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Dedicated to my parents, Edson and Maria.
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ABSTRACT

One type of conditionally convergent series that has long been considered by mathematicians is the Alternating Harmonic Series and its sum under various types of rearrangements. The purpose of this thesis is to introduce results from the classical theory of rearrangements dating back to the 19th and early 20th century. We will look at results by mathematicians such as Ohm, Riemann, Schlömilch, Pringsheim, and Sierpiński. In addition, we show examples of each classical result by applying the Alternating Harmonic Series under the different types of rearrangements, and also introducing theorems by Lévy and Steinitz, and Wilczyński, which are modern extensions of the results of Sierpiński.
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CHAPTER 1

INTRODUCTION

1.1 Preliminaries

Given a sequence \((f_n)\) of real numbers, and a permutation \(\pi : \mathbb{N} \rightarrow \mathbb{N}\), the rearrangement determined by \(\pi\) is the sequence \((f_{\pi(n)})\) whose \(n^{th}\) term is the \(\pi(n)^{th}\) term of the original sequence. A rearrangement of a series \(\sum f_n\) is \(\sum f_{\pi(n)}\) for some such permutation \(\pi\).

We demonstrate this by the following example.

Example 1.1.0.1. Consider the Alternating Harmonic Series (see Section 1.2 below for an evaluation of the series),

\[
\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \cdots = \ln(2).
\]

Now consider another series,

\[
\frac{1}{2} - \frac{1}{4} + \frac{1}{6} - \frac{1}{8} + \frac{1}{10} - \frac{1}{12} + \frac{1}{14} - \cdots = \frac{1}{2} \ln(2).
\]

Note that adding 0 between each term will not change the sum of the series. So we get,

\[
0 + \frac{1}{2} + 0 + \left( -\frac{1}{4} \right) + 0 + \frac{1}{6} + 0 + \left( -\frac{1}{8} \right) + 0 + \frac{1}{10} + 0 + \left( -\frac{1}{12} \right) + 0 + \cdots = \frac{1}{2} \ln(2).
\]
Now let us add this series to the usual Alternating Harmonic Series, term by term, and we have

\[1 + 0 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + 0 + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + 0 + \frac{1}{11} - \cdots\]

\[= 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \cdots = \frac{3}{2} \ln(2).\]

Notice that the resulting series is a rearrangement of the Alternating Harmonic Series, but converges to another sum.

The goal of this thesis is to present some classical results illustrating the extent of this phenomenon.

1.2 Background

1.2.1 Evaluation of the Alternating Harmonic Series

**Theorem 1.2.1.** If \(\sum f_n\) converges, then \(\lim_{n \to \infty} f_n = 0\). In fact, \(\lim_{n \to \infty} \sum_{k=n+1}^{\infty} f_k = 0\) and, for any \(\varepsilon > 0\), there is an \(N\) such that for all \(n < m\) with \(N < m\), we have

\[|\sum_{k=n}^{m} f_k| < \varepsilon.\]

**Proof.** \(\sum f_n\) converges means that the sequence of partial sums \((S_k)\), where

\[S_k = \sum_{n=1}^{k} f_n\]

converges. In other words, the sequence is Cauchy. So for all \(\varepsilon > 0\), there exists an \(N\) for
all \( n < m \) (if \( n > N \), then \( |S_n - S_m| < \varepsilon \)). This is precisely the last statement. The first statement follows from this by taking \( m = n + 1 \). The second statement follows by noting that if \( \sum f_n = s \), then \( S_n \rightarrow S \) if and only if \( S_n - S \rightarrow 0 \). □

**Abel’s Theorem**

Taken from [9, pg. 125], we have the following,

**Theorem 1.2.2** (Abel (1826) [9]). Suppose \( \sum_{n \geq 0} f_n \) converges. We then have that

\[
\lim_{x \to 1^-} \sum_{n \geq 0} f_n x^n = \sum_{n \geq 0} f_n.
\]

Even if \( f(x) = \sum f_n x^n \) has radius of convergence 1, it is still (left) continuous at the endpoint \( x = 1 \).

**Proof.** [18] Suppose \( t_n = f_0 + f_1 + f_2 + \cdots + f_n \) and let \( t_{-1} = 0 \). Then, we have that

\[
\sum_{n \geq 0} f_n x^n = \sum_{n \geq 0} (t_n - t - 1)x^n
\]

\[
= (1 - x) \sum_{n \geq 0} t_n x^n + t_m x^m. \tag{1.1}
\]

Now consider \( |x| < 1 \) and suppose \( m \rightarrow \infty \). We want to show that

\[
\sum_{n \geq 0} f_n x^n = (1 - x) \sum_{n \geq 0} t_n x^n.
\]
Since \((t_n)\) converges, then it is bounded, meaning there is some value \(T\) such that \(|t_n| < T\) for each \(n\). Thus, \(|t_m x^m| < T |x|^m\) for each \(m\). Since \(m \to \infty\), then \(t_n x^n \to 0\). Also, by the fact that \(|x|^m \to 0\), then \(T |x|^m \to 0\). So from (1.1) we now have

\[
= (1 - x) \sum_{n \geq 0}^{m-1} t_n x^n + 0
\]

which results from taking the limit of both sides as \(m \to 0\). Now, let \(t = \lim_{n \to \infty} t_n\), and let \(\varepsilon > 0\). Pick an \(N \in \mathbb{N}\) so that \(n > N\) implies

\[
|t - t_n| < \frac{\varepsilon}{2} \quad (1.2)
\]

Since

\[
(1 - x) \sum_{n \geq 0}^{\infty} x^n = 1 \quad (1.3)
\]

for \(|x| < 1\).

Then, from \(\sum_{n \geq 0}^{\infty} f_n x^n = (1 - x) \sum_{n \geq 0}^{\infty} t_n x^n\) we get

\[
|\sum_{n \geq 0}^{\infty} f_n x^n - t| = |(1 - x) \sum_{n \geq 0}^{\infty} (t_n - t) x^n|
\]

\[
\leq (1 - x) \sum_{n \geq 0}^{N} |t_n - t| |x|^n
\]

\[
= (1 - x) \sum_{n \geq 0}^{N} |t_n - t| |x|^n + (1 - x) \sum_{n \geq N + 1}^{\infty} |t_n - t| |x|^n.
\]

So by (1.2) and (1.3), we have
\[
< \ (1-x) \sum_{n \geq 0} (1-x)^n + (1-x) \sum_{n \geq N+1} \frac{\varepsilon}{2} |x|^n
\]

\[
= \ (1-x) \sum_{n \geq 0} |t_n-t| + \frac{\varepsilon}{2} \sum_{n \geq N+1} |x|^n
\]

\[
\leq \ (1-x) \sum_{n \geq 0} |t_n-t| + \frac{\varepsilon}{2} (1-x) \sum_{n \geq 0} |x|^n.
\]

Since \( |x| < 1 \), then

\[
= \ (1-x) \sum_{n \geq 0} |t_n-t| + \frac{\varepsilon}{2}.
\]

Let \( B = \sum_{n \geq 0} |t_n-t| \) and let there be some \( \delta = \frac{\varepsilon}{2B} > 0 \). If

\[
1 - \delta < x < 1,
\]

then for equation (1.4) we get

\[
| \sum_{n \geq 0} f_n x^n - t | < B(1-x) + \frac{\varepsilon}{2},
\]

and from \( \delta = \frac{\varepsilon}{2B} \) and (1.5),

\[
| \sum_{n \geq 0} f_n x^n - t | < B \cdot \delta + \frac{\varepsilon}{2} = \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.
\]

This proves that

\[
\lim_{x \to 1} \sum_{n \geq 0} f_n x^n = t = \sum_{n \geq 0} f_n.
\]

Geometric Series

We introduce the following theorem for a general geometric series:
**Theorem 1.2.3** ([18]). If $|x| < 1$, then \[ \sum_{n \geq 0} x^n = \frac{1}{1-x}. \] If $|x| \geq 1$, then \[ \sum_{n \geq 0} x^n = \infty. \]

The proof to this theorem can be found in [18, page 61].

Now consider a variation of the geometric series

\[ \sum_{n \geq 0} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \cdots = \frac{1}{1+x}, \]

which converges only when $|x| < 1$, and is divergent for $|x| \geq 1$.

**Theorem 1.2.4.** If $|x| < 1$, then

\[ \sum_{n \geq 0} (-1)^n x^n = \frac{1}{1+x}. \]

If $|x| \geq 1$, then the series diverges.

**Proof.** Suppose we start with

\[ \sum_{n \geq 0} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \cdots. \]

We can rewrite the left side of the equation as

\[ \sum_{n \geq 0} (-x)^n = 1 - x + x^2 - x^3 + x^4 - \cdots. \]

Notice that this is similar to the geometric series from Theorem 1.2.3 with $-x$ in place of $x$, and so we have

\[ \sum_{n \geq 0} (-x)^n = \frac{1}{1-(-x)}. \]
or

\[
\sum_{n=0}^{\infty} (-x)^n = \frac{1}{1 - (-x)} = \frac{1}{1 + x}
\]

such that this series converges for \(|x| < 1\), and diverges for \(|x| \geq 1\). So,

\[
\sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + x^4 - \cdots = \frac{1}{1 + x}
\]

when \(|x| < 1\), and diverges for \(|x| \geq 1\).

Considering only values of \(x\) within the interval of convergence, integrate both sides of

\[
\frac{1}{1 + x} = 1 - x + x^2 - x^3 + x^4 - \ldots
\]

to get

\[
\ln(x + 1) + C_1 = \left( x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \right) + C_2
\]

or

\[
\ln(x + 1) + C_1 - C_2 = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots
\]

(1.6)

for some constants \(C_1\) and \(C_2\). We show term by term that integration gives us \(\sum_{n=0}^{\infty} (-1)^n x^n\):

For \(|-x| < 1\), we have

\[
\frac{1}{1 + x} = \frac{1}{1 - (-x)} = \sum_{n=0}^{\infty} (-x)^n = \sum_{n=0}^{\infty} (-1)^n x^n
\]
for $|x| < 1$. Then,

$$
\int \frac{1}{1+x} \, dx = \int \lim_{k \to \infty} \sum_{n=1}^{k} (-1)^n x^n \, dx = \lim_{k \to \infty} \int \sum_{n=1}^{k} (-1)^n x^n \, dx,
$$

which is a consequence of [18, Theorem 7.16]. So

$$
\int \frac{1}{1+x} \, dx = \sum_{n=0}^{\infty} (-1)^n \int x^n \, dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n+1} x^{n+1} + C,
$$

for $|x| < 1$.

In (1.6), let $C = C_1 - C_2$. Then, we have

$$
\ln(x+1) + C = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad (1.7)
$$

To find $C$, evaluate both sides of (1.7) when $x = 0$. The result is

$$
0 + C = 0 - 0 + 0 - \ldots ,
$$

or $C = 0$. Therefore,

$$
\ln(x+1) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots
$$

for $-1 < x < 1$.

Now, when $x = 1$, we have the following power series

$$
1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \ldots ,
$$

which we recognize as the Alternating Harmonic Series, and converges due to Leibniz’s
Leibniz’s Alternating Series Test

**Theorem 1.2.5** (Leibniz’s Alternating Series Test [9]). If \( \{a_n\} \) is a sequence of positive numbers such that the \( a_n \) monotonically decrease and \( a_n \) converges to 0, then

\[
A = \sum_{n=1}^{\infty} (-1)^{n+1} a_n = a_1 - a_2 + a_3 - a_4 + \cdots
\]

converges.

**Proof.** Let \( A_n \) be the \( n^{th} \) partial sum of \( A \). Since \( \{a_n\} \) are monotonically decreasing, then the following inequalities hold true:

\[
a_1 \geq a_1 - a_2 + a_3 \geq a_1 - a_2 + a_3 - a_4 + a_5 \geq \cdots,
\]

and

\[
a_1 - a_2 \leq a_1 - a_2 + a_3 - a_4 \leq a_1 - a_2 + a_3 - a_4 + a_5 - a_6 \leq \cdots.
\]

We find that \( \{A_{2n}\} \) monotonically increases and is bounded above by \( A_1 \), and that \( \{A_{2n+1}\} \) monotonically decreases and is bounded below by \( A_2 \). Since both sequences are bounded, they are also convergent being that \( \{a_{2n}\} \) and \( \{a_{2n+1}\} \) both converge to 0, or

\[
\lim_{n \to \infty} (A_{2n+1} - A_{2n}) = \lim_{n \to \infty} a_{2n+1} = 0.
\]
It now follows from Abel’s Theorem 1.2.1 that

\[ x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots, \]

converges to \( \ln(1 + x) \) for \( |x| < 1 \), and also converges to \( \ln(2) \) when \( x = 1 \).

### 1.3 History

**Definition 1.3.1.** \( \sum f_n \) is a conditionally convergent series if \( \sum f_n \) converges but \( \sum |f_n| \) diverges.

**Lemma 1.3.2** (pg. 317, [10]). Let \( \sum f_n \) be a conditionally convergent series. Let \( \sum a_n \) be the series of positive terms of \( \sum f_n \) and let \( \sum b_n \) be the series of negative terms of \( \sum f_n \). Then, both \( \sum a_n \) and \( \sum b_n \) diverge.

German mathematician Johann Peter Gustave Lejeune Dirichlet (1805 - 1859) [14] came up with a very important result involving the rearrangement of terms of certain series. Dirichlet was the first to notice that terms in certain series could be rearranged to a sum different from the original series (it was later founded by Bernhard Riemann that this was due to conditionally convergent series). In 1837, Dirichlet published a paper proving that the sum remains the same when rearranging terms in an absolutely convergent series [5].

We proceed to introduce Dirichlet’s result, which we took from the chapter entitled, *Arbitrary Series and Infinite Products* in [7].

**Theorem 1.3.3** ([7]). If \( \sum f_n \) is absolutely convergent and converges to \( \alpha \), then every rearrangement of \( \sum f_n \) also converges to \( \alpha \).

**Proof.** Assume \( \sum |f_n| < \infty \), and that \( \sum f_n = \alpha \). Now let \( (f_{\pi(n)}) \) be a rearrangement of \( (f_n) \). We need to show that \( \sum_{n=1}^{\infty} f_{\pi(n)} = \alpha \).
Fix $\varepsilon > 0$. We must find $N$ such that for all $n > N$,

$$\left| \sum_{k=1}^{n} f_{\pi(k)} - \alpha \right| < \varepsilon.$$ 

First find $N_1$ such that for every $n \geq N_1$,

$$\left| \sum_{k=1}^{n} f_k - \alpha \right| < \frac{\varepsilon}{2},$$

and find $N_2$ such that for every $n > N_2$,

$$\sum_{k=n}^{\infty} |f_k| < \frac{\varepsilon}{2}.$$ 

We may assume that $N_2 \geq N_1$.

Finally, let there be some $N_3$ large enough so that \{1, 2, 3, \ldots, $N_2$\} $\subseteq \{\pi(1), \pi(2), \pi(3), \ldots, \pi(N_3)\}$. We claim that $N = N_3$ works. To see this, note that if $n > N_3$, then $\sum_{k=1}^{n} f_{\pi(k)} = \sum_{k=1}^{N_2} f_k + \sum_{j \in A} f_j$, where $A = \{\pi(1), \pi(2), \pi(3), \ldots, \pi(n)\} \setminus \{1, 2, 3, \ldots, N_2\}$.

Therefore,

$$\left| \sum_{k=1}^{n} f_{\pi(k)} - \alpha \right| \leq \left| \sum_{k=1}^{N_2} f_k - \alpha \right| + \left| \sum_{j \in A} f_j \right|$$

$$< \frac{\varepsilon}{2} + \sum_{j \in A} |f_j|$$

$$\leq \frac{\varepsilon}{2} + \sum_{j=N_2+1}^{\infty} |f_j| < \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon.$$ 

$\square$
In 1839, a German mathematician by the name of Martin Ohm [1] came up with the following rearrangement theorem.

**Theorem 1.3.4 ([11]).** For p and q positive integers, rearrange

\[ \sum_{n \geq 1} \frac{(-1)^{n-1}}{n} \]

by taking the first p positive terms, then the first q negative terms, then the next p positive terms, the next q negative terms, and so on. The rearranged series converges to

\[ \ln(2) + \frac{1}{2} \ln \left( \frac{p}{q} \right). \] (1.8)

First, we introduce some notation:

Denote by \( A(p, q) \) the series resulting from this rearrangement. Let \( S_k \) be the partial sums of \( A(p, q) \) where specifically \( S'_k = S'_k(p, q) \) denotes the \( k^{th} \) partial sum of \( A(p, q) \), and let \( C_n = C_n(p, q) \) be the partial sum of \( A(p, q) \), which is the result in \( S \) from precisely adding \( n \) blocks, where each block consists of \( p \) positive terms and \( q \) negative terms, consecutively.

Denote \( S_n \) to be the partial sums of the usual alternating series, where \( n \) denotes the terms. Call \( C_n \) as \( C_n = S_{f(n)} + R_n \) for some \( f(n) \) and some “remainder” \( R_n \).

In Chapter 2, we analyze Ohm’s rearrangement theorem by splitting up the proof into two parts. In part (1), we look at three possible cases that could happen in the rearrangement: when \( p = q \), \( p < q \), and \( p > q \). We do this by first providing an example for each case, and then proving its generalization. In part (2), we show \( S_n \) and \( C_n = S_{f(n)} + R_n \) converge to \( \ln(2) + \frac{1}{2} \ln \left( \frac{p}{q} \right) \) for each of the three possible cases.

In 1852, German mathematician Bernhard Riemann came up with the explanation for Peter Lejeune-Dirichlet’s discovery [5] that one can change the sum of a conditionally
convergent series by rearranging its terms. His explanation was the following theorem.

**Theorem 1.3.5** (Riemann’s Rearrangement Theorem [10]). A series $\sum f_n$ is conditionally convergent if and only if for each real number $\alpha$, there is a rearrangement of $\sum f_n$ that converges to $\alpha$.

In Chapter 3, we analyze Riemann’s Rearrangement Theorem (also known as the Riemann Series Theorem) by first proving the result, then providing an example of the rearrangement of the Alternating Harmonic Series the sum of which $\frac{3}{2} \ln(2)$. We then compare it to $A(2, 1)$ of Ohm’s rearrangement (Chapter 2).

German mathematician Oscar Schlömilch came up with the following result.

**Theorem 1.3.6** ([22]). Let $f(n) = u_n$ be a decreasing and positive function, such that the asymptotic value $\lim_{n \to \infty} f(n) = 0$. If $K = \lim_{\omega \to \infty} (\omega u_\omega)$ or $K = \lim_{n \to \infty} (nu_n)$, the following holds: If in a series (see note below),

$$s = u_0 - u_1 + u_2 - u_3 + \cdots$$

the terms are rearranged such that always $p$ positive terms are followed by $q$ negative terms, then the sum of the rearranged series is

$$S = s + \frac{\lim(nu_n)}{2} \cdot \log\left(\frac{p}{q}\right).$$

[Note: Schlömilch does not state it directly at this point, but the series that satisfies the theorem must be one of the form $s = \sum_{n=0}^{\infty} (-1)^n u_n$ such that $u_n > 0$ is strictly decreasing and $\lim_{n \to \infty} u_n = 0$, and by the Alternating Series Test it must converge.]

In Chapter 4, we present a translation of Schlömilch’s original paper, and then proceed the rest of the chapter by providing a few examples. Note that the translation of
Schlömilch’s result stems from what he addresses as the “well-known theorem,” also known as the “Mean Value Theorem,” which we also prove. German mathematician Alfred Pringsheim came up with a couple of important results in 1889.

First, we define the following:

**Definition 1.3.7.** Let the number $\alpha$ be the **asymptotic density** of positive terms in $\sum_{n=1}^{\infty} f_n$. That is,

$$\alpha = \lim_{n \to \infty} \frac{p_n}{n},$$

where $p_n$ denotes the number of positive terms in the sequence $(f_n)^k_{n=1}$.

**Theorem 1.3.8 ([2]).** The rearrangement $\sum_{k=1}^{\infty} f_n$ converges to an extended real number if and only if $\alpha$, the asymptotic density of positive terms in $\sum_{k=1}^{\infty} f_n$ exists. Also, the sum of the rearrangement with asymptotic density $\alpha$ is

$$\ln(2) + \frac{1}{2} \ln \left( \frac{\alpha}{1-\alpha} \right).$$

**Theorem 1.3.9 ([2]).** Let $f_1, f_2, f_3, \ldots, f_k$ be a sequence such that each $f_i \in \mathbb{R}$ for $i \in \mathbb{N}$. If $|f_1| \geq |f_2| \geq |f_3| \geq \ldots$, where

$$\lim_{k \to \infty} f_k = 0,$$

and $f_{2j-1} > 0 > f_{2j}$ for $j \in \mathbb{N}$, then the following holds:

1. If $\lim_{k \to \infty} |f_k| = \infty$ and if $S \in \mathbb{R}$, then there exists a rearrangement of $\sum_{j=1}^{\infty} f_j$, call it $\sum_{j=1}^{\infty} f_j^*$, such that $\sum_{j=1}^{\infty} f_j^* = S$, and the asymptotic density of positive terms of the
rearrangement is $\frac{1}{2}$,

whose $\alpha = \frac{1}{2}$, and whose sum is $S$.

2. If $\lim_{k \to \infty} k \cdot f_k = 0$ and if $\sum_{n=1}^{\infty} g_n$ is a rearrangement of $\sum_{n=1}^{\infty} f_n$ for which the asymptotic density $\alpha$ exists where $0 < \alpha < 1$, then

$$\sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} f_n.$$ 

In 1911, Wacław Sierpiński came up with the following theorem.

**Theorem 1.3.10** ([21]). Let $(f_n)$ be conditionally convergent where $U = \sum f_n$ and $V = \sum f_{\pi(n)}$. If $V > U$, there exists an explicitly described rearrangement $\pi$ with the property that each positive term of $f_n$ is left in place (if $f_n > 0$ then $\pi(n) = n$). Similarly, if $V < U$, there exists an explicitly describe rearrangement $\pi$ with the property that each negative term of $f_n$ is left in place (if $f_n < 0$, then $\pi(n) = n$).

[Note that Theorem 1.3.9 is our version of Sierpiński’s original theorem.]

In Chapter 6, we provide a translation of Sierpiński’s paper [21] along with his original theorem. Later, we analyze his result by providing a couple examples.

In Chapter 7, we briefly introduce several modern rearrangements by various mathematicians, which extend Sierpiński’s theorem.
CHAPTER 2

OHM’S THEOREM

2.1 Martin Ohm

Martin Ohm (May 6, 1792 - April 1, 1872) was a German mathematician who earned his doctorate in 1811 at Friedrich-Alexander-University, Erlangen-Nuremberg, under his advisor, Karl Christian von Langsdorf [1]. By the early 1800s he became a professor in the gymnasium at Thorn for mathematics and physics. In 1839, he was chosen to be a professor at the University of Berlin, and delivered lectures at the academy of architecture, and at the school of artillery and engineering [12].

2.2 Ohm’s Rearrangement Theorem

2.2.1 Part I

Theorem 2.2.1 ([11]). For p and q positive integers, rearrange \( \sum_{n \geq 1}^{\infty} \frac{(-1)^{n-1}}{n} \) by taking the first p positive terms, then the first q negative terms, then the next p positive terms, the next q negative terms, and so on. The rearranged series converges to

\[
\ln(2) + \frac{1}{2} \ln \left( \frac{p}{q} \right),
\]  

(2.1)
Denote by $A(p, q)$ the series resulting from this rearrangement. Let $S_k$ be the partial sums of $A(p, q)$ where specifically $S_k' = S_k'(p, q)$ denotes the $k^{th}$ partial sum of $A(p, q)$, and let $C_n = C_n(p, q)$ be the partial sum of $A(p, q)$, which is the result in $S$ from precisely adding $n$ blocks, where each block consists of $p$ positive terms and $q$ negative terms, consecutively.

Denote $S_n$ to be the partial sums of the usual alternating series, where $n$ denotes the terms. Call $C_n$ as $C_n = S_{f(n)} + R_n$ for some $f(n)$ and some “remainder” $R_n$.

It suffices to break the proof of Theorem 2.2.1 into the following:

I Verifying the explicit formula of $C_n(p, q)$ for each of the three types of rearrangements:

(a) $p = q,$

(b) $p > q,$ and

(c) $p < q.$

II Arguing that $S_n$ and $C_n$ converge to $\ln(2) + \frac{1}{2} \ln\left(\frac{p}{q}\right)$.

**Example 2.2.1.1.** Consider $A(2, 2)$. We want to show that $C_n(2, 2) = S_{4n}$. Note that

$$C_1 = 1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{4} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} = S_4 = S_{41}$$

$$C_2 = 1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{4} + \left[\frac{1}{5} + \frac{1}{7} - \frac{1}{6} - \frac{1}{8}\right]$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} = S_8 = S_{42}.$$ 

**Proof.** We proceed by induction with the base case shown in Example 2.2.1.1. So suppose that

$$C_n(2, 2) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{4n-3} - \frac{1}{4n-4} + \frac{1}{4n-1} - \frac{1}{4n} = S_{4n}.$$
Then by the inductive hypothesis,

\[ C_{n+1}(2,2) = C_n(2,2) + \left( \frac{1}{4n+1} + \frac{1}{4n+3} \right) - \left( \frac{1}{4n+2} + \frac{1}{4n+4} \right) \]

\[ = C_n(2,2) + \frac{1}{4n+1} - \frac{1}{4n+2} + \frac{1}{4n+3} - \frac{1}{4n+4} = S_{4n+4}, \]

as wanted, so \( f(n) = 4n \) and \( R_n = 0. \)

\[ \square \]

**General Case:** \( A(p,q), p = q. \)

We now proceed with the more general case:

**Theorem 2.2.2.** For positive integer \( p \), \( C_n(p,p) = S_{2pn} \) for all \( n \).

**Proof.** Consider \( A(p,p) \). We argue that

\[ C_n := C_n(p,p) = S_{2pn} \]

by induction. Note that

\[ C_1 = \left( 1 + \frac{1}{3} + \cdots + \frac{1}{2p-1} \right) - \left( \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2p} \right) \]

\[ = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2p-1} - \frac{1}{2p} = S_{2p}, \]

\[ C_2 = \left( 1 + \frac{1}{3} + \cdots + \frac{1}{2p-1} \right) - \left( \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2p} \right) \]

\[ + \left[ \left( \frac{1}{2p+1} + \frac{1}{2p+3} + \cdots + \frac{1}{4p-1} \right) - \left( \frac{1}{2p+2} + \frac{1}{2p+4} + \cdots + \frac{1}{4p} \right) \right] \]
$$= C_1 + \frac{1}{2p+1} - \frac{1}{2p+2} + \frac{1}{2p+3} - \frac{1}{2p+4} + \cdots + \frac{1}{4p-1} - \frac{1}{4p}$$

$$= S_{2p} + \frac{1}{2p+1} - \frac{1}{2p+2} + \frac{1}{2p+3} - \frac{1}{2p+4} + \cdots + \frac{1}{4p-1} - \frac{1}{4p} = S_{4p}.$$  

Now suppose that

$$C_n(p,p) = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2pn-1} + \frac{1}{2pn} = S_{2pn}.$$

By the inductive hypothesis,

$$C_{n+1}(p,p) = C_n(p,p) + \left( \frac{1}{2pn+1} + \frac{1}{2pn+3} + \cdots + \frac{1}{2pn+2p-1} \right)$$

$$- \left( \frac{1}{2pn+2} + \frac{1}{2pn+4} + \cdots + \frac{1}{2pn+2p} \right)$$

$$= C_n(p,p) + \frac{1}{2pn+1} - \frac{1}{2pn+2} + \frac{1}{2pn+3} - \frac{1}{2pn+4} + \frac{1}{2pn+2p}$$

$$\cdots + \frac{1}{2pn+2p-1} - \frac{1}{2pn+2p}$$

$$= S_{2np} + \frac{1}{2pn+1} - \frac{1}{2pn+2} + \frac{1}{2pn+3} - \frac{1}{2pn+4} + \cdots + \frac{1}{2pn+2p-1} - \frac{1}{2pn+2p}$$

$$= S_{2pn+2p} = S_{2p(n+1)}$$

Therefore, \( f(n) = 2p(n+1) \) and \( R_n = 0. \) \( \square \)
Example 2.2.2.1. Consider $A(3, 1)$. We have

\[ C_1 = 1 + \frac{1}{3} + \frac{1}{5} + \left( -\frac{1}{2} \right) \]

\[ = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \left( \frac{1}{4} + \frac{1}{6} \right) \]

\[ = S_6 + \frac{1}{2} \left( \frac{1}{2} + \frac{1}{3} \right) = S_{6+1} + \frac{1}{2} \left( \frac{1}{2} + \frac{1}{3} \right) \]

\[ C_2 = 1 + \frac{1}{3} + \frac{1}{5} + \left( -\frac{1}{2} \right) + \left[ \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \left( -\frac{1}{4} \right) \right] \]

\[ = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} + \left( \frac{1}{6} + \frac{1}{8} + \frac{1}{10} + \frac{1}{12} \right) \]

\[ = S_{12} + \frac{1}{2} \left( \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \right) = S_{6+2} + \frac{1}{2} \left( \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6} \right) \]

We claim from our analysis of $C_1$ and $C_2$ that

\[ C_n(3, 1) = S_{6n} + \frac{1}{2} \left( \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{3n} \right). \]

So, $f(n) = 6n$ and

\[ R_n = \frac{1}{2} \left( \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{3n} \right). \]

We also find that the last block of $C_n(3, 1)$ is

\[ \frac{1}{6n-5} + \frac{1}{6n-3} + \frac{1}{6n-1} - \frac{1}{2n}. \]
We proceed to verify this.

**Theorem 2.2.3.** (1) The last block of $C_n(3, 1)$ is
\[ \frac{1}{6n-5} + \frac{1}{6n-3} + \frac{1}{6n-1} - \frac{1}{2n}, \]
and
\[ 16n - 5 + 16n - 3 + 16n - 1 - 2n. \]

(2) $C_n(3, 1) = S_{6n} + \frac{1}{2} \left( \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{3n} \right)$.

**Proof.** Let us proceed by induction on $n$. We already have seen that (1) and (2) hold when $n = 1, 2$. Now suppose that (1) and (2) hold for $n$. We need to show that (1) and (2) hold for $n+1$.

**Proof of (1):** By the inductive hypothesis, we know that the positive terms added in the last block of $C_n(3, 1)$ are
\[ \frac{1}{6n-5} + \frac{1}{6n-3} + \frac{1}{6n-1}. \]

Hence, the three positive terms in the last block of $C_{n+1}(3, 1)$ that are added together are
\[ \frac{1}{6n+1} + \frac{1}{6n+3} + \frac{1}{6n+5}. \]

Similarly, by the inductive hypothesis, we know that the term subtracted in the last block of $C_n(3, 1)$ is $\frac{1}{2n}$. Hence, then the term subtracted in $C_{n+1}(3, 1)$ must be $\frac{1}{2n+2}$. We continue to show that these terms added and subtracted in the last block of $C_{n+1}(3, 1)$ by noting that $6(n+1) - 5 = 6n + 1, 6(n+1) - 3 = 6n + 3, 6(n+1) - 1 = 6n + 5, 2(n+1) = 2n + 2$. This proves (1) by induction, and we have
\[ C_{n+1}(3, 1) = C_n(3, 1) + \frac{1}{6n+1} + \frac{1}{6n+3} + \frac{1}{6n+5} - \frac{1}{2n+2}. \]

**Proof of (2):** By the inductive hypothesis,
\[ C_n(3, 1) = S_{6n} + \frac{1}{2} \left( \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{3n} \right). \]

We want to show that
\[ C_{n+1}(3, 1) = S_{6(n+1)} + \frac{1}{2} \left( \frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{3(n+1)} \right). \]

From (1), we have shown that
\[ C_{n+1}(3, 1) = C_n(3, 1) + \frac{1}{6n+1} + \frac{1}{6n+3} + \frac{1}{6n+5} - \frac{1}{2n+2}. \]

Or in other words,
\[ C_{n+1}(3, 1) - C_n(3, 1) = \frac{1}{6n+1} + \frac{1}{6n+3} + \frac{1}{6n+5} - \frac{1}{2n+2}. \]

So, to prove
\[ C_{n+1}(3, 1) = S_{6(n+1)} + \frac{1}{2} \left( \frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{3(n+1)} \right), \]

it is enough to show that
\[
\left( S_{6(n+1)} \right) + \frac{1}{2} \left( \frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{3(n+1)} \right) - \left( S_{6n} + \frac{1}{2} \left( \frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \cdots + \frac{1}{3n} \right) \right) = \frac{1}{6n+1} + \frac{1}{6n+3} + \frac{1}{6n+5} - \frac{1}{2n+2}.
\]

By definition of \( S_k \), we know that
$S_{6(n+1)} - S_{6n} = \frac{1}{6n+1} - \frac{1}{6n+2} + \frac{1}{6n+3} - \frac{1}{6n+4} + \frac{1}{6n+5} - \frac{1}{6n+6},$

and since the expression we are considering equals

$S_{6(n+1)} - S_{6n} + \frac{1}{2} \left( \frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{3(n+1)} - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{3n} \right),$

we need to prove the identity obtained from substituting the expression $S_{6(n+1)} - S_{6n}$:

\[
\frac{1}{6n+1} - \frac{1}{6n+2} + \frac{1}{6n+3} - \frac{1}{6n+4} + \frac{1}{6n+5} - \frac{1}{6n+6} \\
+ \frac{1}{2} \left( \frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{3(n+1)} - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{3n} \right)
\]

\[= \frac{1}{6n+1} + \frac{1}{6n+3} + \frac{1}{6n+5} - \frac{1}{2n+2}.\]

Notice that the terms $\frac{1}{6n+1}, \frac{1}{6n+3}, \frac{1}{6n+5}$ get canceled out and we are left with

\[- \frac{1}{6n+2} - \frac{1}{6n+4} - \frac{1}{6n+6} + \frac{1}{2} \left( \frac{1}{(n+1)+1} + \frac{1}{(n+1)+2} + \cdots + \frac{1}{3(n+1)} - \frac{1}{n+1} - \frac{1}{n+2} - \cdots - \frac{1}{3n} \right) = - \frac{1}{2n+2}.
\]

Expanding and simplifying the left side of the equation, we need to show,

\[- \frac{1}{6n+2} - \frac{1}{6n+4} - \frac{1}{6n+6} + \frac{1}{2} \left( \frac{1}{n+2} + \frac{1}{n+3} + \frac{1}{n+4} + \cdots + \frac{1}{2n+1} + \cdots \right) \\
+ \frac{1}{3n} + \frac{1}{3n+1} + \frac{1}{3n+2} + \frac{1}{3n+3} - \frac{1}{n+1} - \frac{1}{n+2} - \frac{1}{n+3} - \frac{1}{n+4} - \cdots \]
\[-\frac{1}{2n} - \frac{1}{2n+1} - \frac{1}{2n+2} - \frac{1}{2n+3} - \frac{1}{2n+4} - \cdots - \frac{1}{3n}\] = -\frac{1}{2n+2},

or

\[-\frac{1}{6n+2} - \frac{1}{6n+4} - \frac{1}{6n+6} + \frac{1}{2} \left( \frac{1}{3n+1} + \frac{1}{3n+2} + \frac{1}{3n+3} - \frac{1}{n+1} \right) = -\frac{1}{2n+2} \]

\[-\frac{1}{6n+2} - \frac{1}{6n+4} - \frac{1}{6n+6} + \frac{1}{6n+2} + \frac{1}{6n+4} + \frac{1}{6n+6} - \frac{1}{2n+2} = -\frac{1}{2n+2},

which simplifies to \(-\frac{1}{2n+2} = -\frac{1}{2n+2}\), and so (2) follows.
**General Case:** $A(p,q), p > q.$

**Theorem 2.2.4.** Suppose that $p > q.$ (1) The last block of $C_n(p,q)$ is

$$\left(\frac{1}{2np-2p+1} + \frac{1}{2np-2p+3} + \cdots + \frac{1}{2np-1}\right) - \left(\frac{1}{2nq-2q+2} + \frac{1}{2nq-2q+4} + \cdots + \frac{1}{2nq}\right),$$

and

(2) $C_n(p,q) = S_{2np} + \left(\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np}\right).$

**Proof.** We proceed by induction. Note first that

$$C_1 = \left(1 + \frac{1}{3} + \cdots + \frac{1}{2p-1}\right) - \left(\frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2q}\right)$$

$$= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2p-1} - \frac{1}{2p}\right) + \left(\frac{1}{2q+2} + \frac{1}{2q+4} + \cdots + \frac{1}{2p}\right)$$

$$= S_{2p} + \left(\frac{1}{2q+2} + \frac{1}{2q+4} + \cdots + \frac{1}{2p}\right) = S_{2.1.p} + \left(\frac{1}{2q+2} + \frac{1}{2q+4} + \cdots + \frac{1}{2p}\right)$$

$$C_2 = C_1 + \left[\left(\frac{1}{2p+1} + \frac{1}{2p+3} + \cdots + \frac{1}{4p-1}\right) - \left(\frac{1}{2q+2} + \frac{1}{2q+4} + \cdots + \frac{1}{4q}\right)\right]$$

$$= \left(1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{4p-1} - \frac{1}{4p}\right) + \left(\frac{1}{4q+2} + \frac{1}{4q+4} + \cdots + \frac{1}{4p}\right)$$

$$= S_{4p} + \left(\frac{1}{4q+2} + \frac{1}{4q+4} + \cdots + \frac{1}{4p}\right) = S_{2.2.p} + \left(\frac{1}{4q+2} + \frac{1}{4q+4} + \cdots + \frac{1}{4p}\right).$$

This shows that (1) and (2) hold when $n = 1, 2.$

Now suppose that (1) and (2) hold for $n.$ We need to show that (1) and (2) hold for $n + 1.$
Proof of (1): By the inductive hypothesis, we know that the positive terms added in the last block of $C_n(p, q)$ are

$$\left(\frac{1}{2np - 2p + 1} + \frac{1}{2np - 2p + 3} + \cdots + \frac{1}{2np - 1}\right).$$

Hence, the $p$ positive terms in the last block of $C_{n+1}(p, q)$ that are added together are

$$\frac{1}{2np + 1} + \frac{1}{2np + 3} + \cdots + \frac{1}{2np + 2p + 1}.$$ 

Similarly, by the inductive hypothesis, we know that the $q$ terms subtracted in the last block of $C_n(p, q)$ are

$$\frac{1}{2nq - 2q + 2} + \frac{1}{2nq - 2q + 4} + \cdots + \frac{1}{2nq}.$$ 

Hence, the terms subtracted in $C_{n+1}(p, q)$ must be

$$\frac{1}{2nq + 2} + \frac{1}{2nq + 4} + \cdots + \frac{1}{2nq + 2q}.$$ 

This proves (1) by noting that

$$2(n+1)p - 2p + 1 = 2np + 1,$$

$$2(n+1)p - 2p + 3 = 2np + 3, \ldots, 2(n+1)p - 1 = 2np + 2p - 1$$

$$2(n+1)q - 2q + 2 = 2nq + 2,$$

$$2(n+1)q - 2q + 4 = 2nq + 4, \ldots, 2(n+1)q = 2nq + 2q.$$ 

Note also that

$$C_{n+1}(p, q) = C_n(p, q) + \left(\frac{1}{2np + 1} + \frac{1}{2np + 3} + \cdots + \frac{1}{2np + 2p - 1}\right)$$

$$- \left(\frac{1}{2nq + 2} + \frac{1}{2nq + 4} + \cdots + \frac{1}{2nq + 2q}\right).$$
Proof of (2): By the inductive hypothesis,

\[ C_n(p, q) = S_{2np} + \left( \frac{1}{2nq + 2} + \frac{1}{2nq + 4} + \cdots + \frac{1}{2np} \right). \]

We want to show that

\[ C_{n+1}(p, q) = S_{2(n+1)p} + \left( \frac{1}{2(n+1)q + 2} + \frac{1}{2(n+1)q + 4} + \cdots + \frac{1}{2(n+1)p} \right). \]

From (1), we have already shown that

\[
C_{n+1}(p, q) = C_n(p, q) + \left( \frac{1}{2np + 1} + \frac{1}{2np + 3} + \cdots + \frac{1}{2np + 2p - 1} \right) \\
- \left( \frac{1}{2nq + 2} + \frac{1}{2nq + 4} + \cdots + \frac{1}{2nq + 2q} \right).
\]

Since

\[
C_{n+1}(p, q) = C_n(p, q) + \left( \frac{1}{2np + 1} + \frac{1}{2np + 3} + \cdots + \frac{1}{2np + 2p - 1} \right) \\
- \left( \frac{1}{2nq + 2} + \frac{1}{2nq + 4} + \frac{1}{2nq + 2q} \right),
\]

it is enough to show that

\[
C_{n+1}(p, q) = \left( S_{2np} + \left( \frac{1}{2nq + 2} + \frac{1}{2nq + 4} + \cdots + \frac{1}{2np} \right) \right) \\
+ \left( \frac{1}{2np + 1} + \frac{1}{2np + 3} + \cdots + \frac{1}{2np + 2p - 1} \right) - \left( \frac{1}{2nq + 2} + \frac{1}{2nq + 4} + \cdots + \frac{1}{2nq + 2q} \right).
\]

By definition of \( S_k \), we have that
\[ S_{2(n+1)p} - S_{2np} = \left( \frac{1}{2np + 1} + \frac{1}{2np + 3} + \cdots + \frac{1}{2np + 2p - 1} \right) \\
- \left( \frac{1}{2np + 2} + \frac{1}{2np + 4} + \cdots + \frac{1}{2np + 2p} \right) \]

\[ = \frac{1}{2np + 1} - \frac{1}{2np + 2} + \frac{1}{2np + 3} - \frac{1}{2np + 4} + \cdots + \frac{1}{2np + 2p - 1} - \frac{1}{2np + 2p} \]

\[ = \frac{1}{2np + 1} - \frac{1}{2np + 2} + \frac{1}{2np + 3} - \cdots - \frac{1}{2np + 2p}. \]

Notice that \( S_{2np} \) can be written as

\[ S_{2np} = \left( S_{2(n+1)p} - \left( \frac{1}{2np + 1} - \frac{1}{2np + 2} + \frac{1}{2np + 3} - \cdots - \frac{1}{2np + 2p} \right) \right). \]

So substituting \( S_{2np} \) into our expression for \( C_{n+1}(p, q) \), we get

\[ C_{n+1}(p, q) = \left( S_{2(n+1)p} - \left( \frac{1}{2np + 1} - \frac{1}{2np + 2} + \frac{1}{2np + 3} - \cdots - \frac{1}{2np + 2p} \right) \right) \\
+ \left( \frac{1}{2nq + 2} + \frac{1}{2nq + 4} + \cdots + \frac{1}{2np} \right) + \left( \frac{1}{2np + 1} + \frac{1}{2np + 3} + \cdots + \frac{1}{2np + 2p - 1} \right) \\
- \left( \frac{1}{2nq + 2} + \frac{1}{2nq + 4} + \cdots + \frac{1}{2np + 2q} \right). \]

Rewriting this so that the positive and negative terms in

\[ \left( \frac{1}{2np + 1} - \frac{1}{2np + 2} + \frac{1}{2np + 3} - \cdots - \frac{1}{2np + 2p} \right) \]

are grouped together, we have
\[ C_{n+1}(p,q) = \left( S_{2(n+1)p} - \left( \frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2np+2p-1} \right) \right) \]
\[ + \left( \frac{1}{2np+2} + \frac{1}{2np+4} + \cdots + \frac{1}{2np+2p} \right) + \left( \frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np} \right) \]
\[ + \left( \frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2np+2p-1} \right) \]
\[ - \left( \frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2nq+2q} \right). \]

Simplifying we have,

\[ C_{n+1}(p,q) = \left( S_{2(n+1)p} + \left( \frac{1}{2np+2} + \frac{1}{2np+4} + \cdots + \frac{1}{2np+2p} \right) \right) \]
\[ + \left( \frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np} \right) - \left( \frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2nq+2q} \right). \]

Notice by expansion and simplification we get

\[ C_{n+1}(p,q) = S_{2(n+1)p} + \left( \frac{1}{2(n+1)q+2} + \frac{1}{2(n+1)q+4} + \cdots + \frac{1}{2(n+1)p} \right), \]

which is what we wanted. \qed
Example 2.2.4.1. Consider $A(2,3)$. We have

$$C_1 = 1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6}$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} - \left(\frac{1}{5}\right)$$

$$= S_6 - \left(\frac{1}{5}\right) = S_{6.1} - \left(\frac{1}{5}\right)$$

$$C_2 = 1 + \frac{1}{3} - \frac{1}{2} - \frac{1}{4} - \frac{1}{6} + \left[\frac{1}{5} + \frac{1}{7} - \frac{1}{8} - \frac{1}{10} - \frac{1}{12}\right]$$

$$= 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \frac{1}{9} - \frac{1}{10} + \frac{1}{11} - \frac{1}{12} - \left(\frac{1}{9} + \frac{1}{11}\right)$$

$$= S_{12} - \left(\frac{1}{9} + \frac{1}{11}\right) = S_{6.2} - \left(\frac{1}{9} + \frac{1}{11}\right).$$

This leads us to claim general formulas for $C_n(2,3)$, which we proceed to show:

**Theorem 2.2.5.** $C_n(2,3)$ is (1) the last block of $C_n(2,3)$ is $\frac{1}{4n-3} + \frac{1}{4n-1} - \frac{1}{6n-4} - \frac{1}{6n-2} - \frac{1}{6n}$, and (2) $C_n(2,3) = S_{6n} - \left(\frac{1}{4n+1} + \frac{1}{4n+3} + \cdots + \frac{1}{6n-1}\right)$.

**Proof.** Let us proceed by induction on $n$. We already have seen that (1) and (2) hold when $n = 1, 2$. Now suppose that (1) and (2) hold for $n$. We need to show that (1) and (2) hold for $n + 1$.

**Proof of (1):** By the inductive hypothesis, we know that the positive terms added in the last block of $C_n(2,3)$ are $\frac{1}{4n-3}$ and $\frac{1}{4n-1}$. Since we know we need to add two positive terms, then the positive terms added in the last block of $C_{n+1}(2,3)$ are $\frac{1}{4n+1}$ and $\frac{1}{4n+3}$. Similarly,
by the inductive hypothesis, the three terms subtracted in the last block of \( C_n(2,3) \), which are, respectively, \( \frac{1}{6n-4}, \frac{1}{6n-2}, \) and \( \frac{1}{6n} \). So, the three negative terms subtracted in \( C_{n+1}(2,3) \) are \( \frac{1}{6n+2}, \frac{1}{6n+4}, \) and \( \frac{1}{6n+6} \). (1) follows from noting that \( 4(n+1)-3 = 4n+1, 4(n+1)-1 = 4n+3, 6(n+1)-4 = 6n+2, 6(n+1)-2 = 6n+4, 6(n+1) = 6n+6. \)

Also,

\[
C_{n+1}(2,3) = C_n(2,3) + \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{6n+2} - \frac{1}{6n+4} - \frac{1}{6n+6}.
\]

**Proof of (2):** By the inductive hypothesis,

\[
C_n(2,3) = S_{6n} - \left( \frac{1}{4n+1} + \frac{1}{4n+3} + \cdots + \frac{1}{6n-1} \right).
\]

We want to show that

\[
C_{n+1}(2,3) = S_{6(n+1)} - \left( \frac{1}{4n+5} + \frac{1}{4n+7} + \cdots + \frac{1}{6n+5} \right).
\]

From (1), we have shown that

\[
C_{n+1}(2,3) = C_n(2,3) + \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{6n+2} - \frac{1}{6n+4} - \frac{1}{6n+6}
\]

or

\[
C_{n+1}(1,2) - C_n(2,3) = \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{6n+2} - \frac{1}{6n+4} - \frac{1}{6n+6}.
\]

To prove \( C_{n+1}(2,3) = S_{6(n+1)} - \left( \frac{1}{4n+5} + \frac{1}{4n+7} + \cdots + \frac{1}{6n+5} \right) \), it is enough to show
\[
\left( S_{6(n+1)} \right) + \left(-\left( \frac{1}{4n+5} + \frac{1}{4n+7} + \cdots + \frac{1}{6n+5} \right) \right) \\
- \left( S_{6n} + \left(-\left( \frac{1}{4n+1} + \frac{1}{4n+3} + \cdots + \frac{1}{6n-1} \right) \right) \right) \\
= \frac{1}{2n+1} - \frac{1}{4n+2} - \frac{1}{4n+4}.
\]

By definition of \( S_k \), we know that
\[
S_{6(n+1)} - S_{6n} = \frac{1}{6n+1} - \frac{1}{6n+2} + \frac{1}{6n+3} - \frac{1}{6n+4} + \frac{1}{6n+5} - \frac{1}{6n+6},
\]
and since the expression we are considering equals
\[
S_{6(n+1)} - S_{6n} + \left(-\left( \frac{1}{4n+5} + \frac{1}{4n+7} + \cdots + \frac{1}{6n+5} - \frac{1}{4n+1} - \frac{1}{4n+3} - \cdots - \frac{1}{6n-1} \right) \right),
\]
we need to prove the identity obtained from substituting the expression \( S_{6(n+1)} - S_{6n} \):
\[
\frac{1}{6n+1} - \frac{1}{6n+2} + \frac{1}{6n+3} - \frac{1}{6n+4} + \frac{1}{6n+5} - \frac{1}{6n+6} + \\
- \left( \frac{1}{4n+5} + \frac{1}{4n+7} + \cdots + \frac{1}{6n+5} - \frac{1}{4n+1} - \frac{1}{4n+3} - \cdots - \frac{1}{6n-1} \right) \\
= \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{6n+2} - \frac{1}{6n+4} - \frac{1}{6n+6},
\]
or,
\[
\frac{1}{6n+1} - \frac{1}{6n+2} + \frac{1}{6n+3} - \frac{1}{6n+4} + \frac{1}{6n+5} - \frac{1}{6n+6} - \frac{1}{4n+5} - \frac{1}{4n+7} - \cdots
\]
\[ \cdots - \frac{1}{6n+5} + \frac{1}{4n+1} + \frac{1}{4n+3} + \cdots + \frac{1}{6n-1} \]
\[= \frac{1}{4n+1} + \frac{1}{4n+3} - \frac{1}{6n+2} - \frac{1}{6n+4} - \frac{1}{6n+6}. \]

Notice that the terms \(- \frac{1}{6n+2}, - \frac{1}{6n+4}\), and \(- \frac{1}{6n+6}\) get canceled out and we are left with

\[ \frac{1}{6n+1} + \frac{1}{6n+3} + \frac{1}{6n+5} - \frac{1}{4n+5} - \frac{1}{4n+7} - \cdots \]
\[= \frac{1}{4n+1} + \frac{1}{4n+3}. \]

Expanding and simplifying the left side, we need to show

\[ \frac{1}{6n+1} + \frac{1}{6n+3} + \frac{1}{6n+5} - \frac{1}{4n+5} - \frac{1}{4n+7} - \cdots - \frac{1}{6n-1} - \frac{1}{6n+1} - \frac{1}{6n+3} \]
\[= \frac{1}{4n+1} + \frac{1}{4n+3}. \]

This reduces to \( \frac{1}{4n+1} + \frac{1}{4n+3} = \frac{1}{4n+1} + \frac{1}{4n+3} \).

**General Case:** \( A(p, q), p < q. \)

Thus, we come up with the following theorem.
Theorem 2.2.6. Suppose that $p < q$. (1) The last block of $C_n(p, q)$ is
\[
\frac{1}{2np - 2p + 3} + \cdots + \frac{1}{2np - 2p + 1} - \left( \frac{1}{2np - 2q + 2} + \frac{1}{2np - 2q + 4} + \cdots + \frac{1}{2nq} \right),
\]
and
\[
C_n(p, q) = \frac{1}{2np + 1} + \frac{1}{2np + 3} + \cdots + \frac{1}{2nq}.
\]

Proof. We proceed by induction. Note first that
\[
C_1 = \left( 1 + \frac{1}{3} + \cdots + \frac{1}{2p - 1} \right) - \left( \frac{1}{2} + \frac{1}{4} + \cdots + \frac{1}{2q} \right)
\]
\[
= \left( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2p - 1} - \frac{1}{2q} \right) + \left( \frac{1}{2p + 1} + \frac{1}{2p + 3} + \cdots + \frac{1}{2q - 1} \right)
\]
\[
= \frac{1}{2q} + \left( \frac{1}{2p + 1} + \frac{1}{2p + 3} + \cdots + \frac{1}{2q - 1} \right)
\]
\[
= S_{2q} + \left( \frac{1}{2p + 1} + \frac{1}{2p + 3} + \cdots + \frac{1}{2q - 1} \right)
\]
\[
C_2 = C_1 + \left( \frac{1}{2p + 1} + \frac{1}{2p + 3} + \cdots + \frac{1}{2q - 1} \right) - \left( \frac{1}{2q + 2} + \frac{1}{2q + 4} + \cdots + \frac{1}{4q} \right)
\]
\[
= \left( 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots + \frac{1}{2q - 1} - \frac{1}{4q} \right) + \left( \frac{1}{4p + 1} + \frac{1}{4p + 3} + \cdots + \frac{1}{4q - 1} \right)
\]
\[
= S_{4q} + \left( \frac{1}{4p + 1} + \frac{1}{4p + 3} + \cdots + \frac{1}{4q - 1} \right)
\]
\[
= S_{2q} + \left( \frac{1}{4p + 1} + \frac{1}{4p + 3} + \cdots + \frac{1}{4q - 1} \right).
\]

This shows that (1) and (2) hold when $n = 1, 2$. Suppose that (1) and (2) hold for $n$. We need to show that (1) and (2) hold for $n + 1$. 
**Proof of (1):** By the inductive hypothesis, we know that the positive terms added in the last block of $C_n(p, q)$ are

$$\frac{1}{2np - 2p + 1} + \frac{1}{2np - 2p + 3} + \cdots + \frac{1}{2np - 1}.$$

Since we need to add $p$ positive terms, then the $p$ positive terms in the last block of $C_{n+1}(p, q)$ that are added together are

$$\frac{1}{2np + 1} + \frac{1}{2np + 3} + \cdots + \frac{1}{2np + 2p - 1}.$$

Similarly, by the inductive hypothesis, we know that the $q$ terms subtracted in the last block of $C_n(p, q)$ are

$$\frac{1}{2nq - 2q + 2} + \frac{1}{2nq - 2q + 4} + \cdots + \frac{1}{2nq}.$$

and therefore the new terms subtracted in $C_{n+1}(p, q)$ must be

$$\frac{1}{2nq + 2} + \frac{1}{2nq + 4} + \cdots + \frac{1}{2nq + 2q}.$$

(1) follows from noting that $2(n+1)p - 2p + 1 = 2np + 1$, $2(n+1)p - 2p + 3 = 2np + 3$, $\ldots$, $2(n+1)p - 1 = 2np + 2p - 1$, $2(n+1)q - 2q + 2 = 2nq + 2$, $2(n+1)q - 2q + 4 = 2nq + 4$, $\ldots$, $2(n+1)q = 2nq + 2q$. Also,

$$C_{n+1}(p, q) = C_n(p, q) + \left( \frac{1}{2np + 1} + \frac{1}{2np + 3} + \cdots + \frac{1}{2np + 2p - 1} \right)$$

$$- \left( \frac{1}{2nq + 2} + \frac{1}{2nq + 4} + \cdots + \frac{1}{2nq + 2q} \right).$$
Proof of (2): By the inductive hypothesis,

\[ C_n(p, q) = S_{2nq} + -\left(\frac{1}{2np + 1} + \frac{1}{2np + 3} + \cdots + \frac{1}{2nq - 1}\right). \]

We want to show that

\[ C_{n+1}(p, q) = S_{2(n+1)q} + -\left(\frac{1}{2(n+1)p + 1} + \frac{1}{2(n+1)p + 3} + \cdots + \frac{1}{2(n+1)q - 1}\right). \]

From (1), we have already shown

\[ C_{n+1}(p, q) = C_n(p, q) + \left(\frac{1}{2np + 1} + \frac{1}{2np + 3} + \cdots + \frac{1}{2np + 2p - 1}\right) \]
\[ -\left(\frac{1}{2nq + 1} + \frac{1}{2nq + 4} + \cdots + \frac{1}{2nq + 2q}\right). \]

Since

\[ C_{n+1}(p, q) = C_n(p, q) + \left(\frac{1}{2np + 1} + \frac{1}{2np + 3} + \cdots + \frac{1}{2np + 2p - 1}\right) \]
\[ -\left(\frac{1}{2nq + 2} + \frac{1}{2nq + 4} + \cdots + \frac{1}{2nq + 2q}\right), \]

It suffices to show that

\[ C_{n+1}(p, q) = \left(\frac{1}{2np + 1} + \frac{1}{2np + 3} + \cdots + \frac{1}{2nq - 1}\right) \]
\[ +\left(\frac{1}{2np + 1} + \frac{1}{2np + 3} + \cdots + \frac{1}{2np - 1}\right) - \left(\frac{1}{2nq + 2} + \frac{1}{2nq + 4} + \cdots + \frac{1}{2nq + 2q}\right) \]

By definition of \( S_k \), we have that
\[ S_{2(n+1)q} - S_{2nq} = \left( \frac{1}{2nq+1} + \frac{1}{2nq+3} + \cdots + \frac{1}{2nq+2q-1} \right) \]

\[-\left( \frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2nq+2q} \right). \]

Notice that \( S_{2nq} \) can be written as

\[ S_{2nq} = S_{2(n+1)q} - \left( \frac{1}{2nq+1} - \frac{1}{2nq+2} + \frac{1}{2nq+3} - \cdots - \frac{1}{2nq+2q} \right). \]

So, substituting \( S_{2nq} \) into our expression for \( C_{n+1}(p,q) \), we have that

\[ C_{n+1}(p,q) = S_{2(n+1)q} - \left( \frac{1}{2nq+1} - \frac{1}{2nq+2} + \frac{1}{2nq+3} - \cdots - \frac{1}{2nq+2q} \right) \]

\[-\left( \frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2nq+2q} \right) + \left( \frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{2n+2q-1} \right) \]

\[-\left( \frac{1}{2n+2} + \frac{1}{2n+4} + \cdots + \frac{1}{2n+2q} \right). \]

Rewriting this so that the positive and negative terms in \( \left( \frac{1}{2nq+1} - \frac{1}{2nq+2} + \frac{1}{2nq+3} - \cdots - \frac{1}{2nq+2q} \right) \) are grouped together,

\[ C_{n+1}(p,q) = S_{2(n+1)q} - \left( \frac{1}{2nq+1} + \frac{1}{2n+3} + \cdots + \frac{1}{2nq+2q-1} \right) \]

\[ + \left( \frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2nq+2q} \right) - \left( \frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{2nq-1} \right) \]

\[-\left( \frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2nq+2q} \right) + \left( \frac{1}{2n+1} + \frac{1}{2n+3} + \cdots + \frac{1}{2nq-1} \right). \]
Simplifying,

\[ C_{n+1}(p, q) = S_{2(n+1)q} - \left( \frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq+2q-1} \right) + \left( \frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2np+3q-1} \right). \]

By expansion and again simplification,

\[ C_{n+1}(p, q) = S_{2(n+1)q} - \left( \frac{1}{2np+2p+1} + \frac{1}{2np+2p+3} + \cdots + \frac{1}{2nq+2q-1} \right) \]

\[ = S_{2(n+1)q} - \left( \frac{1}{2(n+1)p+1} + \frac{1}{2(n+1)p+3} + \cdots + \frac{1}{2(n+1)q-1} \right). \]

\[ \square \]

### 2.2.2 Part II

Recall \( C_n = C_n(p, q) \) to be the partial sum of \( A(p, q) \), which is the result in \( S \) from precisely adding \( n \) blocks, where each block consists of \( p \) positive terms and \( q \) negative terms, consecutively. Also, recall \( S_n \) to be the partial sums of the usual alternating series, where \( n \) denotes the terms. Call \( C_n \) as \( C_n = S_{f(n)} + R_n \) for some \( f(n) \) and some “remainder” \( R_n \).

We aim to show that \( S_n \) and \( C_n = S_{f(n)} + R_n \) for some \( f(n) \) converging to \( \ln(2) + \frac{1}{2} \ln\left(\frac{p}{q}\right) \). For our argument, we begin by splitting up the proof for each of the three cases. In our argument, we label each case as 1, 2, and 3, respectively.

**Proof.** 1. Consider \( C_n(p, p) = S_{2np} \), where \( f(n) = 2np \) and \( R_n = 0 \).
We show that $S_f(n)$ converges to $\ln(2)$ as $n \to \infty$, and that for each $S'_k$, if $n$ is defined by

$$n(p + q) \leq k < (n + 1)(p + q),$$

then letting $r_k = S'_k - C_n$, we have $r_k \to 0$ as $k \to \infty$. Note that $n \to \infty$ if $k \to \infty$, and so $S'_k \to \ln(2)$.

First, $r_k$ consists of a sum of $x_1 + x_2 + \cdots$, which is at most $p + p - 1 = 2p - 1$ terms where each term is

$$|x_i| \leq \frac{1}{2np + 1}.$$

So, letting $n$ be such that $2np \leq k < 2(n + 1)p$. Then,

$$|x_1 + x_2 + \cdots| \leq \frac{2p - 1}{2np + 1}.$$

But taking the limit of $\frac{2p - 1}{2np + 1}$, we have

$$\lim_{n \to \infty} \frac{2p - 1}{2np + 1} = 0,$$

which means that these extra terms are negligible. Now, since $S_k$ is defined as the partial sums of the usual Alternating Harmonic Series, we know that

$$\lim_{n \to \infty} S_n = \ln(2).$$

By the fact that $S_{2np}$ are the partial sums of the usual alternating series up to $2np$ terms, then
\[
\lim_{n \to \infty} S_{2np} = \ln(2).
\]

2. Consider

\[
C_n(p, q) = S_{2np} + \left( \frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np} \right),
\]

where \( f(n) = 2np \) and

\[
R_n = \left( \frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np} \right).
\]

We have already shown from 1. that \( S_{2np} \) converges to \( \ln(2) \) as \( n \to \infty \). Now we want to show both that

\[
\lim_{n \to \infty} R_n = \lim_{n \to \infty} \left( \frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np} \right) = \frac{1}{2} \ln \left( \frac{p}{q} \right)
\]

and that for each \( S'_k \), if \( n \) is defined by \( n(p+q) \leq k < (n+1)(p+q) \), then, letting \( r_k = S'_k - C_n \), we have that \( r_k \to 0 \) as \( k \to \infty \). But \( n \to \infty \) if \( k \to \infty \), and so \( S'_k \to \ln(2) \).

First, notice that \( r_k \) consists of a sum of \( x_1 + x_2 + \cdots \), which is at most \( p+q-1 \) terms where each term is

\[
|x_i| \leq \frac{1}{2nq+2}
\]

terms. So, for all \( k \), there exists an \( n \) such that \( n(p+q) \leq k < (n+1)(p+q) \), which means

\[
|x_1 + x_2 + \cdots | \leq \frac{p+q-1}{2nq+2}.
\]
But taking the limit of \( \frac{p+q-1}{2nq+2} \), we have

\[
\lim_{n \to \infty} \frac{p+q-1}{2nq+2} = 0.
\]

Therefore, the size of \( r_k \) is 0. In other words, \( r_k \) is so small we can consider these terms to be negligible. Now,

\[
\frac{1}{2} \ln \left( \frac{p}{q} \right) = \frac{1}{2} \ln \left( \frac{np}{nq} \right) = \frac{1}{2} \left( \ln(np) - \ln(nq) \right) = \frac{1}{2} \int_{nq}^{np} \frac{1}{x} \, dx
\]

where

\[
\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np} < \frac{1}{2} \int_{nq}^{np} \frac{1}{x} \, dx.
\]

Also,

\[
\frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np} > \frac{1}{2} \int_{nq+1}^{np+1} \frac{1}{x} \, dx = \frac{1}{2} \left( \ln(np+1) - \ln(nq+1) \right)
\]

\[
= \frac{1}{2} \ln \left( \frac{np+1}{nq+1} \right).
\]

Thus,

\[
\frac{1}{2} \int_{nq+1}^{np+1} \frac{1}{x} \, dx < \frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np} < \frac{1}{2} \int_{nq}^{np} \frac{1}{x} \, dx,
\]

or

\[
\frac{1}{2} \ln \left( \frac{np+1}{nq+1} \right) < \frac{1}{2nq+2} + \frac{1}{2nq+4} + \cdots + \frac{1}{2np} < \frac{1}{2} \ln \left( \frac{np}{nq} \right).
\]

Now taking the limit of the left and right side, applying L'Hopital's rule, we get

\[
\frac{1}{2} \lim_{n \to \infty} \ln \left( \frac{np+1}{nq+1} \right) = \frac{1}{2} \ln \left( \frac{p}{q} \right) \quad \text{and} \quad \frac{1}{2} \lim_{n \to \infty} \ln \left( \frac{np}{nq} \right) = \frac{1}{2} \ln \left( \frac{p}{q} \right).
\]
Therefore,

\[
\frac{1}{2} \lim_{n \to \infty} \ln \left( \frac{np + 1}{nq + 1} \right) = \frac{1}{2} \ln \left( \frac{p}{q} \right)
\]

\[
\leq \lim_{n \to \infty} \left( \frac{1}{2nq + 2} + \frac{1}{2nq + 4} + \cdots + \frac{1}{2np} \right)
\]

\[
\leq \frac{1}{2} \lim_{n \to \infty} \ln \left( \frac{np}{nq} \right)
\]

\[
= \frac{1}{2} \ln \left( \frac{p}{q} \right).
\]

Since both limits converge to \( \frac{1}{2} \ln \left( \frac{p}{q} \right) \) as \( n \to \infty \), then it is true that

\[
\lim_{n \to \infty} R_n = \lim_{n \to \infty} \left[ \frac{1}{2nq + 2} + \frac{1}{2nq + 4} + \cdots + \frac{1}{2np} \right]
\]

\[
= \frac{1}{2} \ln \left( \frac{p}{q} \right).
\]

We conclude that

\[
\lim_{n \to \infty} C_n(p, q) = \lim_{n \to \infty} \left[ S_{2np} + \left( \frac{1}{2nq + 2} + \frac{1}{2nq + 4} + \cdots + \frac{1}{2np} \right) \right]
\]

\[
= \ln(2) + \frac{1}{2} \ln \left( \frac{p}{q} \right)
\]

(by properties of convergent sequences).
3. Consider

$$C_n(p, q) = S_{2nq} - \left( \frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq-1} \right),$$

where $f(n) = 2nq$ and

$$R_n = -\left( \frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq-1} \right).$$

We have already shown from 1. that since $S_n$ is defined as the partial sums of the usual Alternating Harmonic Series, then $S_{2nq}$ are the partial sums of the usual series up to $2nq$ terms, converging to $\ln(2)$ as $n \to \infty$. Now we want to show both that

$$\lim_{n \to \infty} R_n = \frac{1}{2} \ln \left( \frac{p}{q} \right),$$

and that for each $S'_k$, if $n$ is defined by $n(p+q) \leq k < (n+1)(p+q)$, then, letting $r_k = S'_k - C_n$, we have that $r_k \to 0$ as $k \to \infty$. (Note that $n \to \infty$ if $k \to \infty$, and so $S'_k \to \ln(2)$ as well.) First, notice that $r_k$ consists of a sum of $x_1 + x_2 + \cdots$, which is at most $p+q-1$ terms where each term is $|x_i| \leq \frac{1}{2np+1}$ terms. So, for all $k$, there exists an $n$ such that $n(p+q) \leq k < (n+1)(p+q)$, which means that $|x_1 + x_2 + \cdots| \leq \frac{p+q-1}{2np+1}$. But taking the limit of $\frac{p+q-1}{2np+1}$, we have

$$\lim_{n \to \infty} \frac{p+q-1}{2np+1} = 0.$$ 

Therefore, the size of $r_k$ is 0, which means that these extra terms are negligible, and so do
not converge to any value. Note that

\[
\frac{1}{2} \ln \left( \frac{p}{q} \right) = \frac{1}{2} \ln \left( \frac{np}{nq} \right) = \frac{1}{2} \left( \ln(n)p - \ln(n)q \right) = \frac{1}{2} \int_{nq}^{np} \frac{1}{x} \, dx,
\]

where

\[
- \left( \frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq-1} \right) > \frac{1}{2} \int_{nq}^{np} \frac{1}{x} \, dx.
\]

Also

\[
- \left( \frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq-1} \right) < \frac{1}{2} \int_{nq+1}^{np+1} \frac{1}{x} \, dx
\]

\[
= \frac{1}{2} \left( \ln(n)p - \ln(n)q \right)
\]

\[
= \frac{1}{2} \ln \left( \frac{np}{nq} \right).
\]

So,

\[
\frac{1}{2} \int_{nq+1}^{np+1} \frac{1}{x} \, dx > - \left( \frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq-1} \right) > \frac{1}{2} \int_{nq}^{np} \frac{1}{x} \, dx,
\]

or

\[
\frac{1}{2} \ln \left( \frac{np+1}{nq+1} \right) > - \left( \frac{1}{2np+1} + \frac{1}{2np+3} + \cdots + \frac{1}{2nq-1} \right) > \frac{1}{2} \ln \left( \frac{np}{nq} \right).
\]

Now taking the limit of the left and right side, applying L’Hopitals rule, we get
\[
\frac{1}{2} \lim_{n \to \infty} \ln \left( \frac{np + 1}{nq + 1} \right) = \frac{1}{2} \ln \left( \frac{p}{q} \right)
\]

and

\[
\frac{1}{2} \lim_{n \to \infty} \ln \left( \frac{np}{nq} \right) = \frac{1}{2} \ln \left( \frac{p}{q} \right).
\]

Then, it is true that

\[
\frac{1}{2} \lim_{n \to \infty} \ln \left( \frac{np + 1}{nq + 1} \right) = \frac{1}{2} \ln \left( \frac{p}{q} \right) \geq \lim_{n \to \infty} \left[ \frac{1}{2nq} - \left( \frac{1}{2np + 1} + \frac{1}{2np + 3} + \cdots + \frac{1}{2nq - 1} \right) \right]
\]

\[
\geq \frac{1}{2} \lim_{n \to \infty} \ln \left( \frac{np}{nq} \right) = \frac{1}{2} \ln \left( \frac{p}{q} \right).
\]

So, since both limits converge to \( \frac{1}{2} \ln \left( \frac{p}{q} \right) \) as \( n \) tends to \( \infty \), then it is true that

\[
R_n = - \left( \frac{1}{2np + 1} + \frac{1}{2np + 3} + \cdots + \frac{1}{2nq - 1} \right)
\]

converges to \( \frac{1}{2} \ln \left( \frac{p}{q} \right) \) as \( n \to \infty \). So we can conclude that

\[
C_n(p, q) = S_{2nq} + - \left( \frac{1}{2np + 1} + \frac{1}{2np + 3} + \cdots + \frac{1}{2nq - 1} \right)
\]

converges to \( \ln(2) + \frac{1}{2} \ln \left( \frac{p}{q} \right) \) as \( n \to \infty \) (by properties of convergent sequences). \( \square \)
CHAPTER 3

RIEMANN’S THEOREM

3.1 Bernhard Riemann

Bernhard Riemann (17 September 1826 - 20 July 1866) was a German mathematician. In 1846, he attended the University of Göttingen to study theology as his father had encouraged him to do. Eventually, his father gave him permission to study mathematics under Moritz Stern and Carl Friedrich Gauss. In 1852, Riemann began working on the results of Dirichlet involving the Fourier series. Dirichlet found that certain types of series could be rearranged to a sum different from the sum of the original series. Later, Riemann discovered that this works for any conditionally convergent series. This result became known as Riemann’s Rearrangement Theorem in his Fourier series paper, “On the Representation of a Function by a Trigonometric Series,” which he completed in 1853. However, his paper was not published until after his death [13].

3.2 Riemann’s Rearrangement Theorem

Theorem 3.2.1 ([10]). A series $\sum f_n$ is conditionally convergent if and only if for each real number $\alpha$, there is a rearrangement of $\sum f_n$ that converges to $\alpha$.

Proof. ($\Leftarrow$) Note that this direction follows from Dirichlet’s Theorem 1.3.3, which states that an absolutely convergent series converges to the same value no matter how it is rear-
Suppose $\sum f_n$ is conditionally convergent. We want to show that there is a rearrangement, $(f_{\pi(n)})$, of $(f_n)$ whose series converges to the real number $\alpha$.

First, consider the subsequence of positive terms of $(f_n)$, call it $(a_n)$, and the subsequence of negative terms of $(f_n)$, call it $(b_n)$. Then, by Lemma 1.3.2, $\sum a_n = +\infty$ and $\sum b_n = -\infty$.

We know that $\sum a_n = \infty$. This implies that there exists some natural number $N$ such that

$$\sum_{k=1}^{N} a_k > \alpha.$$ 

Now let $N_1 = N$ be the least such number, and consider the partial sum $S_1 = \sum_{k=1}^{N_1} a_k$, so

$$S_1 = \sum_{k=1}^{N_1} a_k > \alpha,$$

but

$$\sum_{k=1}^{N_1-1} a_k \leq \alpha,$$

so that

$$0 < S_1 - \alpha \leq a_{N_1}.$$

Now to $S_1$, add just enough terms from $(b_n)$ (in order) so that the resulting partial sum
\[
\sum_{k=1}^{N_1} a_k + \sum_{i=1}^{M} b_i
\]
is now less than or equal to \(\alpha\). Note that this is possible since \(\sum b_i = -\infty\).

Letting \(M_1\) be the least such number \(M\), and setting

\[
S_2 = \sum_{k=1}^{N_1} a_k + \sum_{i=1}^{M_1} b_i,
\]
we get that

\[
0 \leq \alpha - S_2 < -b_{M_1}.
\]

Continuing this process, we get partial sums that alternate between being larger and smaller than \(\alpha\), and each time choosing the next smallest \(N_k\) or \(M_k\), we get the following rearrangement for \((f_n)\),

\[
a_1, a_2, \ldots, a_{N_1}, b_1, b_2, \ldots, b_{M_1}, a_{N_1+1}, \ldots, a_{N_2}, b_{M_1+1}, \ldots, b_{M_2}, a_{N_2+1}, \ldots
\]

Note that for all odd \(i\), we have that \(|S_i - \alpha| \leq a_{N_i}\), and for all even \(j\), we have that \(|S_j - \alpha| \leq -b_{M_j}\). Now, for any \(n\)

\[
S_{2n+1} > S_{2n+1} + b_{M_n+1} > S_{2n+1} + b_{M_n+1} + b_{M_n+2} > \cdots > S_{2n+2} - b_{M_n+1} > \alpha
\]

and

\[
S_{2n+2} < S_{2n+2} + a_{N_n+1+1} < S_{2n+2} + a_{N_n+1+2} < \cdots < S_{2n+3} - a_{N_{n+2}} \leq \alpha.
\]
So all the partial sums of the rearrangement \((f_{\pi(n)})\) between \(S_{2n+1}\) and \(S_{2n+2} - b_{M_{n+1}}\) are bounded between \(\alpha\) and \(\alpha + a_{N_{n+1}}\), and all partial sums between \(S_{2n+2}\) and \(S_{2n+3} - a_{N_{n+2}}\) are bounded between \(\alpha + b_{M_{n+1}}\) and \(\alpha\).

Since \(\sum f_n\) converges, notice that \((f_n)\) converges to 0. Therefore, \((a_N)\) and \((b_M)\) also converge to 0. Hence, the partial sums of \((f_{\pi(n)})\) converge to \(\alpha\); that is, \(\sum f_{\pi(n)} = \alpha\), as wanted. \(\square\)

### 3.3 Examples

#### Example 3.3.0.1

Consider the usual Alternating Harmonic Series

\[
\sum \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \cdots.
\]

We already know that this sum converges to \(\ln(2)\). However, let us show that this is true by applying Riemann’s Rearrangement Theorem. So first consider the series of positive terms of \(\sum \frac{(-1)^{n-1}}{n}\), call it \(\sum a_n\), and the series of negative terms of \(\sum \frac{(-1)^{n-1}}{n}\), call it \(\sum b_n\).

Now using Riemann’s Rearrangement Theorem, we get the partial sum \(S_1\) by adding just enough terms from \(\sum a_n\) so that \(S_1 > \ln(2) \approx .6931\):

\[
S_1 = 1 > \ln(2).
\]

To get \(S_2\), we add just enough terms from \(\sum b_n\) to \(S_1\) so that \(S_2 < \ln(2)\):
\[ S_2 = 1 - \frac{1}{2} = \frac{1}{2} < \ln(2). \]

To get \( S_3 \), we again add just enough terms from \( \sum a_n \) to \( S_2 \) so that \( S_3 > \ln(2) \):

\[ S_3 = 1 - \frac{1}{2} + \frac{1}{3} \approx .8333 > \ln(2), \]

and likewise, we add just enough terms from \( \sum b_n \) to \( S_3 \) to get \( S_4 \) so that \( S_4 < \ln(2) \):

\[ S_4 = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} \approx .5833 < \ln(2). \]

Continuing this process, we get the following partial sums:

\[
\begin{align*}
S_{10} &\approx .6456 < \ln(2) \\
S_{21} &\approx .7164 > \ln(2) \\
S_{30} &\approx .6768 < \ln(2) \\
S_{41} &\approx .7052 > \ln(2) \\
S_{50} &\approx .6832 < \ln(2) \\
S_{71} &\approx .7018 > \ln(2) \\
S_{80} &\approx .6886 < \ln(2) \\
S_{91} &\approx .7002 > \ln(2) \\
S_{100} &\approx .6898 < \ln(2) \\
S_{111} &\approx .6993 > \ln(2) \\
S_{120} &\approx .6906 < \ln(2) \\
S_{131} &\approx .6986 > \ln(2).
\end{align*}
\]
Each time we add just enough negative terms to get the new corresponding partial sum so that it is less than \( \ln(2) \), the sum increases and approaches the sum of the Alternating Harmonic Series. Likewise, each time we add just enough positive terms to get the new partial sum, the partial sums decrease and approach \( \ln(2) \). In other words, as \( k \) increases, for all \( k \in \mathbb{N} \), \( S_{2k} \to \ln(2)^- \) (from the left), and as \( k \) increases, \( S_{2k+1} \to \ln(2)^+ \) (from the right).

**Example 3.3.0.2.** Consider again the Alternating Harmonic Series. Similar to Example 3.3.0.1., we want to show that there exists a rearrangement that converges to \( \alpha = \frac{3}{2} \ln(2) \approx 1.0397 \). Applying the formula of Riemann’s Theorem we get the following partial sums:

\[
S_1 = 1 + \frac{1}{3} \approx 1.3333 > \frac{3}{2} \ln(2)
\]

\[
S_2 = 1 + \frac{1}{3} - \frac{1}{2} \approx 0.8333 < \frac{3}{2} \ln(2)
\]

\[
S_3 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} \approx 1.1762 > \frac{3}{2} \ln(2)
\]

\[
S_4 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} \approx 0.926190476 < \frac{3}{2} \ln(2)
\]

\[
S_5 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} \approx 1.2191 > \frac{3}{2} \ln(2).
\]

Recall that for every odd partial sum, we are adding just enough positive terms not already used to make the partial sum larger than \( \frac{3}{2} \ln(2) \), and for every even partial sum, we are adding just enough negative terms not already used so that the partial sum is less than \( \frac{3}{2} \ln(2) \). Continuing, we get the following partial sums:
\[ S_6 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} \approx 0.9615 < \frac{3}{2} \ln(2) \]

\[ S_7 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} \approx 1.1051 > \frac{3}{2} \ln(2) \]

\[ S_8 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} \approx 0.9801 < \frac{3}{2} \ln(2) \]

\[ S_9 = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \frac{1}{13} + \frac{1}{15} - \frac{1}{8} + \frac{1}{17} + \frac{1}{19} \approx 1.0916 > \frac{3}{2} \ln(2) \]

\[ S_{10} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} - \frac{1}{4} + \frac{1}{9} + \frac{1}{11} - \frac{1}{6} + \cdots + \frac{1}{17} + \frac{1}{19} - \frac{1}{10} \approx 0.9916 < \frac{3}{2} \ln(2) \]

\[ S_{35} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \cdots - \frac{1}{32} + \frac{1}{65} + \frac{1}{67} - \frac{1}{34} + \frac{1}{69} + \frac{1}{71} \approx 1.0538 > \frac{3}{2} \ln(2). \]

An obvious pattern is found: for every odd partial sum, we need to add the first two positive terms not already used from \( \sum a_n \) so that the partial sum is greater than \( \frac{3}{2} \ln(2) \), and for each even partial sum, the first negative term not already used from \( \sum b_n \) so that the partial sum is less than \( \frac{3}{2} \ln(2) \).

For clarity, we will show sums \( S_{35} \) to \( S_{40} \):

\[ S_{36} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \cdots - \frac{1}{32} + \frac{1}{65} + \frac{1}{67} - \frac{1}{34} + \frac{1}{69} + \frac{1}{71} - \frac{1}{36} \approx 1.02598 < \frac{3}{2} \ln(2) \]

\[ S_{37} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \cdots - \frac{1}{34} + \frac{1}{69} + \frac{1}{71} - \frac{1}{36} + \frac{1}{73} + \frac{1}{75} \approx 1.05301 > \frac{3}{2} \ln(2) \]

\[ S_{38} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \cdots - \frac{1}{34} + \frac{1}{69} + \frac{1}{71} - \frac{1}{36} + \frac{1}{73} + \frac{1}{75} - \frac{1}{38} \approx 1.02669 < \frac{3}{2} \ln(2) \]
\[ S_{39} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \cdots - \frac{1}{36} + \frac{1}{75} + \frac{1}{77} - \frac{1}{38} + \frac{1}{77} + \frac{1}{79} \approx 1.05234 > \frac{3}{2} \ln(2) \]

\[ S_{40} = 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \cdots - \frac{1}{36} + \frac{1}{75} + \frac{1}{77} + \frac{1}{79} - \frac{1}{40} \approx 1.02734 < \frac{3}{2} \ln(2). \]

Notice that if we continue this process, the partial sums eventually converge to \( \frac{3}{2} \ln(2). \)

Thus, \( S_2 \leq S_4 \leq S_6 \leq \cdots \leq S_7 \leq S_5 \leq S_3 \leq S_1. \)

So the rearrangement \( 1 + \frac{1}{3} - \frac{1}{2} + \frac{1}{5} + \frac{1}{7} + \cdots - \frac{1}{36} + \frac{1}{75} + \frac{1}{77} - \frac{1}{38} + \frac{1}{77} + \frac{1}{79} - \frac{1}{40} + \cdots \)

\[ \approx \frac{3}{2} \ln(2). \]

Recall Ohm’s Theorem in Chapter 2 and consider \( A(2,1). \) Using Ohm’s Theorem to look at the partial sums \( A(2,1), \) we get the following first five partial sums:

\[ C_1 = 1 + \frac{1}{3} + \left( -\frac{1}{2} \right) \approx 0.83 = S_2 < \frac{3}{2} \ln(2) \]

\[ C_2 = 1 + \frac{1}{3} + \left( -\frac{1}{2} \right) + [\frac{1}{5} + \frac{1}{7} + \left( -\frac{1}{4} \right)] \approx 0.93 = S_4 < \frac{3}{2} \ln(2) \]

\[ C_3 = 1 + \frac{1}{3} + \left( -\frac{1}{2} \right) + \frac{1}{5} + \frac{1}{7} + \left( -\frac{1}{4} \right) + [\frac{1}{9} + \frac{1}{17} + \left( -\frac{1}{6} \right)] \approx 0.96 = S_6 < \frac{3}{2} \ln(2) \]

\[ C_4 = 1 + \frac{1}{3} + \left( -\frac{1}{2} \right) + \frac{1}{5} + \frac{1}{7} + \left( -\frac{1}{4} \right) + \cdots + [\frac{1}{17} + \frac{1}{17} + \left( -\frac{1}{8} \right)] \approx 0.98 = S_8 < \frac{3}{2} \ln(2) \]

\[ C_5 = 1 + \frac{1}{3} + \left( -\frac{1}{2} \right) + \frac{1}{5} + \frac{1}{7} + \left( -\frac{1}{4} \right) + \cdots + [\frac{1}{17} + \frac{1}{17} + \left( -\frac{1}{10} \right)] \approx 0.99 = S_{10} < \frac{3}{2} \ln(2). \]

Observe that each partial sum \( C_n \) from Ohm’s Theorem is equal to each even partial sum \( S_{2n} \) from Riemann’s Rearrangement Theorem, and we know that by Ohm’s Theorem, \( C_n \) converges to \( \frac{3}{2} \ln(2). \)
CHAPTER 4

SCHLÖMILCH’S THEOREM

In Section 4.3, we provide a modern English-language translation of Schlömilch’s paper [22]. Since the basis of his argument comes from the Mean Value Theorem for Integrals, which he refers to as the “well-known” theorem, we will provide a few preliminaries in Section 4.2 by introducing and arguing the following three results: (1) Rolle’s Theorem, which will be used to prove (2) the Mean Value Theorem, then extend that to (3) the Mean Value Theorem for Integrals.

4.1 Oscar Schlömilch

Oscar Xavier Schlömilch (13 April 1823 - 7 February 1901) was a German mathematician. He studied mathematics and physics primarily in Jena, Berlin and Vienna. Most of his work was strongly influenced by Johann Peter Gustav Lejeune Dirichlet (1805-1859). In 1844, Schlömilch received his doctorate from Friedrich-Schiller-Universität. From 1851 to 1874, he was a professor at Dresden teaching Higher Mathematics and Analytic Mechanics [13].
4.2 Mean Value Theorem

Theorem 4.2.1 (Rolle’s Theorem [18]). Let \( f : [a, b] \to \mathbb{R} \) be a continuous function such that \( f \) is differentiable on \((a, b)\) where \( a < b \). If \( f(a) = f(b) \), then there exists some point \( x \in (a, b) \) where \( f'(x) = 0 \).

Proof. Let \( f(a) = f(b) \). Since \( f \) is continuous on \([a, b]\), then \( f \) attains a maximum at some point \( t \in [a, b] \) and a minimum at some point \( s \in [a, b] \).

Suppose first that \( s, t \) are both endpoints of \([a, b]\). Since \( f(a) = f(b) \), then the maximum and the minimum are equivalent, which means \( f \) is a constant function on \([a, b]\). In other words, \( f(x) = 0 \), then for each \( x \in (a, b) \), \( f'(x) = 0 \), in which case, we are done.

But now consider when (I) \( s \) is not an endpoint of \([a, b]\), or when (II) \( t \) is not an endpoint of \([a, b]\).

(I) If \( s \) is not an endpoint of \([a, b]\), then \( s \in (a, b) \), where \( f \) has a local maximum at \( s \), and therefore \( f'(s) = 0 \).

(II) If \( t \) is not an endpoint in \([a, b]\), then \( t \in (a, b) \) and \( f \) has a local minimum at \( t \), and therefore \( f'(t) = 0 \).

We have proved that in all cases for some point \( x \in (a, b) \), \( f'(x) = 0 \).

Theorem 4.2.2 (Mean Value Theorem [18]). Suppose \( f : [a, b] \to \mathbb{R} \) be a continuous function such that \( f \) is differentiable on \((a, b)\). Then, there is some point \( t \in (a, b) \) such that

\[
f(b) - f(a) = f'(t)(b - a).
\]
Proof. Let

\[ y(x) = \frac{f(b) - f(a)}{b - a} (x - a) + f(a). \]

Notice that this is the slope of the secant of the graph of \( f \) on \([a, b]\). Now let

\[ h(x) = f(x) - \frac{f(b) - f(a)}{b - a} (x - a) - f(a), \]

and note that

\[ h(a) = h(b) = 0 \]

and \( h \) is continuous on \([a, b]\) and differentiable on \((a, b)\). Applying Rolle’s Theorem, there is some \( t \in (a, b) \) such that \( h'(t) = 0 \). But since

\[ h'(t) = f'(t) - \frac{f(b) - f(a)}{b - a} = 0, \]

then

\[ f'(t) = \frac{f(b) - f(a)}{b - a}. \]

\[ \square \]

**Theorem 4.2.3** (Mean Value Theorem for Integrals [8]). If \( f \) is a continuous function on the closed interval \([a, b]\), then there exists some number \( c \) on \((a, b)\) so that

\[ \int_{a}^{b} f(x) \, dx = f(c)(b - a). \]
Proof. Define \( f(x) = f(a) \) for values of \( x < a \) and \( f(x) = f(b) \) for values of \( x > b \). Then, \( f \) has the integral of \( F \) by the Fundamental Theorem of Calculus (see (ii) from [8, pg. 193]). So there is some point \( c \) such that \( a < c < b \) where

\[
F'(c) = \frac{F(b) - F(a)}{b - a}.
\]

By the definition of \( F \), we know that \( F'(c) = f(c) \) and

\[
F(a) = \int_a^a f(x)\,dx = 0,
\]

or

\[
F(b) - F(a) = \int_a^b f(x)\,dx.
\]

Thus,

\[
f(c) = \frac{F(b)}{b - a} = \frac{1}{b - a} \int_a^b f(x)\,dx
\]

or

\[
f(c)(b - a) = \int_a^b f(x)\,dx.
\]

\( \square \)
4.3 On Conditionally Converging Series

[Note: This section is a modern English-language translation of Schlömilch’s original paper [22].]

Dr. Scheibner proved a proposition that a necessary and sufficient condition for absolute convergence of a series (i.e., the series converges to a number no matter how the terms are rearranged) is that the series of absolute values of its terms will also converge to that number.

[In the original paper, Scheibner uses the term “Moduli” to refer to the absolute values of complex numbers. He is stating that this property also holds for series of complex numbers. In particular, when a series of complex numbers is rearranged, it will converge to some limit, and likewise, its absolute value series will also converge to that limit.

Schlömilch points out from the proof of Scheibner’s proposition it is clear that the condition is not a simplification as long as the notion of absolute convergence is defined by arbitrarily reordering of its terms. But consider a special condition where you rearrange the terms only so that \( p \) positive terms are always followed by \( q \) negative terms. We will see that this requirement can, without changing the values, be met even if the absolute value series is not convergent.]

Let \( f(x) \) be a decreasing and positive function, such that the asymptotic value \( \lim_{x \to \infty} f(x) = 0 \) [Schlömilch uses \( f(\infty) = 0 \) to denote \( \lim_{x \to \infty} f(x) = 0 \)]. Then, we have that the following corresponding series converges:

\[
 f(0) - f(1) + f(2) - f(3) + \cdots ;
\]

[The series converges because of the Alternating Series Test (Theorem 1.2.5).]

Let \( f(n) = u_n \) [he uses \( f(x) = u_x \)], and let \( s \) be the sum of the original series; for \( n \to \infty \)
[he uses \( f(\infty) = 0 \),

\[
s_{2m} = u_0 - u_1 + u_2 - u_3 + \cdots + u_{2m-2} - u_{2m-1}.
\]

Let \( S \) denote the sum of the rearranged series defined by \( n = \infty \) [as \( n \) goes to \( \infty \)] for:

\[
S_{(p+q)n} = u_0 + u_2 + u_4 + \cdots + u_{2p-2} - u_1 - u_3 - \cdots - u_{2q-1} + u_{2p} + u_{2p+2} + u_{2p+4} + \cdots + u_{4p-2} - u_{2q+1} - u_{2p+3} - u_{2q+5} - \cdots - u_{4q-1} + u_{4p} + u_{4p+2} - u_{4q+4} + \cdots + u_{6p-2} - u_{4q+1} - u_{4q+3} - u_{4q+5} - \cdots - u_{6q-1} + u_{(2n-2)p} + u_{(2n-2)p+2} + \cdots + u_{2np-2} - u_{(2n-2)q+1} - u_{(2n-2)q+3} - \cdots - u_{2nq-1}.
\]

\([S_{(p+q)n}] \) is the sum of the rearranged series up to \((p+q)n\) terms.

For \( m = nq \), \( s_{2m} = s_{2nq} \) contains all negative terms occurring in \( S_{(p+q)n} \). So we have

\[
S_{(p+q)n} - s_{2qn} = u_{2nq} + u_{2nq+2} + u_{2nq+4} + \cdots + u_{2np-2}, \tag{4.1}
\]

[Note that Schlömilch is taking the difference between \( S_{(p+q)n} \) and \( s_{2qn} \) to compare the two so as to say something about the convergence of the rearranged series. Also note that Schlömilch knows \( s_{2qn} \) converges since it is an alternating series and we assume that it passes the Alternating Series Test.] where in the series \( n(p-q) \) is the number of terms of equation (4.1), assuming \( p > q \). Since \( f(x) \) is decreasing, we have the following inequality
\[
\frac{1}{h} \int_{a}^{a+rh} f(x) \, dx < f(a) + f(a+h) + f(a+2h) + \cdots + f(a+[r-1]h) < \frac{1}{h} \int_{a}^{a+rh} f(x) \, dx + f(a) - f(a+rh).
\]

[Above, he uses the Integral Test.]

If we set \( f(n) = u_n, a = 2nq, h = 2, \) and \( r = n(p-q), \) then the \( f(a) + f(a+h) + f(a+2h) + \cdots + f(a+[r-1]h), \) from above, is equal to the right side of equation (4.1). Thus, we obtain the following inequality [note that \( a + rh = 2nq + n(p-q)2 = 2np \)],

\[
\frac{1}{2} \int_{2nq}^{2np} f(x) \, dx < S_{(p+q)n} - s_{2qn} < \frac{1}{2} \int_{2nq}^{2np} f(x) \, dx + u_{2qn} - u_{2pn},
\]

using the substitution \( x = n\xi \) and \( dx = nd\xi, \)

\[
\frac{1}{2} \int_{2q}^{2p} \frac{n\xi f(n\xi)}{\xi} \, d\xi < S_{(p+q)n} - s_{2qn} < \frac{1}{2} \int_{2q}^{2p} \frac{n\xi f(n\xi)}{\xi} \, d\xi + u_{2qn} - u_{2pn}.
\]

Recall the Mean Value Theorem for Integrals [he refers to this theorem as the “known” theorem], when \( \varphi(\xi) \) is finite and continuous, \( \psi(\xi) \) finally becomes constant and positive for \( \xi = \alpha \) and \( \xi = \beta \) \([\alpha < \mu < \beta],\)

\[
\int_{\alpha}^{\beta} \varphi(\xi) \psi(\xi) \, d\xi = \varphi(\mu) \int_{\alpha}^{\beta} \varphi(\xi) \psi(\xi) \, d\xi, \quad \alpha < \mu < \beta;
\]

and one makes use of this equation [substituting the following values into the equation above] by applying this theorem with \( \varphi(\xi) = n\xi f(n\xi), \psi(\xi) = \frac{1}{\xi}, \alpha = 2q, \beta = 2p, \) so that we can arrive at the following inequality:

\[
\frac{1}{2} n\mu \cdot f(n\mu) \cdot \log\left(\frac{p}{q}\right) < S_{(p+q)n} - s_{2qn} < \frac{1}{2} n\mu \cdot f(n\mu) \cdot \log\left(\frac{p}{q}\right) + u_{2qn} - u_{2pn}.
\]
Now for $n \to \infty$, we have $u_{2qn} \to 0$ and $u_{2pn} \to 0$. Also, $n\mu \to \infty$ since $n\mu > n2q$. Then

$$\lim_{\omega \to \infty} f(\omega) = K.$$ 

[He uses $\lim (\omega f(\omega)) = K$.] So one obtains from the previous inequality

$$S - s = \frac{1}{2} K \cdot \log\left(\frac{p}{q}\right).$$

The same result follows when $p < q$, and we consider the difference is $s_{2qn} - S(p+q)n$, by proceeding analogously.

**Theorem 4.3.1** ([22]). Let $f(n) = u_n$ be a decreasing and positive function, such that the asymptotic value $\lim_{n \to \infty} f(n) = 0$. If $K = \lim_{\omega \to \infty} (\omega u_\omega)$ or $K = \lim_{n \to \infty} (nu_n)$, the following holds:

If in a series (see note below)

$$s = u_0 - u_1 + u_2 - u_3 + \cdots$$

the terms are rearranged such that always $p$ positive terms are followed by $q$ negative terms, then the sum of the rearranged series is

$$S = s + \frac{\lim (nu_n)}{2} \cdot \log\left(\frac{p}{q}\right).$$

[Note: He does not state it directly at this point, but the series that satisfies the theorem must be one of the form $s = \sum_{n=0}^{\infty} (-1)^n u_n$ such that $u_n > 0$ is strictly decreasing and $\lim u_n = 0$, and by the Alternating Series Test it must converge.]
Following amongst other things is the Mean Value Theorem for Integrals for the Alternating Harmonic Series.

If \( \lim (n u_n) = \infty \), as for example for

\[
s = \frac{1}{\sqrt{1}} - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \frac{1}{\sqrt{4}} + \cdots,
\]

notice that the new series is divergent.

If \( \lim (n u_n) = 0 \), then reordering as above does not change the sum. For an example of this, consider the series

\[
s = \frac{1}{2 \log 2} - \frac{1}{3 \log 3} + \frac{1}{4 \log 4} - \cdots,
\]

for which Scheibner’s Criterion does not give a result because the absolute values is divergent.
CHAPTER 5

PRINGSHEIM’S THEOREM

5.1 Alfred Pringsheim

Alfred Pringsheim (2 September 1850 - 25 June 1941) was a German mathematician who studied mathematics in Berlin, receiving his doctorate in 1872. In 1877, he began teaching at the University of Munich as a privatdozent. Two years after he married Hedwig Dohm, an actress from Berlin. The two of them had four sons and one daughter. In 1886, Pringsheim was promoted as an “extraordinary” professor at the Ludwig-Maximilians University of Munich. He continued to work there for the rest of his career starting at the Barvarian Academy of Sciences in 1898. By 1901, he became a full professor, then retired in 1922 [13].

5.2 Pringsheim’s Rearrangement

Let $\sum_{n=1}^{\infty} f_n$ be a rearrangement of the Alternating Harmonic Series. Let $p_k$ denote the number of positive terms in the sequence $(f_n)_{n=1}^{k}$, and let $q_k$ denote the number of negative terms in $(f_n)_{n=1}^{k}$.

Recall the following definition.

Definition 5.2.1. Let the number $\alpha$ be the asymptotic density of the positive terms in $\sum_{n=1}^{\infty} f_n$. That is,
\[ \alpha = \lim_{n \to \infty} \frac{p_n}{n}, \]

where \( p_n \) denotes the number of positive terms in the sequence \( (f_n)_{n=1}^k \).

Note that \( \alpha = \frac{1}{2} \) for the usual Alternating Harmonic Series.

The following is a generalization of Ohm’s Theorem.

**Theorem 5.2.2** ([2]). Let \( \sum f_n \) be a rearrangement of the AHS. The rearrangement \( \sum_{k=1}^{\infty} f_n \) converges to an extended real number if and only if \( \alpha \), the asymptotic density of positive terms in \( \sum_{k=1}^{\infty} f_n \) exists, and in that case

\[ \sum_{n=1}^{\infty} f_n = \ln(2) + \frac{1}{2} \ln \left( \frac{\alpha}{1 - \alpha} \right). \]

**Proof.** Consider \( \sum_{k=1}^{\infty} f_k \), and let \( q_k = k - p_k \) with

\[ \sum_{k=1}^{k} f_k = \sum_{m=1}^{p_k} \frac{1}{2m-1} - \sum_{m=1}^{q_k} \frac{1}{2m}. \]

Then, for each \( k \in \mathbb{N} \), let

\[ a_k = \left( \sum_{n=1}^{k} \frac{1}{n} \right) - \ln(k). \]

So \( (a_k)_{k=1}^{\infty} \) is a decreasing sequence such that

\[ \lim_{k \to \infty} (a_k)_{k=1}^{\infty} = \gamma \]
(known as Euler’s constant). Then,

\[ \sum_{m=1}^{q_k} \frac{1}{2n} = \frac{1}{2} \sum_{m=1}^{q_k} \frac{1}{m} = \frac{1}{2} \ln(q_k) + \frac{1}{2} a_{q_k}, \]

and

\[ \sum_{m=1}^{p_k} \frac{1}{2m-1} = \frac{1}{2} \sum_{m=1}^{p_k} \frac{1}{m} - \frac{1}{2} \sum_{m=1}^{p_k} \frac{1}{2m} = \ln(2p_k) + a_{2p_k} - \frac{1}{2} \ln(p_k) - \frac{1}{2} a_{p_k}. \]

Therefore,

\[ \lim_{k \to \infty} \sum_{n=1}^{k} f_n = \lim_{k \to \infty} \left[ \ln(2p_k) - \frac{1}{2} \ln(p_k) - \frac{1}{2} \ln(q_k) + a_{2p_k} - \frac{1}{2} a_{p_k} - \frac{1}{2} a_{q_k} \right] \]

\[ = \ln(2) + \lim_{k \to \infty} \frac{1}{2} \ln \left( \frac{p_k}{q_k} \right) + \gamma - \frac{1}{2} \gamma - \frac{1}{2} \gamma \]

\[ = \ln(2) + \frac{1}{2} \ln \left( \lim_{k \to \infty} \frac{p_k}{q_k} \right), \]

which is what we wanted. \( \square \)

We introduce another important generalization of Pringsheim’s results:

**Theorem 5.2.3** ([2]). Let \( f_1, f_2, f_3, \ldots, f_k \) be a sequence such that each \( f_i \in \mathbb{R} \) for \( i \in \mathbb{N} \). If

\[ |f_1| \geq |f_2| \geq |f_3| \geq \ldots, \]
\[
\lim_{k \to \infty} f_k = 0,
\]

and \(f_{2j-1} > 0 > f_{2j}\) for \(j \in \mathbb{N}\), then the following holds:

1. If \(\lim_{k \to \infty} k \cdot |f_k| = \infty\) and if \(S \in \mathbb{R}\), then there exists a rearrangement of \(\sum_{j=1}^{\infty} f_j\), call it \(\sum_{j=1}^{\infty} f_j^*\), such that \(\sum_{j=1}^{\infty} f_j^* = S\), and the asymptotic density of positive terms of the rearrangement is \(\frac{1}{2}\).

   whose \(\alpha = \frac{1}{2}\), and whose sum is \(S\).

2. If \(\lim_{k \to \infty} k \cdot f_k = 0\) and if \(\sum_{n=1}^{\infty} g_n\) is a rearrangement of \(\sum_{n=1}^{\infty} f_n\) for which the asymptotic density \(\alpha\) exists where \(0 < \alpha < 1\), then

\[
\sum_{n=1}^{\infty} g_n = \sum_{n=1}^{\infty} f_n.
\]

For the proof, refer to [15].
CHAPTER 6

SIERPIŃSKI'S THEOREM

In Section 6.2 we will provide a modern English-language translation of Sierpiński’s result. In Section 6.3 we show some examples that demonstrate his result.

For clarity, we provide our version of Sierpiński’s original theorem (Theorem 6.2.1):

**Theorem 6.0.4** ([21]). Let \((f_n)\) be conditionally convergent where \(U = \sum f_n\), and let \(V \neq U\) be a real number. If \(V > U\), there exists an explicitly described rearrangement \(\pi\) with the property that each positive term of \(f_n\) is left in place (if \(f_n > 0\), then \(\pi(n) = n\)) (as \(\pi\) is defined in Section 1.1) and \(\sum f_{\pi(n)} = V\). Similarly, if \(V < U\), there exists an explicitly describe rearrangement \(\pi\) with the property that each negative term of \(f_n\) is left in place (if \(f_n < 0\), then \(\pi(n) = n\)) and \(\sum f_{\pi(n)} = V\).

6.1 Wacław Sierpiński

Wacław Sierpiński (14 March 1882 - 21 October 1969) was a Polish mathematician who began his college career in 1899 at the University of Warsaw in the Department of Mathematics and Physics. In 1904, Sierpiński received the gold medal in a prize essay contest regarding Georgy Fedoseevich Voronoy’s contributions to number theory. In 1904, Sierpiński graduated from the University of Warsaw, and immediately began working for some time at an all-girls school in Warsaw teaching mathematics and physics. By 1908,
he received his doctorate and started working at the University of Lvov. Sierpiński became very interested in the study of set theory, and so in 1909 he taught the first lecture on pure set theory [13].

6.2 On a Property of a Non-Absolutely Convergent Series

[Note: This section is a modern English-language translation of Sierpiński’s paper [21]. His paper’s original references come from [16], [20], [19].]

Presented by S. Zaremba in the meeting on March 6, 1911.

After a well-known theorem of Riemann [16], the order of terms under any convergent series that is not absolutely convergent can always be modified so that the sum of the series is a value arbitrarily given in principle. It is possible to demonstrate his theorem by changing appropriately the relative frequency of positive and negative terms in the series given.

I have demonstrated, in a recent note [20], that we can even produce an arbitrary variation of the sum of a series that is not absolutely convergent with the aid of a change in the order of its terms which do not change the disposition of their signs.

We now show that it is enough to modify the order of the terms of a determined sign to experience a change, in advance, which gives the sum of a series that is not absolutely convergent.

**Theorem 6.2.1** ([21]). Let $U$ be the sum of a non-absolutely convergent series. For the series to have an arbitrary sum $V < U$, given in advance, it suffices to modify the order of the positive terms of the series leaving each negative term in its place. On the other hand, it suffices to modify the order of the negative terms, leaving each positive term in its place, for its sum to have a value $V > U$. 
Sierpiński’s theorem may be a bit unclear for the reader. He is defining to two properties that must hold for the rearrangement to occur, but later in his paper he will be defining the explicit rearrangement that is involved.

At the start of his paper, he simply wants to make two claims before he starts defining the process: The series is some conditionally convergent series, which we will denote as \( \sum u_i \), given that this series converges to \( U \) (for \( u_i \in \mathbb{R} \)). Then, Claim (i) states that when the positive terms are rearranged while the negative terms are left in place, the rearranged series of \( \sum +u_i \) converges to a sum \( V \leq U \), and Claim (ii) likewise, when the negative terms are rearranged while the positive terms are left in place, the rearranged series \( \sum -u_i \) converges to some sum \( V \geq U \).

It is sufficient to establish only the first part of the proposal; in fact,

\[
\sum u_i = u_1 + u_2 + u_3 + \cdots
\]

(6.1)

is a series given which is not absolutely convergent, \( U \) its sum, \( T > U \), applying the second part of the theorem to the series (6.1) and the number \( V = T \) is obviously equivalent to the application of the first part of this theorem to the series

\[
\sum -u_i = -u_2 - u_2 - u_3 - \cdots
\]

(6.2)

and the number \( V = -T \).

Let
be a given non-absolutely convergent series, $U$ its sum, $V$ a number $< U$, given in advance, and let

$$a_1 + a_2 + a_3 + \cdots$$ (6.3)

be the series of positive consecutive terms of the series (6.1).

Set

$$U - V = l \quad a_1 + a_2 + a_3 + \cdots + a_n = A_n \quad A_0 = 0;$$

we have:

$$l > 0, \quad \lim_{n \to \infty} a_n = \infty, \quad \lim_{n \to \infty} A_n = 0.$$  

Modifying the order of the terms in the series (6.3), we have a different series

$$c_1 + c_2 + c_3 + \cdots$$ (6.4)

To determine the law of formation of an explicit expression for the terms of (6.4), suppose we have already determined its $n \geq 0$ first terms and set

$$c_1 + c_2 + c_3 + \cdots + c_n = C_n \quad C_0 = 0.$$  

Let $n$ be a positive integer or zero: only one of three following cases may occur:

I. $A_n - C_n \leq l.$
\( \Pi^a. \, A_n - C_n > l \) and the term \( a_{n+1} \) is not in the sum \( C_n \).

\( \Pi^b. \, A_n - C_n > l \) and the term \( a_{n+1} \) is in the sum \( C_n \).

In case I, we choose the index \( r \) to be the smallest, for which \( a_r \) is not part of the sum \( C_n \), and such that

\[
a_r < \frac{1}{2^n} \quad \text{and} \quad a_r < \frac{a_{n+1}}{2}.
\]

Such an index \( r \) exists always since

\[
\lim_{n \to \infty} a_n = 0.
\]

We will let

\[
c_{n+1} = a_r.
\]

In case II\(^a\), we set

\[
c_{n+1} = a_{n+1}.
\]

Now consider case II\(^b\). By virtue of

\[
A_n - C_n > l > 0
\]

we have

\[
A_n > C_n,
\]
which shows that the sum $C_n$ cannot contain all the terms of the series $A_n$. Let $a_r$ be the first term of the sum $A_n$, which does not appear in the sum $C_n$. We set:

$$c_{n+1} = a_r.$$  

The conditions presented define perfectly the series (6.4), and clearly each term in the series (6.3) appears once more in the series (6.4), since we picked $c_{n+1}$ to be the term of (6.3) not in $C_n$. I say that every term in the series (6.3) is in the series (6.4).

Denote by $q_m$ the number that expresses how many indices $n \leq m$ for which we have case II$^b$. The sum $C_{m+1}$ will obviously contain all the terms of the sum $A_{q_m}$. If therefore we show that

$$\lim_{m \to \infty} q_m = \infty,$$

it will follow that (6.4) contains all the terms of the series (6.3).

In the case that the equation

$$\lim_{m \to \infty} q_m = \infty$$

is not satisfied, the number of indices $n$ for which we have the case II$^b$ is finite; $v$ being a fixed number, we have then for $n \geq v$ the case I or II$^a$. Suppose there exists an index $i \geq v$ for which we have the case II$^a$; we have $A_i > C_i$. The number of terms of the sums $A_i$ and $C_i$ being the same, the first may not contain all the terms of the second except in the case $A_i = C_i$; hence, the sum $C_i$ contains terms that do not fall within the sum $A_i$. These terms belonging to the series (6.3) (as all terms of the series (6.4)) but not included in the sum
$A_i$ have in the series (6.3) an index $> i$. The sum $C_i$ contains therefore some terms $a_n$ for which $n > i$. Let $j$ be the smallest index $> i$ for which $a_j$ is in the sum $C_i$. We can easily demonstrate that for the indices $i, i+1, i+2, \ldots, j-1$ we have case II$^a$, while the case II$^b$ will hold for the index $j$. However, this is inconsistent with the hypothesis that for $n \geq v$ we always have case I or case II$^a$. Now this hypothesis requires that we always have case I for $n \geq v$. This being admitted, we have for $n \geq v$ constantly

$$A_n - C_n \leq l \quad \text{and} \quad c_{n+1} < \frac{a_{n+1}}{2},$$

from which for all natural $x$:

$$l \geq A_{v+x} - C_{v+x} > A_v - C_v + \frac{a_{v+1} + a_{v+2} + \cdots + a_{v+x}}{2}.$$ 

Or:

$$a_{v+1} + a_{v+2} + \cdots + a_{v+x} < 2(l - A_v + C_v),$$

which is inconsistent with the divergence of the series (6.3).

We have therefore shown that

$$\lim_{n \to \infty} q_m = \infty.$$ 

Moreover we can consider also as demonstrated that the series (6.4) differs from the series (6.3) only by the order of its terms.

Now denote $p_m$ to be the number expressing how many indices $n \leq m$ there are for which case I is realized. I say that
\[
\lim_{m \to \infty} p_m = \infty.
\]

Suppose this proposal to be inaccurate; \(v\) being a fixed number, we then have case II for all \(n \geq v\). Denote by \(\tau_n\), the number of terms of the sum \(A_n\) not included in the sum \(C_n\). We obviously have

\[
\tau_{n+1} = \tau_n \quad \text{or} \quad \tau_{n+1} = \tau_n - 1,
\]

as case II\(^a\) or II\(^b\) will take place for the index \(n\). As we have shown above, case II\(^b\) is realized for infinitely many indices; so \(\tau_n\) would be negative for sufficiently large values of \(n\), which is clearly absurd. The equality

\[
\lim_{m \to \infty} p_m = \infty
\]

is thus established.

The sequence \(a_n\) tends to 0 and the sequence \(c_n\) differs from the sequence \(a_n\) only by the order of its terms; we therefore also have

\[
\lim_{n \to \infty} c_n = 0.
\]

Imagine a number \(h\) sufficiently large so that we have

\[
\frac{1}{2^{h-1}} < \varepsilon, \tag{6.5}
\]

\(\varepsilon\) a positive number given in advance. As we have

\[
\lim_{n \to \infty} a_n = 0 \quad \text{and} \quad \lim_{n \to \infty} c_n = 0.
\]
and as all the terms of the series (6.4) are at the same time the terms of the series (6.3), we set the number $\varepsilon$ to match a number $v$ such that the inequality $n > v$ implies the inequalities:

$$a_n < \varepsilon, \quad c_i < \varepsilon$$

(Sierpiński (2))

and that the sum $A_v$ contains all the terms of $C_h$.

On the other hand, as we have

$$\lim_{m \to \infty} p_m = \infty, \quad \lim_{m \to \infty} q_m = \infty$$

we can match to the number $v$ a number $\mu$ such as the inequality $m > \mu$ gives:

$$p_m > v \quad \text{and} \quad q_m > v.$$  

(Sierpiński (3))

Now let $m$ be an index $> \mu$. We examine separately the case

$$A_m - C_m \leq l$$

and the case

$$A_m - C_m > l.$$  

Suppose, in the first place,

$$A_m - C_m \leq l. \quad (6.6)$$

Denote by $k$ the largest index $k < m$ for which we have

$$A_k - C_k > l. \quad (6.7)$$
For $n = k + 1, k + 2, \ldots, m$, we evidently have the inequality

$$A_n - C_n \leq l,$$

which corresponds to case I; by the meaning of the symbols $q_k$ and $q_m$, we arrive at the immediate conclusion: $q_k = q_m$. On the other hand, obviously $q_k \leq k$. We have:

$$A_{k+1} - C_{k+1} = A_k - C_k - a_{k+1} - c_{k+1} > A_k - C_k - c_{k+1} > l - c_{k+1}$$

by (6.7); and by (3):

$$k \geq q_k = q_m > v,$$

finally, by (2):

$$c_{k+1} < \varepsilon$$

Consequently:

$$A_{k+1} - C_{k+1} > l - \varepsilon. \quad (6.8)$$

For $n = k + 1, k + 2, \ldots, m$, we have case I; consequently,

$$A_m - C_m - (A_{k+1} - C_{k+1}) > \frac{a_{k+2} + a_{k+3} + \cdots + a_m}{2} > 0,$$

whence:
\[ A_m - C_m > A_{k+1} - C_{k+1} > l - \varepsilon, \]

by (6.8). Following (6.6), we can write:

\[ -\varepsilon < A_m - C_m - l \leq 0. \]  

(6.9)

Now let

\[ A_m - C_m > l. \]  

(6.10)

Denote by \( k \) the largest index \( < m \) for which we have

\[ A_k - C_k \leq l. \]

For \( n = k + 1, k + 2, \ldots, m \), we have do not case I; therefore, \( p_k = p_m \); on the other hand, evidently \( p_k \leq k \). We have:

\[ A_{k+1} - C_{k+1} = A_k - C_k + a_{k+1} - c_{k+1} < A_k - C_k + a_{k+1} \leq l + a_{k+1}; \]

but by (2): \( a_{k+1} < \varepsilon \) since

\[ k \geq p_k = p_m > v. \]

We therefore have

\[ A_{k+1} - C_{k+1} \leq l + \varepsilon. \]  

(6.11)
Denote by \( f_i \) \((i = 1, 2, \cdots, s)\) the indices included in between \( k \) and \( m \) for which case II\(^b\) holds; for the other indices \( n \) between \( k \) and \( m \), we therefore have case II\(^a\). Now:

\[
 a_{n+1} - c_{n+1} = 0.
\]

It follows that

\[
 A_m - C_m - (A_{k+1} - C_{k+1}) = \sum_{i=1}^{s} (a_{f_i+1} - c_{f_i+1}) < \sum_{i=1}^{s} a_{f_i+1}. \quad (6.12)
\]

If for an index \( f \) we have case II\(^b\), the term \( a_{f+1} \) is included in the sum \( C_f \), then we have

\[
 a_{f+1} = c_{g+1} \quad \text{or} \quad g < f.
\]

I say that case I will occur for the index \( g \). Indeed, if we had case II for the index \( g \), it would set

\[
 c_{g+1} = a_{j+1} \quad \text{or} \quad j \leq g;
\]

however, we have

\[
 c_{g+1} = a_{f+1} \quad \text{with} \quad f > g.
\]

We therefore have for the index \( g \) case I; or

\[
 c_{g+1} < \frac{1}{2g}.
\]

For \( f = f_i \) \((i = 1, 2, \cdots, s)\), we obviously have \( f_i > k > v \) and as in the sum \( A_v \) are included all the terms of the sum \( C_h \), the term \( a_{f_i+1} \), having an index greater than \( v \), cannot be in the sum \( A_v \) nor \( C_h \). The index \( g_i + 1 \) of the term

\[
 c_{g_i+1} = a_{f_i+1}
\]
is therefore greater than \( h \) and it follows that: \( g_i \geq h \). We have case II\(^b\) for \( f_i \); consequently, as we have demonstrated above, we have

\[
c_{g_i+1} < \frac{1}{2g_i}.
\]

Thus, we have:

\[
\sum_{i=1}^{s} a_{g_i+1} = \sum_{i=1}^{s} c_{g_i+1} < \sum_{i=1}^{s} \frac{1}{2g_i}; \tag{6.13}
\]

\( g_i \ (i = 1, 2, \cdots, s) \) represents \( s \) different numbers, all \( \geq h \). From that we conclude,

\[
\sum_{i=1}^{s} \frac{1}{2g_i} < \sum_{j=1}^{\infty} \frac{1}{2^j} = \frac{1}{2^{h-1}} < \varepsilon, \tag{6.14}
\]

by (6.5).

We therefore have, by (6.12), (6.13), and (6.14): Hence, by (6.12), (6.13), and (6.14), we get that,

\[
A_m - C_m - (A_{k+1} - C_{k+1}) < \varepsilon;
\]

Thus, by (6.10) and (6.11):

\[
0 < A_m - C_m - l < 2\varepsilon. \tag{6.15}
\]

For all \( m > \mu \), we therefore have one or the other of the inequalities (6.9) or (6.15): so, for all \( m > \mu \), we have:
\[ |A_m - C_m - l| < 2\varepsilon, \]

from which it follows immediately that

\[ \lim_{m \to \infty} (A_m - C_m) = l. \quad (6.16) \]

Now that

\[-b_1 - b_2 - b_3 - \cdots\]

be the series of consecutive negative terms of the series (6.1), \(-B_n\) the sum of the first \(n\) terms.

We have for all natural numbers \(n\):

\[ U_n = A_{r_n} - B_{s_n}, \]

\(r_n\) and \(s_n\) being two non-decreasing sequences and such that

\[ \lim_{n \to \infty} r_n = \infty \quad \text{and} \quad \lim_{n \to \infty} s_n = \infty. \quad (6.17) \]

Now form a new series

\[ v_1 + v_2 + v_3 + \cdots \quad (6.18) \]

by replacing each positive term \(u_n = a_{r_n}\) of the series (6.1) by the term \(c_n = c_{r_n}\), and maintaining without any modification the negative terms of this series.
The series (6.18) differs from (6.1) only by the order of its positive terms. Upon designating by \( V_n \) to be the sum of the first \( n \) terms of the series (6.18), we evidently have:

\[
V_n = C_{r_n} - B_{s_n} = U_n - (A_{r_n} - C_{r_n}).
\]

By (6.16) and (6.17), we have:

\[
\lim_{n \to \infty} (A_{r_n} - C_{r_n}) = l,
\]

Therefore,

\[
\lim_{n \to \infty} V_n = \lim_{n \to \infty} U_n - \lim_{n \to \infty} (A_{r_n} - C_{r_n}) = U - l = V.
\]

Our theorem is thus demonstrated.

Note again that by suitably modifying the order of the terms of a single sign one could obtain a divergent series starting with a non-absolutely convergent series; this results almost immediately from the following theorem I demonstrated in a recent publication [19]:

“By changing the order of the terms of a divergent series whose terms are positive and tend to 0, we can always get a series which diverges more slowly than a divergent series, given in advance, whose terms are also positive.”

### 6.3 Examples

**Example 6.3.0.1.** Consider again the Alternating Harmonic Series
\[ \sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \frac{1}{5} - \frac{1}{6} + \frac{1}{7} - \frac{1}{8} + \cdots \]

whose sum is \( U = \ln(2) \). Suppose \( V = \frac{\ln(2)}{2} \). Then, \( I = U - V = \ln(2) - \ln(2) = \frac{\ln(2)}{2} \).

Now we have that \( a_1 = 1, a_2 = \frac{1}{3}, a_3 = \frac{1}{5}, a_4 = \frac{1}{7}, a_5 = \frac{1}{9}, a_6 = \frac{1}{11}, \ldots \).

Given that \( C_0 = 0, \) and \( A_0 = C_0, \) then \( A_0 - C_0 = 0 < \frac{\ln(2)}{2} \). So we are in case \( I \) and we choose index \( r \) to be the smallest, for which \( a_r \) is not part of the sum \( C_0 \), and such that \( a_r < \frac{1}{20} = 1 \) and \( a_r < \frac{a_1}{2} = \frac{1}{2} \). Set \( c_1 = a_r \). Since \( a_r = \frac{1}{3} \), then \( c_1 = \frac{1}{3} \).

Now consider \( A_1 = 1 \) and \( C_1 = \frac{1}{3} \). Then, \( A_1 - C_1 = \frac{2}{3} > \frac{\ln(2)}{2} \), and since \( a_2 = \frac{1}{3} \) is in the sum \( C_1 \), then we are in case \( II \). So let \( a_r \) be the first term of the sum \( A_n \), which is not in the sum \( C_1 \). We set \( a_r = c_2 \). Since \( a_r = 1 \), then \( c_2 = 1 \).

Now

\[ A_2 = 1 + \frac{1}{3} = \frac{4}{3} \]

and

\[ C_2 = \frac{1}{3} + 1 = \frac{4}{3}. \]

So

\[ A_2 - C_2 = \frac{4}{3} - \frac{4}{3} = 0 < \frac{\ln(2)}{2}. \]
So we are in case I. Thus, we choose \( r \) to the smallest for which \( a_r \) is not part of the sum \( C_2 \), and \( a_r < \frac{1}{2} = \frac{1}{4} \) and \( a_r < \frac{a_3}{2} = \frac{1}{10} \). Since 1 and \( \frac{1}{4} \) are in the sum \( C_2 \), then \( a_r = \frac{1}{11} \), which means \( c_3 = \frac{1}{11} \).

Then

\[
A_3 = 1 + \frac{1}{3} + \frac{1}{5} = \frac{4}{3} + \frac{1}{5} = \frac{23}{15}
\]

and

\[
C_3 = \frac{1}{3} + 1 + \frac{1}{11} = \frac{47}{33}.
\]

So

\[
A_3 - C_3 = \frac{23}{15} - \frac{47}{33} = \frac{6}{55}.
\]

so we are in case I, which means we will choose \( r \) to be the smallest for which \( a_r \) is not part of the sum \( C_n \), and such that \( a_r < \frac{1}{23} = \frac{1}{8} \), and \( a_r < \frac{a_4}{2} = \frac{1}{14} \). So let \( c_3 = a_r \). Since \( a_r = \frac{1}{15} \), then \( c_4 = \frac{1}{15} \).

Next, we have

\[
A_4 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7}.
\]

and
\[ C_4 = \frac{1}{3} + 1 + \frac{1}{11} + \frac{1}{15}. \]

Hence, we are in case I again. So choose \( r \) to be the smallest for which \( a_r \) is not part of the sum \( C_n \), and such that \( a_r < \frac{1}{16} \) and \( a_r < \frac{a_5}{2} = \frac{1}{18} \). Let \( c_5 = a_r \), which means \( c_5 = \frac{1}{19} \).

Note we are in case I until \( n = 7 \) (or \( C_7 \)). For the sake of time, we will skip the evaluation of \( c_6 \) and \( c_7 \), but do note that \( c_6 = \frac{1}{23} \) and \( c_7 = \frac{1}{65} \).

Thus,

\[ A_7 = 1 + \frac{1}{3} + \frac{1}{5} + \frac{1}{7} + \frac{1}{9} + \frac{1}{11} + \frac{1}{13}, \]

and

\[ C_7 = \frac{1}{3} + 1 + \frac{1}{11} + \frac{1}{15} + \frac{1}{19} + \frac{1}{23} + \frac{1}{65}. \]

Notice we have

\[ A_7 - C_7 > \frac{\ln(2)}{2}, \]

and that \( a_8 = \frac{1}{13} \) is in the sum \( C_7 \). So we are in case \( II^b \). Now let \( a_r \) be the first term of \( A_7 \) not in the sum \( C_7 \). Set \( c_8 = a_r \). Since \( a_r = \frac{1}{5} \), then \( c_8 = \frac{1}{5} \).

So

\[ A_8 - C_8 > \frac{\ln(2)}{2}. \]
and \(a_9 = \frac{1}{17}\) is not in the sum \(C_8\). So we are in case \(II^a\), thus since we let \(c_9 = a_9\), then 
\(c_9 = \frac{1}{17}\).

Now 
\[A_9 - C_9 < \frac{\ln(2)}{2},\]

which means we are in case \(I\). So we choose \(a_r\) to be the smallest, for which it is not in \(C_9\) and 
\(a_r < \frac{1}{2^r} = \frac{1}{312}\) and \(a_r < \frac{a_{10}}{2} = \frac{1}{38}\). Since \(a_r = \frac{1}{313}\), then 
\(c_{10} = \frac{1}{313}\).

We note that case \(I\) continues again for a while before it switches to another case. Nonetheless, the process for the formation of the rearrangement is the same.

Taking the \(c_1, c_2, c_3, c_4, \ldots\) terms we found and using them to replace the positive terms of the original series, respectively, we get the following new series:

\[
\frac{1}{3} - \frac{1}{2} + \frac{1}{4} - \frac{1}{6} + \frac{1}{11} - \frac{1}{8} + \frac{1}{19} - \frac{1}{10} + \cdots + \frac{1}{17} - \frac{1}{16} + \frac{1}{513} - \frac{1}{18} + \frac{1}{1025} - \frac{1}{20} + \cdots
\]

**Example 6.3.0.2.** Consider the series,

\[
\sum (-1)^n \frac{1}{\sqrt{n}} = \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{7}} + \cdots,
\]

where \(a_1 = \frac{1}{\sqrt{2}}, a_2 = \frac{1}{\sqrt{4}}, a_3 = \frac{1}{\sqrt{6}}, a_4 = \frac{1}{\sqrt{8}}, \ldots\).

Consider again that \(V = \frac{\ln(2)}{2}\). Given that \(C_0 = 0\) and \(A_0 = 0\), then
\[ A_0 - C_0 < \frac{\ln(2)}{2}. \]

So we are in case I. Let us choose \( r \) to be the smallest index, for which \( a_r \) is not part of the sum \( C_0 \), and \( a_r < \frac{1}{2^0} = 1 \) and \( a_r < \frac{a_1}{2} = \frac{1}{2\sqrt{2}} \). Since \( a_r = \frac{1}{\sqrt{10}} \), then \( c_1 = \frac{1}{\sqrt{10}} \).

Since \( A_1 = \frac{1}{\sqrt{2}} \) and \( C_1 = \frac{1}{\sqrt{10}} \), then \( A_1 - C_1 > \frac{\ln(2)}{2} \), and because \( a_2 = \frac{1}{\sqrt{4}} \) is not in \( C_1 \), then we are in case \( II^a \). Let \( c_2 = a_2 \). Since \( a_2 = \frac{1}{\sqrt{4}} \), then \( c_2 = \frac{1}{\sqrt{4}} \).

Next, we have that

\[ A_2 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} \]

and

\[ C_2 = \frac{1}{\sqrt{10}} + \frac{1}{\sqrt{4}}. \]

Then, we get that

\[ A_2 - C_2 > \frac{\ln(2)}{2}, \]

and \( a_3 = \frac{1}{\sqrt{6}} \) is not in \( C_2 \). Hence, we are in case \( II^a \) again. Since \( a_3 = \frac{1}{\sqrt{6}} \), then \( c_3 = \frac{1}{\sqrt{6}} \).

Then

\[ A_3 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{6}} \]

and
\[ C_3 = \frac{1}{\sqrt{10}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{6}}. \]

Since

\[ A_3 - C_3 > \frac{\ln(2)}{2}, \]

and \( a_4 = \frac{1}{\sqrt{8}} \) is not in \( C_3 \), then we are in case \( II^a \). So we have that \( a_4 = c_4 = \frac{1}{\sqrt{8}} \).

Now notice that

\[ A_4 - C_4 > \frac{\ln(2)}{2} \]

and \( a_5 = \frac{1}{\sqrt{10}} \), which is in \( C_4 \). Thus, we are in case \( II^b \). So we choose \( a_r \) to be the first term in \( A_n \) that is not in \( C_4 \). Let \( a_r = c_5 \). Since \( a_r = \frac{1}{\sqrt{8}} \), then \( c_5 = \frac{1}{\sqrt{8}} \).

Since

\[ A_5 = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{10}} \]

and

\[ C_5 = \frac{1}{\sqrt{10}} + \frac{1}{\sqrt{4}} + \frac{1}{\sqrt{6}} + \frac{1}{\sqrt{8}} + \frac{1}{\sqrt{2}}, \]

then it is easy to see that

\[ A_5 - C_5 = 0 < \frac{\ln(2)}{2}. \]
So we are in case $I$. We pick the index $r$ to be the smallest for which $a_r$ is not in $C_5$, and $a_r < \frac{1}{2^5} = \frac{1}{32}$ and $a_r < \frac{a_6}{2} = \frac{1}{2\sqrt{12}}$. Let $c_6 = a_r$. Since $a_r = \frac{1}{\sqrt{1020}}$, then $c_6 = \frac{1}{\sqrt{1020}}$.

Note that we stay in case $I$ for a while. Continuing, the process for the formation of the new rearranged series is the same. Then, taking the $c_1, c_2, c_3, c_4, \ldots$ terms and replacing them with all the positive terms of the original series, we get the following rearrangement:

\[
\frac{1}{\sqrt{10}} - \frac{1}{\sqrt{3}} + \frac{1}{\sqrt{4}} - \frac{1}{\sqrt{5}} + \frac{1}{\sqrt{6}} - \frac{1}{\sqrt{7}} + \frac{1}{\sqrt{8}} - \frac{1}{\sqrt{9}} + \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{11}} + \frac{1}{\sqrt{1020}} - \cdots = \ln 2.
\]
CHAPTER 7

ADDITIONAL RESULTS

In this chapter, we introduce some modern results complementing the classical theory we have presented throughout this thesis.

7.1 Lévy and Steinitz

We know that by Riemann’s Theorem a conditionally convergent series of real numbers can be rearranged to sum to any \( \alpha \in \mathbb{R} \).

In [17], Rosenthal formulated Riemann’s result another way:

*The set of all sums of rearrangements of a given series of real numbers is either empty, a single point (if the series is absolutely convergent), or spans the entirety of \( \mathbb{R} \).*

We can extend this idea to complex numbers:

**Theorem 7.1.1** ([17]). *The set of all sums of rearrangements of a given series of complex numbers is the empty set, a single point, a line in the complex plane, or the whole complex plane.*

More generally, the Levy-Steinitz Theorem gives a similar result in \( n \) dimensions.
Theorem 7.1.2 (Lévy-Steinitz Theorem [17]). Let $\sum T$ be a given series in $\mathbb{R}^n$. Let $R$ be the set of all sums of rearrangements of $\sum T$ in $\mathbb{R}^n$. Then, $R$ is either $\emptyset$ or a translate of a subspace (that is, $R = v + M$ for some vector $v$ and some linear subspace $M$).

For a proof, refer to [6, pg. 54-61].

We proceed with a few examples.

Example 7.1.2.1. Consider the following in $\mathbb{R}^2$: $\sum \left( \frac{(-1)^{n-1}}{n}, 0 \right)$, in which we are considering the sums of all the rearrangements of this particular series of vectors. Notice that the every $x$-coordinate is a sum from a rearrangement of the alternating harmonic series, which we know to be conditionally convergent.

By Riemann’s Theorem, since we can get rearrangements that converge to any real number, then the set of all the $x$-coordinates in this series of vectors is $\mathbb{R}$. Since the $y$-coordinate is always 0, then it is easy to see that the set of $\sum \left( \frac{(-1)^{n-1}}{n}, 0 \right)$ spans all of $\mathbb{R}$ along the $x$-axis for every $y = 0$. In other words, $\sum \left( \frac{(-1)^{n-1}}{n}, 0 \right)$ in $\mathbb{R}^2$ is the line $y = 0$, which is an affine space$^1$.

Example 7.1.2.2. Now consider the following series in $\mathbb{R}^2$: $\sum \left( \frac{(-1)^{n-1}}{n}, \frac{1}{2^n} \right)$.

Again, we see that every $x$-coordinate is a sum from a rearrangement of the AHS. From the previous example, we have already verified that the set of all the $x$-coordinates in this series of vectors is the set $\mathbb{R}$. Notice that the $y$-coordinates come from $\sum \frac{1}{2^n}$, where

$$\sum \frac{1}{2^n} = \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1.$$  

$^1$An Affine Space is a translation of a subspace. For example, in $\mathbb{R}^2$, a line in a one-dimensional subspace is the line $y = mx$ or the vertical line $x = 0$, while a one-dimensional affine space is the line $y = mx + b$ or the line $x = b$. 


Also note that
\[
\sum |\frac{1}{2^n}| = |\frac{1}{2}| + |\frac{1}{4}| + |\frac{1}{8}| + |\frac{1}{16}| + \cdots \\
= \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \cdots = 1
\]

By definition, \( \sum \frac{1}{2^n} \) is absolutely convergent. It follows from Dirichlet’s theorem that every rearrangement of \( \sum \frac{1}{2^n} \) converges to 1. Thus, in \( \mathbb{R}^2 \), \( \sum \left( \frac{(-1)^{n-1}}{n}, \frac{1}{2^n} \right) \) is the line \( y = 1 \), which is an affine space.

**Example 7.1.2.3.** Last, take the example \( \sum \left( \frac{(-1)^{n-1}}{n}, \frac{1}{2^n} \right) \). We already know that the set of all the x-coordinates in this series of vectors spans \( \mathbb{R} \). So now consider the y-coordinates, \( \sum \left( \frac{1}{2^n} - \frac{(-1)^{n-1}}{n} \right) \). Since we know that the sum of every rearrangement of \( \frac{1}{2^n} \) is always 1, then the difference \( \frac{1}{2^n} - \frac{(-1)^{n-1}}{n} \) in the set of sums of all rearrangements produces the set \( \mathbb{R} \). In other words, the y-coordinates spans \( \mathbb{R} \), just as the x-coordinates do.

Important to note is that for a chosen rearrangement yielding some particular sum in the x-coordinate, that same sum is used in the difference of \( \frac{1}{2^n} - \frac{(-1)^{n-1}}{n} \) of the y-coordinate. In other words, say we let \( V_n = \left( \frac{(-1)^{n-1}}{n}, \frac{1}{2^n} \right), A_n = \frac{(-1)^{n-1}}{n} \), and \( B_n = \frac{1}{2^n} \).

Then, for a permutation \( \pi : \mathbb{N} \rightarrow \mathbb{N} \),

\[
\sum V_{\pi(n)} = \sum \left( \frac{(-1)^{n-1}}{n}, \frac{1}{2^n} - \frac{(-1)^{n-1}}{n} \right) \\
= \left( \sum A_{\pi(n)}, \sum B_{\pi(n)} - \sum A_{\pi(n)} \right).
\]

Now let \( \sum A_{\pi(n)} = S \).

Then, we have \((S, 1 - S)\). So \( x + y = 1 \), which means that \( \sum \left( \frac{(-1)^{n-1}}{n}, \frac{1}{2^n} - \frac{(-1)^{n-1}}{n} \right) \).
Lévy and Steinitz also came up with the following rearrangement theorem in terms of vectors, which serves as an important part in proving the Lévy-Steinitz Theorem.

**Theorem 7.1.3** (The Rearrangement Theorem [17]). In \( \mathbb{R}^n \), if a subsequence of the sequence of partial sums of a series of vectors converges to \( S \), and if the sequence of terms of the series converges to 0, then there is a rearrangement of the series that sums to \( S \).

For a proof of this, refer to [17, pg. 346].

### 7.2 Extensions of Pringsheim’s Results

In this section, we will introduce some additional generalization, due to Scheepers [23], of Pringsheim’s rearrangement results [15].

First recall the definition of asymptotic density from Chapter 5.

**Definition 7.2.1.** A number \( \alpha \) is the asymptotic density of the positive terms in \( \sum_{n=1}^{\infty} f_n \), provided that

\[
\alpha = \lim_{n \to \infty} \frac{P_n}{n}
\]

where \( P_n \) denotes the number of positive terms in the sequence \( (f_n)_{n=1}^{k} \).

Let us provide an example to make the concept of asymptotic density more clear to the reader.
Example 7.2.1.1. Consider the usual Alternating Harmonic Series. By [2],

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n},$$

has $\alpha = \frac{1}{2}$. Again, consider the rearrangement $A(2, 1)$. We will illustrate that $A(2, 1)$ has $\alpha = \frac{2}{3}$ by showing that quotients $\frac{p_{3n}}{3n}$ for $n \leq 5$ actually equals $\frac{2}{3}$:

$$C_1 = 1 + \frac{1}{3} + (-\frac{1}{2}) \implies \frac{p_3}{3} = \frac{2}{3},$$

$$C_2 = 1 + \frac{1}{3} + (-\frac{1}{2}) + \frac{1}{5} + \frac{1}{7} + (-\frac{1}{4}) \implies \frac{p_6}{6} = \frac{4}{6} = \frac{2}{3},$$

$$C_3 = 1 + \frac{1}{3} + (-\frac{1}{2}) + \frac{1}{5} + \frac{1}{7} + (-\frac{1}{4}) + \frac{1}{11} + (\frac{1}{6}) \implies \frac{p_9}{9} = \frac{6}{9} = \frac{2}{3},$$

$$C_4 = 1 + \frac{1}{3} + (-\frac{1}{2}) + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + (-\frac{1}{4}) + \cdots + \frac{1}{13} + \frac{1}{15} + (\frac{1}{8}) \implies \frac{p_{12}}{12} = \frac{8}{12} = \frac{2}{3},$$

$$C_5 = 1 + \frac{1}{3} + (-\frac{1}{2}) + \frac{1}{5} + \frac{1}{7} + \frac{1}{11} + \frac{1}{13} + \frac{1}{15} + (\frac{1}{8}) + \cdots + \frac{1}{17} + \frac{1}{19} + (-\frac{1}{10}) \implies \frac{p_{15}}{15} = \frac{10}{15} = \frac{2}{3}.$$

Next let us bring forth some notation from [23] [Note that some of our notation may be slightly different]:

Let $\sum f$ be a conditionally convergent series if it is convergent. Then, both $\{n : f(n) > 0\}$ and $\{n : f(n) < 0\}$ are infinite sets.

Let $a_1, a_2, a_3, \cdots$ be positive terms of $f$ in the order that they occur, and let $-b_1, -b_2, -b_3, \cdots$ be the negative terms of $f$ in the order that they occur.

Now suppose $f$ to be signwise monotonic. Suppose $S$ to be an infinite subset of $\mathbb{N}$ where $T = \mathbb{N} \setminus S$ is an infinite set. Let $f_S$ be such that
\[ f_S(n) = \begin{cases} 
  \text{\(j\)th positive term of } f \text{ if } n \text{ is the } j\text{th element of } S, \\
  \text{\(j\)th negative term of } f \text{ if } n \text{ is the } j\text{th element of } T. 
\end{cases} \]

So \( f_S \) is a Riemann (Chapter 3) rearrangement of \( f \). For any set \( X \), let \(| X |\) denote the size of \( X \). Then, for some subset \( S \) of \( \mathbb{N} \), let \( \pi_S(n) = | S \cap \{1, 2, 3, \ldots, n\} | \) be the pre-density of \( S \), let \( d_+(S) = \limsup \frac{\pi_S(n)}{n} \) be the upper density of \( S \), and let \( d_-(S) = \liminf \frac{\pi_S(n)}{n} \) be the lower density of \( S \).

Note that \( 0 \leq d_-(S) \leq d_+(S) \leq 1 \) [23], and if the equality of \( d_+(S) \) and \( d_-(S) \) exists, then we can let \( d(S) = d_+(S) = d_-(S) \) be the asymptotic density of \( S \). Now suppose that \( S \) is a subset of the natural numbers and for any \( \sum f_S \) is a conditionally convergent series. Let \( R \) be a subset of the natural numbers such that for any finitely many \( n \), we have \( \pi_S(n) < \pi_R(n) \).

So for each \( n \), pick some \( k_n \) small enough so that

\[ n - \pi_R(n) = k_n - \pi_S(k_n). \]

Then, for each \( n \) we have

\[ \sum_{m \leq n} f_R(m) - \sum_{i \leq k_n} f_S(i) = a_{\pi_S(k_n)+1} + \cdots + a_{\pi_R(n)}. \]

Note that \( \sum f_S \) and \( (a_{\pi_S(k_n)+1} + \cdots + a_{\pi_R(n)}) \), \( n \in \mathbb{N} \), determines whether \( \sum f_R \) converges, and if the convergence exists, \( \sum f_S \) and \( (a_{\pi_S(k_n)+1} + \cdots + a_{\pi_R(n)}) \) also determines the limit of \( \sum f_R \). Also note that for each \( n \), the \( n^{th} \) term of \( (a_{\pi_S(k_n)+1} + \cdots + a_{\pi_R(n)}) \) is bounded from below by

\[ (\pi_R(n) - \pi_S(k_n) - 1) \cdot a_{\pi_R(n)}, \]
and bounded from above by

\[(\pi_R(n) - \pi_S(k_n) - 1) \cdot a_{\pi_S(k_n)}.\]

Next, the following are results (taken from [23]) of Pringsheim, which are extensions of Schlömilch’s results (Chapter 4) and Ohm’s theorem (Chapter 2).

Let \(S \subset \mathbb{N}\). Define

\[\omega_f = \{x \in (0,1) : (\exists S \subset \mathbb{N}) (d(S) = x \text{ and } \sum f_S(n) \text{ converges})\} \text{.}\]

**Lemma 7.2.2** ([23]). Let \(f\) be signwise monotonic. If \(|\omega_f| > 1\), then for all \(A, B \subset \mathbb{N}\) where \(d(A) = d(B)\) and \(\sum f_A\) converges, \(\sum f_B\) converges and

\[\sum f_A = \sum f_B.\]

**Theorem 7.2.3** ([23]). Let \(f\) be signwise monotonic, converging to 0. Let \(0 < x < 1\) be given. Then, the following are equivalent:

1. \(x \in \omega_f\) (asymptotic density) and \(\lim n \cdot a_n = \infty\).
2. For each set \(B\), such that \(\sum f_B\) converges, \(d(B) = x\).

**Statement 7.2.3.1** (Pringsheim 1 [23]). Let \((n \cdot a_n), n \in \mathbb{N}\), diverge to \(\infty\). Then \(\sum f_R\) converges if and only if \(a_{\pi_S(k_n)} \cdot (\pi_R(n) - \pi_S(k_n))\) converges. Also, if

\[\lim (a_{\pi_S(k_n)} \cdot (\pi_R(n) - \pi_S(k_n))) = a,\]

then

\[\sum_{n<\infty} f_R(n) = \sum_{n<\infty} f_S(N) + a.\]

**Note:** This statement is false. A counterexample is described in [23, pg. 422].
Theorem 7.2.4 (Pringsheim II [23]). Suppose \( \lim_{n \to \infty} n \cdot a_n = 0 \). If \( (a_{\pi_S(k_n)} \cdot (\pi_R(n) - \pi_S(k_n))) \) is bounded, then

\[
\sum_{n \to \infty} f_R = \sum_{n \to \infty} f_S.
\]

Theorem 7.2.5 (Pringsheim-Schlömilch [23]). Suppose \( (n \cdot a_n) \to t \) such that \( t \in \mathbb{R} \setminus \{0\} \). Then, \( \sum f_R \) converges if and only if \( (a_{\pi_S(k_n)} \cdot (\pi_R(n) - \pi_S(k_n))) \) converges. Also, if

\[
\lim_{n \to \infty} a_{\pi_S(k_n)} \cdot (\pi_R(n) - \pi_S(k_n)) = a,
\]

then

\[
\sum_{n \to \infty} f_R = \sum_{n \to \infty} f_S + t \cdot \ln(1 + \frac{a}{t}).
\]

In summary, mathematicians found: (1) the convergence criteria of \( \sum f_B \) when \( \lim n \cdot a_n = \infty \), (2) the convergence criteria of \( \sum f_B \) when \( \lim n \cdot a_n = 0 \), and (3) the change in value of \( \sum f_B \) for all \( B \) with \( 0 < d(B) < 1 \) when \( \lim n \cdot a_n = t \neq 0 \) [23].

7.3 Władysław Wilczyński

Recall that Riemann’s Rearrangement Theorem states that for each conditionally convergent series \( \sum f_n \) (of real numbers) and every real number \( \alpha \), there is a permutation \( \pi : \mathbb{N} \to \mathbb{N} \), such that \( \sum f_{\pi(n)} = \alpha \). A problem addressed in [4, pg. 64] is whether or not one can extend this result of Riemann by always taking a permutation \( \pi \) so that it only changes a small set of terms in \( \sum f_n \).

Let us begin by introducing some notation from [4].
Let $|X|$ be the cardinality of a set $X$. Let $(a_n)_{n\in\omega}$ be a given sequence and define $a_n^+ = \max\{a_n, 0\}$ and $a_n^- = \min\{a_n, 0\}$. Consider the series $\sum_{n\in\omega} a_n$ and $A \subset \omega$, by $\sum_{n\in A} a_n$ we denote the series $\sum_{n\in\omega} \chi_A(n) \cdot a_n$.

An ideal on $\omega$ is a family $I \subset P(\omega)$ ($P(\omega)$ being the power set of $\omega$) such that it is closed under taking subsets of finite unions. Assume that all considered ideals are proper (unless stated otherwise) and contain all finite sets. We are able to talk about ideals on any countable set by identifying this set with $\omega$ in terms of a fixed 1–1 and onto occurrence.

The ideal of all finite sets of natural numbers is denoted Fin. An ideal $I$ is dense if each $A \notin I$ has an infinite subset that belongs to the ideal.

The notion of smallness of the set of terms in a series can be considered in terms of ideals of subsets of $\mathbb{N}$, denoted $\mathcal{I}$. Then, for $I \subset \mathcal{P}(\mathbb{N})$, where $\mathcal{P}(\mathbb{N})$ denotes the powerset of $\mathbb{N}$, there is a permutation $\pi : \mathbb{N} \to \mathbb{N}$ that changes a small set of terms of $\mathbb{N}$ when $\{n \in \mathbb{N} : \pi(n) \neq n\} \in I$.

**Definition 7.3.1 ([3]).** An ideal of the form

$$I_f = \{A : \sum_{n\in A} f(n) < \infty\}$$

is a summable ideal, where $f$ is some positive function, such that $\sum_n f(n) = \infty$.

Now we proceed to introduce two properties introduced in [4].

**$(R)$ Property:** $I \subset \mathcal{P}(\mathbb{N})$ has the $(R)$ property if for each conditionally convergent series $\sum_n f_n$ and $r \in \mathbb{R}$, there is a permutation $\pi_r : \mathbb{N} \to \mathbb{N}$ where $\sum_n f_{\pi_r(n)} = r$ and $\{n \in \mathbb{N} : \pi_r(n) \neq n\} \in I$. 
(W) **Property:** \( \mathcal{I} \subset \mathcal{P}(\mathbb{N}) \) has the (W) property if for each conditionally convergent series \( \sum_n f_n \) there exists an \( A \in \mathcal{I} \) where \( \sum_{n \in A} f_n \) is conditionally convergent.

Now we introduce a couple of applicable theorems by Wilczyński.

**Theorem 7.3.2** ([4]). Let \( \mathcal{I} \) be an ideal in \( \mathbb{N} \). Then, \( \mathcal{I} \) has the (R) property if and only if \( \mathcal{I} \) cannot be extended to a summable ideal if and only if \( \mathcal{I} \) has the (W) property.

**Theorem 7.3.3** ([24]). If \( \sum_{n=1}^{\infty} f_n \) in a conditionally convergent series, then there is a set \( A \subset \mathbb{N} \), where the asymptotic density of \( A \) is equal to zero and \( \sum_{n=1}^{\infty} \chi_A(n) \cdot f_n \) is conditionally convergent.

A detailed proof of Theorem 7.3.2 and Theorem 7.3.3 can be found in [24, pg. 80].

Thus, there are many results regarding rearrangements of series.
REFERENCES


<http://www-groups.dcs.st-and.ac.uk/history/Biographies/Dirichlet.html’>.


