in another sense. For example, the rational numbers have infinite cardinality and are dense in the real line, but the set of rational numbers is "small" in comparison to the real line. That is, almost every point in the real line is not rational. For another example, the continuous functions have the same cardinality as the real line, but almost every function from $\mathbb{R}$ to $\mathbb{R}$ has a discontinuity. The qualifier "almost every" can be strictly defined, usually by defining a measure on the space.

Let $C[0,1]$ be the set of continuous functions from $\mathbb{R} \to \mathbb{R}$. Our aim is to find a superset $\mathcal{S} \supseteq \mathcal{C}$ that is similar to $\mathcal{C}$. We want the functions in $\mathcal{S}$ to behave like functions in $\mathcal{C}$ at almost every point. To begin, we need to make our definition of "almost every" rigorous. To illustrate the goal, choose a set $I \subseteq \mathbb{R}$. Let A be a subset of I. Choose a point x from A using a uniform probability distribution. If x an element of A with probability 1, then A contains "almost every" element of I.

In fact, the probability example given above is equivalent to the idea of Lebesgue measure. If a point in $I \setminus A$ is chosen with probability 0, then we will say the *measure* μ of $I \setminus A$ is 0. Measure theory is introduced in Chapter 2, and the theorems which are necessary for this paper are presented. Measure theory is a tool for comparing the size of two sets which may have the same cardinality. Measure theory works in an intuitive way on metric spaces; Chapter 2 showcases measure theory in the realm of metric spaces. We will begin by considering the metric space $\mathbb{R}$ with the absolute value metric, following Oxtoby's *Category and Measure* [11]. We will also consider the Cantor space and the space of continuous functions in Chapters 5 and 6 respectively.

Return for a moment to the goal: we want to find functions which are 'almost continuous.' That means that for any point x, almost all of the points y near s should have $f(y)$ close to $f(x)$. In topological terms, a function is continuous if the pre-image of every open set is itself an open set. For our purposes, we will be content if the pre-image of an open set contains 'almost every' point of an open set. Chapter 2 discusses the basic topological notions that are needed.

Armed with the tools of Measure Theory and Topology, Chapter 3 makes rigorous our definition of 'almost every' nearby point. A point x is a density point of a set A if

$$\lim_{h\to 0} \frac{\mu\big((x-h,x+h)\cap A\big)}{\mu\big((x-h,x+h)\big)} = 0. \tag{1.1}$$

That is, as the intervals around x shrink, the ratio of the measure of those intervals intersected with A approaches 1. When defining continuous functions, it is common practice to define continuity at a point, then to declare a function continuous if it is continuous at every point. In the same way, we define functions which are approximately continuous at a point. If a function is approximately continuous at each point, we will call the function *approximately continuous*.

In Chapter 4, we develop the *density topology*, a space in which each open set A has the property that all points in A are density points of A. As a convenience, the density function Φ is defined as the function which takes a set A as input and returns all the density points of that set. Chapter 4 demonstrates that a function is approximately continuous if and only if it maps open sets in the density topology to open sets in the Euclidean topology.

Of course, other spaces with metric lend themselves to a similar approach. We define a measure on the Cantor Space in Chapter 5 using the a common metric. In fact, the density topology can be defined on the Cantor Space once the measure is in place. The topology has many similarities to the density topology on $\mathbb{R}$. After defining the density topology on the Cantor Space, we demonstrate some properties using only the closure and interior topological functions. These properties are applicable to both density topologies: one on the Cantor Space and the other on $\mathbb{R}$.

At this stage, it may seem that each metric space can be used to generate a density topology on its underlying space. With this idea in mind, we begin Chapter 6. The target space is the space of continuous functions on an interval. We take the usual supremum norm as our metric. In fact, it is impossible to develop a meaningful density topology on this space. Any open metric ball in the space will have measure zero or an undefined measure. This is a result of the infinite dimensionality of the space of continuous functions. To work around this obstacle, we introduce the theory of prevalent and shy sets. A prevalent set is analogous to a set of full measure, but the definition is stricter. A *prevalent set* is a set A such that *every* compactly supported measure on the space has positive measure intersection with A. In contrast, a full measure set satisfies the same property but for only one particular measure, not *all compactly supported measures on that space.* The scope of this work only allows us to introduce the theory of prevalence and shyness as a density topology analogue for infinite-dimentional spaces.

CHAPTER 2

PRELIMINARIES

This paper will make use of Lebesgue measure, set theory, topology, and some analysis. As most readers will be familiar with these topics, references are included for general notions and a few important theorems are stated for later use.

2.1 Topology

Definition 2.1.1. *A collection of sets $\mathcal{A}$ is the collection of* open sets *of a topology on a space X if the following properties are satisfied:*

1. $\emptyset, X \in \mathcal{A}$
2. *For any* $A, B \in \mathcal{A}, \quad A \cap B \in \mathcal{A}$.
3. *For any collection* $\{B_i\}_{i \in I}$ *with* $B_i \in \mathcal{A}$ *for all* $i \in I$, *we have* $\bigcup B \in \mathcal{A}$.

The basic open sets in the usual topology on $\mathbb{R}$ are the open intervals. The intervals are called *open balls* or just *balls.* Let (a, b) be an open ball. Note that collection of open intervals satisfies all properties above. Let $\mathcal{T}$ be the collection of open intervals, their finite intersections, and arbitrary unions. The collection $\mathcal{T}$ is the usual topology on $\mathbb{R}$.

The *open sets* of a topology X are precisely the members of $\mathcal{A}$. If a set is the complement of an open set, it is called *closed.* In a topological space, a set A is called *dense* if it intersects every open set. This is equivalent to saying that each element of the space is either in A or a limit point of A. A set A is *nowhere dense* if for any open set, an open subset is contained in the complement of A. An *open cover* of a set A is a collection of open sets $\mathcal{U} = \{U \subseteq X\}$ such that $A \subseteq \bigcup U$. In metric spaces, the notion of *bounded sets* exists. A set A is bounded if the set is contained in a ball of finite radius. On the reals, this means that a set is bounded if it is contained in some finite interval.

Theorem 2.1.2 (Heine-Borel)**.** *A subset $A \subseteq \mathbb{R}$ is a closed, bounded set if and only if every open cover of A has a finite subcover.*

Theorem 2.1.2 is true in $\mathbb{R}$ with the usual topology but does not necessarily hold in other topologies. For example, the Heine-Borel theorem requires a different formulation to hold on the space of continuous functions $C[0,1]$ with the absolute value norm. A set A is *compact* if every open cover of A has a finite subcover.

Theorem 2.1.3 (Bolzano-Weierstrass)**.** *In $\mathbb{R}$ with the usual topology, a set is closed and bounded if and only if it is compact.*

Note that the Bolzano-Weierstrass theorem is specific to Euclidean space. If a open set A contains a point x, then A is a neighborhood of x. We will use N_x to denote an arbitrary neighborhood of x.

Definition 2.1.4 (Perfect Set)**.** *A* perfect set *F is a closed set with no isolated points. Any neighborhood of a point x in F must intersect F at a point other than x.*

Any closed interval $[a, b]$ is a perfect set, as is the Cantor Set (see Example 4.1.4). All of the above properties are given for individual sets. However, properties of the topological space may be defined by the properties of sets in that space.

Definition 2.1.5. *A* separable topology *has a countable set which has nonempty intersection with every open set.*

The balls formed by a metric may or may not generate a topology on a space. A topological space may have many metrics which are unrelated to each other and to the topological properties. However, if a metric exists such that the metric generates a topology, the topology is said to be *completely metrizable.* As metrizability and separability are both useful properties, spaces which are both metrizable and separable are of interest.

Definition 2.1.6. *A* Polish space *is a topological space that is separable and completely metrizable.*

The real numbers with the usual metric form a Polish space.

Dense sets are very dependent on the topology. So are the *first category* sets which are constructed from the dense sets.

Definition 2.1.7. *A set is said to be first category (or meager) iff it is a countable union of nowhere dense sets.*

Equivalently, a first category set is contained in a countable union of closed, nowhere dense sets. Baire proved the following theorem about $\mathbb{R}$, which will be vital to demonstrate that some sets have positive Lebesgue measure.

Theorem 2.1.8 (Baire)**.** *If a set $A \subseteq \mathbb{R}$ is a first category set, then A^c (the complement of A) is dense. No interval of $\mathbb{R}$ is first category. The intersection of any sequence of dense open sets is dense.*

The proof of Baire's theorem is found in [11]. We will consider the same conditions on other topological spaces throughout the paper.

2.2 Borel Sets

Topological spaces are characterized by the collection of sets which are open. Once we have defined which sets are open, it is natural to ask which sets can be constructed using only the open sets. That is, "Which sets can be made by taking unions and intersections of open sets?" Let $\mathcal{A}$ be a collection of sets. The collection of sets which can be formed by finite unions, intersections, and complements of elements $\mathcal{A}$ is called an *algebra*.

We only require an algebra to be closed under finite unions and complements. Note that if an algebra is closed under finite unions and complements, it is also closed under finite intersections as well. Any intersection can be written the following way:

$$A \cap B = (A^c)^c \cap (B^c)^c = (A^c \cup B^c)^c.$$

The definition of an algebra of sets is a natural analogue to the definition of an arithmetic algebra. In the usual algebra of real numbers, we define two operations (addition and multiplication). The algebra consists of all finite combinations using these two operations and the elements of $\mathbb{R}$. In fact, polynomials are algebraic functions which take an input and perform defined addition and multiplication operations.

Algebras are often described by the generating collection of subsets. For example, take the collection $G = \big\{\{n\} : n \in \mathbb{Z}\big\}$, the integer singletons. We can define A to be the algebra of sets generated by elements of G. Note that the sets $\{2,3\}, \emptyset$, and $\{1, 2, \ldots 100\}$ are all contained in A. Then see that $2\mathbb{Z}$, the collection of even numbers, is not contained in A because it would require infinitely many unions. If we want to include sets which are generated by infinitely many unions, we will have to change our definition from algebra to σ-algebra.

Definition 2.2.1 (σ-algebra)**.** *If a class of subsets of X is closed under countable unions, complementations, and contains X itself, the class is a* σ-algebra.

To continue the example above, take the collection G as before. This time, countable unions and complements are permitted. Let A' be the algebra generated in this way. Notice that the union of all members of G is countable and $\bigcup\{n\} = \mathbb{Z}$, so the set A' is in fact a σ-algebra. The definition of σ-algebra covers the set-theoretic operations of intersection, complement, and union. The σ-algebra generated by the open sets of any topology is the collection of *Borel sets* of that topology.

Definition 2.2.2 (Borel Sets)**.** *Let $\mathcal{T}$ be a topology. The collection of Borel sets of $\mathcal{T}$ is the smallest σ-algebra containing the open sets of $\mathcal{T}$.*

The Borel sets can be characterized as the σ-algebra generated by the open sets. Thus, we can talk about Borel sets on any topology. In addition, the intersection of σ-algebras is also a σ-algebra. If a set is Borel, it has a quantifiable complexity. The open sets are the simplest Borel sets. Let the open sets be denoted as $\underset{\sim}{\sum}{}^0_1$. The closed sets are simply complements of open sets. Denote the closed sets $\underset{\sim}{\prod}{}^0_1$. From these two collections, all Borel sets can be made. The following naming conventions track the complexity of the Borel sets:

Characterization 2.2.3.

Open Sets	Countable Unions of Closed Sets	etc.	
$\underset{\sim}{\sum}{}^0_1$	$\underset{\sim}{\sum}{}^0_2$ or F_σ	$\underset{\sim}{\sum}{}^0_3$ or $G_{\delta\sigma}$	$\ldots$
Closed Sets	Countable Intersections of Open Sets	etc.	
$\underset{\sim}{\prod}{}^0_1$	$\underset{\sim}{\prod}{}^0_2$ or G_δ	$\underset{\sim}{\prod}{}^0_3$ or $F_{\sigma\delta}$	$\ldots$

Consider the set $\mathbb{Q}$ of rational numbers. The set is not open, as any interval contains irrational numbers. The set is not closed, as we can construct a sequence of rationals which approach $\sqrt{2}$, an irrational number. However, each singleton $\{q\}, q \in \mathbb{Q}$ is closed, and the countable union $\bigcup_{q\in\mathbb{Q}}\{q\} = \mathbb{Q}$. So $\mathbb{Q}$ is an F_σ or $\sum_2^0$ set. Interestingly, the rational numbers are not a G_δ set. The proof below is adapted from an example which appeared on math.stackexchange.com [14].

Proof. Assume $\mathbb{Q} = \bigcap\limits_{k\in\mathbb{N}} U_k$ for open U_k. Enumerate the rationals $\mathbb{Q} = \{q_k : k \in \mathbb{N}\}$. Define the open sets $W_k = U_k \setminus \{q_k\}$. There exists an open interval (a_1, b_1) such that $[a_1, b_1] \subseteq W_1$. For each k, let $r_k = \frac{1}{4}(b_k - a_k)$. Then $a_k < a_k + r_k < b_k - r_k < b_k$. Note that W_{k+1} is dense and open, so there exists an open (a_{k+1}, b_{k+1}) such that

$$(a_{k+1}, b_{k+1}) \subseteq [a_{k+1}, b_{k+1}] \subseteq W_{k+1} \cap (a_k, b_k)$$

For each k, $[a_{k,+1}b_{k+1}] \subseteq [a_k, b_k]$, so $\big([a_k, b_k] : k \in \mathbb{N}\big)$ is a decreasing sequence of closed intervals. So $\bigcap_k [a_k, b_k] \neq \emptyset$. The following containments hold:

$$\bigcap_k [a_k, b_k] \subseteq \bigcap_k W_k \subseteq \bigcap_k U_k, \subseteq \mathbb{Q}$$

But $\bigcap_k W_k$ does not contain any element of $\mathbb{Q}$. Thus we have a contradiction, and $\mathbb{Q}$ cannot be written as $\bigcap_k U_k$ for open U_k. $\mathbb{Q}$ is not G_δ. $\square$

There exist many more interesting examples of Borel sets which are outside the scope of this paper. The interested reader is directed to Kuratowski and Mostowski's "Set Theory" [8] for a more thorough presentation of Borel sets. For the most part, we will work with the Borel sets of the density topology. Most importantly, it will be shown that the measurable sets are precisely the Borel sets of the Density Topology.

2.3 Measure Theory

Measure theory is central to this thesis. Section 2.3 will quickly cover some of the important terms and definitions. For an in depth treatment, the reader is directed to Oxtoby's book, *Measure and Category.* Chapters 1 and 3 cover the topics of metric spaces, outer measure, and Lebesgue measure. The following definitions and theorems will be used throughout the paper. Proofs may be found in Oxtoby [11] except where otherwise noted.

2.3.1 Metric Spaces

Choose any two points $x, y \in \mathbb{R}$. Intuitively, there exists a *distance* between x and y. The distance between x and y is $d(x, y) = |x - y|$. This distance is called the *Euclidean metric.* A distance function (or metric) d satisfies the following requirements for each pair $x, y \in \mathbb{R}$:

1. $d(x, y) \geq 0$

2. $d(x, y) = 0 \Leftrightarrow x = y$

3. $d(x, y) = d(y, x)$

4. $d(x, z) \leq d(x, y) + d(y, z)$

The distance function $d(x, y) = |x - y|$ satisfies these four properties. Let $I \subseteq \mathbb{R}$ be any interval (a, b). Using d from above, we define the *length* of an interval to be the distance between its endpoints. If I is an interval with endpoints $a < b$, the length of I is $\ell(I) = d(a, b)$. Note that $[a, b]$, (a, b), $[a, b)$, and $(a, b]$ all have the same length. Given any point $x \in \mathbb{R}$, we define an *open ball of radius ρ around x* as

$B_\rho(x) = \{y \in \mathbb{R} : d(x, y) < \rho\}$. Every interval (a, b) is an open ball of radius $(b-a)/2$ centered at the midpoint $(a+b)/2$. Length is one way of measuring the "size" of a set. All nontrivial intervals have the same cardinality as the real line. However, length satisfies the intuition that the interval $(0, 1)$ is "smaller" than the interval $(0, 10)$.

Choose any subset $A \subseteq \mathbb{R}$. The set A may not be connected, so the idea of "length" may not apply. For any set A, there exists a countable collection $\{U_i : i \in \mathbb{N}\}$ of open intervals such that $A \subseteq \bigcup_i U_i$. So the size of A is somehow "smaller than or equal" than the size of $\bigcup_i U_i$. Find a collection of finite intervals $\mathcal{I} = \{I : I \text{ interval}\}$ such that $A \subseteq \bigcup I$.

Definition 2.3.1. *The* outer measure μ^* *of a set* A *is defined as*

$$\mu^*(A) = \inf\big(\sum_i \ell(I_i)\big)$$

where $\ell(I_i)$ *is the length of interval* I_i *and* $A \subseteq \bigcup_i I_i$.

Every set has an outer measure. To see this, consider the intervals $\big\{(i-1, i+1) : i \in \mathbb{Z}\big\}$ as a covering for any set. It is worth noting that both length and outer measure of a set may be infinite, as in the case of the interval $(0, \infty)$. If a set has a defined length, that length is the outer measure of the set. This is not immediately clear. The proof relies on the Heine-Borel Theorem. That is, any cover of I has a finite subcover. The Heine-Borel Theorem is not proved here, but the proof is standard in many analysis textbooks.

Lemma 2.3.2. *Let* I *be an interval. Then* $\mu^*(I) = \ell(I)$.

Outer measure is monotonic. Let $B \subseteq A$ and note that any covering of A is also a covering of B. Outer measure is a measurement limiting the size of a set from above.

If a set has outer measure m it is "small enough" to be contained in a collection of intervals whose lengths sum to m. Next we define the *inner measure* of a set, a way of measuring a set's "size" from below.

Definition 2.3.3. *The* inner measure μ_* *of a set* A *is*

$$\mu_*(A) = \sup\{\mu^*(K) : K \subseteq A, K \textit{ is compact}\}$$

A set is *compact* in $\mathbb{R}$ if it is closed and bounded. This statement depends on the Heine-Borel theorem and is only applicable in $\mathbb{R}$. If the inner and outer measure of a set A agree, the set is *measurable.* Not all sets are measurable, but many more measurable sets exist than intervals.

Definition 2.3.4. *A set* A *is measurable if* $\mu_*(A) = \mu^*(A)$*. If* A *is measurable, then the measure of* A *is* $\mu^*(A) = \mu_*(A) = \mu(A)$.

Oxtoby characterizes the definition of measurability in a different way. Note that the open cover is replaced by an open superset. This is because the union of open sets is open, see Section 2.1 for details.

Characterization 2.3.5. A set A is measurable if and only if for any $\varepsilon > 0$, there exists an open set $U \supseteq A$ and a closed set $F \subseteq A$ such that $\mu^*(U) - \mu^*(A) < \varepsilon$ and $\mu^*(A) - \mu^*(F) < \varepsilon$.

Let A be a set such that $\mu(A) = 0$. Then A is a *measure zero* set, also called a *nullset.* A measure zero set is "small" in the sense that it can be covered by the countable union of arbitrarily small intervals. A measure zero set may still be unbounded or have large cardinality. For example, the rational numbers $\mathbb{Q}$ are a nullset, but $|\mathbb{Q}| = \omega$ and $\mathbb{Q}$ is unbounded.

Not all subsets of $\mathbb{R}$ are measurable. Unless specifically stated otherwise, all sets demonstrated in this thesis will be measurable. Measurable sets have many nice set-theoretic properties, including the following:

Theorem 2.3.6. *Every interval is measurable.*

Proof. Let I be an interval with endpoints $\alpha < \beta$. By Lemma 2.3.2, $\mu^*(I) = \ell(I)$. The inner measure of I can be found by $\mu_*(I) = \lim_{n\to\infty} \ell([\alpha + \frac{1}{n}, \beta - \frac{1}{n}])$. It may be necessary to choose a large starting n so that $[\alpha + \frac{1}{n}, \beta - \frac{1}{n}] \subseteq I$ for all n. Then $\mu_*(I) = \mu^*(I) = \mu(I) = \ell(I)$. □

Lemma 2.3.7. *If A is measurable, then A^c is measurable.*

Proof. Fix $\varepsilon > 0$. By Characterization 2.3.5, there exists an open U and a closed F such that $F \subseteq A \subseteq U$, $\mu^*(A) - \mu^*(F) < \varepsilon$ and $\mu^*(U) - \mu^*(A) < \varepsilon$. Then $F^c \supseteq A^c \supseteq U^c$. F^c is open and U^c is closed. Also, $U \setminus F = F^c \setminus U^c$, so $\varepsilon/2 > \mu^*(U \setminus F) = \mu^*(F^c \setminus U^c)$. As ε goes to 0, the outer measures of F^c and U^c approach A^c. So A^c is measurable. □

Lemma 2.3.8. *If A and B are measurable, then $A \cap B$ is measurable.*

Proof. Since A and B are measurable, there exist sets F_1, F_2 closed and U_1, U_2 open such that $F_1 \subseteq A \subseteq U_1$ and $F_2 \subseteq B \subseteq U_2$. In addition, F_1, F_2, U_1, U_2 can be chosen such that

$$\mu(A - F_1) \leq \varepsilon/2, \qquad \mu(B - F_2) \leq \varepsilon/2$$
$$\mu(U_1 - A) \leq \varepsilon/2, \qquad \mu(U_2 - B) < \varepsilon/2.$$

Define $F = F_1 \cap F_2$ and define $U = U_1 \cap U_2$. Then

$$F = F_1 \cap F_2 \subseteq A \cap B \subseteq U_1 \cap U_2 = U$$

$$U \setminus F \subseteq (U_1 \setminus F_1) \cup (U_2 \subseteq F_2).$$

So $\mu^*(U \setminus F) \leq \mu^*(U_1 \setminus F_1) + \mu(U_2 \subseteq F_2) < \varepsilon$. The sets U and F satisfy the criteria for Caracterization 2.3.5, and $A \cap B$ is measurable. $\square$

Measurability of sets is a key concern in the study of the density topology. As demonstrated later, the measurable sets are precisely the sets constructed by countable unions and intersections of open sets in the density topology. Because of this, we want as many ways as possible to characterize which sets are measurable. Oxtoby gives some criteria which are sufficient for a set to be measurable.

Theorem 2.3.9. *Let A be a bounded set. If, for any $\varepsilon > 0$, there exists a closed $F \subseteq A$ such that $\mu^*(F) > \mu^*(A) - \varepsilon$, then A is measurable.*

Theorem 2.3.10. *Let $\{A_i\}$ be a countable sequence of disjoint measurable sets. Then $A = \bigcup A_i$ is measurable and $\mu(A) = \sum \mu(A_i)$.*

Theorem 2.3.11. *A set A is measurable iff it can be represented by the union of an F_σ set and a nullset.*

Proof. ($\Rightarrow$) Let A be measurable. Then, for any $n \in \mathbb{N}$, we can find a compact F_n and an open G_n such that $F_n \subseteq A \subseteq G_n$ and $\mu^*(G_n \backslash F_n) < 1/n$. Let $E = \bigcup F_n$. Since it is the countable union of closed sets, we have E is an F_σ set. Then let $N = A \backslash E$. Since $\mu^*(G_n \backslash F_n) < 1/n$ and $N \subseteq G_n \backslash F_n$, we conclude that N is a nullset. That is, $\mu^*(N) < 1/n$ for each n.

($\Leftarrow$) Let A be represented by $E \cup N$ where E is an F_σ set and N is a nullset. Since the class of measurable subsets is a σ-algebra, then $\bigcup F_i = E$ is measurable. By Theorem 2.3.6, every nullset is measurable. So $E \cup N$ is measurable. □

We explore further properties of Lebesgue measure which will be necessary throughout the paper. First, Lebesgue measure is *translation invariant.*

Theorem 2.3.12. *Let A be a measurable set. Let $v \in \mathbb{R}$ and $A + v = \{x + v : x \in A\}$. Then $\mu(A) = \mu(A + v)$.*

Note that this is not true for all measures. Let p be the probability measure given by the normal distribution centered at zero. Then the measure p is clearly not translation-invariant.

The real line has uncountable cardinality. But, given any collection of pairwise disjoint sets, only countably many may have positive measure. This fact is necessary to build the density topology; it forms the key to proving that any union of open sets is open.

Theorem 2.3.13. *The Lebesgue measure μ on $\mathbb{R}$ satisfies the countable chain condition. That is, let C be a collection of sets $C = \{A_t\}_{t \in T}$, where $A_t \subseteq \mathbb{R}$, such that for each $t \in T$, A_t is measurable, $\mu(A_t) > 0$, and $\mu(A_s \cap A_r) = 0$ whenever $s \neq r$.*

Proof. Let C be a collection of sets, each with positive measure, such that the intersection of two distinct elements of C has measure zero. Assume that C is uncountable to show a contradiction. Consider $\mathbb{R}$ as the union of countably many intervals $[n, n + 1]$. Since we have uncountably many sets in C, we know that one of the intervals $[i, i+1]$ has uncountably many sets $A_j \in C$ such that $\mu(A_j \cap [i, i+1]) > 0$. Without loss of generality, let $[i, i+1] = [0, 1]$, since the Lebesgue measure is invariant under translation (Theorem 2.3.12). Also, we may disregard the sets which have a measure zero intersection with the interval $[0,1]$.

Now, we have uncountably many sets $A_j \subseteq [0,1]$, each of which has a nonempty intersection with $[0,1]$. Take a sequence $(\varepsilon_k) \to 0$. For some ε_k, there must be uncountably many A_j such that $\mu(A_j) \geq \varepsilon_k$. Otherwise, the cardinality of $\{A_j\}$ would be countable. Let $\{A_{j'}\}$ be the collection of sets which have measure greater than ε_k. We assumed that the pairwise intersections of elements of $\{A_j\}$ have measure zero. So $\mu([0,1]) \geq \sum_j \mu(A_j) \geq \sum_{j'} \mu(A_{j'})$. But $\sum_{j'} \mu(A_{j'}) = \infty > \mu([0,1])$. This is a contradiction. So any collection of sets such that the pairwise intersection has measure zero must be a countable collection. That is, Lebesgue measure satisfies the countable chain condition on $\mathbb{R}$. $\square$

Similarly to the countable chain condition, it is a useful fact to know how measure behaves with respect to a nested descending sequence of measurable sets.

Theorem 2.3.14. *Let $A_1, A_2, \ldots$ be a descending sequence of measurable sets such that $\mu(A_i) < \infty$ for some i. That is, $A_j \subseteq A_i, j > i$. Then $\lim_{n\to\infty} \mu(A_n) = \mu(\bigcap_n A_n)$.*

Proof. Without loss of generality, we may assume $\mu(A_1) < \infty$. Note that $A_1 \setminus \bigcap_i A_i = \bigcup_{i=1} \left(A_i \setminus A_{i+1}\right)$. The union is a disjoint union, so

$$\mu\left(A_1 \setminus \bigcap_i A_i\right) = \sum_{i=1}^{\infty} \mu\left(A_i \setminus A_{i+1}\right) \tag{2.1}$$

$$= \lim_{m\to\infty} \sum_{i=1}^{m-1} \mu\left(A_i \setminus A_{i+1}\right) \tag{2.2}$$

$$= \lim_{m\to\infty} \mu\left(A_i \setminus A_m\right) \tag{2.3}$$

$$= \mu\left(A_1\right) - \lim_{m\to\infty} \mu(A_m). \tag{2.4}$$

Compare the left side of 2.1 with line 2.4. We see that $\mu\left(\bigcap_i A_i\right) = \lim_{m\to\infty} \mu(A_m)$. $\square$

CHAPTER 3

DENSITY

One of the most useful properties of continuous functions is that behavior is locally similar. By varying the input values a small amount, we observe only small perturbations in the outputs. Put another way, a function $f : \mathbb{R} \to \mathbb{R}$ is continuous if and only if

$$\forall x \in \mathbb{R}, \quad \forall \varepsilon \in (0, \infty), \quad \exists \delta \in (0, \infty) \text{ such that}$$

$$|f(x) - f(x+y)| < \varepsilon, \quad \forall y \in (-\delta, \delta)$$

A natural question arises; what happens if "almost every" point near x exhibits this behavior? We interpret "almost" to mean every point near x except for a measure zero set. This definition is natural because an "almost continuous" function should satisfy the following notion: Given a function f and $x, \varepsilon \in \mathbb{R}, \varepsilon > 0$, there exists a δ such that any $y \in (-\delta, \delta)$ satisfies $|f(x) - f(x+y)| < \varepsilon$ with probability 1. To formalize the concept of "almost every," we introduce the concept of density points of a set. This concept will naturally lead to a function which returns the density points of a set. This density function allows us to define a topology where open sets are "almost" open in the usual topology, discussed in Chapter 4.

3.1 Definition and First Examples

Recall that the pre-image of an open set in a continuous function is an open set. Take any open set $U \subseteq \mathbb{R}$ and any $x \in U$. Shrinking intervals centered at x will eventually be contained in U. To define density in a set, we will also take shrinking intervals about a point. But, rather than requiring the intervals to be eventually contained in U, we loosen the restriction to say that the measures of the set around the point must approach the measure of the interval. We adopt the notation and terminology used by Wilczyński [17].

Definition 3.1.1 (Density). *Let $A \subseteq \mathbb{R}$ be a measurable set. The density of x in A is:*

$$d_A(x) = \lim_{h \to 0} \frac{\mu(A \cap (x-h, x+h))}{\mu\big((x-h, x+h)\big)} = \lim_{h \to 0} \frac{\mu(A \cap (x-h, x+h))}{2h}$$

if the limit exists. When $d_A(x) = 1$, we say that x is a density point *of A.*

As an illustration, consider the following example. Note that I is not an open set.

Example 3.1.2. Let I be the interval $[0, 1)$.

1. $d_I(0) = \lim_{h \to 0} \frac{\mu(I \cap (x-h, x+h))}{2h} = \lim_{h \to 0} \frac{h}{2h} = \frac{1}{2}$
2. Similarly, $d_I(1) = \frac{1}{2}$
3. For all x in $(0, 1), d_I(x) = 1$
4. For all x not in $[0, 1], d_I(x) = 0$

In the example above, every point x in $(0, 1)$ has density 1 in I. The shrinking intervals around x are eventually contained fully in I. This behavior holds for any open set.

Lemma 3.1.3. *Let A be an open set. Then $d_A(x) = 1$ for all $x \in A$.*

Proof. Choose x. Since A is open, there exists an open interval centered at x, contained in A. Then shrinking open intervals around x are eventually contained in A, so $d_A(x) = 1$. □

Note that closed sets fail to enjoy this property. The closed interval $A = [0, 1]$ has two points which are not density 1 in A. The points 0 and 1 have density $\frac{1}{2}$. As seen in Example 3.1.2, it is relatively simple to construct sets which have points with density 0, 1, and $\frac{1}{2}$. For any point $c \in [0, 1]$, a simple construction creates a set A and a point x such that $d_A(x) = c$.

Claim 3.1.4. Let $x \in \mathbb{R}$ be given. For any $c \in [0, 1]$, a set A exists such that $d_A(x) = c$.

Proof. The choice of A is obvious for $c = 0, 1$. Let $c \in (0, 1)$. For each n in $\mathbb{N}$, define the set $I_n = [x - \frac{1}{n}, x - \frac{1}{n+1}] \cup [x + \frac{1}{n+1}, x + \frac{1}{n}]$. For each I_n choose a subset $A_n \subseteq I_n$ such that $\mu(A_n) = c \cdot \mu(I_n)$. For example, A_n could be chosen as

$$A_n = \left[x - \frac{1}{n+1} - c\left(\frac{1}{n} - \frac{1}{n+1}\right), x - \frac{1}{n+1}\right] \cup \left[x + \frac{1}{n+1}, x + c\left(\frac{1}{n} - \frac{1}{n+1}\right)\right].$$

Once A_n is chosen for each n, let $A = \bigcup_n A_n$. It is clear that for any n, we have $\mu(A \cap (x - \frac{1}{n}, x + \frac{1}{n})) = c \cdot \frac{2}{n}$ by the choice of A_n. We calculate the density $d_A(x)$:

$$d_A(x) = \lim_{h \to 0} \frac{\mu(A \cap (x - h, x + h))}{2h}.$$

As $h \to 0$ we may assume $h < 1$. Let n be such that $\frac{1}{n+1} \leq h \leq \frac{1}{n}$. We have the following upper bound:

$$\frac{\mu(A \cap (x-h, x+h))}{2h} \leq \frac{\mu(A \cap (x-\frac{1}{n}, x+\frac{1}{n}))}{2h} = \frac{2c}{2hn} \leq \frac{c(n+1)}{n}.$$

Similarly, the lower bound is:

$$\frac{\mu(A \cap (x-h, x+h))}{2h} \geq \frac{\mu(A \cap (x-\frac{1}{n+1}, x+\frac{1}{n+1}))}{2h} = \frac{2c}{2h(n+1)} \geq \frac{cn}{n+1}.$$

Then

$$\lim_{n\to\infty} \frac{cn}{n+1} \leq d_A(x) = \lim_{h\to 0} \frac{\mu(A \cap (x-h, x+h))}{2h} \leq \lim_{n\to\infty} \frac{c(n+1)}{n},$$

and we conclude that $d_A(x) = c$. □

If a point x has high density (close to 1) in a set A, it means that points near x will be members of A with high probability. As discussed in the beginning of the chapter, if $d_A(x) = 1$, we can conclude that points near x lie in A with probability 1. Of course, this may only be probability 1 at the limit of the shrinking intervals about x. So we need to be careful of the word "near." In practice, the points of density 1 are of high importance. If a point has density 1 in a set A, it is said to be a *density point of* A. We introduce a function which returns all of the density points of a set.

Definition 3.1.5. *For a measurable set* $A \subseteq \mathbb{R}$ *the set of density points of* A *is* $\Phi(A) = \{x \in \mathbb{R} : d(x, A) = 1\}$. *The function* Φ *is called the* density function. *That is,* Φ *takes any measurable set as input and returns the set of all points that have density 1 in* A.

When Φ is applied to a set A, it may add or remove points (or both). Points in A may or may not be density points, and points in A^c may or may not be density points of A.

Example 3.1.6. Let $A = \big([0,1) \cup (1,2) \cup \mathbb{Q}\big) \setminus \{q + \sqrt{2} : q \in \mathbb{Q}\}$. We find $\Phi(A)$. For each $x < 0$, note that shrinking intervals around x eventually intersect only with $\mathbb{Q}$. Since $\mu(\mathbb{Q}) = 0$, $x < 0$ cannot be a density point. For $x = 0$, note that shrinking intervals intersected with A eventually have measure $\frac{1}{2}$. So $0 \notin \Phi(A)$. For $x \in (0,2)$, shrinking intervals eventually intersect A everywhere except the points $q + \sqrt{2}$. Since there are only measure 0 many points missing, the measure of the intervals around x is 1. So $(0,2) \subseteq \Phi(A)$. For $x = 2$, the same argument applies as $x = 0$. So $2 \notin \Phi(A)$. For $x > 2$, we use the same argument as $x < 0$. Then $\Phi(A) = (0,2)$.

Intuitively, the density function Φ removes the points of a set which are separated from the main "body" where the set has full measure and "fills in" the small holes. In example 3.1.6, the hole at 1 is filled in, and the measure zero set $\{q + \sqrt{2}\}$ is removed. In this way, the density function cleans up a set. Each point of the resulting set is surrounded by points of density 1.

3.2 Lebesgue Density Theorem

In Example 3.1.6, both $\Phi(A) \setminus A \neq \emptyset$ and $A \setminus \Phi(A) \neq \emptyset$. The next obvious question to ask is: how are A and $\Phi(A)$ related? In fact, there are limitations on how many points Φ can add or remove from a set. In Example 3.1.6, $\mu(\Phi(A) \setminus A) = 0$ and $\mu(A \setminus \Phi(A)) = 0$. In fact, the application of Φ can only add a measure zero set. Similarly, Φ can only remove a set of measure zero. The symmetric difference $A \Delta \Phi(A)$ is the collection of all points where A differs from $\Phi(A)$. The symmetric difference of any measurable set with its set of density points is a nullset (measure zero set). This is a famous theorem of Lebesgue (and the origin of the name *density topology*). The following proof of Lebesgue's density theorem was given by Faure [5].

Theorem 3.2.1 (Lebesgue Density Theorem). *Let $A \subseteq \mathbb{R}$ be a measurable set. Then* $\mu\big(A\Delta\Phi(A)\big) = 0$.

Proof. We want to show that $A \setminus \Phi(A)$ is a nullset. If $\mu(A \setminus \Phi(A)) = 0$ for all A, then $\Phi(A) \setminus A \subset A^c \setminus \Phi(A^c)$ and A^c is measurable. Then we can use our result again to show $A^c \setminus \Phi(A^c)$ is a nullset.

We will first prove that the theorem holds in the bounded case, then show the unbounded case. So we assume that A is bounded.

For $n \in \mathbb{N}$ let

$$E_n = \left\{ x \in A : \liminf_{h \to 0} \frac{\mu\big(A \cap [x-h, x+h]\big)}{2h} < 1 - \frac{1}{n} \right\}$$

Note that $A \setminus \Phi(A) = \bigcup_{n=1}^{\infty} E_n$. If we can show that each E_n is a nullset, that $A \setminus \Phi(A)$ is contained in a countable union of nullsets, so $A \setminus \Phi(A)$ is a nullset. Let $E = E_n$ and assume E is not a nullset to show a contradiction.

If E is not a nullset, then $\mu^*(E) > 0$. Then we can choose an open set G such that $E \subset G$ and $\mu(G) < \mu^*(E)/(1 - \frac{1}{n})$, that is, we choose an open set with outer measure arbitrarily close to (but larger than) E. The inequality can also be written $(1 - \frac{1}{n}) \cdot \mu(G) < \mu^*(E)$. Now consider all closed subintervals $I \subset G$. Let $\mathcal{A}$ be the collection of closed subintervals $I \subset G$ such that $\mu(A \cap I) \leq (1 - \frac{1}{n}) \cdot \ell(I)$. The intervals in $\mathcal{A}$ cover E. To see this, choose an $x \in E$. There exists an $h_1 > 0$ such that $(x - h_1, x + h_1) \subseteq G$. By the definition of E, there exists an $h, 0 < h < h_1$ such that $\frac{\mu(A \cap (x-h, x+h))}{2h} > 1 - \frac{1}{n}$. Then $I = (x-h, x+h) \in \mathcal{A}$.

Let I_i be a sequence of disjoint intervals from $\mathcal{A}$. Then,

$$\mu^*(E \cap \bigcup I_i) \leq \sum \mu(A \cap I_i) \leq (1 - \frac{1}{t}) \sum \ell(I_t) \leq (1 - \frac{1}{t})\mu(G) < \mu^*(E)$$

So for any disjoint sequence $\{I_i\}$ of members of $\mathcal{A}$, we have (ii) $\mu^*(E\backslash \bigcup I_i) > 0$.

Next we construct one such sequence $\{I_i\}$ of disjoint intervals. Choose I_1 arbitrarily. Then choose I_2 such that $I_2 \cap I_1 = \emptyset$. Continue for m choices. Let $I_1 \ldots I_m$ be the sequence, and let $\mathcal{A}_m$ be the members that are disjoint to the ones in the sequence. Then we use the properties (i) and (ii) from above to show that $\mathcal{A}_n$ is nonempty. Since A is bounded, there is an upper bound for lengths of members of $\mathcal{A}$. Let δ_n be the least such upper bound. Then we choose some $I_{n+1} \in \mathcal{E}$ such that $|I_{n+1}| > \delta_n/2$. (I_{n+1} is disjoint from the $\{I_n\}$ of lesser index and its addition to the sequence will make $\mathcal{A}_{n+1}$ significantly smaller than $\mathcal{A}_n$). The sequence can be extended this way countably many times. So let $B = E\backslash \bigcup^\infty I_n$. Using property (ii), again we have $\mu^*(B) > 0$. We can choose some N such that

$$\sum_{N+1}^{\infty} |I_n| < \mu^*(B)/3$$

Then for each interval I_n, let J_n be the interval with the same center but a radius more than 3 times larger. That is, $|J_n| > 3|I_n|$. Using the inequality above, we know that $\bigcup_{N+1}^\infty J_n$ does not cover B. Let x be a point of E that is not covered by some J_n for $n > N$. Then $x \in E \setminus \bigcup_{N+1}^\infty$. We use the property (i) above to note that some interval I exists with center x, $I \in \mathcal{A}_N$. The interval I intersects some interval I_n for $n > N$. Let k be the first integer such that I_k intersects I. $k > N$ and $|I| \leq \delta_{k-1} < 2|I_k|$. Then $I \cap I_k \neq \emptyset$, and $|J_k| > 3|I_k|$, so $x \in I \subset J_k$. This contradicts that $x \notin \bigcup_{N+1}^\infty J_i$. So $A\backslash\Phi(A)$ is a nullset.

If we take any unbounded set H, we can decompose it into a countable union of disjoint bounded sets H_i, each of which give the property that $\mu(H_i) = \mu(\Phi(H_i))$. Then $H_i\backslash\Phi(H_i)$ is a nullset. So $\bigcup \Phi(H_i) + K = \phi(H)$, where K is a set of countable cardinality (the boundary points of the sets H_i). Then $\sum \mu(H_i) + \mu(K) = \mu\big(\bigcup H_i \cup K\big) = \mu(H) = \mu(\Phi(H)) = \sum \mu(\Phi(H_i))$. $\square$

By Lebesgue's theorem, each measurable set A differs from $\Phi(A)$ by a nullset. It is a natural question to wonder which other sets B have the same density set. That is, for which B does $\Phi(B) = \Phi(A)$? Partition all subsets of the reals into equivalence classes $\{\mathcal{V}_\epsilon\}$ such that $A, B \in \mathcal{V}_\epsilon \Leftrightarrow \mu(A \Delta B) = 0$. Then $\Phi(A)$ and $\Phi(B)$ differ by a nullset. We will show that $\Phi(A) = \Phi(B)$. So Φ takes a subset of $\mathbb{R}$ as input and returns the same output for each element $\mathcal{V}_\epsilon$. For all $A \in \mathcal{V}_\epsilon$, $\Phi(A) \in \mathcal{V}_\epsilon$ as well. So Φ gives us a way to choose a canonical element from each such equivalence class.

Theorem 3.2.2. *If A, B measurable and $\mu(A \Delta B) = 0$, then $\Phi(A) = \Phi(B)$.*

Proof. Let $x \in \Phi(A)$. For all $h > 0$, $\mu(A \cap (x-h, x+h)) = \mu(B \cap (x-h, x+h))$. These two sets differ by a nullset, so

$$\lim_{h \to 0} \frac{\mu\left(A \cap (x-h, x+h)\right)}{2h} = \lim_{h \to 0} \frac{\mu\left(B \cap (x-h, x+h)\right)}{2h}.$$

Thus $x \in \Phi(B)$. The proof that $\Phi(B) \subseteq \Phi(A)$ is similar. $\square$

The empty set and all measure zero sets, by the theorem above, have empty density set. All complements of measure zero sets have all of $\mathbb{R}$ as density set. Here is the first hint that a topology might be somehow embedded in the density properties. We will develop this idea further in the next chapter.

3.3 Properties of the Density Function

Since the density function operates on sets, we will take some time to develop the interaction between the density function and basic set-theoretic operations. First, note that $\Phi(\emptyset) = \emptyset$ and $\Phi(\mathbb{R}) = \mathbb{R}$. Next, we note that the density function distributes across set intersection, but not across unions.

Theorem 3.3.1. *The density function distributes across set intersection. That is,* $\Phi(A \cap B) = \Phi(A) \cap \Phi(B)$.

Proof. First, note that $\Phi(A \cap B) \subseteq \Phi(A)$ and $\Phi(A \cap B) \subseteq \Phi(B)$, so $\Phi(A \cap B) \subseteq \Phi(A) \cap \Phi(B)$. To show the other inclusion, choose an interval I.

$$\mu(I \cap A) + \mu(I \cap B) \leq \mu(I) + \mu(I \cap A \cap B).$$

Divide each side by the measure $\mu(I) = |I|$:

$$1 - \frac{\mu(I \cap A \cap B)}{|I|} \leq 1 - \frac{\mu(I \cap A)}{|I|} + 1 - \frac{\mu(I \cap B)}{|I|}.$$

Some algebra gives the following:

$$\frac{\mu(I \cap A) + \mu(I \cap B)}{|I|} - 1 \leq \frac{\mu(I \cap A \cap B)}{|I|}.$$

Next we let $|I| \to 0$. Then $d_A(x) + d_B(x) - 1 \leq d_{A \cap B}(x)$. If $x \in \Phi(A)$ and $x \in \Phi(B)$, we have $d_A(x) = d_B(x) = 1$. Then, by the above inequality,

$$d_A(x) + d_B(x) - 1 = 1 \leq d_{A \cap B}(x).$$

The density $d_{A \cap B}(x)$ is bounded above by 1. So $d_{A \cap B}(x) = 1$. Then $\Phi(A) \cap \Phi(B) = \Phi(A \cap B)$. $\square$

This theorem gives another immediate result: Φ is monotonic.

Lemma 3.3.2. *Given a set $A \subseteq B$, then $\Phi(A) \subseteq \Phi(B)$.*

Proof. Note that $A \subseteq B$ implies that $A \cap B = A$. Using Theorem 3.3.1, we see that $\Phi(A) = \Phi(A \cap B) = \Phi(A) \cap \Phi(B) \subseteq \Phi(B)$. □

It is not necessary for $A \subseteq B$ in order for $\Phi(A) \subseteq \Phi(B)$. It is sufficient for $A \setminus B$ to have measure zero.

Lemma 3.3.3. *Let $A, B \subseteq \mathbb{R}$ such that $\mu(A \setminus B) = 0$. Then $\Phi(A) \subseteq \Phi(B)$.*

Proof. Assume $\Phi(A) \not\subseteq \Phi(B)$. Then there exists an $a \in \Phi(A)$ such that $a \notin \Phi(B)$.

Case 1:

$$\lim_{h \to 0} \frac{\mu\big((A \setminus B) \cap (x-h, x+h)\big)}{2h} > 0.$$

Then there exists a $G \subseteq A$ such that $G \cap B = \emptyset$ and $\mu(G) > 0$. Then $\mu(A \setminus B) \geq \mu(G) > 0$. This contradicts the original assumption $\mu(A \setminus B) = 0$.

Case 2:

$$\lim_{h \to 0} \frac{\mu\big((A \setminus B) \cap (x-h, x+h)\big)}{2h} = 0. \tag{i}$$

Since $a \notin \Phi(B)$,

$$\lim_{h \to 0} \frac{\mu\big(B \cap (x-h, x+h)\big)}{2h} = 0. \tag{ii}$$

Combining (i) and (ii), we see

$$\lim_{h \to \infty} \frac{\mu\big(A \cap (x-h, x+)\big)}{2h} = 0.$$

So $a \notin \Phi(A)$, contradicting the assumption. □

Not only is Φ monotonic, it is idempotent. Repeated operations return the same result.

Lemma 3.3.4. *Let $A \subseteq \mathbb{R}$. Then $\Phi(A) = \Phi(\Phi(A))$.*

Proof. Choose $x \in \Phi(A)$. Then

$$\lim_{h\to 0} \frac{\mu(A \cap (x-h, x+h))}{2h} = 1.$$

By the Lebesgue density theorem 3.2.1, $\mu(A\Delta\Phi(A)) = 0$. So $\mu(A \cap (x-h, x+h)) = \mu(\Phi(A) \cap (x-h, x+h))$, and

$$d_{\Phi(A)}(x) = \lim_{h\to 0} \frac{\mu(\Phi(A) \cap (x-h, x+h))}{2h} = \lim_{h\to 0} \frac{\mu(A \cap (x-h, x+h))}{2h} = d_A(x).$$

So $d_{\Phi(A)}(x) = 1$ if and only if $d_A(x) = 1$. Then $\Phi(\Phi(A)) = \Phi(A)$. □

As noted in Lebesgue's density theorem, a set A differs from $\Phi(A)$ by only a measure zero set. A measurable set differs from its complement by a set of measure larger than zero. Do $\Phi(A)$ and $\Phi(A^c)$ share any members? If so what does the intersection look like?

Theorem 3.3.5. *Let A be measurable. Then $\Phi(A) \cap \Phi(A^c) = \emptyset$.*

Proof. Assume to the contrary that $\Phi(A) \cap \Phi(A^c) \neq \emptyset$. Then there exists $x \in \Phi(A) \cap \Phi(A^c)$. Let $I_n = (x - \frac{1}{2n}, x + \frac{1}{2n})$ for each $n \in \mathbb{N}$.

Since $x \in \Phi(A)$, then $\frac{\mu(A\cap I_n)}{\mu(I_n)} \to 1$. So there exists an N_1 such that for $n \geq N_1$, $\mu(A \cap I_n) > \frac{2}{3}\mu(I_n)$. Similarly, there exists an N_2 such that for any $n \geq N_2$, $\mu(A^c \cap I_n) > \frac{2}{3}\mu(I_n)$. Let $N = \max\{N_1, N_2\}$. Then for $n \geq N$, the following inequalities hold:

$$\mu(A \cap I_n) > \frac{2}{3}\mu(I_n) \quad \text{and} \quad \mu(A^c \cap I_n) > \frac{2}{3}\mu(I_n).$$

Note that $(A \cap I_n) \cap (A^c \cap I_n) = \emptyset$. So $\mu(I_n) \geq \mu(A \cap I_n) + \mu(A^c \cap I_n) > \frac{4}{3}\mu(I_n)$. This is a contradiction. So $\Phi(A) \cap \Phi(A^c) = \emptyset$. $\square$

The question remains whether $\Phi(\bigcup A_t) = \bigcup \Phi(A_t)$ for every collection $\{A_t\}$. In fact, the equality does not hold for every collection. Consider the collection $\{\mathbb{Q} + r : r \in [0, 1]\}$ where $\mathbb{Q} + r = \{q + r : q \in \mathbb{Q}\}$. Each $\mathbb{Q}_r$ has measure zero, so $\Phi(\mathbb{Q}_r) = \emptyset$. But $\Phi(\bigcup \mathbb{Q}_r) = \Phi(\mathbb{R}) = \mathbb{R}$.

The density function may even fail to distribute across finite unions. Consider the following example.

$$\Phi\big((0,1)\big) \cup \Phi\big((1,2)\big) = (0,1) \cup (1,2) \neq (0,2) = \Phi\big((0,1) \cup (1,2)\big).$$

In some cases, however, the equality $\Phi(\bigcup A_t) = \bigcup \Phi(A_t)$ holds, and it is these cases which give rise to the density topology.

CHAPTER 4

DENSITY TOPOLOGY

4.1 Defining a Topology using Density

In the previous chapter, we noted that for any arbitrary collection $\{A_t\}$, $\Phi(\bigcup_t A_t)$ is not necessarily equal to $\bigcup_t \Phi(A_t)$. However, if $A_t \subseteq \Phi(A_t)$ for each t, then $\bigcup_t A_t \subseteq \Phi\big(\bigcup_t A_t\big)$. The containment holds for any arbitrary collection of sets with this property. Demonstrating this fact is key to building a topology where the open sets are those sets A such that $A \subseteq \Phi(A)$. We call this topology the *density topology.*

Definition 4.1.1. *A set A is open in the density topology if A is measurable and $A \subseteq \Phi(A)$, where $\Phi(A)$ is the set of density points of A. We use $\mathcal{T}$ to denote the collection of open sets. The topology is denoted $(\mathbb{R}, \mathcal{T})$, or just $\mathcal{T}$ if the space is clear from context.*

That such a collection of open sets forms a topology is not immediately clear. The difficulty lies in the fact that a topology must be closed under arbitrary unions, and arbitrary unions of measurable sets are not necessarily measurable. The proof follows Wilczyński's presentation [17].

Theorem 4.1.2. $(\mathbb{R}, \mathcal{T})$ *is a topology.*

Proof. Note that $\Phi(\emptyset) = \emptyset$, $\Phi(\mathbb{R}) = \mathbb{R}$, and $\emptyset$ and $\mathbb{R}$ are measurable. So $\emptyset, \mathbb{R} \in \mathcal{T}$. To show that $\mathcal{T}$ is closed under arbitrary unions, choose a collection of open sets. We want to show that $\bigcup_t A_t \subseteq \Phi\left(\bigcup_t A_t\right)$ and $\bigcup_t A_t$ is measurable. Since $A_t \in \mathcal{T}$, we have $A_t \subseteq \Phi(A_t)$, and it immediately follows that $\bigcup_t A_t \subseteq \bigcup_t \Phi(A_t)$. Choose some x in $\bigcup_t \Phi(A_t)$. Then a T exists such that $x \in \Phi(A_T)$. Note that $A_T \subset \bigcup_t A_t$, so $\Phi(A_T) \subseteq \Phi\big(\bigcup_t A_t\big)$ (by Lemma 3.3.2). Then $x \in \Phi\big(\bigcup_t A_t\big)$. Since this holds for all $x \in \bigcup_t A_t$, we have the result $\bigcup_t A_t \subseteq \Phi\big(\bigcup_t A_t\big)$.

It remains to show that arbitrary unions of open sets are measurable. Let $\{A_t\}_{t\in T}$ be a collection of open sets. Linearly order the elements of T. Choose a sequence $(t_n : n \in \mathbb{N})$ in the following way. Choose the first element of T to be t_0. Following the linear order on T compare each element $A_{t'}$ with A_{t_0}. If $\mu(A_{t'} \setminus A_{t_0}) \geq 0$, let $T_1 = t'$. If no such t' exists, let the sequence be $(t_0, t_0, \dots)$. Once t_1 is chosen, search through T (starting after t_1) to find an A_{t_2} such that $\mu(A_{t_2} \setminus \bigcup\limits_{n=0}^{1} A_{t_n}) \geq 0$. Continue for each n. If at any step m, no A_{t_m} can be found such that $\mu\left(A_{t_m} \setminus \bigcup\limits_{n=0}^{m-1} A_{t_m}\right)$, let the sequence be $(t_0, t_1, \dots t_{m-1}, t_{m-1} \dots)$. Whether or not a unique t'_n can be found for each n, Theorem 2.3.13 shows that the sequence may be at most countably long.

The result is a countable sequence $(t_n :\in \mathbb{N})$ such that for any $r \in T$, we have $\mu\left(A_r \setminus \bigcup\limits_{n=0}^{\infty} A_{t_n}\right) = 0$. Since $(A_{t_n} : n \in \mathbb{N})$ is a countable sequence of measurable sets, $\bigcup\limits_{n=0}^{\infty} A_{t_n}$ is measurable. To show that the arbitrary union $\bigcup\limits_{t\in T} A_t$ is measurable, consider each t. Using Lemma 3.3.3,

$$\mu\left(A_t \setminus \bigcup_{n=0}^{\infty} A_{t_n}\right) = 0 \quad \Rightarrow \quad \Phi(A_t) \subseteq \Phi\left(\bigcup_{n=0}^{\infty} A_{t_n}\right).$$

Using the Lebesgue density theorem (Theorem 3.2.1), $\bigcup\limits_{n=0}^{\infty} A_{t_n}$ and $\Phi\left(\bigcup\limits_{n=0}^{\infty} A_{t_n}\right)$ differ by a nullset. So $\Phi(\bigcup_{n=0}^{\infty} A_{t_n})$ is measurable. Lastly, $\bigcup\limits_{t\in T} A_t$ is measurable because it differs from $\bigcup\limits_{n=0}^{\infty} A_{t_n}$ by a nullset. Since $\bigcup\limits_{t\in T} A_t \subseteq \Phi\left(\bigcup\limits_{t\in T} A_t\right)$ and is measurable, we conclude that $\bigcup_{t\in T} A_t \in \mathcal{T}$. Therefore, $\mathcal{T}$ is closed under arbitrary unions.

Lastly, let $\{A_t\}$ be a finite collection of sets $A_t \in \mathcal{T}$. Since $A_t \in \mathcal{T}$, then $A_t \subseteq \Phi(A_t)$. By Theorem 3.3.1, $\bigcap_t A_t \subseteq \bigcap_t \Phi(A_t) = \Phi(\bigcap_t A_t)$. So $\mathcal{T}$ is closed under finite intersections. $\square$

As an immediate consequence, it can be shown that the topology $\mathcal{T}$ is finer than the usual (Euclidean) topology on $\mathbb{R}$. To see this, take any open set A in the usual topology. Choose a point $x \in A$ and see that there exists an open interval around x contained in A. Using Definition 3.1.1, note that $d_A(x) = 1$. Since this holds for each $x \in A$, we conclude that $A \subseteq \Phi(A)$. In addition, each open set is measurable, so for all A open in the usual topology, $A \in \mathcal{T}$. This shows that $\mathcal{T}$ is at least as fine as the Euclidean topology.

The topology $\mathcal{T}$ is strictly finer than the usual topology T. Consider the following set which is open in $\mathcal{T}$ but is not open in the usual topology. Let $A = \mathbb{R}\backslash\mathbb{Q}$. The set A is not open in the Euclidean topology; for any open ball B, there exists a point $x \in \mathbb{Q} = A^c$ such that $x \in B$. Next, we want to see that A is open in $\mathcal{T}$. Choose any point $z \in A$. Let I be any interval centered on z with diameter $2h$. The measure of I is $2h$. Removing $\mathbb{Q}$ (countably many points), the measure of $I\backslash\mathbb{Q}$ is still $2h$. So, using Definition 3.1.1, it is shown that $d_A(x) = 1$ and $A \in \Phi(A)$. Also note that $\mathbb{R}$ is measurable, so the removal of a measure zero set $\mathbb{Q}$ gives a measurable set A. Thus A is open in $\mathcal{T}$.

This example can be generalized as follows. If A is open in the Euclidean topology and M has measure zero, then $A \setminus M$ is open in the Density Topology, though it is not necessarily open in the Euclidean Topology. From these examples, a few wider lemmas can be drawn. First, countable sets and measure zero sets cannot have density points.

Lemma 4.1.3. *If a set A is countable and $A \neq \emptyset$, then $A \notin \mathcal{T}$.*

Proof. Assume A is countable (possibly finite) and nonempty. The measure of a countable set is zero by Theorem 2.3.6. At each point $x \in A$, the density $d_A(x)$ is

$$d_A(x) = \lim_{h \to 0} \frac{\mu(A \cap (x-h, x+h))}{2h} = 0.$$

So $\Phi(A) = \emptyset$. Then $A \notin \mathcal{T}$. $\square$

If a set has uncountable cardinality, it may still fail to have positive measure (and therefore fail to have any density points). Consider the Cantor set, an uncountable set of zero measure, and therefore zero density.

Example 4.1.4. The Cantor set $\mathcal{C}$ is uncountable and has measure zero. The Cantor set is defined as the intersection of sets A_n:

$$\begin{aligned}
A_0 &= \left[0, 1\right] \\
A_1 &= \left[0, \frac{1}{3}\right] \cup \left[\frac{2}{3}, 1\right] \\
A_2 &= \left[0, \frac{1}{9}\right] \cup \left[\frac{2}{9}, \frac{1}{3}\right] \cup \left[\frac{2}{3}, \frac{7}{9}\right], \left[\frac{8}{9}, 1\right] \\
&\vdots
\end{aligned}$$

Set A_{n+1} is constructed by removing the middle third of each interval comprising set A_n. Define the Cantor set $\mathcal{C} = \bigcap_{n \in \mathbb{N}} A_n$.

We show that $\mathcal{C}$ is uncountable by showing that injections exist between $\mathcal{C}$ and $[0, 1]$. The identity function from $\mathcal{C}$ to $[0, 1]$ is injective. It remains to show that an injection f exists from $[0, 1]$ to $\mathcal{C}$. Each $x \in [0, 1]$ has a binary representation $x = 0.b_1 b_2 \dots$. The function f maps all x with $b_1 = 0$ into the left interval of A_1.

Similarly f maps all x with $b_1 = 1$ into the right interval of A_1. Note that A_{n+1} contains two intervals inside each interval of A_n. So b_{n+1} determines whether f will map into the left or right subinterval of A_n. So for any $x, y \in [0,1], x \neq y$, we have $f(x)$ and $f(y)$ mapping into different intervals. Then $f : [0,1] \to \mathcal{C}$ is injective. So the cardinality of $\mathcal{C}$ is uncountable.

To show that $\mu(\mathcal{C}) = 0$, note that $\mu(A_n) = (\frac{2}{3})^n$. By Theorem 2.3.14, we conclude

$$\mu(\mathcal{C}) = \lim_{n\to\infty} \mu(A_n) = \lim_{n\to\infty} \left(\frac{2}{3}\right)^n = 0.$$

So the density set $\Phi(\mathcal{C}) = 0$.

The density topology requires open sets to be measurable, but not all measurable sets are open in $\mathcal{T}$. It is natural to ask about the relationship of the other measurable sets to the open sets in $\mathcal{T}$. Scheinberg [13] gives a characterization: the Borel sets of $\mathcal{T}$ are the Lebesgue measurable sets.

Theorem 4.1.5. *The Lebesgue measurable sets are precisely the sets which are Borel in the density topology* $\mathcal{T}$.

Proof. Let A be a Borel set in $\mathcal{T}$. Then A is formed from countable unions and countable intersections of sets in $\mathcal{T}$. Each element of $\mathcal{T}$ is measurable, so A is measurable.

Let A be measurable. Then, by the Lebesgue Density Theorem, A can be written as the union of a set of density points C and a measure zero set F. The set C is open in $\mathcal{T}$. Since $\mu(F) = 0$, $\Phi(\mathbb{R} \setminus F) = \mathbb{R}$. So F^c is open, and F is closed. Then $A = C \cup F$ is the union of an open and a closed set. So A is a Borel set. $\square$

Even though a set is Borel, it may be quite complicated to describe. Fortunately, for any measurable set, there exists a subset with the same measure from the F_σ level of the Borel hierarchy, that is, a countable union of closed sets. The theorem is mentioned by Darji [3].

Theorem 4.1.6. *Let A be a measurable set. Then there exists a set $G \subseteq A$ such that G is F_σ with respect to the Euclidean Topology, $\mu(A \setminus G) = 0$, and G is open in $\mathcal{T}$.*

Proof. Assume A is bounded. By Theorem 3.2.1, the Lebesgue Density Theorem, the set of points of A which are not density points has measure zero. Let $Z = \{x \in A : d_A(x) \neq 1\}$. Then $A \setminus Z$ is measurable and has measure equal to A. Since $A \setminus Z$ is measurable, the inner measure satisfies

$$\mu_*(A \setminus Z) = \sup\{\mu^*(K) : K \subseteq A \setminus Z, \quad K \text{ compact}\} = \mu(A \setminus Z).$$

For each $n \in \mathbb{N}$, choose a compact K_n such that $K_n \subseteq A \setminus Z$ and $\mu(A \setminus K_n) < \frac{1}{n}$. Such a K_n exists because A is bounded. Let $G = \bigcup_{n=0}^{\infty} K_n$, and define $N = ((A \setminus Z) \setminus G)$. It is clear that $\mu(N) = 0$. Since Z is a nullset, $\mu(A \setminus G) = \mu\left((A \setminus Z) \setminus G\right) = \mu(N) = 0$. The set G is a union of compact sets, so G is F_σ.

Lastly, we need to show that $G \in \mathcal{T}$. By Theorem 3.2.2, $\mu(A \setminus Z) = \mu(A)$, shows that $\Phi(A \setminus Z) = \Phi(A)$. Then $G \subseteq A \setminus Z \subseteq A \subseteq \Phi(A)$, and $\mu(G \Delta A) = 0$. So $G \subseteq \Phi(A) = \Phi(G)$, and $G \in \mathcal{T}$.

If A is not bounded, for each positive integer n let $G_n \subseteq A \cap [-n, n]$ such that G_n is F_σ with respect to the Euclidean topology, $\mu(A \cap [-n, n] \setminus G_n) = 0$, and G_n is open in $\mathcal{T}$. Then $G = \bigcup G_n$ satisfies the conclusion of the statement. □

Thus the density topology provides several tools for working with measurable sets. For any class of subsets which differ by measure zero, the density function finds a canonical representative of that class. Any measurable set can be written as a Borel set in $\mathcal{T}$, Theorem 4.1.5. Each measurable set has an F_σ subset which differs from it by measure zero, Theorem 4.1.6. Combining these statements, we can say the following. Let $\{A_i\}$ be a collection of subsets of $\mathbb{R}$ such that $\mu(A_i \Delta A_j) = 0$ for all $i \neq j$. Then there exists an F_σ set B such that $\mu(A_i \Delta B) = 0$ for all i.

4.2 Approximate Continuity

Up to this point, we have developed the density topology as it arises from Lebesgue measure. Historically, however, the topology was first described as a "loosening" of the conditions defining continuous functions. According to Wilczyński [17], A. Denjoy first described "approximately continuous" functions in 1915. Wilczyński gives an equivalent definition to Denjoy's original formulation.

Definition 4.2.1. *A function f is* approximately continuous *at a point x iff there exists a measurable set A_x such that*

$$x \in \Phi(A_x) \quad \textit{and} \quad \lim_{t \to x,\ t \in A_x} f(t) = f(x).$$

The definition above uses the density function to say that "almost all" points near x behave as one would expect values from a continuous function to behave.

Approximate continuity is a relaxation of the conditions of continuity. Of course, if a function is continuous at x, it is also approximately continuous. As expected, if a function is approximately continuous at every point, the function is approximately continuous.

Example 4.2.2. The function f defined below is continuous on $\mathbb{R} \setminus \{\frac{1}{n}, n \in \mathbb{N}\}$. The function f is approximately continuous on the same interval.

$$f(x) = \begin{cases} \frac{1}{n} & x = \frac{1}{n}, \quad n = 1, 2, \dots \\ 0 & \text{otherwise} \end{cases}$$

Example 4.2.3. The characteristic function $\chi_{\mathbb{Q}}$ is discontinuous everywhere, but approximately continuous at $\mathbb{R} \setminus \mathbb{Q}$. Notice that $\mathbb{Q}$ is a measure zero set, so the removal of $\mathbb{Q}$ does not affect the measure of any interval around x, when x is irrational.

$$\chi_{\mathbb{Q}}(x) = \begin{cases} 1 & x \in \mathbb{Q} \\ 0 & x \notin \mathbb{Q} \end{cases}$$

Note that the same may be said for any characteristic function χ_M where M is a measure zero set.

Example 4.2.4. Let k be the function

$$k = \begin{cases} \sin\left(\frac{\pi}{x}\right) & x > 0 \\ 0 & x \leq 0 \end{cases}$$

Note that k is continuous on $\mathbb{R} \setminus \{0\}$, so k is approximately continuous on $\mathbb{R} \setminus \{0\}$. Next, consider $x = 0$. Fix any $0 < \varepsilon < 1$, and let $A = \{x \geq 0 : \sin\left(\frac{\pi}{x}\right) > \varepsilon\}$. The roots of $\sin\left(\frac{\pi}{x}\right)$ are $\{\frac{1}{n} : n = 1, 2, \dots\}$. Fix some n and let $I_n = (\frac{1}{n+1}, \frac{1}{n})$. We claim that as $\varepsilon \to 0$, the measure of $A \cap I_n$ approaches the measure of I_n.

Note that $0 < \varepsilon < 1$, $k(x)$ is continuous with no roots on I_n, and $|k(y)| = 1$ for some $y \in I_n$. So there exists an a, b such that $|k(a)| = |k(b)| = \varepsilon$ and for all $x \in (a, b)$,

$|k(x)| > \varepsilon$. Let $d = \max\{|a - \frac{1}{n+1}|, |\frac{1}{n} - b|\}$. Then at most measure d of I_n is not in A on the left of a and on the right of b. So we propose the following lower bound.

$$\frac{\mu\left(I_n\right) - 2d}{\mu(I_n)} \leq \frac{\mu\left(\left(A \cap \left(I_n\right)\right)\right.}{\mu(I_n)}$$

Then let $\varepsilon \to 0$. So $a \to \frac{1}{n+1}$ and $b \to \frac{1}{n}$. As a result, $d \to 0$. Then the lower bound $\frac{\mu(I_n)-2d}{\mu(I_n)} \to 1$. So as $\varepsilon \to 0$, the measure of points x such that $|k(x)| < \varepsilon$ gets arbitrarily small. So there does not exist a density set B such that $0 \in \Phi(B)$ and $\lim\limits_{t \to 0, t \in B} f(t) = 0$. Therefore, $k(x)$ is not approximately continuous at 0.

It seems that a function which is everywhere approximately continuous must be continuous everywhere. However a simple counterexample can be constructed. The following appeared on math.stackexchange.com [12].

Theorem 4.2.5. *There exist functions f which are approximately continuous for all $x \in \mathbb{R}$ but are discontinuous on a dense set of $\mathbb{R}$.*

Construction. For each $n = 3, 4, 5 \ldots$, let $c_n = \frac{(-1)^n}{n}$. Define the function f:

$$f(x) = \begin{cases} \text{linear} & \text{from 0 at } c_n - \frac{1}{n^n} \text{ to 1 at } c_n \\ \text{linear} & \text{from 1 at } c_n \text{ to 0 at } c_n + \frac{1}{n^n} \\ 0 & \text{elsewhere} \end{cases}$$

For any $\varepsilon > 0$, there exists a c_n such that $|c_n - 0| < \varepsilon$, and $f(c_n) = 1$. But $f(0) = 0$. So f is discontinuous at $x = 0$.

The function f is continuous at every $x \neq 0$. To show that f is approximately continuous at 0, let $Z = \{x : f(x) = 0\}$. Of course, $\lim\limits_{t \to x,\ t \in Z} f(t) = 0 = f(t)$. It remains to show that $0 \in \Phi(Z)$. Note that Z is measurable, since it is the countable union of disjoint intervals. Let I_h denote the interval $(x-h, x+h)$. Use the definition of density.

$$d_Z(0) = \lim_{h\to 0} \frac{\mu\left(Z \cap I_h\right)}{2h} = \lim_{h\to 0} \frac{\mu(I_h) - \mu(I_h \setminus Z)}{2h}$$

Then note that the measure of $\mu(I_h \setminus Z)$ is bounded above by $\sum\limits_{\frac{1}{n}\leq h}^{\infty} \frac{2}{n^n}$. So the density is

$$d_Z(0) = \lim_{h\to 0} \frac{\mu(I_h) - \mu(I_h \setminus Z)}{2h} \geq \lim_{h\to 0} \frac{2h - \sum\limits_{n\geq\frac{1}{h}}^{\infty} \frac{2}{n^n}}{2h} \geq \lim_{h\to 0} \frac{2h - \sum_{n\geq\frac{1}{h}}^{\infty} \frac{2}{2^n}}{2h} \geq \lim_{h\to 0} \frac{2h - 2^h}{2h} = 1$$

So f is discontinuous at 0 but is approximately continuous at 0. Construct a function which is discontinuous on $\mathbb{Q}$ but approximately continuous on $\mathbb{R}$ as follows. Fix an enumeration of the rationals: $\{q_i \in \mathbb{Q} : i = 1, 2, 3\ldots\}$. For each i, define f_i to be $f(x - q_i)\cdot 2^{-i}$. Note that each f_i is approximately continuous at q_i but is not continuous there. Let $g(x) = \sum\limits_{i=1}^{\infty} f(x)$. Let $\restriction_B$ denote the restriction of a function to the domain B, and let $g_i = f_i \restriction_{\mathbb{R}\setminus\{q_i\}}$. Then $g \restriction_{\mathbb{R}\setminus\mathbb{Q}} = \sum_{i=1}^{\infty} g_i$. Each g_i is continuous on its domain, so g_i is continuous on the domain $\mathbb{R}\setminus\mathbb{Q}$. Then $g\restriction_{\mathbb{R}\setminus\mathbb{Q}}$ is the absolutely convergent sum of continuous functions, so $g\restriction_{\mathbb{R}\setminus\mathbb{Q}}$ is continuous. Since $\mathbb{R}\setminus\mathbb{Q}$ is open in the density topology, g is approximately continuous on $\mathbb{R}\setminus\mathbb{Q}$.

To show g is discontinuous on $\mathbb{Q}$, choose some $q_i \in Q$ and fix an $0 < \varepsilon < \frac{1}{2^i}$. We will show that arbitrarily close to q_i, there exist $x \in \mathbb{R}$ such that $|g(q_i) - g(x)| > \varepsilon$. There are only finitely many $n < i$.

Let $h = \sum\limits_{n=1}^{i-1} f_n$ and note that each f_n is continuous at q_i. So h is continuous at q_i. Let J be an interval centered at q_i such that $\forall x \in J, |h(x) - h(q_i)| < 1/2^{i+1}$. Let $k = \sum\limits_{n=i+1}^{\infty} f_n$ and note that $|k(x)| < 1/2^{i+1}$. So g is the function $h + f_i + k$. The function

h is continuous and k is bounded above by $1/2^{i+1}$. Recall that f_i is discontinuous at q_i because there are points arbitrarily close to q_i such that $|f_i(q_i) - f_i(y)| > 1/2^i$. Let y be one such point. Then $|g(y) - g(q_i)| > 1/2^i > \varepsilon$. □

Let f be an approximately continuous function, defined on the domain $D \subseteq \mathbb{R}$. Then for each x in the domain D, it is true that $x \in \Phi(A)$ for some $A \subseteq D$. Taking this insight to its conclusion, we see that approximately continuous functions are precisely the functions $f : D \to \mathbb{R}$ mapping $\mathcal{T}$-open sets to open sets in the usual topology.

Theorem 4.2.6. *A function f is approximately continuous if and only if the pre-image of every usual open set is $\mathcal{T}-$open.*

Proof. Let U be an open set in the usual topology such that $U \subseteq \text{Range}(f)$ and $f^{-1}(U) \in \mathcal{T}$. Fix some $y \in U$ and let $x \in f^{-1}(y)$. Fix an $\varepsilon > 0$. Since U is open, there exists an open interval I centered at y with radius $\varepsilon/2$. Let $D = f^{-1}(I)$. Note that $f^{-1}(y) \in D$ and $D \in \mathcal{T}$ by assumption. Let J_n be the open interval centered at $f^{-1}(y)$ with radius $\varepsilon/2$. For any $x \in D \cap J$, $|f(x) - y| < \varepsilon$.

$(\Leftarrow)$: Let U-open describe sets which are open in the usual topology, and let $\mathcal{T}$-open describe open sets in the density topology. Let f be a function such that the pre-image of every U-open set is $\mathcal{T}$-open. Choose some x in the domain of f. Choose a shrinking sequence of open intervals $\{I_i\}$ centered at $f(x)$ such that $|I_i| \to 0$. For each i, the pre-image $f^{-1}(I_i) = U_i$ is $\mathcal{T}$-open, so $U_i \subseteq \Phi(U_i)$. So at each step i:

$$x \in \Phi(U_i) \quad \text{and} \quad y \in I_i \Rightarrow f^{-1}(y) \in U_i$$

This satisfies Definition 4.2.1, and f is approximately continuous.

$(\Rightarrow)$: Let f be approximately continuous. Then at each x in the domain, there exists a set A_x such that $x \in \Phi(A_x)$ and $\lim_{t\to x,\ t\in A_x} f(t) = f(x)$. Since $\Phi(A_x) \neq \emptyset$, then $\Phi(A_x)$ is $\mathcal{T}$-open. So f maps the $\mathcal{T}$-open set $\Phi(A_x)$ into a usual-open set around $f(x)$. $\square$

4.3 Topological Properties of the Density Topology

Since the density topology is finer than the usual topology, sets which are dense in the usual topology may be nowhere dense in $\mathcal{T}$. For example, in the usual topology, $\mathbb{Q}$ is dense. But take any open interval $I \subseteq \mathbb{R}$. Let $D = I\backslash\mathbb{Q}$. I is an open set in $\mathcal{T}$. As we saw previously, the removal of countably many points does not impact the density of the remaining points. So D is open in $\mathcal{T}$. Then we have found an open set $D \subset I$ such that $D \cap \mathbb{Q} = \emptyset$. Then $\mathbb{Q}$ is not a dense set in $\mathcal{T}$. In fact the measure zero sets characterize the nowhere dense sets in $\mathcal{T}$. Wilczyński [17] presents a proof.

Lemma 4.3.1. *A set A has measure zero iff A is nowhere dense in $\mathcal{T}$.*

Proof. $(\Rightarrow)$: Let $\mu(A) = 0$. Then $\Phi(A^c) = \mathbb{R}$. By Theorem 3.3.5, $\Phi(A) = \emptyset$. But $A^c \subseteq \Phi(A^c)$, so $A^c \in \mathcal{T}$. That is, A^c is open and A is closed in $\mathcal{T}$. Also, $Int(A) \subseteq \Phi(A) \cap A = \emptyset$. So A is a closed set with empty interior, and A is nowhere dense.

$(\Leftarrow)$: Let A be nowhere dense. Then $\Phi(A) = \emptyset$. Using the Lebesgue Density Theorem: $\mu(A\Delta\Phi(A)) = 0$ gives us $\mu(A) = 0$. $\square$

There are more open sets in the density topology than in the usual topology. However, each new open set introduced by the density topology must still be somewhere dense. This result is expected, as every nonempty open set in $\mathcal{T}$ must have positive measure at some point.

Theorem 4.3.2. *A set A cannot be nowhere dense in the usual topology and at the same time open in the density topology.*

Proof. $(\Leftarrow)$: Let $\mu(A) = 0$, and assume A is somewhere dense. Then there exists an open interval I such that for every $U \subseteq I$, U open in the usual toplogy, we have $U \cap A \neq \emptyset$. Define $G = I \setminus A$. Since $\mu(A) = 0$, then G is measurable and $\Phi(G) = I$. So $G \in \mathcal{T}$. By definition $G \cap A = \emptyset$, so A is not dense on I. Then A is nowhere dense.

$(\Rightarrow)$: Let A be nowhere dense in T. Then A is nowhere dense in $\mathcal{T}$. By theorem 4.3.1, $\mu(A) = 0$, and A is not open in $\mathcal{T}$. □

The usual topology on $\mathbb{R}$ is a Hausdorff space. That is, given any two points x, y, there exist open sets U, V such that $x \in U$, $y \in V$ and $U \cap V = \emptyset$. This is a direct result of the fact that the usual topology is metrizable. The density topology $\mathcal{T}$ has no obvious metric associated with it. However, $\mathcal{T}$ is Hausdorff, because it is a refinement of the usual topology. The density topology is also a regular space.

Definition 4.3.3. *A topological space is completely regular if given any closed set F, a point x not in F, there exists a continuous function f: $X \to \mathbb{R}$ such that $f(x) = 0$, and $f(y) = 1$, $\forall y \in F$.*

To show that the space is regular, we will need two lemmas. These lemmas, presented by Goffman, Neugebauer, and Nishiura, lead to the Lusin-Menchoff theorem. The theorem states that given a Borel set A and a closed subset $X \subseteq \Phi(A)$, there exists a perfect set P such that $X \subseteq P \subseteq A$ and $X \subseteq \Phi(P)$. This result in turn is used to show the regularity of $\mathcal{T}$. We begin with the lemma for one density point.

Lemma 4.3.4. *Let $B \subseteq \mathbb{R}$ be a Borel set, and let $x \in B$ such that $d_B(x) = 1$. Then there exists a perfect set K such that $x \in K$ and $K \subseteq B$.*

Proof. Choose such an x. For each $i \in \mathbb{N}$, there exists an interval I_i such that the following properties hold.

1. $x \in I_i$
2. $I_i \supset I_{i+1}$
3. $\lim_{i \to \infty} \mu(I_i) = 0$
4. $\lim_{i \to \infty} \frac{\mu(B \cap I_i)}{\mu(I_i)} = 1$

Then we note that each $\mu(B \cap (I_n \backslash I_{n+1})) > 0$, and is a Borel set. Since every Borel set contains a perfect set of the same measure, let $P_n \subseteq B \cap (I_n \backslash I_{n+1})$ be a perfect set with positive measure. Each P_n is perfect. The union of closed sets P_n is not necessarily closed, since a limit point may be introduced in the union which is not contained in any P_n. However, the only possible limit point introduced by $\bigcup_n^\infty P_n$ is x because of the shrinking intervals I_n. So $\bigcup_n^\infty P_n \cup \{x\}$ is closed. For each P_n, choose an element $p_n \neq x$. The sequence p_n approaches x because of the shrinking intervals I_n. So x is not an isolated point, and $\bigcup_n^\infty P_n \cup \{x\}$ is perfect. $\square$

Lemma 4.3.4 shows that an individual density point is contained in some perfect subset. Goffman, Neugebauer, and Nishiura [6] expand the lemma to countable subsets of Borel sets.

Lemma 4.3.5. *Let $B \subset \mathbb{R}$ be a Borel set. Let C be a countable subset of B with $cl(C) \subseteq B$ and each $x \in C$ is a density point of B. Then there exists a perfect set K such that $C \subseteq K \subseteq B$.*

Proof. Let $C = \{x_i : i = 1, 2, 3, \ldots\}$. For each x_i, choose a perfect K_i such that $x_i \in K_i \subseteq B$ and K_i can be covered by an interval of length $1/i$. Lemma 4.3.4 shows that such a K_i exists for each x_i. Let $K = cl(C) \cup \bigcup_i K_i$. We claim that K is perfect. Choose a point $z \in K$ to show that z is not isolated. First, suppose that $z \in Cl(C)$. Every open set containing z must intersect $Cl(C)$ at some other point. So z is not isolated. Next suppose $z \in K_j$ for some j. But K_j is perfect so none of its elements are isolated. Therefore, K is perfect. □

Goffman, Neugebauer, and Nishiura detail a proof of the Lusin-Menchoff theorem [6]. This theorem allows us to prove that the the density topology is both completely regular and Hausdorff.

Theorem 4.3.6 (Lusin-Menchoff Theorem)**.** *Let $A \subseteq \mathbb{R}$ be a Borel set. Let $X \subseteq A$ be a closed set such that $X \subseteq \Phi(A)$. Then there exists a perfect set P such that $X \subseteq P \subseteq A$ and $X \subseteq \Phi(P)$.*

Proof. Let such an X be given. Since X is a closed set, then $X = Cl(X)$. Note that X is the closure of a countable set. We can construct such a countable set C by choosing an element from $X \cap (q_1, q_2)$ for $q_1, q_2 \in \mathbb{Q}$ whenever the intersection is non-empty. For every $n \in \mathbb{N}$, find the subset $R_n \subseteq A$ such that

$$R_n = \left\{ x \in A : \frac{1}{n+1} < d_A(x) < \frac{1}{n}, n \in \mathbb{N} \right\}$$

Then $\bigcup_n^\infty R_n \cup X \subseteq A$. Using Lemma 4.3.5, for every n, there exists a perfect set $P_n \subseteq R_n$ such that $\mu(R_n \setminus P_n) < 2^{-n}$. Let $P = \bigcup_{n=1}^\infty P_n \cup X$. Similarly to the union described in Lemma 4.3.5, P is a perfect set. Also, $X \subseteq P \subseteq A$. It remains to be seen that $X \subseteq \Phi(P)$.

Note that $\mu(A) \leq \mu(P_n) + 2^{-n}$, so we have

$$\mu(A \setminus P) = \mu\left(\bigcup_{n=1}^{\infty} R_n \cup X \setminus \bigcup_{n-1}^{\infty} P_n \cup X\right) \tag{4.1}$$

$$= \mu\left(\bigcup_{n=1}^{\infty} R_n \setminus \bigcup_{n} P_n\right) \tag{4.2}$$

$$= \bigcup_{n=1}^{\infty} \mu(R_n \setminus P_n) \tag{4.3}$$

$$< \sum_{n=1}^{\infty} 2^{-n} \tag{4.4}$$

By choice of initial P_n, we can make (4.4) arbitrarily small, say $\sum_{n=1}^{\infty} r^{-n} < \varepsilon$ for some small r. Then

$$\lim_{i\to\infty} \frac{\mu(A \cap I_i)}{\mu(I_i)} \leq \lim_{i\to\infty} \frac{\mu(P \cap I_i)}{\mu(I_i)} + \varepsilon$$

Since $x \in X \subseteq \Phi(A)$, we conclude $\lim_{i\to\infty} \frac{\mu(A\cap I_i)}{\mu(I_i)} = 1$. Therefore, as ε goes to 0, $\lim_{i\to\infty} \frac{\mu(P\cap I_i)}{\mu(I_i)} = 1$ and $x \in \Phi(P)$. $\square$

The Lusin-Menchoff Theorem allows us to prove that $(\mathbb{R}, \mathcal{T})$ is a completely regular and Hausdorff space. We follow Wojdowski's proof [18], which cites a lemma from Bruckner [2], stated below without proof.

Lemma 4.3.7. *Let A be an open set in $\mathcal{T}$. There exists a $\mathcal{T}-$continuous function f such that $0 < f(x) \leq 1$ for all $x \in A$ and $f(x) = 0$ for all $x \notin A$.*

Theorem 4.3.8. *The space $\mathbb{R}$ with the density topology $\mathcal{T}$ is a $T_{3\frac{1}{2}}$ or Tychonoff space. It is both completely regular and Hausdorff.*

Proof. As shown above, the space is Hausdorff. To show that it is completely regular, let F be a closed set and let $x \in F^c$. Note that $\mathbb{R} \setminus \{y\}$ is Borel and $\mathcal{T}$-open. By

Theorem 4.3.6, there exists a perfect P such that $F \subseteq P \subseteq \big(\mathbb{R} \setminus \{y\}\big)$. Then P^c and $\mathbb{R} \setminus \{y\}$ are open. From Lemma 4.3.7, there exists an f, g such that

$$0 < f(x) \leq 1 \text{ for } x \in P^c \text{ and } f(x) = 0 \text{ for } x \in P$$

$$0 < g(x) \leq 1 \text{ for } x \in \mathbb{R} \setminus \{y\} \text{ and } g(y) = 0$$

Let $h(x) = \frac{f(x)}{f(x)+g(x)}$. Then $h(y) = 1$ and $h(x) = 0$ for all $x \in P$. $\square$

This chapter explored some of the topological properties of the Density Topology, starting from the definition of the density function. For a deeper look into the Density Topology on $\mathbb{R}$, the reader is directed to Wilczyński [17], Scheinberg [13], Tall [15], and Zahorski [19].

CHAPTER 5

DENSITY TOPOLOGY ON THE CANTOR SPACE

Chapter 3 discussed the density topology on the space of real numbers using the Euclidean metric. As we will see, the density topology can be defined on other spaces. In fact, the density topology can be defined on any finite-dimensional Euclidean space $\mathbb{R}^n$, but these spaces require additional definitions, as the naive approach becomes ambiguous in the multi-dimensional case. For an in-depth treatment of the density topologies which arise on a given $\mathbb{R}^n$, see [6] and [18]. When considering which spaces have a density topology, it simplifies things if a metric exists on that space. But just having a metric is not a sufficient condition to give a density topology which makes sense. For a counterexample, see chapter 6.

Next, we turn our attention to another space which has a metric and a measure, the Cantor Space. Our development of the theory follows Andretta and Camerlo [1].

5.1 A Metric on the Cantor Space

The Cantor set $\mathcal{C}$ is formed by taking the interval $[0,1]$ and removing the middle open third, then removing the middle open third of each of the two remaining intervals. Continue the process for countably many steps, removing the middle third of each remaining interval at each step. The Cantor set is the intersection of the sets produced in this way. The Cantor set has a bijection to the interval $[0,1]$ as follows: Write each number $x \in [0, 1]$ using its binary representation. This representation corresponds to a series of choices. If the first digit is 0, then x will be mapped to the left interval defined in step 1 of the construction. If the first digit is 1, x is mapped to the right interval. Each digit decides the "left" or "right" mapping. If x is a finite decimal of length n, consider it as an infinite decimal which is constantly zero after the nth position. This gives a map from $[0, 1] \to \mathcal{C}$. To see the other direction, just note that the left endpoint of an interval is mapped to a binary sequence which is eventually zero, and the right endpoints to sequences which are eventually constant 1. Any other point has a binary representation corresponding to the intervals in the construction.

A Cantor Space is any topological space that is homeomorphic to the Cantor set. We denote our Cantor Space C or $2^{\leq\omega}$, since each point can be written as a sequence of choices in a binary decision tree. We will say that a sequence is *finite* iff it is eventually constant. Note that there are countably many such sequences. All other sequences will be called *infinite*. Note that finite and infinite sequences are included in $2^{\leq\omega}$. Let $2^{<\omega}$ denote the set of sequences of finite length and 2^{ω} denote the set of sequences of infinite length. Finite length sequences of length n correspond to endpoints of intervals removed in step n during the usual construction of the Cantor Set in $\mathbb{R}$. If a sequence has finite length, it may be extended by another sequence of finite or infinite length.

Notation 5.1.1. Let $s \in 2^{<\omega}$ be a finite sequence $s = (s_0, s_1, \ldots s_n)$, and let $x \in 2^{\leq\omega}$ be an infinite sequence $(x_0, x_1, \ldots)$. Concatenate the sequences as follows: $s^\frown x = (s_0, s_1, \ldots s_n, x_0, x_1, \ldots)$. Then we say that $s^\frown x$ *extends* s.

If $s \in 2^{<\omega}$, then the length of s is $\ell(s) < \infty$. Otherwise, $\ell(s) = \infty$.

The set of sequences which extend a finite sequence s is the basic neighborhood of s. Given a set A, the *localization* of A with respect to s is the intersection of the basic neighborhood of s with the set A.

Definition 5.1.2. *Let A be a subset of C. Let $s \in 2^{<\omega}$ be a point (a finite sequence). We define the* localization of A to s *as* $A_{\lfloor s \rfloor} = \{x \in 2^{\leq\omega} | s^\frown x \in A\}$.

The localization finds all points $a \in A$ such that the sequence of a begins with s. The Cantor Space comes equipped with a metric, defined below. It is important to note that all basic neighborhoods can be written as open balls using this metric. Let x be a point. The basic neighborhood of x is $N_x = B_{2^{-\ell(x)}}(x)$. If x has infinite length, the basic neighborhood is just $\{x\}$.

Definition 5.1.3. *Let $x, y \in 2^{\leq\omega}$. Define the distance d between x and y as*

$$d(x,y) = \begin{cases} 0 & x = y \\ 2^{-n} & n \textit{ is the first position where } x \neq y \end{cases}$$

The function d is indeed a metric, but the proof is left to the reader. The metric can be seen in other contexts as an ultrametric. Using this metric, open balls are defined in the usual way. Let x be a point. The open ball of radius r is denoted $B_r(x)$ and is the set of all points of with distance less than r from x using the metric defined above. The topology T_C is constructed using the open balls as a basis. In fact, the topology T_C is the same as the subspace topology with the Cantor set as a subset of the reals.

5.2 A Measure on the Cantor Space

When deciding how to define a measure on the Cantor Space, it is natural to think of the space as the collection of binary sequences. This leads to the definition of a probability measure on the space. Choose a sequence $s \in C$. The measure of the basic neighborhood N_s should be equal to the probability that a random finite binary sequence is in N_s.

Definition 5.2.1. *Let $s \in C$, and let N_s be the basic neighborhood of s. Define the measure of N_s as $\mu(N_s) = 2^{-\ell(s)}$.*

This approach is also called the "coin-tossing" or Bernoulli measure. Will still need to prove that μ is in fact a measure. The name "coin-tossing" is appropriate, as μ measures the chances a random binary sequence will begin with a particular finite subsequence. For example, let $s = 0001$. Then $\mu(N_s) = \frac{1}{16}$, as an infinite binary sequence begins with 0001 with probability $\frac{1}{16}$. A few more facts about basic neighborhoods are pertinent to our discussion.

Lemma 5.2.2. *Let $s \in C, \ell(s) < \infty$. Then for any $x \in C$ which extends s, $N_x \subseteq N_s$.*

Proof. Take s, x as above. Note that since x extends s (x may be infinite) we have:

$$x = s_1, s_2, \ldots s_{\ell(s)}, x_1, x_2, \ldots x_n, \ldots$$

If x is an infinite sequence, then $N_x = \{x\}$ and $N_x \subseteq N_s$. Assume x is finite. To show $N_x \subseteq N_s$, it suffices to show that y extends s whenever y extends x.

Let y be a point which extends x. We have $y = s_1, s_2, \ldots s_{\ell(s)}, x_1, x_2, \ldots x_n, y_1, y_2$. The $y_1, y_2, \ldots$ subsequence may or may not be infinite, but it extends s, and the proof is complete. $\square$

Remark 5.2.3. Let s be a point, and let N_s be the basic neighborhood of s. We can consider the neighborhood of s as the whole space C localized to the point s. That is, $N_s = C_{\lfloor s \rfloor}$. So $\mu(N_s) = \mu\big(C_{\lfloor s \rfloor}\big) = 2^{-\ell(s)}$.

Note that when $\ell(s) = 0$, $C_{\lfloor s \rfloor} = C$, and $\mu(C_{\lfloor s \rfloor}) = \mu(C) = 2^0 = 1$. So the measure of the entire space is 1, as required for μ to be a probability measure. This definition is intuitive for basic neighborhoods, but to be defined as a measure, it should be defined on more sets. We extend the above definition by defining the measure on the space.

Definition 5.2.4. *The* Lebesgue measure μ *on* C *is the Borel measure such that for each basic neighborhood* N_s, $\mu(N_s) = 2^{-\ell(s)}$.

It remains to be shown which sets are measurable. A set is measurable if and only if all of its localizations are measurable.

Characterization 5.2.5. A set A is measurable if and only if for each $s \in 2^{<\omega}$

$$\mu(A_{\lfloor s \rfloor}) = \frac{1}{2}\big(\mu(A_{\lfloor s^\frown 0 \rfloor}) + \mu(A_{\lfloor s^\frown 1 \rfloor})\big).$$

It is clear that this property holds at least for the basic neighborhoods. Using the basic neighborhoods as the generating sets, the Lebesgue measure μ is defined on all the Borel sets. Not only do we want the measure to hold on the basic neighborhoods, but each metric ball should be measurable. Also, each measurable set should have a sensible measure on all of its localized subsets.

Lemma 5.2.6. *Let* $A \subseteq C$ *be measurable. Then*

$$\mu(A) = \sum_{s \in 2^{<\omega}} 2^{-2\ell(s)-1} \mu(A_{\lfloor s \rfloor}).$$

Proof. Start with a measurable $A \subseteq C$. By Characterization 5.2.5, we know

$$\mu(A) = \frac{1}{2}\big(\mu(A_{\lfloor 0 \rfloor}) + \mu(A_{\lfloor 1 \rfloor})\big).$$

Each of the localizations can be localized again. Note that A is measurable, and a localization is the intersection of A with a basic neighborhood. So each localization is measurable.

$$\mu(A) = \frac{1}{2}\big(\mu(A_{\lfloor 0 \rfloor}) + \mu(A_{\lfloor 1 \rfloor})\big) = \frac{1}{2}\Big(\frac{1}{2}\big(\mu(A_{\lfloor 00 \rfloor}) + \mu(A_{\lfloor 01 \rfloor})\big) + \frac{1}{2}\big(\mu(A_{\lfloor 10 \rfloor}) + \mu(A_{\lfloor 11 \rfloor})\big)\Big)$$

This process can be iterated any finite number of times. At stage n, the localizations simplify as follows:

$$\mu(A) = 2^{-n} \sum_{s \in 2^n} \mu(A_{\lfloor s \rfloor}), \quad n < \infty$$

If we let n go from $0 \to \infty$, we see that $\sum_n 2^{-n-1} = 1$. We give an equality for $\mu(A)$:

$$\mu(A) = 1 \cdot \mu(A) = \sum_{n=0}^{\infty} 2^{-n-1} \mu(A)$$

Then we substitute the previous derived equality for $\mu(A)$. Combining the exponents:

$$\mu(A) = \sum_{n=0}^{\infty} 2^{-2n-1} \sum_{s \in 2^n} \mu(A_{\lfloor s \rfloor})$$

$$\mu(A) = \sum_{s \in 2^{<\omega}} 2^{-2\ell(s)-1} \mu(A_{\lfloor s \rfloor})$$

$\square$

If $\ell(s)$ is infinite, then $\mu(A_{\lfloor s \rfloor}) = 0$. No elements extend s, so $A_{\lfloor s \rfloor} = \{s\}$. Let $s = (s_1, s_2, \dots)$ be s written as a sequence. Then for each $i = 1, 2, 3, \dots$, $\{s\} \subseteq$

N_{s_i}. So $\mu(\{s\}) \le \mu(N_{s_i})$ for each i. The limit of measures of neighborhoods is $\lim_{i\to\infty} \mu(N_{s_i}) = \lim_{i\to\infty} 2^{-\ell(s_i)} = 0$. So $\mu(A_{\lfloor s \rfloor}) = \mu(\{s\}) = 0$.

5.3 Density in the Cantor Space

The density topology is defined using the measure given in Definition 5.2.4. In the space of real numbers, density at a point is measured in relationship to the measure of constricting intervals around that point. In the Cantor Space, the basic neighborhoods take the place of constricting intervals.

Definition 5.3.1. *Let A be a measurable set, and let $s \in C$. Let s_i be the first i terms of s. The* density of s in A *is defined as*

$$\begin{aligned} d_A(s) &= \lim_{i\to\infty} \frac{\mu(A \cap N_{\lfloor s_i \rfloor})}{\mu(N_{\lfloor s_i \rfloor})} && (5.1) \\ &= \lim_{r\to 0} \frac{\mu\left(A \cap B_r(s)\right)}{\mu(B_r(s))} && (5.2) \\ &= \lim_{n\in\mathbb{N}} \frac{\mu\left(A \cap B_{2^{-(n+1)}}(s)\right)}{\mu\left(B_{2^{-(n+1)}}(s)\right)} && (5.3) \end{aligned}$$

A point s is a density point of A iff $d_A(s) = 1$. The density function is defined $\Phi(A) : \mathcal{P}(C) \to \mathcal{P}(C), \Phi(A) = \{x \in C : d_A(x) = 1\}$.

Line 5.1 is the density with respect to localization. Line 5.2 uses the fact that basic neighborhoods are also open balls. Line 5.3 comes from Definition 5.2.4. The theory of the density topology is underpinned by the Lebesgue Density Theorem. The Lebesgue Density Theorem states that the symmetric difference between a set and its density points has measure zero. To see a proof of the Lebesgue Density Theorem for $\mathbb{R}$, see Theorem 3.2.1. Given an arbitrary measure on a measure space, the Lebesgue

Density Theorem may not hold. To go any farther in our definition of the density topology on C, it is necessary to show that the Lebesgue Density Theorem holds on C. Benjamin Miller [9] showed a set of criteria which is sufficient for the Lebesgue Density Theorem to hold on a given space. We present the criteria without proof.

Theorem 5.3.2 (Proposition 2.10 of [9])**.** *Let X be a Polish space with an ultrametric. Let μ be a probability measure on X, and let $A \subseteq X$ be a Borel set. Define $B_t(x)$ to be the open ball of radius t around x using the ultrametric. Then*

$$\lim_{t\to 0} \frac{\mu\big(A \cap B_t(x)\big)}{\mu\big(B_t(x)\big)} = 1$$

for almost every $x \in A$.

It is clear that $\mathcal{C}$ is Polish. The metric on $\mathcal{C}$ is an ultrametric, see note after Definition 5.2.1. The measure μ is a probability measure on $\mathcal{C}$, and the criteria are satisfied. Continuing our exposition of the density topology, we show that the easily constructable sets are in fact measurable.

Theorem 5.3.3. *Let $s \in 2^{<\omega}$. Then N_s is measurable and $N_s \subseteq \Phi(N_s)$.*

Proof. N_s is measurable and $\mu(N_s) = 2^{-\ell(s)}$ by Definition 5.2.1 for each s. Let $\ell(s) = n < \infty$. Choose some $x \in N_s$ to show that $x \in \Phi(N_s)$. Consider the density of x in N_s:

$$d_{N_s}(x) = \lim_{i\to\infty} \frac{\mu(N_s \cap N_{x_i})}{\mu(N_{x_i})}.$$

Since $x \in N_s$, then x extends s (or $x = s$). So $N_{x_i} \subseteq N_s$ for all $i \geq \ell(s)$, and the density $D_{N_s}(x)$ is

$$d_{N_s}(x) = \lim_{i\to\infty} \frac{\mu(N_{x_i})}{\mu(N_{x_i})} = 1.$$

So $N_s \subseteq \Phi(N_s)$ for every s of finite length. $\square$

The above theorem cannot be extended to points of infinite length. If $\ell(s) = \omega$, then $\mu(N_s) = 0$. Then $N_s \not\subseteq \Phi(N_s) = \emptyset$.

Once the Lebesgue Density Theorem is in place, we may define the density topology. The definition is the same as stated in Chapter 3. It is not immediately clear that the collection of sets $\mathcal{T}_C = \{A \text{ is measurable: } A \subseteq \Phi(A)\}$ is in fact a topology.The proof follows similarly to Theorem 4.1.2, so it is omitted here. Open sets in the density topology have an interesting characterization which is presented here.

Characterization 5.3.4. The topology can also be characterized by

$$\mathcal{T}_C = \{\Phi(A) \setminus N : A \text{ is measurable and } N \text{ is a nullset}\}.$$

Proof. Choose a set $T = \Phi(A) \setminus N$ as above. Then $\Phi(T) = \Phi(\Phi(A) \setminus N)$. But the removal of a nullset does not change the measure, so $\Phi(\Phi(A) \setminus N) = \Phi(\Phi(A)) = \Phi(A)$. But $T = \Phi(A) \setminus N$ so $\Phi(T) = \Phi(\Phi(A)) = \Phi(A) \supseteq T$. So T is open in $\mathcal{T}_C$.

Let U be an open set in $\mathcal{T}_C$. Then $U \subseteq \Phi(U)$. Then $U = \Phi(U) \setminus \big(\Phi(U) \setminus U\big)$. Now, notice that $\mu(\Phi(U) \setminus U\big) = 0$ by the Lebesgue Density Theorem (Miller [9]). Then every open set in $\mathcal{T}_C$ can be written as the density points of a set minus a nullset. $\square$

5.4 Properties of the Density Topology on C

We explore the properties of the topological space $\mathcal{T}_C$. Recall that $\mathbb{R}$ is a separable space in the usual topology, but it is not separable in the density topology. The same is true on the Cantor Space C.

Theorem 5.4.1. *C is separable with the topology generated by the metric d (Definition 5.1.3). However, $\mathcal{T}_C$ is not separable.*

Proof. Since $2^{<\omega} \subseteq C$, C is separable with the usual topology. Assume $\mathcal{T}_C$ is separable, then there exists a countable dense W. Since W is countable, $\mu(W) = 0$. Then $C \setminus W$ is measurable and $C \setminus W \subseteq \Phi(C \setminus W) = C$. So $C \setminus W \in \mathcal{T}_C$, and $\mathcal{T}_C$ is not separable. $\square$

Lemma 5.4.2. *In the usual topology, for any basic neighborhood of a point x, Int$(N_x) = N_x$ and Cl$(N_x) = N_x$. Therefore, N_x is open and closed. Let $\mathcal{N}$ be a finite collection of basic neighborhoods. Then $\bigcap \mathcal{N}$ is clopen and $\bigcup \mathcal{N}$ is clopen.*

Note also that C is locally compact. Every $x \in C$ has a compact neighborhood, namely N_x. All points $y \in N_x$ are within distance $(\frac{1}{2})^{\ell(x)}$, and N_x is closed.

Every basic neighborhood can be represented by a element x of finite length. The basic neighborhood is comprised of all elements which extend x. The basic neighborhoods provide a natural way of analyzing sets in the space. Since all points can be represented as binary sequences, the basic neighborhood is a binary test for sequence extension.

Lemma 5.4.3. *Let $x, y \in 2^\omega$. If y is not an initial segment of x and x is not an initial segment of y, then $\forall z \in N_x, z \notin N_y$.*

Proof. Recall that $N_x = \{z \in \mathcal{C}; \exists t \in \mathcal{C}, z = x^\frown t\}$. If z is an element of N_y, z can be written $z = y^\frown t', t' \in \mathcal{C}$. But $z = x^\frown t$, and x does not agree with y (neither is an initial segment of the other). So z cannot be written as $y^\frown t'$. Therefore, $z \notin N_y$. $\square$

If a set of elements can be arranged in a sequence $\{x_1, x_2, x_3, \dots\}$ such that x_j extends x_i for all $j > i$, we call the sequence a chain.

Definition 5.4.4. *A* chain *is a set A of points such that one point is an initial segment for all the other points. After removing that point, there exists a point in the remaining set which is an initial segment for all the others.*

If every pair of points in a set disagrees in at least one position, the set is an anti-chain. *That is, no points extends any other point.*

For example, the set $A = \{001, 00101, 001010, 00101011\}$ is a chain. Chains will be key to understanding basic neighborhoods of C.

Theorem 5.4.5. *Let $A \subset 2^{<\omega}$ be an anti-chain. Then $N_x \cap N_y = \emptyset$, for $x, y \in A, x \neq y$. As a result, $\mu\big(\bigcup_{x \in A} N_x\big) = \sum_{x \in A} \mu(N_x) \leq 1$.*

Proof. Let A be an anti-chain. By the previous lemma, $N_x \cap N_y = \emptyset$ for all $x, y \in A, x \neq y$. Since measure is countably subadditive and each neighborhood is disjoint, $\mu(\bigcup_{x \in A} N_x) = \sum_{x \in A} \mu(N_x)$. Each point is measured at most once, so $\sum_{x \in A} \mu(N_x) \leq \mu(\mathcal{C}) = 1$. $\square$

As demonstrated above, anti-chains are useful for characterizing disjoint sets. This allows easy computation of measure. This principle can be carried to any clopen set. Let D be a clopen set and let $L = \{s : N_s \subseteq D\}$. Next, restrict L to $T = \{s \in L : s^\frown 0,\ s^\frown 1 \notin L\}$. The set T is a "tree" which is identified by D. We claim that each clopen D is uniquely identified by a tree T and the cardinality of T is finite. Let elements of T be called "leaves."

Theorem 5.4.6. *Let $D \subseteq 2^\omega$ be a clopen set. Then D can be uniquely identified with a tree $T = \{s : N_s \subseteq D\}$ such that the elements of T define the component neighborhoods of D. That is, $D = \bigcup\{N_t : t \in T\}$.*

Proof. Let T be a finite subset of 2^ω. In fact, the elements of T form an anti-chain. Define the set $D = \bigcup_i N_{x_i}$ where $x_i \in T$. To show that D is clopen, we will demonstrate that D and D^c are open. D is open, since it is the union of open sets N_{x_i}. Let $T = \{a_1, a_2, \dots\}$. A point y is an element of D^c if and only if y does not extend some a_j. So $D^c = \bigcup_h N_{y_h}$, where $y_h \in C$ and y_h does not extend any element of T. Then D^c is the union of open sets and is therefore open. So D is clopen.

To show that each D has a unique T, assume there exist two such trees T_1 and T_2. Since T_2 is distinct from T_1, one of them must have an element which is not in the other. Assume $e \in T_2, e \notin T_1$. Note that the elements of T_1 define all the component neighborhoods of D. Since the elements of T_2 form an anti-chain, T_2 must define a component neighborhood N_e which T_1 does not. Note that $N_e \nsubseteq D$, and the proof is complete. □

The density topology in C and the density topology in $\mathbb{R}$ share many similar properties. We list several properties where the proofs follow exactly as in $\mathbb{R}$.

Lemma 5.4.7. *The density function Φ is monotonic.*

Proof. Let A, B be sets such that $A \subseteq B$. If $x \in \Phi(A)$, then $d_A(x) = 1$. But $d_B(x) \geq d_A(x)$, so $d_B(x) = 1$. □

Lemma 5.4.8. *Let $\mathcal{I}$ denote the collection of nullsets of C. Then $\mathcal{I}$ is a $\sigma-$ideal of measurable sets and $\mathcal{I} = \{A \subseteq C : A$ is nowhere dense in $\mathcal{T}\}$.*

Proof. Let $\{N_i\}$ be a countable collection of nullsets. We proved earlier (Claim 2.12) that $\mathcal{I}$ is closed under countable unions. Also, any subset of a nullset is a nullset, so $\mathcal{I}$ is closed under intersections. If we take any measurable set A, then $A \cap I$ is a nullset for each $I \in \mathcal{I}$.

Let $I \in \mathcal{I}$. Then $\mu(I) = 0$. Consider the set $C \setminus I$. Note that the removal of a nullset does not affect the measure at any point, so $\Phi(C \setminus I) = \Phi(C) = C$. So $C \setminus I$ is open and I is closed in $\mathcal{T}$. Then $Int(I) \subseteq I \cap \Phi(I) = I \cap \emptyset = \emptyset$, so I is nowhere dense (empty interior).

Let J be nowhere dense. Then $Cl(J)$ is also nowhere dense and closed. So

$$Int(Cl(J)) \subseteq Cl(J) \cap \Phi(Cl(J)) = Cl(J) \cap \emptyset = \emptyset$$

Then since $\mu(A \Delta \Phi(A)) = 0$ for all A, then $\mu(Cl(J)) = \mu(\Phi(Cl(J))) = 0$. Also note that $J \subseteq Cl(J)$ so $J \in \mathcal{I}$. $\square$

Theorem 5.4.9. *A is a nullset if and only if it is meager in $\mathcal{T}$. That is $\mu(A) = 0 \Leftrightarrow A = \bigcup_i (C_i)$, for a countable union of nowhere dense sets.*

Proof. Let A be a meager set. Then $A = \bigcup_i A_i$ where A_i is nowhere dense. Then A_i is a nullset, by the previous lemma. The collection of nullsets is a σ-ideal, so $\bigcup_i A_i = A$ is a nullset.

Let N be a nullset. Then, using the previous lemma, N is a nowhere dense set. So N can be written (trivially) as the union of nowhere dense sets. Then N is meager. $\square$

Recall that a space is Baire if every union of countably many closed nowhere dense sets has empty interior.

Theorem 5.4.10. *$(C, \mathcal{T})$ is a Baire space.*

Proof. Let $N_1, N_2, \ldots$ be a countable collection of closed nowhere dense sets. By the Lemma 5.4.8, each N_i is also a nullset, and the union of countably many nullsets is a nullset. So $\bigcup_i N_i$ is a nullset, and is therefore nowhere dense. So $Int\Big(\bigcup_i N_i\Big) = \emptyset$, and $(C, \mathcal{T})$ is a Baire space. $\square$

Like the Density Topology on the $\mathbb{R}$, the Density Topology on C is not separable. Both are refinements of the usual topologies, and both are Baire spaces. In the next chapter, we discuss other properties that both topologies share.

CHAPTER 6

GENERAL PROPERTIES OF THE DENSITY TOPOLOGY

The topological functions of interior and closure can be defined with reference to the collection of open sets in a topology. This chapter explores some of the properties which are the same in the density topology on $\mathbb{R}$ and the density topology on the Cantor Space. The results only require interior, closure, and density functions.

6.1 Interior and Closure Properties

We begin by defining the interior and closure operators.

Definition 6.1.1. *The* interior *of a set A is the largest open set contained in A.*

The interior can also be characterized as $Int(A) = \bigcup_i U_i, U_i \subseteq A, U_i$ open.

Definition 6.1.2. *The* closure *of a set A is the smallest closed set containing A.*

The exterior can also be characterized as $Cl(A) = \bigcap_i C_i, C_i \supseteq A, C_i$ closed. Both $\mathbb{R}$ and C have metric functions which are used to define Lebesgue measure on the respective space. These metric functions give rise to the "usual" or Euclidean topologies on $\mathbb{R}$ and C. Throughout this chapter, we will assume that an *open* or *closed* set is open or closed with reference to the density topology. However, a *metrically open* or *metrically closed* set will be open or closed with respect to the usual topology on each space.

If a set is contained in its density points, it is an open set. For a set which is not open, it can often be helpful to find the interior and closure of that set. The interior gives the largest open set contained in the original set, and the closure gives the smallest closed set which contains the original set. Thus, we can think of interior and closure as 'approximations' of some set with certain desired properties (open or closed). In a similar way, the density function gives a close approximation of a set. It returns a set that differs from the original set by measure zero such that points in the new set each have density 1. As we will see, the density function Φ interacts nicely with the Int and Cl operators. This section aims to prove that $Int(A) \subseteq \Phi(A) \subseteq Cl(A)$. We follow Andretta and Camerlo [1] throughout this section except where noted. The notation $A \equiv B$ denotes that the sets A, B have the property $\mu(A \Delta B) = 0$, where Δ is symmetric set difference. The first proof demonstrates that $Int(A) \subseteq \Phi(A)$. For any open A the proof takes one line:

$$Int(A) \subseteq A \subseteq \Phi(A)$$

But this is not immediately true for A which is not open.

Lemma 6.1.3. *For any measurable set $A \subseteq C$, $Int(A) \subseteq \Phi(A)$.*

Proof. $Int(A)$ is open, so $Int(A) \subseteq \Phi(Int(A))$. Also, for any A, we have $Int(A) \subseteq A$. Use Lemma 5.4.7 and the fact that Φ preserves containment:

$$Int(A) \subseteq A \quad \Rightarrow \quad \Phi(Int(A)) \subseteq \Phi(A)$$

Then we have $Int(A) \subseteq \Phi(Int(A)) \subseteq \Phi(A)$. □

The interior of a set A is an open set contained in A. The interior $Int(A)$ may be missing points from A. How close is the containment to equality? The answer depends on how 'close' A is to being measurable. For any set A, define a *measurable kernel* B of A as an open set $B \subseteq A$ such that $\mu(B) = \mu_*(A)$, where μ_* is the inner measure, if such a B exists.

Theorem 6.1.4. *For any $A \subseteq C$, the interior $Int(A) = A \cap \Phi(B)$, where B is a measurable kernel of A.*

Proof. Let $x \in Int(A)$. Note $x \in A$, and there exists an open set $U \subseteq A$ such that $x \in U$. Since U is open, $x \in \Phi(U)$. Next, $U \setminus B \subseteq A \setminus B$ and $U \setminus B$ is measurable. Also, $\mu^*(A \setminus B) = 0$, so $\mu(U \setminus B) = 0$. So $\Phi(U) = \Phi(U \cap B)$, because U and $U \cap B$ differ by a nullset. Then $\Phi(U) = \Phi(U \cap B) \subseteq \Phi(B)$. So $x \in A \cap \Phi(B)$.

Let $x \in A \cap \Phi(B)$. Since B is an open subset of A, and $x \in A$, $B \cup \{x\} \subseteq A$. Then consider $\Phi(B \cup \{x\})$. The addition of one point does not change the set of density points so $\Phi(B \cup \{x\}) = \Phi(B)$. We define S as:

$$\big(B \cup \{x\}\big) \cap \Phi(B) = \big(B \cup \{x\}\big) \cap \Phi\big(B \cup \{x\}\big) = S$$

Then find the density points $\Phi(S)$:

$$\Phi(S) = \Phi\Big(\big(B \cup \{x\}\big) \cap \Phi\big(B \cup \{x\}\big)\Big) = \Phi\big(B \cup \{x\}\big) \cap \Phi\big(\Phi(B \cup \{x\})\big)$$

$$= \Phi(B \cup \{x\}) \cap \Phi(B \cup \{x\}) = \Phi(B \cup \{x\}) \supseteq S$$

Finally, we conclude that $S \subseteq \Phi(S)$, so S is open in $\mathcal{T}$. Since S is open, $x \in S$, and $S \subseteq A$, we conclude that $x \in Int(A)$. □

For a given set A with a measurable kernel B, the density set $\Phi(B)$ is always measurable. By the Lebesgue Density Theorem, $\mu(\Phi(B)) = \mu(B)$. Since B is a measurable kernel, $\mu_*(A) = \mu(B) = \mu(\Phi(B))$. So the measure of the interior of A is bounded above by the inner measure of A. The next step is consider the closure and its relationship to Φ.

Lemma 6.1.5. *For any closed set A, we have $\Phi(A) \subseteq A$.*

Proof. Consider a closed set A. Then A^c is open. By definition of open, $\Phi(A^c) \subseteq \big(\Phi(A)\big)^c$. Then,

$$A^c \subseteq \Phi(A^c) \subseteq \big(\Phi(A)\big)^c.$$

Compare the first and last terms, then take the complement. The result: $\Phi(A) \subseteq A$ when A is closed. □

Lemma 6.1.6. *For any measurable set A, $\Phi(A) \subseteq Cl(A)$.*

Proof. For any A, $A \subseteq Cl(A)$. Since Φ is monotonic (Lemma 5.4.7), $\Phi(A) \subseteq \Phi(Cl(A))$. Also $Cl(A)$ is closed, so Lemma 6.1.5 gives $\Phi(Cl(A)) \subseteq Cl(A)$. Joining these containments gives

$$\Phi(A) \subseteq \Phi(Cl(A)) \subseteq Cl(A).$$

□

Using Lemmas 6.1.3 and 6.1.6, we conclude that for any measurable set $A \subseteq C$, the relation $Int(A) \subseteq \Phi(A) \subseteq Cl(A)$ holds. The density function maps many sets to each density set. An inspection of the density set gives some information about possible pre-images.

Theorem 6.1.7. *A set A is the density set of a closed set B if and only if $A = \Phi(Cl(A))$.*

Proof. $(\Leftarrow)$: Note that $Cl(A)$ is a closed set. So A is the density set of a closed set.

$(\Rightarrow)$: $A = \Phi(B) \subseteq B$ because B is closed. $Cl(A) \subseteq B$. B is not necessarily the smallest closed set that contains A, but the set of points in $B \setminus A$ is a nullset (by Lebesgue Density Theorem). So we assume that $Cl(A) = B$.

$$A \subseteq Cl(A) = B$$

$$\Phi(A) \subseteq \Phi(Cl(A)) = \Phi(B)$$

Since $A = \Phi(B)$, then we conclude that $A = \Phi(Cl(A)$. □

Similarly, a set A is the density set of an open set if the following conditions are met.

Theorem 6.1.8. *A set A is the density set of an open set U if and only if $A = \Phi(Int(A))$.*

Proof. $(\Leftarrow)$: Note that $Int(A)$ is an open set. So A is the density set of an open set.

$(\Rightarrow)$: Let $A = \Phi(U)$ for some open U. $\Phi(U)$ differs form U by a nullset, so $Int(\Phi(U))$ differs from $Int(U)$ by a nullset. Therefore, we assume that $Int(U) = A$.

$$A = Int(U) \subseteq U$$

$$\Phi(A) = \Phi(Int(U)) \subseteq \Phi(U)$$

Since $A = Int(U)$, we can say $\Phi(A) = \Phi(Int(A))$. But $\Phi(A) = \Phi(\Phi(A)) = \Phi(U) = A$. So $A = \Phi(Int(A))$. □

Finally, we can characterize the density function as a function of the interior and closure of a set. The theorem is detailed by Wilczyński [17].

Theorem 6.1.9. *Let A be any set. Then $A = \Phi(A)$ if and only if $A = Int(Cl(A))$.*

Proof. $(\Rightarrow)$: Let $A = \Phi(A)$. The containment $A \subseteq Cl(A)$ is true for any set, so $\Phi(A) \subseteq Cl(\Phi(A))$. Recall that the interior of a set can be characterized as the union of all open subsets. Since A is open, we have $A = \Phi(A) \subseteq Int(Cl(\Phi(A)))$. To prove the opposite containment,

$$\begin{aligned} Int(Cl(A)) &= Cl(A) \cap \Phi(Cl(A)) && (6.1)\\ &\subseteq Cl(A) \cap \Phi(A) && (6.2)\\ &= \Phi(A) = A && (6.3)\end{aligned}$$

The proof of (6.1) follows from Theorem 6.1.7. Set (6.2) follows from $A \subseteq Cl(A)$ and the monotonicity of Φ. Lastly, $\Phi(A) \subseteq Cl(A)$ is a consequence of Lemma 6.1.6.

$(\Leftarrow)$: Let $A = Int(Cl(A))$. Using Theorem 6.1.4, $A = Int(Cl(A)) = Cl(A) \cap \Phi(B)$, where B is a measurable kernel of $Cl(A)$. Since B is a measurable kernel of $Cl(A)$, $B \subseteq \Phi(Cl(A))$. By monotonicity, $\Phi(B) \subseteq \Phi(Cl(A))$. Lemma 6.1.6 shows that $Cl(A)$ closed implies $\Phi(Cl(A)) \subseteq Cl(A)$. Then $\Phi(B) \subseteq Cl(A)$. Note that $A \subseteq Cl(A)$ and $\Phi(B) \subseteq Cl(A)$, so

$$A = Cl(A) \cap \Phi(B) \Rightarrow A = \Phi(B).$$

Find the density points of A and $\Phi(B)$ to complete the proof.

$$\Phi(A) = \Phi(\Phi(B)) = \Phi(B) = A$$

Thus $A = Int(Cl(A))$ implies that $A = \Phi(A)$. □

If a set A is both closed and open, it is simple to find the density points of A using the theorem above. Let A be a clopen set, then $Cl(A) = A$, and $Int(Cl(A)) = Int(A) = A$. By Theorem 6.1.9, $A = \Phi(A)$. All results in this section are applicable to both the density topology on $\mathbb{R}$ and the density topology on C. In the interest of the widest applicability of these results, it is natural to ask whether a Density Topology can be defined on every space. In the next section, we show that the approach given in Chapters 4 and 5 does not produce a density topology.

6.2 A Naive Approach to the Density Topology

We would like to develop a density topology on the space of continuous functions $\mathcal{C}[a, b]$, sometimes abbreviated $\mathcal{C}$. The first approach would be to use a metric on $\mathcal{C}$ and develop a Lebesgue measure. Unfortunately, this approach fails. Lebesgue measure cannot exist on any infinite dimensional space. This chapter shows why Lebesgue measure cannot be defined on $\mathcal{C}$ in the usual way. Then we explore the ideas of *prevalence* and *shyness* as another way of talking about "almost every" function in $\mathcal{C}$

The naive approach to developing the density topology follows Chapters 3 and 5. Beginning with a metric, define inner and outer measure using basic neighborhoods. The sets on which inner and outer measure agree will be the measurable sets. Then define density in the usual way. We we will see, this approach fail on the space of continuous functions when basic open neighborhoods fail to be measurable.

To begin, start with the the usual max metric $\delta(f, g) = \sup\{|f(x) - g(x)| : x \in [0, 1]\}$ for $f, g \in \mathcal{C}$. Using this metric, define open balls as follows:

$$B_r(f) = \{g \in \mathcal{C} : \delta(f,g) < r\}.$$

We use open balls to find the outer measure of a set.

Definition 6.2.1. *Let $A \in C[0,1]$. The outer measure of A is defined as*

$$\mu^*(A) = \inf \left\{ \sum_{i=1}^{\infty} \rho_i : \bigcup_{i=1}^{\infty} B_{\rho_i}(f_i) \text{ is a countable open cover of } A \right\}.$$

By the Stone-Weierstrass Theorem, the set of polynomials with rational coefficients is a countable, dense subset of $C[0,1]$. So a countable open cover of A always exists, namely $\mathcal{O} = \{B_1(p) : p \text{ is a polynomial with rational coefficients}\}$. It needs to be shown that μ^* is monotonic and subadditive. To see monotonicity, let $A \subseteq B$. Notice that any countable open cover of B is a countable open cover of A as well, so $\mu^*(A) \leq \mu^*(B)$.

To show countable subadditivity, take A_i to be a disjoint countable collection of open sets. For each A_i, let $U_i = \{V_{i,j} : j < \infty\}$ be a countable open cover. Now consider $A = \bigcup_i A_i$. We discuss the containment

$$A_i \subseteq \bigcup_j V_{i,j}, \text{ so } A = \bigcup_i A_i \subseteq \bigcup_i \left(\bigcup_j V_{i,j} \right)$$

Then we conclude that

$$\mu^*(A) = \mu^* \left(\bigcup_i A_i \right) \leq \mu^* \left(\bigcup_i \bigcup_j V_{i,j} \right) = \sum_i \mu^* \left(\bigcup_j V_{i,j} \right) = \sum_i \mu^*(U_i)$$

and the function μ^* is countably subadditive. So μ^* is an outer measure on $C[0,1]$. The inner measure is defined as on other spaces. A set is *compact* in $\mathcal{C}$ if every open cover has a finite subcover.

Definition 6.2.2 (Measurable Set)**.** *The inner measure μ_* of a set A is*

$$\mu_*(A) = \sup\{\mu^*(K) : K \text{ is compact, } K \subseteq A\}$$

If $\mu_*(A) = \mu^*(A)$, then A is measurable and $\mu(A) = \mu^*(A)$.

The Heine-Borel theorem fails in $C[0,1]$. That is, a closed, bounded set is not necessarily compact. To illustrate, let A be the set of functions f_n which satisfy the following definition:

$$f_n(x) = \begin{cases} 0 & x \in [0, \frac{1}{n+1}] \\ \text{linear} & \text{from } (\frac{1}{n+1}, 0) \text{ to } (\frac{1}{n}, 1) \\ 1 & x \in [\frac{1}{n}, 1] \end{cases}$$

The set A is bounded. The functions f_n converge pointwise to $2f$ where $f(0) = 0$ and $f(x) = 1$ elsewhere. But $f \notin C[0,1]$, so A is closed. Then choose an open cover of open balls around each function with radius $\frac{1}{2}$. The open cover is countable, but no finite subcover exists, as each function is distance 1 from each other function. So each element of the open cover is necessary, and the Heine-Borel theorem fails.

We need to find a sequence of compact sets contained in an open ball such that the outer measures of the sequence approach the diameter of the open ball. The following proof demonstrates such a sequence of compact sets.

Lemma 6.2.3. *Let $A = B_f(\rho)$ be an open ball in $C[0,1]$. Then $\mu_*(A) \geq \rho$.*

Proof. Define the following sets K_i for $i \in \mathbb{N}$.

$$K_i = \left\{ g(x) = f(x) + ax : \forall a \in \left[-\rho + \frac{1}{i}, \rho - \frac{1}{i} \right] \right\}$$

It is clear that $K_i \subseteq A$ for all i. If we can show K_i is compact and $\lim_{i\to\infty} \mu^*(K_i) = 2\rho$, then $\mu(A) \geq 2\rho$. Fix some i. The set K_i is isometric to $\left[f(1) - (\rho - \frac{1}{i}), f(1) + (\rho + \frac{1}{i})\right]$ on the real line with the regular topology. Since closed intervals are sequentially compact on the real line, K_i is sequentially compact and therefore compact.

Since $K_i \subseteq A$, then $\mu_*(A) \geq \mu^*(K_i)$. To compute the outer measure, choose find a countable open covering $\mathcal{U}$ of K_i. Assume that the sum of interval diameters of $\mathcal{U}$ is less than $2\rho - \frac{2}{i}$. Since each $g \in K_i$ is covered, then there is a countable open cover of the interval $\left[f(1) - (\rho - \frac{1}{i}), f(1) + (\rho - \frac{1}{i})\right]$ in the usual topology such that the sum of interval diameters is strictly less than $2\rho - \frac{2}{i}$. This contradicts the fact that an interval has measure equal to its length. So the outer measure $\mu^*(K_i) \geq 2\rho - \frac{2}{i}$. Taking the limit as $i \to \infty$:

$$\lim_{i\to\infty} \mu^*(K_i) = 2\rho.$$

Therefore, $\rho = \sup\{\mu^*(K_i)\} \leq \mu_*(A) \leq \rho$. $\square$

At this stage, it seems that open balls are measurable. In the reals and in the Cantor Space, every set is outer-measurable. However, we have not proved that to be the case in $\mathcal{C}$. Michael Taylor [16] presents a sufficient condition for a set to be μ^*-measurable.

Fact 6.2.4. A set A is μ^*-measurable iff $\mu^*(Y) = \mu^*(Y \cap A) + \mu^*(Y \setminus A), \forall Y \subseteq \mathcal{C}$.

We prove that open balls are not outer-measurable with the measure μ^*.

Theorem 6.2.5. *Let $f \in \mathcal{C}$, and let $B_\rho(f)$ be an open ball centered at f. Then $B_\rho(f)$ is not μ^*-measurable.*

Proof. Without loss of generality, let $A = B_1(f)$ for the constant function $f(x) = 0$. Fix some $0 < \varepsilon < 1$. For each n, construct the following equation

$$f_n(x) = \begin{cases} \text{linear} & \text{from 0 at } x = \frac{1}{n} - \frac{1}{n(n+1)} \text{ to 1 at } x = \frac{1}{n} \\ \text{linear} & \text{from 1 at } x = \frac{1}{n} \text{ to 0 at } x = \frac{1}{n} + \frac{1}{n(n+1)} \\ -\varepsilon & x \text{ elsewhere} \end{cases}$$

Each $f_n \in B_1(f)$. Let $\lambda = \min\{\varepsilon, 1 - \varepsilon\}$, and let $C_n = B_\varepsilon(f_n)$. Then $C_n \subseteq B_1(f)$. Note that for each $n \neq m$, we have $C_n \cap C_n = \emptyset$ since $\delta(f_n, f_m) = 1 > \varepsilon$. By monotonicity of μ^*

$$\bigcup_{n=1}^{\infty} C_n \subseteq B_1(f) \quad \Rightarrow \quad \mu^* \left(\bigcup_{n=1}^{\infty} C_n \right) \leq \mu^*(B_1(f)).$$

Since the C_n are pairwise disjoint, $\mu^* \left(\bigcup_{n=1}^{\infty} C_n \right) \geq \sum_{n=1}^{\infty} \mu^*(C_n) \geq \sum_{n=1}^{\infty} \varepsilon = \infty$. Therefore, the outer measure of every open ball is infinite. □

If open balls do not have finite outer measure, then *most* of the sets in $\mathcal{C}$ are infinite outer measure, and μ does not tell us much about the space. In fact, Lebesgue measure cannot be defined on any separable Banach space, unless the measure is 0 for every set.

Theorem 6.2.6. *Let X be a separable Banach space. No non-zero Lebesgue measure exists on X.*

Proof. Suppose a non-zero Lebesgue measure μ exists on X. Let $B_\varepsilon(x)$ be an open ball centered at some $x \in X$ such that $\mu(B_\varepsilon(x)) = c, 0 < c < \infty$. Within $B_\varepsilon(x)$, choose an infinite sequence of disjoint open balls $B_{\varepsilon/4}(x_i)$ of radius $\frac{\varepsilon}{4}$. Such a sequence exists because X is infinite dimensional. Since μ is a Lebesgue measure, it is translation invariant and $\mu(B_{\varepsilon/4}(x_i)) = \mu(B_{\varepsilon/4}(x_j))$ for all i, j. Then, by countable additivity of Lebesgue measure,

$$\sum_{i=0}^{\infty} B_{\varepsilon/4}(x_i) \leq \mu(B_\varepsilon(x)) = c < \infty$$

So each $B_{\varepsilon/4}(x_i)$ has measure zero. But since X is separable, each subset $A \subseteq X$ can be covered by a countable collection of balls of radius $\varepsilon/4$, each with measure zero. So $\mu(A) = 0$ for each $A \subseteq X$. $\square$

6.3 A Different Approach: Prevalence

Suppose we want to make a statement about "almost all" elements of an infinite-dimensional topological space X. As seen in Section 6.2, there is no Lebesgue measure analogue on infinite dimensional spaces. To circumvent this, William Ott and James Yorke [10] define the notion of *prevalence.* They begin by listing some properties of measure zero sets.

1. A measure zero set has no interior.
2. Every subset of a measure zero set also has measure zero.
3. A countable union of measure zero sets also has measure zero.
4. Every translate of a measure zero set also has measure zero.

Ott and Yorke give a definition of *shyness* to sets which satisfy the properties above, replacing "measure zero" with "shy". The definition is applicable to all complete metric linear spaces. That is, a space with a complete metric, and addition and multiplication are continuous on that space. Fortunately, $\mathcal{C}$ has a complete metric (the sup norm), and is continuous with respect to addition and multiplication. The definition of shyness is a natural extension of finite-dimensional measure.

Definition 6.3.1. *Let m be a measure on $\mathcal{C}$. The measure m is* transverse *to a set A if the following two conditions are met:*

1. *There exists a compact $K \subseteq \mathcal{C}$ such that μ is supported on K. That is, if $A \subseteq \mathcal{C} \setminus 0$, then $\mu(A) = 0$.*

2. *For all $f \in \mathcal{C}$, $\mu(\{x + f : x \in A\}) = 0$.*

This definition forces m to be compactly supported. Also, by item 2, all possible translations of the set have to have measure zero in order for the measure to be transverse.

Definition 6.3.2. *Let $A \subseteq \mathcal{C}$ be a set. If a measure m on $\mathcal{C}$ exists such that m is transverse to A, then A is a* shy *set.*

As an immediate corollary, if A is a shy set and $B \subseteq A$, then B is shy. It is clear that shyness is a stronger condition than being measure zero. A shy set must have measure zero on all translations *on all compactly supported measures!*

Definition 6.3.3. *Let A be a shy set. Then A^c is a prevalent set.*

The theory of prevalence is introduced by Hunt, Sauer, and Yorke [7] and developed further by Ott and Yorke [10]. As the theory is rich enough to warrant another full paper, we list some of the directly applicable results without proof.

Lemma 6.3.4. *Prevalent sets are dense.*

Lemma 6.3.5. *Countable unions of shy sets are shy.*

At this juncture, it seems that prevalence and shyness act as stricter analogues of full measure and measure zero respectively. Hunt, Sauer, and Yorke confirm this intuition by proving the following theorem on $\mathbb{R}^n$.

Theorem 6.3.6. *A set $A \subseteq \mathbb{R}^n$ is* shy *if and only if A has Lebesgue measure zero.*

Chapters 4 and 5 dealt with the density topology on one-dimensional spaces. As seen in Section 6.2, the density topology cannot naively be defined on an infinite-dimensional space $\mathcal{C}$. There exists a large body of work covering the density topology on finite dimensional spaces, particularly $\mathbb{R}^n$. On the finite-dimensional spaces, there exist multiple definitions of density, including *strong* and *ordinary* density. The interested reader is directed to Goffmann, Neugebauer, and Nishiura [6]. It is clear that the approach presented by Yorke, et. al. can be applied to $\mathcal{C}$. The question remains, however, which sets are prevalent? Which are shy? The presence of differing definitions of density in $\mathbb{R}^n$ suggests that the prevalence approach would be fruitful in the infinite-dimensional case, as it requires *all* measures to satisfy some property.

REFERENCES

[1] Andretta, Alessandro and Riccardo Camerlo. "Descriptive Set Theory of the Lebesgue Density Theorem." Adv. Math. vol. 234, 2013, pp 1-42.

[2] Bruckner, Andrew. "Differentiation of Real Functions." Lecture Notes in Mathematics, Springer-Verlag, New York. 1978.

[3] Darji, Udayan B. "Limits of Differentiable Functions". Proc. Amer. Math. Soc., vol. 124 (1), 1996, pp. 129-134.

[4] Denjoy, A. "Sur les fonctions dérivées sommables." Bull. Soc. Math. France. vol 43, 1915, pp 161-248.

[5] Faure, Claude-Alain. "A Short Proof of Lebesgue's Density Theorem." American Mathematical Monthly, vol. 109 (2), 2002. pp 194-196.

[6] Goffman, Casper; Neugebauer, J.; and Nishiura, T. "Density Topology and Approximate Continuity" Duke Mathematical Journal, vol. 28 (4), 1961, pp. 497-505.

[7] Hunt, Brian R., Tim Sauer, and James Yorke. "Prevalence: A Translation-Invariant "Almost Every" on Infinite-Dimensional Spaces." Bull. Amer. Math. Soc., vol. 27 (2), 1992. pp. 217-238.

[8] Kuratowski, Kazimierz and Andrzej Mostowski. "Set Theory." North-Holland Publishing Company, Amsterdam. 1976.

[9] Miller, Benjamin. "The Existence of Measures of a Given Cocycle, I: Atomless, Ergodic σ-finite Measures." Ergod. Theory and Dynam. Sys., 28, (2008). no. 5.

[10] Ott, William and James A. Yorke. "Prevalence", Bulletin of the American Mathematical Society, vol. 42 (3), 2005, pp. 263-290.

[11] Oxtoby, John C. "Measure and Category," second edition (1980). Graduate Texts in Mathematics, Springer-Verlag, New York. 1980.

[12] Richards, Trevor. "Is an everywhere approximately continuous function everywhere continuous? Request for a counter example." Mathematics Stack Exchange. http://math.stackexchange.com/q/1174420. Published March 4, 2015. Viewed Nov. 22, 2015.

[13] Scheinberg, Stephen. "Topologies Which Generate a Complete Measure Algebra." Advances in Mathematics, vol. 7 (3), 1971, pp. 231-239.

[14] Scott, Brian M. "How to show that $\mathbb{Q}$ is not G_δ." Mathematics Stack Exchange. http://math.stackexchange.com/q/69458. Published October 3, 2011. Viewed Jan. 23, 2015.

[15] Tall, Franklin D. "The Density Topology". Pacific Journal of Mathematics, vol. 62 (1), 1976, pp. 275-284.

[16] Taylor, Michael E. "Measure Theory and Integration." Graduate Studies in Math. No. 76, American Math. Society, Providence, R.I., 2006.

[17] Wilczyński, Wladyslaw. "Density Topologies." Handbook of Measure Theory. Elsevier Science. New Holland, 2002, pp. 675-702.

[18] Wojdowski, Wojciech. "A Generalization of the Density Topology." Real Analysis Exchange, vol. 32 (2), 2006/2007, pp. 349-358.

[19] Zahorski, Z. "Sur la Première Dérivée." Trans. Amer. Math. Soc. vol. 69, 1950, pp. 1-54 (French).