CONCEPTUAL UNDERSTANDING OF FRACTIONS AND DECIMALS FOR FOURTH GRADE STUDENTS

by

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DEDICATION

I would like to dedicate this study to:

My Dad and Mom- who believed I could, and my children Christian, Reese and Sydney, for understanding ‘short term sacrifice equals long term gain’

To Sam- for repeated readings, math discussions, and diagrams on napkins in our favorite places, and who inspires determination and dedication

To the staff at Chandlers Steak House- the best place to write in the city…any city.

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ABSTRACT

A study was conducted in two 4th grade classrooms based on current research of foundational concepts of fraction and decimal knowledge, socio-cultural learning theory, cognition and international mathematics education. The goal of this study was for students to acquire conceptual and procedural knowledge of fraction and decimal concepts. When students have multiple experiences delving into rigorous tasks with fractions and decimals, researchers (Lamon, 2006; Siegler & Alibali 2005) suggest students will show an increase in understanding. Cognition and developmental stages were examined and incorporated within the suggested tasks of the instructional unit. With assistance from current research, this study demonstrated students showed significant gains from pre to post assessment. This study provided information that determined students acquired a stronger foundation and a deeper understanding of decimals and fractions, preparing them for middle and high school mathematics.

TABLE OF CONTENTS

DEDICATION iv

ACKNOWLEDGEMENTS v

ABSTRACT vi

LIST OF TABLES x

LIST OF FIGURES xi

LIST OF ABBREVIATIONS xii

CHAPTER ONE: BACKGROUND AND PURPOSE OF THE STUDY 1

Introduction: Background and Focus of the Study 1

Purpose of the Study 4

Goals for the Study 6

Conceptual Understanding vs. Procedural Knowledge 6

Research Question: 12

CHAPTER TWO: LITERATURE REVIEW 13

Literature Review 13

Foundational Concepts 14

Unit and Unitizing 14

Partitioning and Iterating 15

Equivalence 18

Rival Explanation 19

Conceptual Understanding vs. Procedural Knowledge 20
Role of Cognitive Theories 23
Shifting Instructional Practices 28
Assessment Results 29
Trends in International Research 31
CHAPTER THREE: DESIGN OF THE STUDY 35
Research Design 35
Threats to Internal Validity 36
Threats to Generalizability 36
Trustworthiness of Results 37
Participants and Context 37
Participant Characteristics 38
Procedure 38
CHAPTER 4: RESULTS AND METHODS OF ANALYSIS 40
From Pre to Post Assessment 40
Item Analysis 41
Discussion 43
Trends in Assessment 43
CHAPTER 5: RESULTS, RECOMMENDATIONS, AND CONCLUSION 45
Summary 50
Contributing Factors and Limitations 51
CONCLUSION 53
REFERENCES 55

APPENDIX A 63

Suggested Sequence of Tasks 63

APPENDIX B 101

Pre/Post Test 101

LIST OF TABLES

Table 4.1 Period 1: Greatest Item Gain 41

Table 4.2 Period 2: Greatest Item Gain 41

Table 5.1 Period 1 Smallest Item Gain 42

Table 5.2 Period 2 Smallest Item Gain 42

LIST OF FIGURES

Figure 1. Student 1 Work Sample 44

Figure 2. Student 2 Work Sample 44

LIST OF ABBREVIATIONS

DMT	Developing Mathematical Thinking
MTI	Mathematical Thinking for Instruction

CHAPTER ONE: BACKGROUND AND PURPOSE OF THE STUDY

Introduction: Background and Focus of the Study

After discussing fractions with my 4th grade students, I realized the same type of discussion would follow: I often posed questions to find whether my students have conceptual understanding of difficult rational number ideas. Most students have mixed understandings of early, informal ideas associated with fractions and decimals (Mack, 1999). I noticed a small percentage of students arrive in fourth grade accurately naming and understanding relative size, and performing computations with these numbers**.** I found many other students were not able to understand and place rational number concepts into their long term memories. Some of my students were not able to accurately communicate their understanding and struggled to actively participate in tasks that focused on rational number concepts. After discussing students' struggles with them, it seemed they used their whole number understandings for rational numbers in error-prone ways, such as adding both the numerator and denominator when joining fractions. Depending on their prior conceptual or procedural experiences (or a mixture of both), it appeared students gravitated to methods that produced limited success when these students had used the methods for whole number computation.

In my class, students appeared to be able to successfully use manipulatives to begin building their understanding; most could even be guided to draw a model of the physical pieces they worked with**.** When it was time for students to use symbolic notation with fraction computation and equivalence, the understanding seemed to frequently slip away. For example, when comparing fractions with physical models, students could easily see the largest fraction. When the physical model was not being used, some students still had to draw the model to compare size of fractions. This lack of conceptual knowledge is what I would like to understand more deeply as well as how to build it within my students. It is noted that students who use newly attained concepts by then applying their understanding to new situations, are likely to be successful (Siegler, & Alibali, 2005). When finding equivalent fractions, their discussions were rich with what "sounded like" understanding. However, this understanding seemed to end when they attempted to access how they used a diagram of physical models and attempted to decipher the vocabulary necessary for understanding a fraction or decimal task. When examples of diagrams were not available, or they could not remember how to draw the model they needed to solve the task, students struggled to make sense of the relative size of the unit fraction (any fraction with a numerator of "1", e.g. ¼) or decimal. Sometimes there was confusion about what to name the unit fraction. While it is perceived that students have understanding during class discussions, there are some students being left behind (Tzur, 2007). Some students need much more conceptual practice but are forced to procedural methods, and some students have language deficiencies that create barriers when concepts are discussed.

In past school years, with practice, the top 20% of students in the class demonstrate success. They were able to reason and explain how to find the solution to equivalence with computation and comparison ideas. Most were even able to apply these ideas to a new situation. These students accessed a way to draw a diagram and most times, when asked, could discuss and write the explanation of their process. My concern and focus is for students who have no point of access and lack the skills to decide how to approach novel tasks. These students also struggle to use models, diagrams, and fraction number names to explain their understanding. The students I am concerned about are those who are lacking language and experience with number. These students will undoubtedly need a carefully chosen path to help them gain understanding with rational number concepts.

In past years, fraction lessons in my classroom ranged from pure problem solving investigations to rote computation practice from a textbook. If I am measured as a highly qualified teacher by my students' standardized test scores, then most years I have attained what my district determines successful. Even so, I do not feel my students have adequately gained experience and knowledge of lasting rational number concepts. Reviewing the content-knowledge expectations for my students as they moved to 5th and 6th grade left me concerned that they were unlikely to be as successful in later grades as they had been in my 4th grade classroom. Knowing this made me begin to wonder how to teach fractions and decimals with a balance of conceptual understanding and procedural knowledge. Unfortunately, there is evidence that typical curriculum in U.S. schools is not effective in helping students conceptually understand the difficult ideas associated with rational number topics (Empson, 2003). There tends to be an emphasis on procedures being taught before students are able to understand why the procedures, and later, standard fraction or decimal algorithms, can be useful and efficient. Instead of investigating why these algorithms are used or why they work, many of my previous students were delivered lessons that forced the students to use a model or method that was foreign to them. In comparison, the computer programs that run in schools for intervention, counter-intuitively teach procedures and algorithms to students who need the most help with acquisition of conceptual understanding. Often, these programs force students to perform computation with rational number, which are unfamiliar and seem not to fit how students naturally think about different fraction and decimal relationships.

Another concern was making sure students leave fourth grade (and subsequent grades) with in-depth knowledge of the concepts taught. My goal is to give students a solid foundation in beginning rational number sense to better assist them in middle school and high school. According to the Common Core State Standards (NGACCSSO, 2010), fifth grade students should be ready to begin middle school with knowledge of fractions that is thorough and flexible, in order to operate using multiplication and division with rational numbers and to make connections to various ways of thinking about these numbers and operations. Fifth grade students should also be able to apply fraction models to add and subtract fractions with unlike denominators. Sixth grade students should be able to interpret and compute quotients of fractions. In order to create this level of understanding, students need rich, rigorous tasks promoting a deep insight into fractions and decimals (NGACCSSO, 2010).

Purpose of the Study

The purpose of this study was to develop a sequence of instructional tasks based on foundational concepts of fraction and decimals, cognitive theories, and instructional theories. The sequence of tasks is meant as guide to build rational number sense and to encourage students' conceptual knowledge. Rational number sense can be built from students' understanding of whole number (Mitchelmore & White, 2000). From my reading of Mitchelmore and other researchers such as Streefland (1985), Mack (2001), Lamon (2006) and Tzur (2007), when students utilize the same decomposing strategies and understanding of the repackaging and grouping of whole number, their rational number sense increases. When students experience rational number in a developmental sequence, perhaps their use of diagrams and models will not be as difficult but will follow their physical construction of rational number more accessibly.

Making fraction and decimal understanding even more problematic, programs in schools teach students ‘how to’ procedures. This often leads students to a misunderstanding and loss of the relative size and conceptual foundation needed for acquisition. Struggling students need carefully organized tasks, embedded with rigorous, real life situations (Lamon, 1996). They also need opportunities with language usage and experience based on conceptual understanding research (Empson, Junk, Dominguez & Turner, 2006). With these carefully sequenced tasks, struggling students may have a decrease in time needed to acquire these difficult concepts (Van de Walle, 2007). When this occurs, perhaps there will be less time spent re-teaching and more time extending concepts for students in order to apply and synthesize rational number ideas. Instead of giving a prescription on how things should be completed or forcing an algorithm on students who may not understand, teachers might be better off helping students make the connections between rational number ideas (Empson, 2003). Finally, when students leave upper elementary, the goal is to have given a strong foundation in beginning rational number sense and for students to have experienced thorough ideas and tasks. With each of these pieces in place, students are hypothesized to have a strong base in fractions, decimals, and percents.

Goals for the Study

Perhaps because of the available curriculum resources or students' prior experiences, teachers seem to focus their instruction on procedural knowledge, while researchers suggest we should include much more conceptual understanding during instructional time. In the case of rational number, Moss and Case (1999) suggest instruction emphasizing procedural knowledge of rational number as opposed to conceptual understanding, will ultimately discourage students from understanding rational numbers in a meaningful way. Streefland (1985) suggests the notation children devise themselves will lead to multiplicative reasoning instead of additive reasoning: a critical element for adequate understanding of rational numbers. Multiplicative reasoning involves reconceptualization of the unit. When multiplicative reasoning is evident, students are using equivalence to change the unit to use it in another way (Harel & Confrey, 1994). When students only have additive reasoning, each unit is continually joined, and students do not understand the new amount has equivalent quantities (Hiebert & Behr, 1988). Teachers may ask how to instruct students to maintain the balance between conceptual understanding and procedural knowledge of typical fractions and decimals. The purpose of this proposal suggests instruction with a stable balance of conceptual understanding and procedural practice will increase 4th grade students' knowledge in rational number.

Conceptual Understanding vs. Procedural Knowledge

Many topics of concern for conceptual understanding and procedural knowledge arise when students are acquiring fraction and decimal knowledge. Use of procedural understanding and efficiency may be the teachers' end goal for students. Using knowledge of whole number can help focus students' beginning fraction and decimal knowledge (Mitchelmore & White, 2000). A common perception of a student who is successful with fraction and decimal concepts is that the student can replace whole number ideas when they are no longer applicable to rational number thinking.

Teachers' goal of procedural understanding and efficiency leads to other important ideas and concerns. One question is whether students will be able to access their newly acquired knowledge when instructional support, such as models and diagrams, are not readily available. Finding the instructional tasks to support this is a difficult puzzle to decipher. The experiences and tasks provided in my class attempted to build strong conceptual understanding, which then would be applied to more difficult concepts in middle and high school. Teachers might question how to integrate these ideas and conceptual understanding into mathematics instruction. This is a compelling question and is argued by researchers such as Lamon (2006). Will our students benefit from traditionally taught procedures for lasting understanding, or will a basis and experience of conceptual tasks offer a better approach?

Research suggests students can and should use some ideas from their whole number knowledge to begin to understand fractions. Mitchelmore and White (2000) discuss when children have an understanding of whole number and fractions, their ideas of these complex concepts can be abstracted. For example, students can use their understanding of decomposition of number to help partition a whole into fair shares and unit fractions. When students learn to count unit fractions, they are using the same whole number counting strategies with enumerating the unit. To illustrate this, students can use the one-fourth unit fraction to enumerate one-fourth, three times. A problem arises when students reach one whole and try to count with a unit fraction beyond it (Van de Walle, 2007). Students typically are not sure how to symbolically notate more than one and are unsure as to what the new quantity is called. As students grasp understanding of rational number, they often are able to exchange whole number ideas, for rational number ideas. Teachers can implement techniques related to learning theories from the field of cognition such as helping students make connections to what they already know and building a mental schema for fractions. Rational number concepts can begin to feel like the natural next step for students as multiplicative reasoning is acquired and understanding of when whole number ideas are no longer needed (Moss and Case, 1999).

The second question in the research relates to the difficulty students have transferring their understanding of fractions and decimals with physical models and diagrams to symbolic notation. The study involved students learning to retain and apply rational number knowledge gained from previous instruction. Because children have such strong experience and rules formed with numbers, this causes difficulties with relative sizes of fractions. Students tend to mis-transfer their knowledge of whole number concepts to fractions. Knowledge of whole number will be helpful, but might become a road block when students begin to compare fractions and decimals (Van de Walle, 2007). For example, when ordering fractions, students may be able to reason correctly which unit fraction is larger with a physical model such as fraction paper strips or fraction rods. When the physical models are removed or students forget how to model the task with manipulatives and other resources, they may reason because ten is larger than four, tenths should be larger than fourths. In contrast, using the statement, ‘larger numbers on the bottom mean smaller fractions,’ is dangerous and inappropriate (Mitchelmore & White, 2000). Students need practice ordering unit fractions, 1/b as well as non-unit fractions in the form – a/b. Often, available curriculum resources may press students to find common denominators and may teach cross multiplication, but this gives minimal attention to the relative size of the fractions being compared. For students to be able to use physical models and diagrams and to transfer their ideas to symbolic notation, they must have multiple ways to practice doing so. Siegler and Alibali (2005) suggest they must also revisit concepts and become reflective of their methods and ideas. A procedural method of choosing the answer is not the goal when first grasping fractions and decimals. When students are taught to compare and critically analyze different models and strategies, they will be likely to apply their knowledge to new situations and easily transfer to procedures with understanding (Siegler & Alibali, 2005).

The third question related to this study is to carefully organize tasks using developmental practices. I considered how students acquire mathematical vocabulary and concepts. Opportunities ranged from the time to discuss and reason about fraction and decimal ideas to being able to justify and apply new concepts (Tzur, 2007). Students should be asked to justify their results and think with carefully led questions (Brendefur & Frykholm, 2000). Tasks and pressing questions lead students to acquire new understanding of fractions and decimals. Another method proposed is to press students' conceptual knowledge by putting limits on how students solved tasks. Solving without paper and pencil or using another student's idea are two ways students can justify their thinking (Tzur, 2007).

Simon (1995) suggests students may have conceptual regress from one day to the next. This does not imply students lose the information, but some may need to be given a prompt and framework for the new task. When students are asked to contemplate and represent realistic fraction situations with notation they devise themselves and with guidance from their instructor, they are more likely to apply ideas to new situations. Within the unit were models students could use for this purpose. Embedding mathematical problems in context is essential for students to learn to use and operate with rational numbers effectively. Without them, rational number concepts are very difficult to conceptualize and quickly lose meaning (Bay-Williams & Martinie, 2003). In order to support understanding of fraction and decimal knowledge, specifically equivalence, the building up strategy is one that works well for children (Lamon, 2006). This unit encouraged multiplicative reasoning for problem solving. The tasks were framed to decrease conceptual regress, teaching students to build on their prior understanding.

The fourth and final question related to this study is how to address ways to support the next related mathematical concepts students will face when they leave elementary school. Some researchers suggest that instruction in rational number concepts cannot be limited to what students will encounter in the upper elementary grades (Lamon, 2006). It is critical the concepts students receive instruction on while in elementary school should have underlying principles that support rational number understanding in middle school and high school. Students will face more complex rational number, ratio and proportion, and algebraic ideas in upcoming years (NGACCSSO, 2010). For example, knowledge of equivalent fractions and iterating and partitioning fractions and decimals help support the understanding of proportions and algebraic ideas (Streefland, 1985).

Streefland (1985) suggests lesson facilitation will increase long term learning processes when using intuitive notions leading to abstraction, discovering applicability within related concepts, and connecting many approaches (ratio) with a variety of structure and context. Teaching consciousness of concepts learned with an element of conflict and reflection will benefit students in all mathematical concepts. Streefland (1985) suggests the multiplicative structure of fractions must not be divorced from ratio instruction. All work within supporting multiplicative reasoning will benefit long term learning.

In conclusion, some studies suggest students' experience and informal understanding of the previous concept domains should play a critical role in their mathematical development (Baroody, Ginsburg, & Waxman, 1982). The questions related to this study focus on how to help students use their whole number knowledge to support newly acquired fraction and decimal concepts. Students should investigate at which point whole number knowledge must be modified for an accurate understanding of rational numbers. This study was framed by research on learning and cognition with special emphasis on the ways students may be able to transfer understanding of fraction and decimals gained using a progression of representational models (Bruner, 1966). To assist students in communicating and applying their understanding, a carefully sequenced suggested selection of tasks involving fraction and decimal ideas was created based on relevant research. Using physical models, diagrams, and symbolic notation and pairing of these representations with opportunities for students to write about their knowledge, the goal for this study was for students to gain a meaningful conceptual understanding of fraction and decimals and to instill student understanding that could be applied later to upper elementary, middle, and high school mathematics.

Research Question:

Will instructional practices and tasks emphasizing reasoning and conceptual understanding have an effect on 4th grade students' understanding of fraction and decimals?

CHAPTER TWO: LITERATURE REVIEW

Literature Review

The purpose of this thesis was to develop a sequence of instructional tasks based on foundational concepts of fraction and decimals, cognitive theories, and instructional theories. The sequence of tasks is meant as a guide to build rational number sense and encourage students' conceptual knowledge. Rational number sense can be built from students' understanding of whole number (Mitchelmore & White, 2000). From my reading of Mitchelmore and White and other researchers such as Streefland (1985), Mack (2001), Lamon (2006) and Tzur (2007), when students utilize the same decomposing strategies and understanding of the repackaging and grouping of whole number, their rational number sense increases. In order for fraction and decimal number sense to be acquired, there are three foundational concepts agreed upon by researchers (Barnett-Clarke, Fisher, Marks & Ross, 2010). These concepts along with conceptual understanding and procedural knowledge, cognitive theories, and instructional theories will be discussed.

When these pieces are carefully considered to create a suggested path of tasks to develop fraction and decimal number understanding, students gain understanding for long term application (Van de Walle, 2007; Watanabe, 2006). The goal is also for the knowledge gained to be available for transfer, application and synthesization towards more complex rational number ideas.

Foundational Concepts

Fraction and decimal knowledge is based on several foundational concepts. Researchers agree on three critically important ideas that should be developed for true fraction understanding. These foundational concepts are **units and unitizing, partitioning and iterating, and equivalence** (Barnett-Clarke, et al., 2010). These concepts are defined and discussed below with implications for teaching, learning and assessing in classroom settings.

Unit and Unitizing

The first foundational concepts are units and unitizing. According to Barnett-Clarke, et al. (2010) units can be discrete or countable. Units can also be part of a whole or continuous and measurable as in pizza, brownies, ribbon, and miles. A unit fraction is the size of the counting piece. Determining the unit is key to interpretation and is important because it describes the size of some quantity with rational number (p. 19). The first step is to determine what is the unit or whole (Behr, Lesh, Post & Silver, 1983; Carraher 1992; Kieren 1992; Lamon, 2007).The unit is used within all of these foundational concepts as it is the most fundamental aspect of rational number understanding. Lamon (2007) and Kieren (1992) claim students must be given tasks that help develop their idea of the counting unit and tasks must also give students the opportunity to learn and apply the idea of a unit fraction. For example, when students understand the ‘one’ can be decomposed into 1/b units, they will be able to count past ‘one’ with the unit fraction and be able to understand how many 1/b fractions compose the whole (Lamon, 2007).

In earlier work, Lamon (1996) noted unitizing is the renaming of the pieces or combining of units, for the purpose of counting in a new group. This idea is found throughout fraction and decimal understanding and is vitally important to students' understanding of why fractions can and will be renamed with other number names. Renaming is significant when finding equivalent fractions as well as with fraction computation of unlike denominators (p. 171). As the fraction is renamed, the number of counting pieces in the unit increases. The opposite can be said for the size of the counting pieces; they will decrease in size. Steffe, Cobb and Von Glasersfeld (1988) suggest implications for instruction can be students investigating tasks that help develop their idea of the counting unit (p. 13). Tasks must also give students the opportunity to learn and apply the idea of a unit fraction. For example, when students understand the 'one' can be decomposed or partitioned into 1/b units, they will be able to count past 'one,' with the unit fraction, and be able to understand how many 1/b fractions compose the whole (Lamon, 2007; Kieren, 1992).

Partitioning and Iterating

The second and third ideas are partitioning and iterating. Susan Lamon (2006) describes partitioning as breaking or fracturing of a whole. It can also be described as dividing an object or objects into a number of disjoint and exhaustive parts. In addition partitioning is discussed as parts not overlapping. When a whole is partitioned, each of the parts is of equal area. Mack (2001) describes a necessary skill as reconceptualizing the whole, when partitioning. The knowledge of piecing the whole back together is a large consideration when deciding how many pieces to cut and how large or small the pieces will be.

Susan Lamon (2006), gives ground rules for partitioning and iterating:

- Each unit is equal.
- If a unit consists of more than one item, they must be the same size.
- When shares are *equal*, this means in amount, but shares do not always have to have the same number of pieces.
- Equal shares do not have to be the same shape.

Instructors can begin building ideas of partitioning with children at an early age. When tasks are meaningful for children, they are naturally curious enough to solve and consider ways to share amounts fairly. Equal sized amounts are important for children, and fair sharing tasks should be used throughout an elementary students' experience with fractions (Lamon, 2006). This idea is important because instructors can build on students' prior experiences. Teachers can extend knowledge by helping students begin to fair share in other ways beyond splitting each piece in halves. Students will begin by partitioning each whole into the amount of people sharing then move to finding more sophisticated ways to share (Van de Walle, 2007).

When students begin to partition, it is introduced visually (Lamon, 2006). Students will begin by sharing in one-half pieces, but should be pressed to begin sharing as efficiently as possible. An idea student should consider while fair sharing is the amount of the shares given to each and the amount each receives. These ideas involve anticipating, estimating, and visualizing the relative size of each share before cutting the whole, or 'one.'

Students can experience determining which fraction is larger. Students should work to discover by how much more the largest fraction is. The comparison should move beyond a qualitative comparison to quantitative. When beginning to compare quantitatively, Lamon (2006) suggests students should reason about two different bar models divided into the specific, different fractional parts. Students can be asked to divide the pieces in each whole so they have the same size pieces. This allows for the two fractions to be compared by same sized pieces. Students can begin to reason by how much more the largest fraction is, compared to the smallest fraction.

Partitioning plays an even more important role when students are determining equal sized pieces when adding and subtracting fractions, as well as comparing fractions to discover which fraction is larger and by how much. These concepts may take several years to acquire. Partitioning involves students understanding that as the number of pieces increase in the whole or one; the smaller the pieces become (Mack, 2001).

Iterating of fractions is related but is the ‘building up’ of the unit piece. It is another way to make sense of fractions and improper fractions. When a unit is copied to create the one or whole, the unit has been iterated (Lamon, 2006). Barnett-Clarke, et al. (2010) suggest a whole can be subdivided into units for example, into four equal-sized pieces. Each of these pieces is thought of as ¼. An example of iteration is using four ¼ pieces to create one. When given an amount of one-fourth pieces, such as five ¼ pieces, it is notated in this way: 5/4. This means five copies of the unit fraction ¼. Experiences with both partitioning and iterating will help clear up confusion between the number of pieces in the share and the name of the share (Van de Walle, 2007). When the number of counting pieces in the unit increase, the opposite can be said for the size of the counting pieces; they will decrease in size (Steffe, Cobb & von Glasersfeld, 1988; Lamon, 1996).

Equivalence

Equivalence of fractions and decimals is the last foundational concept when developing understanding of rational number ideas (Smith, 2002). Lamon (2006) defines equal in part-whole fractions as the same in number, length, and area. In other words, many different fractions can name the same amount. Equivalence is an important idea throughout mathematical development and should not be put off during instruction. Instruction focusing on equivalence should begin with students partitioning the entire unit into smaller pieces, renaming the pieces into more pieces, or combining to make fewer pieces, by chunking. As students are acquiring the vocabulary to describe their models and thinking, they should be led to understand the difference between parts and pieces (Lamon, 2006). One part is not the same as one piece. A part may have more than one piece included within it.

When students begin to have a firm understanding of equivalence, they may use this knowledge to determine which fraction is larger, how much larger, or if they are equal (Van de Walle, 2007). Leinwand and Ginsberg (2007) report countries such as Singapore use pictorial and concrete representations, along with abstract symbolism, to build a sound understanding of equivalence. Students and teachers use multiple representations to build conceptual understanding in this foundational concept (Kamii & Clark, 1999). When teachers begin fraction instruction, often tasks include shading a fractional amount. Teachers can use this opportunity to encourage students to investigate ways to rename the shaded piece. An approach used by teachers to help students understand equivalence is using different models to find different number names for a fraction (Mack, 1999). For some students, this may be the first time they are discovering and investigating the possibility that fractions can have many different names.

With an approaching understanding of fraction and decimal equivalence, students can adjust how a fraction looks or ***rename*** the number and use this understanding to make sense of the comparison. In a 5th grade classroom, while comparing 6/8 and 4/5, a student decided to change 4/5 to 8/10, understanding equivalence. This made comparing the two fractions conceptually easier, as each fraction (6/8 and 8/10) were now two pieces away from making one. Since eighths are larger than tenths, then they can determine 8/10 is greater than 6/8 (Saxe, Gearhart & Seltzer, 1999). Whether students are able to reason with a concrete representation or abstractly, using their knowledge of equivalence can be helpful with comparing and later with finding equal-sized pieces for fraction computation (Van de Walle, 2007). With multiple experiences renaming common fractions, students will gain the knowledge to find equivalences to common fractions. Research suggests the algorithmic rules should not be taught or used until the students are able to understand how the steps in the procedures relate to what they know conceptually or what the solution means (Lamon, 2006; Saxe, Taylor, McIntosh & Gearhart, 2005).

Rival Explanation

A potential rival explanation would be current curriculum proposes students spend a few days on procedures for operating with fractions and decimals. Direct instruction of concepts and procedures is believed to increase computational skills (Whitehurst, 2003). Teachers may also argue standardized test scores are satisfactory, using procedures. The National Governors Association and the Council of Chief State

School Officers' Common Core Standards (2010) suggest instead, students will acquire procedural knowledge with understanding of underlying mathematic principals of the procedure. Within the standards is a framework for students to learn underlying mathematical principles from Kindergarten through 6th grade. The principles include knowledge of place value, as well as magnitude of whole number and rational numbers. The Standards suggest it may take students up to three years or longer to acquire the understanding needed to perform procedural computations with the standard algorithm. In order to support both procedural and conceptual knowledge, a problem-based classroom is suggested to help students develop an understanding of equivalent fractions, as well as support an understanding of a conceptually-based algorithm (Moss and Case, 1999).

Conceptual Understanding vs. Procedural Knowledge

Conceptual understanding begins with the real-life, self-constructed knowledge that may be correct or incorrect based on the student's understanding and experience. This knowledge can be drawn upon and used to apply to similar situations, as well as used to build knowledge of other new, related mathematical ideas. It is suggested this informal knowledge may be unrelated to symbols and procedures (Mack, 1999). Students' understanding of operating on fractions consists of rote procedures without connections to other mathematical concepts and is most often incorrect. Researchers Gunderson and Gunderson (1957) and Leinhardt (1988) have shown that students can use informal knowledge to reason about joining and separating fractional quantities when real-life situations are presented.

Moss and Case (1999) and Saxe, et al. (1999) suggest instruction emphasizing procedural knowledge of rational number, as opposed to conceptual understanding, will ultimately discourage students from understanding rational numbers in a meaningful way. Streefland (1985) and Mack (2001) suggest the notation children devise themselves will lead to multiplicative reasoning instead of additive reasoning; this is a critical element for adequate understanding of fractions and decimals. Teachers may ask how to instruct students to maintain the balance between conceptual understanding and procedural knowledge of typical fractions and decimals. The purpose of this study is to suggest that instruction in both conceptual understanding and procedural practice will increase 4th grade students' knowledge in rational number.

Use of procedural knowledge and efficiency may be teachers' end goal for students. They may question how to integrate conceptual understanding into everyday mathematics instruction. This is a compelling question and is argued by many (Lamon, 2006). Will our students benefit from traditionally taught procedures for lasting understanding, or will a basis and experience of conceptual tasks offer a better approach? Research suggests a focus on conceptual understanding for instruction will develop into a deeper understanding for procedural knowledge (Kieren, 1992).

Students' conceptual understanding of fractions and decimals can begin from some ideas from their whole number knowledge. Mitchelmore and White (2000) discuss children having two unrelated concepts when understanding whole number and fractions, but when a sub concept for fractions is formed within whole number, children have a greater understanding of whole number and fractions. Students with experience of conceptual tasks will begin to decide when to use whole number ideas and when to shift from whole number to fractions and decimals. For example students can use their understanding for decomposition of number to help partition a whole into fair shares and unit fractions. When students learn to count unit fractions, they are using the same whole number counting strategies with enumerating the unit fraction. To illustrate this, students can use the one-fourth unit fraction to enumerate one-fourth, three times. A problem arises when students reach one whole, and try to count with a unit fraction beyond it (Van de Walle, 2007). Students may join the amount already counted with more of the unit fraction. Students will often add the numerators and the denominators instead of remembering which unit or size of piece they are counting in. A focus on conceptual tasks and understanding will assist students with grasping rational number ideas. They often are able to exchange whole number ideas for rational number ideas (Mitchelmore and White, 2000).

Students' knowledge with real-life experiences and tasks constructed can play a sizable part when they are learning fractions and decimals (Brown, Collins, & Duguid, 1989; Carpenter & Fennema, 1988). Informal mathematics, intuitive knowledge (Leinhardt, 1988), and prior experiences, whether correct or incorrect, can be drawn upon when the student is faced with real-life situations in mathematics. Informal knowledge of fractions and decimals may begin with students' formal knowledge and experiences with whole number (Mack, 1999), but when teachers use cognition and are able to help students make connections to what they already know, rational number concepts can begin to feel like the natural next step for students (Moss and Case, 1999).

Role of Cognitive Theories

The roles that cognitive theories play in learning are important to conceptual understanding in a variety of ways. Significant aspects of a learner's concept attainment, according to a cognitive theoretical perspective, involve concrete and abstract understanding, the use of prior knowledge, models chosen for solving tasks, one's schema of a concept, and modes of representation (Battista, 2004; Anderson, 1977; Bruner, 1966). When a student is developing a concept, they are more than likely to begin with an understanding that is limited to a concrete or physical representation (Battista, 2004). Siegler and Alibali (2005) suggest students with prior knowledge of a concept may remember more of the concept than those students with little to no prior knowledge. Sometimes though, prior knowledge might lead students to remember incorrectly, but this is less common than students forgetting material they have only recently learned which they have little previous knowledge to connect with.

A learner's prior knowledge comes from experiences that lead them to draw correct inferences about a concept or topic. In mathematical problem-solving, students often encounter the need to model their thinking or solution strategy. Models can take the form of symbolic notation, visual diagrams, or physical materials (Bruner, 1966). Siegler and Alibali (2005) indicate that if students are given choices involving the model they use to solve problems and demonstrate their understanding, the model used is representative of the student's knowledge about particular concepts. The model chosen can also be representative of the process the student followed and can even influence the very same process (Gravemeijer, 2004). The connections between models and potential solutions can vary in degree and those models that align to the students' own processes are more likely to be used in future situations, and lead to greater understanding of difficult concepts (Siegler & Alibali, 2005).

R. C. Anderson (1977) suggests a learner creates and then adds to his or her schema. A schema is a central concept and all of the related ideas. As the learner increases experiences to the created schema, the connections between ideas within the schema become stronger. The learner has an increased understanding of the concepts. The ideas within the concepts, if illustrated, would look much like a web with the beginning concept drawn in the middle. For example, if a student is familiar with fraction and decimals such as the number one-fourth, the student might have the symbolic notation in his or her mind. Perhaps the student also has experience with the part-whole concept of four, one-fourth pieces and one of those pieces. The student may also have beginning experiences with fair sharing. When a student has to decide how to share at least one of something with four people, he or she must decide how to make sense of how much each person will get and what to name the share. The model the student chooses to describe his or her thoughts and communicate understanding will reflect the student's current understanding of one-fourth. Adding to the student's understanding of one-fourth may be physical models and diagrams the teacher provided for in-class experiences. Perhaps the student begins to understand the idea of 'quarter' is the same and equal to one-fourth of one or the whole. The student may recognize and connect the fraction and decimal notation of one-fourth to twenty-five hundredths and twenty-five percent.

When a student has the opportunity to investigate other one-fourth equivalences, their understanding begins to expand further. They might begin to understand as the whole can be different quantities, so will one-fourth of the newly named amount.

Experiences may also include the student beginning to understand the one-fourth quantity can be iterated, or copied. After the opportunities to make sense of one-fourth, the student may be able to accurately name and diagram *n* copies of one-fourth, its newly named value, and equivalence to other fractional amounts.

As a student leaves 4th and 5th grades, they might have experience with the ratio one to four. His or her schema may shift slightly because of the new meaning of the symbolic notation with the meaning ‘one for every four’ (Lamon, 2006).

All of the related experiences of one-fourth build an interesting schema for the student. Because each student’s experience and view may be different, their schema will be as well. There may stronger connections of understanding based on prior knowledge, tasks experienced, and the focus on conceptual understanding or procedural knowledge. Fractions and decimals are very intricately related and have many difficult sub-concepts within and between them. R. C. Anderson (1977) stated the learner begins to sort their ideas into different groups. The groups may be unequal, depending on current experiences and prior knowledge. Experiences with ideas may seem difficult at first, but will lessen and become something else the learner knows well. The learner may have another new experience with the concept, which will completely change the ideas and piles sorted. Until the learner can make sense of these ideas again, the piles of ideas in the learner’s mind may be rough and uneven. The whole process is repeated with each new experience of the concept and the connections between the ideas within the schema grow stronger (Anderson, 1977).

Jerome Bruner (1966) presented ways learners choose to represent ideas when problem-solving and communicating knowledge. When learners first experience a concept depending again on prior knowledge and experiences, he or she will naturally find a way to represent acquired understanding. The modes of representation are: **enactive representation, iconic representation, and symbolic representation.** Each is described in detail below.

Enactive representation is the physical model or image of what is real. This may take the shape of an image, action, or physical model. When a learner is familiar with what is real pertaining to the concept, he or she can easily reconstruct these ideas. In the classroom, teachers may use manipulatives to aid students in their understanding. A possible rival explanation is Piaget's Stage Theory (Piaget 1951, 1969). Piaget described stages of development that happen in a fixed sequence, and transitions occur at certain approximate ages. The operations of each stage are more complex and adaptive than the previous stage. Piaget would consider it unnecessary and ineffective to teach a subject or concept requiring the learner to demonstrate something where the operations have not been developed (Driscoll, 2005). Bruner, in contrast, believed the sequence of stages through which learners pass are not influenced by their age, but instead are influenced by their environment. It is important to note teachers may begin instruction with the enactive representation but may not need physical models for some students. If students are able to adequately demonstrate understanding of the concept with diagrams and drawings, the images, actions, and physical models may only be needed briefly (Bruner, 1966).

The second mode of representation is iconic representation. This representation enables the learner to "summarize events by the selective organization of percepts and images." The learner transforms their understanding of either their own enactive representation or their perception and understanding (Bruner, 1966). When a learner can accurately diagram a scale or model of the magnitude of a number, or the joining or separating of two or more numbers and their relative size, the learner is adequately describing their understanding. Also within iconic representation is the learner's ability to describe their recollection of an event with a drawing or diagram (Bruner, 1966). Implications for instruction are making sure students are able to diagram or draw their understanding after constructing the physical representation.

The final mode of representation according to Bruner (1966) is symbolic representation. When a student is able to demonstrate his or her understanding of the task by using only numerals to describe the magnitude and quantity of the numbers involved, he or she may have symbolic understanding. Sometimes, a student may have been instructed how to solve problems procedurally and may lose the value of the numbers in the task. If students are taught symbolic representations before understanding the underlying mathematic principles, it is more likely for students to misunderstand the standard algorithm. This misunderstanding is apparent as students are involved in more difficult mathematics. When they are unable to understand the underlying principles of their notations, they may not be able to apply their knowledge to new mathematical situations. When students are able to use their symbolic notations, diagram and label using the symbolic notations, students may be more likely to understand the symbolic notation and mathematical principles. As students' experience and knowledge increase about a concept, their ideas may progress through Bruner's examples of representation, becoming progressively abstract in thinking and understanding (Bruner, 1966).

Shifting Instructional Practices

When teachers in the United States begin to shift towards teaching for conceptual understanding, there might be a struggle to discover students' levels of sophistication. Battista (2004) suggests this issue may be due to the fact teachers aren't aware of the obstacles students face when in the midst of gaining understanding of these topics. Teachers should have a method for deciding which level of sophistication their students have when beginning to understanding a concept. Battista has determined tasks which demonstrate the levels of sophistication of understanding, to assist with assessment. He uses 5th grade work to reveal students enumerating the unit, with all available counting units showing to solve the task. Students do not seem to struggle when they are able to count visible units. In the second task, the units are not all available to count. Students struggling with these tasks and who need to see all of the spatial units represented fall into Battista's internalized level. He describes this as a student abstracting the concept, so that the idea may be re-created, but the student only understands perceptually, or just on the surface (Battista, 2004). The same may be said for acquisition of fraction and decimal concepts as well.

Teachers begin to pay close attention to levels of sophistication as students move towards conceptual understanding. It is noted that students who use newly attained concepts by then applying their understanding to new situations, are likely to be successful (Siegler & Alibali, 2005). In reality there is a disparity of understanding. Some students might be perceived as having understanding of the topic but when asked to apply their knowledge to a new situation may forget the framework provided and discussions with classmates. Because of this, there are a larger percentage of students who are getting left behind (Tzur, 2007).

Assessment Results

Unfortunately, there is evidence the typical curriculum in U.S. schools is not effective in helping students conceptually understand the difficult ideas with rational number (Empson, 2003). In one report, students in middle school and high school were not able to correctly reason and answer a multiple choice assessment item about estimated sums of fractions. Some chose not to answer at all (Carpenter, Corbitt, Kepner, Lindquist, & Reys, 1980).

According to the Program for International Assessment (PISA), students in the United States ranked 24th out of 29 countries in the Organization for Economic Cooperation and Development (OECD). The United States is falling below Poland, Hungary, and Spain in the three years since the previous assessment. Trends in International Mathematics and Science Studies (TIMSS) report a somewhat different story. Student achievement improved from the last assessment taken. The TIMSS show the United States are beginning to close the gap between white and black students. The difference lies in knowing the United States has maintained its mathematical achievement and closed the gap between Caucasian and African-American students (Glasgow, Ragan, Fields, Reys & Wasman, 2000). Other countries have maintained and surpassed the United States. It is also important to note the TIMSS assessment asks students to recall information of straightforward questions (Bybee & Stage, 2005).

In contrast, the PISA asks students to recognize and interpret mathematical and scientific problems, apply their knowledge, translate problems into mathematical contexts, and justify their solutions. The PISA assesses students' knowledge and whether or not they have the ability to solve tasks. The results of the 2003 PISA indicate that students in the United States were below two-thirds of the countries in the OECD. The United States had more students test at the basic level and fewer in the advanced level. White students alone were below the OECD average, while African American and Hispanic students are significantly lower. In order to support the level of problem solving students face on international assessments, the need for a shift of instruction is vital (Glasgow, et al. 2000).

There tends to be an emphasis on procedures being taught before students are able to understand why the traditional algorithms can be useful and efficient. Instead of investigating why they are used, the lessons force students into a model that is foreign to them. In comparison, the computer programs that run in schools for intervention counter-intuitively teach procedures and algorithms to students who need the most help with acquisition of conceptual understanding. Often, these programs force students to perform computation with rational number, which are unfamiliar and seem not to fit how students naturally think about different fraction and decimal relationships.

If results of national and state assessments indicate students are not able to grasp fraction and decimal concepts conceptually and their understanding is shallow, a shift in thinking and instruction may be the solution. For students with learning difficulties to average and advanced students, changing instruction and teachers' thinking about fraction and decimal ideas is vital (Vanhille & Baroody, 2002). Some students are lacking in knowledge and experience to construct conceptual understanding and aren't able to abstract procedures that are connected to concrete experiences. Some students are unfamiliar with multiplicative reasoning, which makes concepts of equivalence difficult (Lamon, 2006).

Trends in International Research

Leinwand and Ginsburg (2007) report internationally math instruction has a different focus. Instead of requiring students to learn the standard algorithm, students in Singapore encounter in-depth mathematical concepts first. Textbooks in Singapore classrooms devote pages and pages to one concept. Students focus only on one to three concepts per week. The result is fewer topics covered in one school year, but each topic is covered thoroughly and in-depthly. Singapore teachers feel the iconic representation is so crucial, they require students to diagram their understanding before moving onto symbolic representation of the task. Students are to master such diagrams as the bar model to represent their thinking. Without thorough understanding of a diagram, students are not allowed to move to symbolic representation (Leinwand & Ginsburg, 2007).

Students in European countries such as the Netherlands spend time investigating topics and devising their own models to represent their mental strategies. Students are given contextual tasks to solve (Gravemeijer, 2004). Instruction is based on guided reinvention, which is an instructional theory suggested by Hans Freudenthal (1991). **Guided reinvention** is based on the idea that students are not passive recipients but active participants in mathematics they can reinvent themselves. Students are investigating and discovering their own strategies and the model for notation to solve. Not only are students using strategies and models to represent them they are flexible with which strategy and model is used. They find the best strategy and model to complete each task. The teacher poses questions to help students gradually develop a more formal level of understanding. This can be loosely defined as 'mathematizing.' Mathematizing is the level of understanding of students. Students are able to use a model for solving because the context follows a sequence. A deeper level of mathematizing is when students can demonstrate understanding with multiple strategies and flexible thinking to successfully communicate their understanding of the presented tasks (Treffers and Bishop, 1987).

Teachers worry less about how many are topics are covered in one school year, but focus instead on in-depth student understanding. Students are communicating, comparing, and contrasting their ideas and models. Teachers are facilitators; students do not look for acknowledgement of the 'right answer' but the solution that is mathematically correct. Student conversations involve mathematical principles and the levels of student understanding. Instead of the 'wrong answer,' teachers look for student work and discussion that is approaching understanding and nearing sophistication. Teachers anticipate most student responses and are ready with next steps of instruction. Incorrect answers are used to look for misunderstandings and lack of sophistication to build upon.

Specific to fractions and decimals, teachers in Japan use measurement to introduce fractions. The primary reason for Japanese textbooks to introduce fractions as measurement is for students to understand fractions as quantities. Teachers' manuals discuss when students are introduced to part-whole concepts, this may contribute to the confusion that fractions are quantities. Therefore fractions are introduced as a measure less than one. Furthermore, non-unit fractions are considered as a collection of unit fractions. Students are able to grasp the concept of fractions greater than one. Another major difference is the lack of the area model commonly seen in textbooks published in the United States. Fifth and sixth grade students in Japan are introduced to the area model as numbers and not measured quantities, except for instances of liquid measurement. Finally, discrete fractions are absent in Japanese curriculum. This concept is introduced in later grades with ratio and proportion ideas (Watanabe, 2006).

Instructional Theory Guiding the Shift

The shift in classrooms begins with facilitating an inquiry-based classroom. Students are problem-solving as they investigate tasks with underlying mathematical principles. Each task will lead them to acquire knowledge that will deepen their understanding. In order for an inquiry-based classroom to be successful, a classroom culture is established where students are able to explain and justify solutions, attempt to make sense of other students' solutions, and offer suggestions when interpretation of the task is misinterpreted.

Research suggests teachers utilize an instructional theory that does not require a set of instructional tasks, but instead uses the idea that the instructional tasks could work. Because each class of students is different, the local instruction theory provides teachers with a framework of reference from which to build lessons on. Teachers use the framework and knowledge of their students to decide what fits the needs of the students (Gravemeijer, 2004).

A central struggle with reform mathematics is between the openness of one's own construction of understanding, which may or may not be correct, and the obligation to reach the end goal. The focus should not be on teaching a set of strategies and models. Instead teachers can help guide students to learn a set of number relations in order to be flexible with mental computations (Gravemeijer, 2004). Teachers can support learners by pressing their thinking to becoming more mathematically sophisticated. Therefore a prescribed set of instructional tasks, strategies, and models is merely ideas. The tasks suggested all give the focus to the learner and the paths he or she may take.

Threads found within successful international classrooms and classrooms in the United States with long term conceptual understanding and procedural knowledge are these: students are given tasks that have been researched and well thought out. Teachers use their knowledge of student learning, mathematical principles, and possible solution strategies. Students are given the time to investigate, determine, and compare and contrast their ideas with others. Standard algorithms are not introduced until students have developed the mathematical insight into what their strategies and models connect to.

CHAPTER THREE: DESIGN OF THE STUDY

Research Design

A study was conducted in the semi-rural city of Caldwell, Idaho. Two fourth grade classrooms were instructed for four weeks with carefully sequenced tasks based from current research on fraction and decimal knowledge acquisition, cognition, and international mathematics education. Both classrooms received these tasks because there was no control group, as other fourth grade teachers in the district use the same teaching and learning ideas this study is based on. The research question that was the basis of this study was whether or not instructional practices and tasks emphasizing reasoning and conceptual understanding have an effect on 4th grade students' understanding of fraction and decimals. This research was a quasi-experimental design, using a mixture of quantitative and qualitative methodology. The instrument used in this design was a twelve task pre/post assessment. The assessment items were modified tasks from the literature review, which were based on the foundational concepts of fraction and decimal knowledge (Barnett-Clarke, et al. 2010). The assessment required students to demonstrate their acquired knowledge of fractions and decimals with a mixture of symbolic solutions, requiring students to defend their symbolic solutions or models with written justification and diagrams.

The pre/post assessment was coded using the foundational concepts (Strauss, 1987). Each concept was given an equal amount of tasks to measure understanding. Here is the coding system for each concept: I/P represents Iterating and Partitioning**,** E/R represents Equivalence and Relationships, U represents Units and Unitizing, and R/S represents Representations and Situations.

The qualitative changes in students' reasoning were measured by comparing pre to post assessment and reviewing whether or not their written mathematical vocabulary and understanding became more sophisticated between pre and post assessment items.

Threats to Internal Validity

There are two threats to internal validity to this study. The first threat to internal validity is that I was the teacher for both fourth grade classrooms. This factor may jeopardize the credibility in results, as there was not another teacher instructing. The second threat to internal validity is the lack of control group for the study. Other than the statistically significant gains from pre to post assessment, there is nothing that proves the treatment gains were not the result of natural maturation. A control group, although difficult to find, might have helped determine that instruction based on conceptual understanding would result in a greater significant gain than the control group taught with traditional instruction.

Threats to Generalizability

This study included two fourth grade classrooms from one school. The results should generalize to other fourth grade students in Caldwell and possibly to other schools with similar demographics.

Trustworthiness of Results

The sample size of the group to be studied is small with only 49 students in the treatment group and no comparison group. Although I looked for significant changes in pre to post test scores, the results (because I was the teacher in both groups and with the small n), will be suspect. The study is exploratory and does not demonstrate cause and effect.

Participants and Context

I was the teacher for both groups and have seven years of Developing Mathematical Thinking (DMT) professional development. In addition, I have been recruited to teach the Mathematical Thinking for Instruction (MTI) course, which is built from the DMT framework. Teachers included in the DMT grant participate in a 3 year grant cycle. Each year has a different focus. Teachers receive professional development in a week long institute on the specific focus for the year, as well as guided meetings called Unit Studies based on the same yearly focus, as well as data analysis and classroom practices, every 6 weeks. Observations are made twice a year by DMT instructors. The observations are guided by observation rubrics. Teachers are given a level that indicates their ability to include five critical elements in their classroom instruction that were presented during the professional development sessions. These elements are: taking students' ideas seriously, pressing students conceptually, encouraging multiple strategies and representations, addressing misconceptions, and understanding the relational structure of mathematics (Brendefur, Strother, & Peck 2010).

This instructional unit was preceded with a twelve question pre assessment. The suggested tasks within the instructional unit used include researched tasks and developmental practices with fractions and decimals. Suggested tasks have been constructed with extension ideas, support for struggling students, and questions and mathematical content vocabulary to assist with clarification. After the four week unit, students were given the twelve question post test to measure gain in understanding.

Participant Characteristics

There were 49 fourth grade students who received the beginning of treatment. Of the original group of students, only forty-five students received the pre assessment, the treatment, and the post assessment. Four students moved from the classroom to another school. Of the forty-five students, there were twenty-two Hispanic students and twenty-three Caucasian students**.** Of these students, four were English Language Learners. The English Language Learners' primary language in the classroom was English. At the current time of the treatment, there were forty-four 10 year olds, three 11 year olds, and two 9 year olds. In the classrooms, 71% of students qualified for free and reduced lunch.

Procedure

After careful study and research of rational number ideas and developmental practices, a pre assessment was constructed to reflect the most important concepts within this domain. The students received instruction of a four week unit on fractions and decimals. Each class period was one hour. Students who were not showing an understanding of the tasks were given an intervention time of 20 minutes daily. Students were involved in constructing, diagramming, discussion, and reflection. They were graded on their understanding of concepts based on a level of sophistication in diagramming and symbolic usage. Because I was the fourth grade students' mathematics teacher, I developed the instructional tasks and delivered the pre assessment, treatment, intervention, and post assessment. Students were assessed using paper and pencil.

CHAPTER 4: RESULTS AND METHODS OF ANALYSIS

From Pre to Post Assessment

A paired-sample t-test sought to discover if there were significant gains from pre assessment to post assessment. If significant gains from pre assessment to post assessment were found, I may be able to conclude the treatment of the carefully sequenced tasks based on reasoning and conceptual understanding lead to the significant gains.

All 4th grade students were measured, and results showed a significant gain from pre to post assessment. Students who received conceptual based tasks on fraction and decimals showed a significant gain from pre assessment (*M= 5.00, S.D. = 4.05)* to post assessment (*M= 13.3, S.D =2.75), t (44) = 16.22, p=*.00.

Each of the 4th grade classes (Period 1 and Period 2) were independently tested to find gains from pre to post assessment. Period 1 students who received conceptual based tasks on fraction and decimals showed a significant gain from pre assessment (*M= 5.00, S.D. =3.63)* to post assessment (*M= 13.04, S.D.=2.62), t(23)= 15.48, p < .001*.

Period 2 students who received conceptual based tasks on fraction and decimals showed a significant gain from pre assessment (*M= 5.81, S.D.= 4.53)* to post assessment (*M= 13.71, S.D.= 2.91), t(20)= 8.92, p= <.001.*

Item Analysis

The qualitative measurement included inspecting percentages of pre/post assessment items that related to the foundational concepts: **units/unitizing, iterating/partitioning, and equivalence.** The pre/post assessment items related to foundational concepts that had the most gains are listed below for each period:

Table 4.1 Period 1: Greatest Item Gain

Item Number	Foundational Concept	Pre Assessment%	Post Assessment%	Gain
1	Iterating/Partitioning	37	100	63
4	Equivalence	29	95	66
12	Equivalence	12	85	73

Table 4.2 Period 2: Greatest Item Gain

Item Number	Foundational Concept	Pre Assessment%	Post Assessment%	Gain
5	Unitizing	13	89	76
12	Equivalence	26	89	63
13	Iterating/Partitioning	30	94	64

For each period, the items that repeatedly showed the most gain were **equivalence** and **iterating/partitioning**. Not only were students' answers correct, but their sophistication in the use of diagrams and/or explanations increased as well. Students were able to correctly use fraction and decimal vocabulary and symbolic notation. When looking at the Common Core Standards for 5th grade (NGACCSSO, 2010) it was noted students should be prepared for such tasks as using equivalent fractions as a strategy to add and subtract fractions, and they should be able to apply and extend previous understandings of multiplication and division to multiply and divide fractions.

While the gains in understanding the foundational concepts of **iterating/partitioning** and **equivalence**, the minimal gains or low percentage of students correct on the following items for each class period was concerning.

Table 5.1 Period 1 Smallest Item Gain

Item Number	Foundational Concept	Pre Assessment%	Post Assessment%	Gain
9	Iterating/Partitioning	25	23	-2
5	Unitizing	25	76	51
13	Unitizing	25	76	51

Table 5.2 Period 2 Smallest Item Gain

Item Number	Foundational Concept	Pre Assessment%	Post Assessment%	Gain
9	Iterating/Partitioning	21	57	36
2	Iterating/Partitioning	13	68	56
11	Unitizing	13	73	60

These items were chosen because of the extremely low gains compared to other items, as well as the items THAT did not have an item average above 80%. Item #9 showed quite a difference in % correct and very low to negative gains. In Period 1 there was a loss of 2%. In Period 2 there was a gain of only 36%.

Discussion

Trends in Assessment

Several trends were noticed when students discussed classroom tasks, diagrammed understanding, and were being assessed. One misconception noted was students' ideas of the task "find the greatest fraction." Their experiences with the concept of unit fractions helped with how to measure unit fractions conceptually. Students were adept at understanding of comparing unit fractions. Only ten percent of students were confused when tasks asked them to compare fractions with the same numerator, but different denominators (each fraction 1/b piece from making one). Sixty-four percent of students were also confused with numerators that were different and had different denominators. The confusion was discovered with classroom discussion. Students were misunderstanding the need to consider the fraction a/b's total area, or placement on the number line. Item number 9 in the pre/post assessment follows this trend.

Which fraction is larger $\frac{4}{5}$ *or* $\frac{8}{9}$*? Explain your thinking without a diagram.*

The goal of item 9 was to determine whether students could reason that each of these fractions is a unit fraction away from one. When looking at post assessment results I realized it did not matter whether students did or did not use a diagram. I counted this item correct if a student could draw or reason correctly. The most noticeable trend was students discussing or diagramming each fraction with equal-length bar models, but answering the item after only looking at the size of each unit fraction; students found $\frac{1}{5}$

was greater than $\frac{1}{9}$. The item instead asks students to iterate four $\frac{1}{5}$'s and to iterate eight $\frac{1}{9}$'s. The thirty-six percent of students who correctly thought about this item either discussed that each fraction was unit fraction away from one, the $\frac{1}{9}$ fraction took the least amount of space to iterate again to one, or students showed these fractions on a number line and notated the the smallest fraction would take less space to reach one.

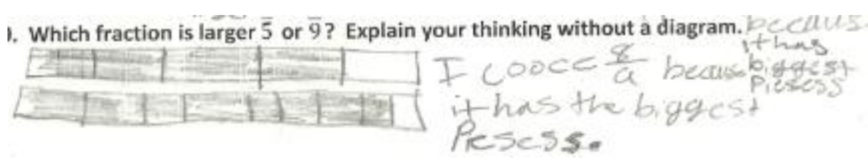

Figure 1. Student 1 Work Sample

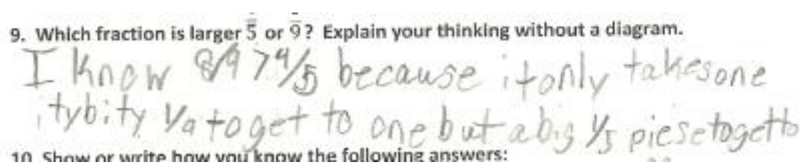

Figure 2. Student 2 Work Sample

CHAPTER 5: RESULTS, RECOMMENDATIONS, AND CONCLUSION

When I began this study, my overall goal was to guide students to develop conceptual understanding of fraction and decimal concepts. A review of the research lead to finding carefully sequenced tasks to encourage conceptual understanding. The research also provided a framework in which to utilize learning theories in order to bridge students' conceptual understanding related to procedural and symbolic knowledge. There seemed to be a delicate balance of how to encourage students to use their own model, yet recognize their knowledge in a more formalized notation. The following sub goals and outcomes are discussed with recommendations for further studies and instruction.

The first goal of this study was to help students access what they knew about whole number and guide them to use this knowledge for beginning decimal and fraction understanding. When students were able to iterate a unit fraction, they used the same enumerating idea as they would have if they had counted with whole numbers. When students were able to decompose '1' or a whole number other than one, they used the decomposition of number. More specifically, when students decomposed a whole number into its fractional parts, they equipartitioned the '1' or whole number. When students joined, separated, and multiplied a fraction by a whole number, they used iteration. Items on the pre/post assessment that assessed these concepts bridging whole number to fraction and decimals, resulted between 85-100% of students' showing correct understanding depending on the specific item. This is important because at least 85% of students showed they could use whole number knowledge when necessary but could modify their thinking to fraction and decimal understanding.

As mentioned in the literature review, students sometimes struggled with how to notate and compute fractions that were being joined together but that would result in a sum greater than one. The items on the pre/post assessment that assessed this concept resulted in 73-100% correct student answers. These data indicate there were students who understood how to notate and reason about numbers greater than 1. There were other students who could join fractions resulting in less than ‘1’ but were not sure how to diagram or symbolically notate the solution for numbers greater than ‘1.’ Based on these results, students should be working to intertwine and utilize their knowledge of iterating and partitioning whole numbers greater than ‘1.’ Students need many experiences with enumerating past ‘1.’ A number line would be a valuable tool to help students make jumps as they enumerate the unit fraction. A valuable question a teacher could ask would be, “How many of the unit fractions are there in (insert a whole number greater than 1)?” Students could also practice counting out loud forward and backward with the unit fraction. Much like counting on a number line, as a unit fraction is added or taken away, the number of unit fractions increase or decrease and could be notated or voiced by the class.

The second goal of this study was to help students transfer their understanding of fractions and decimals with physical models and diagrams to symbolic notation. This was an important step when students began to reason with very small unit fractions such as 20ths or 100ths. Finding a model other than a hundredths grid to model these small unit fractions lead to students trying to draw 100/100, when they could have reasoned about which fractions would be close to 0, ½ or 1. Items on the pre/post assessment that assess these concepts averaged between 75-90% of students showing correct answers. Referring back to the literature review, classroom tasks should include time for students to build and manipulate physical models. This is helpful for students to understand the underlying mathematical principals involved in the model they represent (Bruner, 1966). The suggested instructional unit suggested time for students to do so, but there could be modified tasks and extensions for students to build and then diagramming their construction and understanding. Students should also justify why they should use one model over another. With the sequenced tasks created, there was not as much time left for students to have a written justification for their chosen model. Another possibility would be to have more time for student work to be shared, visibly examined, and discussed in small group and whole class settings. Students could compare and contrast methods and models, discussing why they would use one student's idea over another. In this examination process, efficiency and mathematical clarity may become more evident (Gravemeijer & Van Galen, 2003).

Another important idea in this study was to focus the sequence of tasks and discussions based mainly on Jerome Bruner's learning theory of modes of representations. The modes of representation are enactive, iconic, and symbolic. Students spent time during each task experiencing concepts using physical models, diagramming what they constructed or manipulated, and notating their thinking symbolically. There was some disconnect found among tasks where students had to name pieces of a bar model (if they chose to use that representation). This is significant because students need many experiences with partitioning and iterating. These foundational concepts cannot be taught in isolation. If a bar model is used to equipartition to the unit fractions (1/b) then a number line can be used to compare jumps of the unit fraction (1/b) (Lamon, 2006). Students were able to verbalize the names and could accurately diagram their thinking. When it came time to symbolically name partitioned or iterated pieces, students would name them incorrectly. I think this is because students focused mostly on the bar model and the fraction (a/b). It was not until I discovered this misconception that I really explicitly had students diagram a number line counting in the unit fraction and asked the question "How many (1/b) are in (a/b)?" Students would benefit from continued experience with diagramming their own ideas as well as investigating other students' understandings. This was a process we used with most lessons, but I cannot emphasize the importance of continuing student comparison, discussion, and justification. The most success was found when students problem solved, discussed, compared and analyzed other students' work together, and then were able to write about their new understanding. Students would also benefit from experiencing tasks that support iterating and partitioning ideas, as these foundational concepts occur most often at the same time during the problem-solving process. This is an important understanding because iterating and partitioning cannot be separated and compartmentalized. For example, as a unit fraction is repeated, students can voice and notate the change happening on the number line. As the quantity of the unit fraction increases, the numerator increases. When the goal has been reached, students can discuss how many of the unit fractions are included in the whole number, or the repeated enumerated unit fraction. Representations of student thinking could come in the form of bar models and number lines. Careful thought and anticipation of the model that best fits given tasks should be considered so students can experience many representations. This is an example of mathematizing, referred to in the literature review. When a student has a deep understanding of a concept, they will choose which representation fits the task and the expected solution (Treffers and Bishop, 1987). Justifying students' own thoughts and understandings, as well as investigation of other students' ideas, should help with supporting iconic and symbolic representation.

The final goal was to sequence tasks with a balance of conceptual understanding and procedural knowledge in order to support more difficult concepts that students will encounter in 5th grade and middle school. While all suggested tasks in the instructional unit support multiplicative reasoning, the only device to measure whether or not students are ready for more difficult concepts was to look at the same teaching standards found in later grades. The Common Core Standards for 5th grade state students should use equivalent fractions as a strategy to add and subtract fractions (NGACCSS, 2010). Pre/post test item number 8 measured students' understanding of equivalent fractions. This item resulted in 95% of students correctly showing their understanding. Equivalence seemed to be difficult for students until classroom discussions emphasized the need to to partition each unit fraction into the same quantity of pieces. Some students were not aware of the necessity to partition each unit fraction into the same quantity of pieces until a student mentioned 'sharing equally' means to divide each piece equally, like 'sharing brownies.' When this realization was made, students were able to verbalize, diagram, and discuss their representations of the parts which were equal. If one unit fraction was partitioned, the other unit fractions were partitioned in the same way. The students also remembered the ratio table as a way to show the relationship of the unit changing multiplicatively.

The Common Core Standard in 5th grade addressing multiplication of fractions states students should apply and extend previous understandings of multiplication and division to multiply and divide fractions (NGACCSS, 2010). Pre/post assessment item number 11 measured students' understanding of multiplication of a fraction by a whole number. This item resulted in 73-80% of students correctly showing their understanding of multiplication by a whole number or repeated addition. My recommendation would be to give students multiple experiences and tasks to iterate fractions greater than one. This will be beneficial because students can iterate and count in the unit fraction then discuss how many of the unit fractions are in the new quantity. Students would benefit by using physical models to show the quantity increasing as they are multiplying or repeatedly adding. After using the physical models, students could diagram the increased quantity and rename the mixed number as an improper fraction. They could state how many of the multiplied fraction there are within the new quantity. Symbolically, students would benefit by notating iterations showing multiplicative understanding with the ratio table. These small adjustments to the suggested tasks or tasks teachers create themselves will help support conceptual understanding and procedural knowledge in 4th grade and beyond.

Summary

In conclusion, during this study students experienced tasks pressing their understanding of whole number. Most students became able to determine when to begin interchanging whole number ideas for fraction and decimal ideas. Problem solving and class discussions became relevant when learning to transfer students' newly acquired understanding gained from constructing and diagramming to symbolic notation. Students became adept at the challenge of constructing and diagramming the mathematics of presented tasks, choosing the model which best fit their thinking and the mathematical situation. Most students were strengthening their skills to mentally decipher understanding that didn't require diagramming. Some students still need much more practice with symbolically notating their understanding. Finally, students were not just given procedures to memorize, but gained adequate understanding and skills to support and tackle concepts they will see in grades ahead.

Contributing Factors and Limitations

This study involved a student population that could have been much larger. However, the students in these two classrooms had a wide range of mathematical levels and a mixture of low to mid range socio-economic status (SES). This is important to note as this student population has a very strong skill level of computation, but many lack the language ability to be able to describe understanding. Future studies may benefit from a much larger student population keeping the SES range as similar as possible.

Further studies may also benefit from having a control and a treatment group. Because I was the mathematics teacher, it was difficult to find a similar population in the same district. This was difficult because all elementary schools were included in the Developing Mathematical Thinking grant. There was no 4th grade teacher in the district who does not incorporate at least some of the DMT ideas at some point during instruction. I would recommend finding a control group outside of the district or state.

Based on the results and significant gains from pre assessment to post assessment, it seems students conceptually grasped an understanding of fraction and decimal ideas. According to specific items from the pre/post assessment, students were able to reason conceptually about their solutions. They also were able to use some procedures and symbolic notation to show their understanding. Students also showed gains with concepts that will support their learning in 5th and 6th grades.

CONCLUSION

Deciding which tasks to facilitate in the classroom is a mathematics teacher's most important job. Finding the sequence and balance between conceptual understanding and procedural knowledge and all of the contributing research, should be the most important consideration when deciding on these tasks. The classroom should be a place where students are able to problem solve tasks together and individually, be given an opportunity to construct meaning for themselves, and determine the best method for progression to formalization through the concepts. There is much that should be planned with and for before sequencing instructional units and individual lessons. Students' language level and ability should also be a major factor in how the teacher assesses his or her students. A mathematics teacher should have structural mathematics knowledge as well as an understanding of how students acquire a concept in order to retain, apply, and synthesize to other more difficult concepts.

The results of this study indicate when all of the previously mentioned factors are in place, gains in student achievement are significant. The results of this study show the necessity for students to be given the time and tools to grapple with the unknown. As the mathematics standards that guide instruction become more focused on justification and proof of understanding, students need experiences that will help them become flexible thinkers to determine the best possible solution strategy and why it is so. Instructional practices in mathematics' classrooms that assist students to become *thinkers* should be encouraged so they may acquire a deep understanding of mathematics throughout their school career and beyond.

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APPENDIX A

Suggested Sequence of Tasks

Suggested Sequence of Tasks

<u>Fraction and Decimal Unit: Tasks and Descriptions</u>

This unit is designed and focused on understanding of fractions for 3rd and 4th grade students, and as a remedial tool for 5th and 6th grade students. Ratio and proportion were not included.

Throughout this instructional unit, there are several essential understandings described. As teachers engage their class in these tasks, the Essential Understandings section will guide them to keep in mind the most important elements of fraction and decimal understanding. The four Essential Understandings of fractions and decimals are: units and unitizing, iterating and partitioning, equivalence and relationships, and representations:

Day 1: Pre-Test and Introduction to Fair Shares

Give students approximately 45 min to complete the Pre-test

Concepts	-Fractions as 'fair shares' (e.g. quotient and part-whole meanings for fractions) -No specific decimal concepts are appropriate at this point. However, focusing on tenths or hundredths may support later decimal learning.
Key Developments and Understandings	**Units and Unitizing** **Partioning and Iteritating** -When a whole number is 'split' into more parts than available units (e.g. 3 apples shared by 4 friends), the result is a fractional part of units. -Students will want to use the 'halving' strategy- they need to move to 3 and 6 sharers, (tasks 4-6) -When 1 is split into equal parts, the result is a unit fraction (e.g. 1 meter of string cut into 4 pieces of equal length creates four ¼ meter lengths of string).

	Equivalence and Relationships **Representations** Teacher can fold paper strips into equal parts, and cut, demonstrating students' thinking and discussion. Teacher may also draw a rectangular brownie on the board and divide into equal parts.
Materials Needed	Pictures of a brownies, 4-5 paper strips per student, math journals or notebook paper
Lesson Duration:	**Lesson (after Pre-test):** 30 minutes, **Share out:** 15 minutes
Task: Fair Shares **Process**	**Warm up** Teacher begins this session with the fair share warm up tasks listed above. As students become familiar with 'fair shares,' teacher may then press students to try the following: **Task:** 1. 5 brownies shared by 4 kids 2. 4 brownies shared by 6 kids 3. 7 brownies shared by 6 kids 4. 5 brownies shared by 3 kids These tasks may be written on the board with work space underneath for student strategies later during the class period. The teacher may also use the document camera to demonstrate student work. Students are drawing brownies, folding paper and cutting apart to show number of pieces each would get. Students can begin to name the pieces as they split equally.
Questions to elicit student understanding	"How big is this piece?" "What is the name of the piece?"
Notes	-Students will try to split pieces in half, as their experience with fractions might only be splitting in halves and fourths. Push students to think of and verbalize how many pieces are needed to share with the amount of people

Days 2-3Creating and Adding Unit Fractions with Pipe Cleaners and Paper Strips

Concepts and Vocabulary	-Fractions as numbers -Numerator: How many counted -Denominator: The size of the pieces being counted, "what is being counted"
Key Developments and Understandings	**Units and Unitizing** -Non-unit fractions are composed of unit fractions. The numerator indicates the number of unit fractions of the given denominator needed to compose the fraction (e.g. 4/5 is four 1/5 unit fractions). **Partitioning and Iterating** Students are partitioning the pipe cleaner or paper strips to find equal pieces, and to find the number of number of equal pieces in the '1' or whole Students are using the equal-sized pieces or unit fractions to iterate, finding non-unit fractions **Equivalence and Relationships** Students may begin to recognize that some fractions can be renamed into double or half the denominator or counting piece **Representations and Situations**
Materials Needed	4 to5 paper strips per student, 4 to5 pipe cleaners per student, math journals or notebook paper *Some student may prefer the pipe cleaners, because they unfold and refold easily without leaving a crease. Other students may be successful with paper strips. The teacher could model using both.
Lesson Duration	**Warm up:** 10 minutes **Lesson:** 30 minutes, **Share out with the class:** 15-20 min
Task: Folding Pipe Cleaners and Paper Strips **Process:**	**Warm up** Teacher hands out a paper strip and pipe cleaner to the students, while modeling using the paper strip and pipe cleaner. Studenst are investigating how to: Find 1/2 Find ¼ Find 2/4 *Be explicit about adding ½ equivalences to the ½ chart- have students show how they can make ½ with two, ¼ pieces

	Teacher can model student thinking and discussion using paper strips to equal '1' by folding into parts, these parts can then be labeled ¼, to show that ¼ +1/4= 2/4 **Task:** Find 1/8 Find 4/8, (this should be with the same paper strip, or an equal size paper strip to show ½ is also = to 2/4, 4/8 Find 1/3 Find 2/3 Students pair up to find 9/8, 4/3, 3/2 Extensions: 1/5, 2/5, 4/5 Students pair up to find 6/5, 7/5 These tasks are written one at a time on the board, as the teacher says each task. Students are working and discussing how to partition the paper or pipe cleaner to find each unit fraction, and then how to iterate to find the unit fractions added together. Students are folding, then diagramming what they have constructed, using a bar model, linear model or number line to show each task. Students may feel comfortable coming and notating their strategy on the board to explain, or the teacher may notate for them. At the end of the session, students are explaining ways to partition, and how to iterate the unit fraction with a sentence starter, "An idea that helps me split '1' or more than "1" into fractional parts is:"
Questions to Elicit Student Understanding	"How many equal sized pieces make the '1'?" "How many of this piece will fit into the '1'?" "How many pieces to cover? "What is similar about these models?" What is different?"
Notes	Teacher should be helping students to find ways to compare strategies and notations-informal to formal: paper folding to bar models, to linear models, to the number line. This can be done by having students demonstrating these models with diagrams on the board or under the document camera

Day 4-Renaming the Unit with Pipe Cleaners or Paper Strips

Concepts and Vocabulary	-Fractions as numbers -Numerator: How many counted -Denominator: The size of the pieces being counted or the name that tells you what unit is being counted
Key Developments and Understandings	**Units and Unitizing** -Non-unit fractions are composed of unit fractions. The numerator indicates the number of unit fractions of the given denominator needed to compose the fraction (e.g. 4/5 is four 1/5 unit fractions). **Partitioning and Iterating** The unit is repeated or iterated to construct the whole **Equivalence and Relationships** **Representations and Situations** The referent whole or 1 can change sizes.
Materials Needed:	4-5 pipe cleaners per student, math journals or notebook paper
Lesson Duration	1 day, Lesson 45 min, 10-15 min for students to write what they learned or extension for ticket out
Task: Building with unit fractions, and breaking into unit fractions	**Warm up** Using 1 pipe cleaner, without folds, teachers asks: • "If this is ½, what does the whole look like?" (Student pairs join their pipe cleaners together to make '1'.) • Following with, "How many one-half pieces to make '1'?" • "What are other pieces that = ½? • Draw a picture which models other pieces which = ½. (These can be put under the document camera) look for 2/4, 4/8, 3/6 Using the same pipe cleaner, teacher asks: • "If this is 1/3, what does the whole look like?" (Students may join 3 pipe cleaners to make '1'.) • Following with, "How many one-third pieces to make '1'?" Teacher can model student thinking and discussion using pipe cleaners or paper strips (unfolded),

	notating 1/2 +1/2 =1, 1/3+1/3+1/3=1 : **Task** Teacher should press students to try the following: Using 1 pipe cleaner, without folds, teachers asks: • "If this is 2/4 what would the whole look like" • Following with, "How many two-fourth pieces to make '1'?" Teacher writes: 2/4 +2/4 = 1, students can diagram on white boards or math notebooks, partitioning the fractional piece to show understanding of 2/4=1/4+1/4 Using the same pipe cleaner, teacher asks, • "If this is ¾ what would the whole look like?" (Students would use 1 piper cleaner for ¾, and partition another pipe cleaner into 3 pieces, to show 1 piece more can be named 1/4 • Following with, "How many one-fourth pieces to make '1'?" Teacher writes: 3/4 +1/4 = 1, students can diagram on white boards or math notebooks, partitioning the fractional piece to show understanding of¾=1/4+1/4+1/4 Using the same pipe cleaner, unfolded, teacher asks: • "if this is 2/8 what would the 1 look like?" (Students can put 4 pipe cleaners together to make 1) • Following with, "How many two-eighths pieces to make 1?" Teacher writes: 2/8 +2/8+2/8+2/8=, "explain why we represent this fraction with 2/8"? Using the same pipe cleaner, unfolded, teacher asks: • "If this is 4/8, what would the 1 look like?" (Students can put 2 pipe cleaners together to show 1, they may also recognize that 4/8 +4/8 =1, just as ½+1/2=1. Ask, "4/8=1/2. "Explain why this works" Students write what they understand, and share with a math partner from another pairing. Using the same pipe cleaner, unfolded, teacher asks: • "If this is 2/3, what would the 1 look like?" (Students can partition the pipe cleaner in ½ (2/3 of 3/3) and partition another pipe cleaner in ½ to add to 2/3

	• Following with, "How many 1/3 pieces make 1?" Teacher writes: 2/3 + 1/3 = 3/3. The teacher can also ask, "What are the parts of 2/3?" Answer 1/3 +1/3 Extension -If this is 4/3, what would the whole look like, then asks, "How many 1/3 pieces in 4/3?" (4), "How many 1/3 pieces in 1?" (3), When drawing, the teacher can complete 3/3 to make the '1', and 1 more 1/3 piece, to show 1 1/3. These tasks are written one at a time on the board, as the teacher says each task. Students are working and discussing how to partition the pipe cleaner to find each unit fraction, and then how to iterate to find the unit fractions added together. Students are folding, or drawing using a bar model, linear model or number line to show each task. Students are finding similarities and differences in student models. At the end of the session, students are explaining ways to partition, and how to iterate the unit fraction
Questions to Elicit Student Understanding:	"How many equal sized pieces make the '1'?" "How many pieces to cover?" "Are there different-sized pieces that are equal or equivalent?"
	• Teacher should be helping students to find ways to compare strategies and notations by connecting informal ideas to formal: paper folding to bar models, to linear models, to number line • Teacher should be pressing students to move from enactive representations to the iconic representations.

Quiz: Students will demonstrate and be rated on partitioning, iterating and fair share, and equivalences to 1/2.

Day 5-Comparing and Ordering Unit Fractions

Concepts and Vocabulary:	-Comparing/ordering Unit Fractions -Fractions as numbers -Numerator: How many counted -Denominator: The size of the pieces being counted, "what is being counted"
Key Developments and Understandings	**Units and Unitizing:** -Non-unit fractions are composed of unit fractions. The numerator indicates the number of unit fractions of the given denominator needed to compose the fraction (e.g. 4/5 is four 1/5 unit fractions). **Equivalence and Relationships** -Equivalent fractions -Understanding the relative size of fractions as students begin to understand that the smaller the denominator, the more pieces it takes to create the whole or 1. **Representations and Situations:** When students are given realistic situations, they are able to begin to make sense of fractions and can demonstrate their ideas by their representations such as paper folding, drawings and number lines.
Materials Needed	4 to5 paper strips per student, math journals or notebook paper
Lesson Duration	1 day, 30 minutes, with a 15 minute 'share out' session with individual student strategies and thoughts
Task: Comparing and Ordering Unit Fractions Process"	****Begin using a numberline to ask students to place one given fraction on the number line. Students will begin with one fraction per session, and adding an additional fraction each time. Suggestions: ½, or renamed as 2/4, ¼ and 4/4, ¼ and ¾ , then 1/3** **-Students are asked if fractions are closer to 0, ½ or 1** **Warm Up** Ask students: "If you wanted the largest piece of rectangular pizza would you rather have?" • ½ or ¼? (Students should reason that since it takes 2 one-half pieces to make the whole or '1' pizza, ½ would be the better choice, and it takes 4, ¼ pieces to make the whole or '1' pizza, ¼ would be smaller than ½ **Task**

	Ask students: "If you had these pieces of pizza, how could you prove to me that you know how to put them in order from the least to the greatest sized pieces? - build, draw write to justify how you know Can I start with comparing 2 of the pieces? Justify your order of least to greatest sized pieces with a model, drawing, or diagram in your math journal." 1/3, 1/8, 1/5, 1/10- leave models on the board from class discussion as a way to justify which is larger
Questions to Elicit Student Understanding	"How many equal sized pieces make the '1'?" "If my '1' is cut into 10 pieces, will the pieces be bigger or smaller than a '1' cut into 8 pieces?' Students will use the sentence starter to explain: "I know the largest pieces will be from the '1' cut into ____pieces, because…" "How many of this piece to cover the '1'? "Are the size of the pieces bigger or smaller than____?" "What are your reasons for ordering this way? Use models/diagrams to explain your ideas."
Notes	Teacher should be helping students to find ways to model or notate each of these fractions to compare the size of each unit fraction. Teacher should be pressing students to move from enactive representations to iconic and symbolic representations..

Day 6-7 Cuisenaire Rods: What is the number name?

Concepts and Vocabulary	-Comparing/ordering Unit Fractions -Fractions as numbers -Numerator: How many counted -Denominator: The size of the pieces being counted, "what is being counted" -The referent whole can change
Key Developments and Understandings	**Units and Unitizing:** -Non-unit fractions are composed of unit fractions. The numerator indicates the number of unit fractions of the given denominator needed to compose the fraction (e.g. 4/5 is four 1/5 unit fractions). -As students work with the Cuisenaire rods, they will be asked to think of the pieces as different fractions. For example, **Equivalence and Relationships** -Equivalent fractions- when students understand the size of fractional pieces, and the relationship they have with the whole or '1', as well as the other pieces, they are able to reason with needing to change the size of the denominator and how to see fractional parts differently when asked to compare and order.
Materials Needed	**Cuisenaire rod set for each group of 2-3 students, math journals or notebook paper, chart paper for class notes**
Lesson Duration	2 days, 10-15 min warm up, 30-45 minutes, teacher lead instruction, with a 'share out' session with individual student strategies and thoughts
Process: Cuisenaire rods: What is the number name?	**Warm up** Teacher will say "Tell your neighbor which sized piece of pizza you would rather have if you wanted the largest piece. Justify your answer with writing/telling if the fraction is closer to 0, ½ or 1. (Teacher writes each pair down one at a time, with discussion after.) Would you rather have: ½ or ¼? 1/3 or ½? 1/6 or ¼? 1/8 or 1/10? *Teacher says, " what is in common with each of these whole pizzas? (They have to be the same size to compare which is larger)

	Challenge: Teacher draws 2 different sized pizzas on the board, making the 1/3 piece of the 1st pizza bigger than the ½ of the 2nd pizza, then asks, "Which would you rather have 1/3 or ½?" Students should decide and discuss that the size of the piece depends on the size of the whole or '1.' **Task:** As each task is given to the class, one at a time, students are building and discussion, the teacher is circulating and watching for student understanding and misconceptions. After each task, there is a discussion of what students found, and are justifying their thoughts. The teacher is notating as students discuss. Students are notating each finding in their math journal. Students will build, diagram and justify their understanding. Teacher asks: 1)If orange is the whole, what number name would we give the yellow? 2) if blue is the whole, what is the number name for light green? 3)If brown is the whole, what is red? (1/4) 4)If brown is the whole, what is the name for pink? (1/2)- Day 7 Warm up: ****Begin using a numberline to ask students to place one given fraction on the number line. Students will begin with one fraction per session, and adding an additional fraction each time. Suggestions: ½, or renamed as 2/4, ¼ and 4/4, ¼ and ¾ , then 1/3** **-Students are asked if fractions are closer to 0, ½ or 1** Start with 1/3 pieces 5)If blue is the whole, then what is Light green? (1/3) 6)What is the number name for white? (1/9) 7) If blue is the whole then Dark green is (6/9- using whites, and (2/3 using light green) 8) if blue is the whole, then 1 red is (2/9 because 3R =2/3 (3 red = 2 light green or 1 dg 9)If blue is the whole the brown is (8/9) 10)If blue is the whole the name for pink is (4/9) b/c it takes 2 reds to make pink (2/9 +2/9=4/9 11) If whole is blue, what is the name for black?

	If students are struggling, use these rods to continue to see relationships with the '1' or whole, and the number of pieces that cover, and equivalence **Use the pink, brown, red for ½, 1/8, 1/4 **Use blue, light green, white for 1, 1/3, 1/9, red=2/9 **Use dark green, light green, red and white for 1, ½ and 3/6, 1/3, 2/6, 1/6 Task: extension 1) 1 red rod= whole, what number is dark brown 2) 1 red rod =1/5, how many to get to the whole altogether 3)dark green =3/5, which rod =1/2 (yellow) 4. if dark green is the whole, what fraction (or number name) is the yellow rod? 5. If dark green rod is one whole what fraction is the blue rod? Ticket out: use the bar model or two numberlines to show which is larger, ¾ or 3/6
Questions to Elicit Student Understanding	"How many equal sized pieces make the '1'?" "How many of this piece to cover the '1'? "Are the size of the pieces bigger or smaller than____?" "Are there different sized pieces that are equivalent?" "What are your reasons for thinking this way? Use models/diagrams to justify responses."
Notes:	• Teacher should be helping students to find ways to build the given fraction as well as encouraging the bar model or linear model for notating • Teacher should be pressing students to move from enactive representations to iconic and symbolic representations. For example from tracing the rods, to the bar model, to the number line

****Quiz: Name all the pieces which equal ½ (students may build, and diagram with a bar model, or justify with a number line, partitioned into appropriate parts, to show equivalence to 1/2**

Equivalence and Sharing and Comparing

(Susan Lamon, 2006)

Day 8"Cut the cakes to show pieces of the same size", shade each length to show which is larger (pgs 94-95)

Concepts and Vocabulary	**-Equivalence** **-Comparing fractions with a given bar model** **-Extension: How much larger?- understanding 'how much larger is a long process (Lamon, 2006), but students are given a model to begin to compare which fraction is larger, and by how much**
Key Developments and Understandings	**-Units and Unitizing:** Students will need to know how the '1' is partitioned to create unit fractions. For example, understanding that a unit fraction such as ¼ is created by partitioning the '1' into four, ¼ pieces. ¼ is one piece of the four. **_Partitioning and iterating:** Students will use the previously mentioned understanding to determine which of the given fractions is largest **Equivalence and Relationships:** When students have the opportunity to show equivalence with the model given to them, they can generate 'same sized pieces' in each bar model or 'cake pan'. Understanding equivalence will help students

	when they begin to operate with fractions. Finding common denominators will be less difficult when students understand relationships between fractions.
Materials Needed:	**-Blackline master with fraction bar models (BLM #___)(Susan Lamon, 2006)- making the pieces in the cake equivalent, which fraction is larger and by how much** **-Paper strips for folding**
Lesson Duration	**Warm up: 10 min (unit fraction comparison- 1 pair)** **“Equivalent cake pieces”: 30 min** **“Which fraction is larger, and by how much?”: 30 min**
Process:	**Warm up:** Pick a unit fraction pair for students to justify which is larger -Ask students: “If you had these pieces of pizza, how could you prove to me that you know how to put them in order from the least to the greatest sized pieces? - build, draw write to justify how you know Can I start with comparing 2 of the pieces? Justify your order of least to greatest sized pieces with a model, drawing, or diagram in your math journal.” 1/3, 1/8, 1/5, 1/10- leave models on the board from class discussion as a way to justify which is larger

	Task: • Students will practice shading bar models to show equivalent pieces • When students are working with 'which fraction is larger, and by how much,' it may be important for modeling on the first several tasks. • Students may not be able to determine how much larger the largest fraction is, but using equivalence, can be lead to begin understanding. • **Ticket out:** Students are asked to determine how to use given fractions in bar models, to make equal size pieces • **Or,** they may choose to determine which fraction is larger, and by how much. Students will justify their thinking with a sentence starter: • **"I understand _____ fraction is larger because….I know it is larger because…"**
Extension	**Use the previous tasks to determine if students can use the numberline with the bar model to justify equivalence or fraction is larger**

Day 9- Renaming Fractions as Decimals with 10x10 grids

Concepts and Vocabulary	-Renaming Fractions as Decimals -Decimals are similar to 'part-whole' fractions and represent parts of 1. With decimals, the denominators are always powers of 10 and follow a similar sequence to whole number place value. -tenths, hundredths, thousandths
Key Developments and Understandings	**Units and Unitizing:** -Students will need to recognize the fraction name and be able to make its decimal comparison by recognizing equivalence or by renaming to a new place value. --*'Tenths, hundredths,* and *thousandths'* in the context of decimals represent parts of 1 that are decreasing in size. Each unit of the larger place value is 'split' into 10 of the next smaller place value. For example, 3 tenths are composed of 30 hundredths, therefore .3 and .30 are the same portion of 1 but are measured in different units. **Partitioning and Iterating:** Students will be partitioning or splitting up a unit, the 10x10 grid into equivalent portions. For example, students will be asked to find ½ of the grid and ¼ of grid. They will need to portion the grid into the amount of pieces indicated by the denominator. **Equivalence and Relationships:** -If students are able to recognize the relationship between equivalent fraction and decimals, they will be able to solve and justify ordering fractions and decimals, and will have multiple ways of determining the equivalence of rational number.
Materials Needed:	10x10 grid squares (4 on each page), 2 per student, or plastic sheets and expo markers can be for reuse and erasing, math journals or notebook paper, chart paper for class notes, overhead copy or projection of 10x10 grid sheet
Lesson Duration	10-15 minutes for warm up, 30-45 minutes for the task, 10 minutes for a 'share out' session with individual student thoughts
Task: <u>Renaming</u> <u>Fractions as</u>	**Warm up:** **Task:** • Students should be given some time to familiarize themselves with the 10x10 grids. For example, each

Decimals with **10x10 grids** **Process:**	grid represents the number '1'. • From practicing multiplication facts, students should be familiar with the product of 10x10. Therefore, they should understand that there are 100 small grid squares. Teacher should begin by saying: "Just like we have 2 names, fractions can be named something else too. We are going to look at another way to name a fraction. Use Van de Walle's meter and decimeter reference. "How many squares are in our 10x10 grids? Let's pretend each grid is representing the number '1'." Tasks: Students will be finding equivalent fractions and decimals. Teacher presents the tasks one at a time, and asks: • "Shade ½ the grid. "How many squares is ½ of the grid? So, another name for ½ is 50/100. When we rename this in decimals we say 50, one-hundredths or .50" • "Each column has how many 100ths? How many rows of 10? How many rows did we shade? The decimal name is also 5/10 or .5" • "On the next grid, shade 25/100ths. How many 10's did you color?" 2. "are there pieces shaded that aren't in a group of 10? What are those pieces called?" 100ths. "So we colored 2 10ths and 5 100ths. That decimal name is .25 or 25/100ths or 2 10ths and 5 hundredths." • "What did we do with the area of the grid when we cut it in ½? What can we say we did when we found 25/100ths?" Cut the 50/100 in half. "What would be the name of the fraction that is ½ of a ½?" 1/4 • "On the next grid, show me 100, one-hundredths. What is the fraction name?" Answer:100/100. "Are there are other names we can call it?" Answer: 10/10, 1. "Do we have a decimal or fraction?" "Show me how you know." • "On the next grid, shade 75/100. How many 10's did you color?" 10. "Are there pieces shaded that aren't in a group of 10? What are those pieces called?" 100ths. "So we colored 7 10ths and 5 100ths. That decimal name is .75 or 75/100ths or 7 10^{th}s and 5 hundredths." • "Does anyone recognize where we might see 100ths of something?" 100 pennies in a dollar, 10ths are a

	dime, 5/10 is a nickel, ½ of a 10th. We also talk about percents almost every day. Percents are a comparison to 100 and are really measured in 100ths.” • “On the next grid, shade 1/10th. How many 100ths are shaded?” 10. “How many groups of 10 are shaded?”1. “So this fraction name is 1/10 or one-tenth. Does anyone know how we can write it as a decimal? How did we write the others?” .1 • Use the ratio table to show 2/10=1/5, combined with the 10th grid to show 4/10= 2/5, 6/10= 12/20. As the number of pieces in the ‘1’ increase, they are getting smaller. Justification: pieces get smaller because the same ‘1’ must fit into the same area • “Try coloring four 10ths. Show me how to write it as a fraction, and different ways to rename as a decimal.” • “Try 6/10, 8/10, 9/10. Justify your fraction and decimal.” • What would 11/10 look like? Would that be less than one grid, equal to 1 grid or more than one grid? What about 13/10? How much more than 1 is that? • What would another fraction be that would describe how much we had? What would a decimal be to describe how much we have?” 1 3/10, 1.3 What would the decimal name and fraction name be if I wanted to split 25/100 or ¼ in half? Justify your answer with your 10x10 grid. (1/8 or 12 ½ hundredths) decompose 12 ½ hundredths into 10/100, 2/100, ½ of 100th
Questions to Elicit Student Understanding	“Find all the ways to make______” Insert fraction. “How many equal sized pieces make the ‘1’?” “Rename this as fraction or rename it as a decimal.” “What are your reasons for thinking this way? Use models/diagrams to justify responses.” “How are decimals like fractions?” “How are fractions and decimals different?” “What is another name for fractions and decimals greater than 1?”
Notes	Teacher should be pressing students to make connections between the grids and the method they used to shade the grid as well and the number names and symbols used to describe the shaded portion. This will connect the iconic and symbolic representational modes.

Day 9-10-Using 0, ½ and 1 to compare fractions

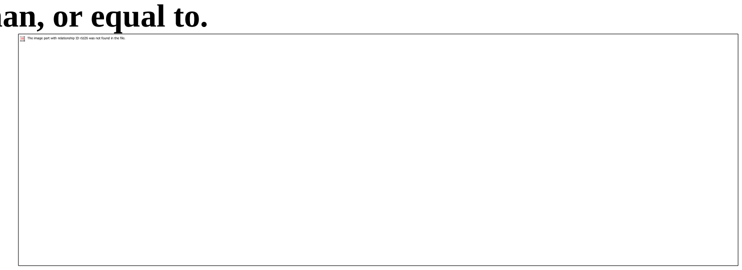
Concepts and Vocabulary:	-Equivalence comparisons to landmarks 0, 1/2, and 1 -greater than, less than, equal to: 0, ½ and 1
Key Developments and Understandings	-Fractions can be compared by means of either common denominators, common numerators, or land mark numbers such as 0, ½, and 1. **Units and Unitizing:** **-**When judging the size of fractions, unit fractions and how many it takes to make the whole or 'one' is important to consider. **Equivalence and Relationships:** --The 'size' of the denominator must be considered when converting to equivalent fractions, specifically how this affects the numerator. For example, 3/6 = 6/12 because sixths are twice the size of twelfths (or twelfths are half the size of sixths). Therefore, it should take twice the number of twelfths to create a fraction equivalent to 3/6. -The 'size' of the denominator must be considered to gain understanding of how close the fraction is to 0, ½ 1. How many of the unit fractions to get close to the fraction landmarks? -If the numerator of two fractions are the same, the students need to conceptually understand which denominator, or the measuring piece is the largest.
Materials Needed:	3x5 cards, 6-7 per student group, 10x10 grid paper (4 on a page) 2 sheets per student, math journals or notebook paper, chart paper for class notes, overhead copy of the 10x10 grid sheet, paper strips, 2-3 per student available
Lesson Duration:	2 days: each day, 10-15 min warmup, 30-45 minutes for tasks, 10-15 minutes for student 'share out' session as a class or in math journals
Task: **<u>Using 0, ½ and 1 to compare fractions</u>**	**Day 9:** **Warm up** **Process:** • Write fractions on note cards for each group of 3-4 students- fractions that are: greater than 1 (9/8, 11/10, 12/11, 6/5, 4/3) with the others ranging from 0-1 such as 3/12 2/10, 2/3, 1/5, 7/8, 3/6, 6/7, 7/12 , but with denominators of 12 or less. • Students sort into 3 groups. The groups are:

	less than half, greater than half, or more than 1. If the numbers are less than half, students can determine if the number is closer to 0 or ½. • Fractions close to close to 0 are: 1/5, 2/10, 3/12. Fractions close to ½ are: 2/3, 3/6, 7/12. Fractions close to 1 are: 7/8,6/7. • Some students may need to use the fraction rods, bar models, linear model and number line to judge the size of the fraction. Teacher should frequently ask: "How do you know this works? Are there other ways to describe/rename this fraction? • Students should try writing number sentence to compare fractions. For example, if the fractions 2/3 and 2/10 are being compared, students can write "2/3 +1/3= 3/3, 2/3 is 1/3 piece away from 1, but 2/3 can be renamed as 4/6. 4/6 is 1 more piece than 3/6. 1/6 pieces are smaller, so 4/6 or 2/3 is closer to ½." "2/10 -2/10= 0, 2/10 is closer to 0." Students work in math partnerships. The teacher may ask students to justify how they know a fraction from each pile (0, ½, and 1), is close to each landmark number **Day 10:** **Process** Continue with: use fractions with denominators greater than 12.. Press students to find the fraction equivalences that are close to ½, as well as fractions that have larger denominators that are close to 0 and 1. Fractions that can be used are: 53/100, 12/100, 79/100, 18/40, 15/30, 7/14, 19/20, 24/50, 2/50, 90/100- suggestion: give one or two fractions to different groups and have them justify why they believe fractions are close to the benchmark numbers • Write fractions on the board, students find fractions that are close to the landmarks- no sorting, but students can use models, manipulatives, **number lines** to help Students are continuing to prove the size of each fraction by writing number sentences.
Questions to Elicit Student Understanding	"How many equal sized pieces make the '1'?" "Can you rename this as an equivalent fraction or rename it as a decimal." "How many of the unit fraction will it take to

	reach 0, ½ or 1?" "Draw a diagram or write a number sentence that reflects your thinking and describes which landmark number you think its closest to." "What are your reasons for thinking this way? Use models/diagrams to justify responses." "Can you write a number sentence to prove this fraction is greater than, less than or closer to the landmark?"
Notes:	• Teachers should be asking students to rename fractions as decimals, or equivalent fractions to help decide if they are closer to 0, ½ or 1. Students should be involved in a discussion about the size of fractions being compared to 0, ½ or 1, and how they know the size by using written justifications (e.g. paragraphs), diagrams, models, or number sentences.

Day 11-Comparing Fractions- Which is greater?

Concepts and Vocabulary	Which fraction is larger? Which landmark can it be compared to? Using 0, ½, 1 , equivalent fractions, and decimal names of fractions to compare
Key Developments and Understandings	- **Equivalence and Relationships** -The 'size' of the denominator must be considered when converting to equivalent fractions, specifically how this affects the numerator. For example, 3/6 = 6/12 because sixths are twice the size of twelfths (or twelfths are half the size of sixths). Therefore, it should take twice the number of twelfths to create a fraction equivalent to 3/6. -The 'size' of the denominator must be considered to gain understanding of how close the fraction is to 0, ½ 1. How many of the unit fractions to get close to the fraction landmarks? -If the numerator of two fractions are the same, the students need to conceptually understand which denominator, or the measuring piece is the largest. -The referent whole needs to be known before being able to compare fractions. -Fractions can be compared easily if the size of 1 is equal.

Materials needed	10x10 grid paper (4 on a page) 2 sheets per student, math journals or notebook paper, chart paper for class notes, overhead copy of the 10x10 grid sheet, fraction rods
Lesson Duration	1 day, 10-15 minutes for warm up, 30-45 min for the lesson, 10 min for student 'share out' or writing about what students have learned
Task: **Comparing** **Fractions- Which** **is greater?**	**Process:** **Warm-up:** Teacher writes on the board: "Order from smallest to largest: 1/4,1/5, 1/3. prove your answer with a diagram or number sentence" When students finish, have several justify their thoughts, look for students that used a diagram and those who used number sentences, can the students compare how they are similar? **Task:** Teacher writes on the board: "How much of a pizza would you rather have (if you wanted the 'most')?" The first one or two fractions can be discussed as a class. Some students might be ready to move ahead but should always be proving with diagrams or number sentences. Other students may need to use paper folding, have help with the bar model and the number line. **Teacher can introduce the double number line: same-sized number line to placing the landmarks 0, ½ and 1 on each line. Students can place each fraction on a number line in its correct place to compare greater than, less than, or equal to.**  • 5/12 or ¾: Are they equal or is one less? • 4/5 and 9/10: Are they equal or is one less? • Others to try: 4/5 or 4/9 4/7 or 5/7 3/8 or 4/10 5/3 or 5/8

	¾ or 9/10 7/12 or 5/12 5/8 or 6/10 4/6 or 7/12	3/8 or 4/7 3/5 or 3/7 9/8 or 4/3 8/9 or 7/8
Questions to Elicit Student Understanding	“How many equal sized pieces make the ‘1’?” “Can you rename this as an equivalent fraction or rename it as a decimal.” “How many of the unit fraction will it take to reach 0, ½ or 1?” “Where on the number line does each fraction fit? How do you know?” “Draw a diagram or write a number sentence that reflects your thinking and describes which landmark number you think its closest to.” “What are your reasons for thinking this way? Use models/diagrams to justify responses.”	
Notes	• Teacher should be pressing students to utilize a variety of representations to justify their conclusions. Enactive, Iconic, and Symbolic representations should be encouraged along with connections between each mode. • Students should be involved in a discussion about the size of fractions being compared, and how they know the size by using written justifications (e.g. paragraphs),diagrams, models, or number sentences.	

Day 12-Roll Out Fractions

Concepts	Which fraction is larger? Using 0, ½, 1 , equivalent fractions, and decimal names of fractions to compare
Key Developments and Understandings	**Units and Unitizing:** -The referent whole needs to be known before being able to compare fractions. **Equivalence and Relationships** --Fractions can be compared by means of common denominators, common numerators, or land mark numbers such as 0, ½, and 1. --The 'size' of the denominator must be considered when converting to equivalent fractions, specifically how this affects the numerator. For example, 3/6 = 6/12 because sixths are twice the size of twelfths (or twelfths are half the size of sixths). Therefore, it should take twice the number of twelfths to create a fraction equivalent to 3/6. -Equivalence doesn't change the size of the whole or 1 or the part, the pieces get smaller=more parts -The 'size' of the denominator must be considered to gain understanding of how close the fraction is to 0, ½ 1. How many of the unit fractions to get close to the fraction landmarks? -If the numerator of two fractions are the same, the students need to conceptually understand which denominator, or the measuring piece is the largest. -Fractions can be compared easily if the size of 1 is equal.
Materials Needed	2 Dice per student pair, math journals or notebook paper, chart paper for class notes Have available: 10x10 grid paper (4 on a page) 2 sheets per student, fraction rods, overhead of 10x10 grid
Lesson Duration	1 day, 10 minutes for warm up, 30 minutes for students to play and discuss, 10-15 minutes for student 'share out' time
Task: Roll Out Fractions	**Process:** **Warm up:** Teacher asks, " if you didn't want a big piece of a candy bar, which sized piece would you choose? Prove your thinking by using diagrams or number sentences." Fractions are: ½ or 1/3 4/5 or 4/9 2/6 or 1/3 **Task:** The Fraction Roll Out Game • Students are trying to create the smallest fraction with the 2 dice rolled. Students are working together, one rolls the dice, trying to make the smallest fraction, one dice is the

	numerator the other is the denominator. The other partner rolls the dice and creates the smallest fraction- partners compare which is smaller- proven with double number lines, equal sized rectangular pieces drawn out. The students' discussion of how to create a smaller fraction is very important, as well as how they justify why one fraction is smaller than the other.  • This can be used as an assessment for student understanding- students can keep track of their rolls and their fractions created on a note sheet of paper to be handed in at the end of the session.
Questions to Elicit Student Understanding	"How many equal sized pieces make the '1'?" "How many pieces to cover?" The smaller the piece (or unit fraction), the more of these unit fractions it will take to make 1. The larger the piece (or unit fraction) the fewer of these pieces it will take to make 1. "Can you rename this as an equivalent fraction or rename it as a decimal." "How many of the unit fraction will it take to reach 0, ½ or 1?" "Draw a diagram or write a number sentence that reflects your thinking and describes which landmark number you think its closest to." "What are your reasons for thinking this way? Use models/diagrams to justify responses."
Notes	• Students should be involved in a discussion about the size of fractions created, and how they know by written justification with diagrams, models, or number sentences.

Day 13-14-Addtion and Subtraction of Fractions and Decimals

Concepts	-Addition and subtraction of fractions and decimals -As students are adding and subtracting fractions, the teacher should focus on making sure students aren't adding denominators; instead, the teacher should be helping students understand that when adding or subtracting fractions, the same 'measure' or 'sized piece' is being added or subtracted. The sum or difference represented by adding or subtracting the numerators represents a quantity of the same unit. -If the denominator is different, students are finding a same sized piece in common with both denominators to be able to add or subtract, using equivalence .
Key Developments and Understandings	-Estimating and knowing the magnitude of the fractions being added or subtracted together is crucial! **Units and Unitizing** -When adding decimals the rules of whole number place-value still apply. As you 'fill' one place value unit with 10 of those units, you compose 1 of the next larger place value unit. **Partitioning and Iterating:** -Just as with whole numbers, the ability to partition (decompose) fractions and decimals is important when adding or subtracting, so that parts of the fraction can be joined or separated using strategies students use with whole numbers. **Equivalence and Relationships:** -When adding or subtracting fractions, the need for a common denominator should be explained as converting to the same 'measure' or 'sized piece' so that the sum represented by adding the numerators represents a quantity of the same unit. - Adding and subtracting with like and unlike denominators: If students are familiar with adding of fractions with like denominators and finding equivalence, students should be able to add fractions with unlike denominators easily. -Fractions can be compared by means of common denominators, common numerators, or land mark numbers such as 0, ½, and 1. -The 'size' of the denominator must be considered when converting to equivalent fractions, specifically how this affects the numerator. For example, 3/6 = 6/12 because sixths are twice the size of twelfths (or twelfths are half the size of sixths). Therefore, it should take twice the number of twelfths to create a fraction equivalent to 3/6.

	-Equivalence doesn't change the size of the whole or 1 or the part, the pieces get smaller=more parts. -The 'size' of the denominator must be considered to gain understanding of how close the fraction is to 0, ½ 1. How many of the unit fractions to get close to the fraction landmarks? -If the numerator of two fractions are the same, the students need to conceptually understand which denominator, or the measuring piece is the largest. -The referent whole needs to be known before being able to compare fractions. -Fractions can be compared easily if the size of 1 is equal.
Materials Needed:	**Day 13:** Math journals or notebook paper, chart paper for class notes Have available: 10x10 grid paper (4 on a page) 2 sheets per student, fraction rods, overhead of 10x10 grid **Day 14** 5 colors of construction paper, cut into strips, 12 inches in length, scissors, fraction dice, Math journals or notebook paper, chart paper for class notes Have available: 10x10 grid paper (4 on a page) 2 sheets per student, fraction rods, overhead of 10x10 grid
Lesson Duration	2 days: each day, 10- 15 minutes for warm up, 30 minutes for the lesson, 10-15 minutes of student 'share out' or writing about their fraction knowledge from the lesson
Task: Addition and Subtractions of Fractions and Decimals	**Day 13** **Process:** **Warm up:** The teacher writes on the board: 1/4 +1/4+ 1/4= ¾, is the same as ¼+1/2= ¾, why? Then the teacher writes and asks, students can justify and prove: "How is ¼ +1/4= ½ and the previous example the same? What are other combinations that have the same idea happening?" (e.g. 1/6 + 1/6+1/6=1/2, etc.) When added together, can we rename these pieces as equivalent fractions?" **Task:** **2 problems, the first written, while teacher gives a**

	few minutes for students to work on their own. The students can discuss, and write their justification on the board for comparison to other students' ideas. **#1:** Kate used 3/5 meters of cardboard for her project, and Joe used 4/5 of cardboard for his project. How many meters of cardboard did they use together? **Teacher asks: How many 1/5 pieces will you need to make 3/5 and 4/5?** **#2** Jim filled his container with 4/6 of a gallon of water. He also filled another container with 5/6 of a gallon of water. How much of a gallon, or how many gallons of water does he have? **Challenge:** Jim filled his container with 4/6 of a gallon of water. He also filled another container with ½ a gallon. How much of a gallon, or how many gallons of water does he have now? **Teacher may need to remind students of several ways to find fraction equivalence, such as the paper folding, bar model and double number line.** **Day 14** **Process:** **Warm up:** Teacher writes on the board: "Prove if this is correct or incorrect: 1/3 + 1/3 =2/6 +2/6. Justify with a diagram, number lines, or number sentences." Then the teacher can write: "2/10 +2/10= 2/5. Why? Justify with a diagram, number lines or number sentences The class can discuss and diagram: ¼ +1/4 =1/2, and discuss why this example and the previous use similar ideas?" Students can also relate 1/6 +1/6 +1/6 = ½ as well. **Task: Fraction Cover up and Uncover Game** Cover Up and Uncover: - whole, ½, ¼, 1/8,1/16) Teacher makes the game pieces with the students (no labeling of fraction parts): • The class can discuss that the first strip is '1'- "can anyone rename as a decimal or percent?" • The 2nd strip is folded and cut into 2 equal pieces," the name of these 2 pieces is '1/2.'"

	• The 3rd piece is folded in 1/2, two times. Students can discuss how many pieces make the '1'. " There are four equal pieces to cover, so each piece is called ¼. It takes four, ¼ pieces to make '1'. ¼ is what of ½? ¼ is half of ½." • The 4th piece is folded in ½, three times. Students can discuss how many pieces now make the '1'. "There are eight equal pieces to cover, so each piece is called 1/8. It takes eight 1/8 pieces to make '1'."Also, students should be pressed to name 1/8 is half of ¼. Or 1/8 +1/8= ¼. • The last strip is folded in half, four times. Students can discuss how many pieces now make the '1. "There are 16 equal pieces to cover, so each piece is called 1/16. It takes sixteen 1/16 pieces to make '1'." Students should also be pressed to name 1/16 is half of 1/8. Or 1/16+1/16= 1/8. Cover Up Fraction Game 1. 2 players each use their own (1 strip) as a 'game board' 2. P1 rolls the fraction die (w/only the four fractions written on it...2 sides blank: ½, ¼, 1/8, and 1/16) 3. P1 places the appropriate fraction strip on top of 1 starting on the left and covering to the right. 4. P2 repeats 5. The first player to 'cover 1' is the winner. But, players can't go over 1. If they roll more than they have left to cover, they skip a turn. 6. Then, play 'Cover 2' so kids can go over 1 and use mixed numbers and improper fraciotns. After players are familiar with the game play, they should draw pictures and write number sentences matching their game boards. They can 'exchange' pieces to trade for larger pieces...eventually ending

	at 1/2 + 1/2=1 Uncover 1. After covering 1 w/the cover up game, P1 rolls and removes the fraction rolled starting from right to left. Players cannot skip around the 1 and must uncover from right to left. They will likely need to exchange pieces to remove the appropriate portion of their game board while leaving the correct remaining portion of their game board. It may be helpful to think of the removing the area that is rolled and not necessarily the exact piece. For example, if ¼ is rolled, students will need to remove ¼ amount of space or area, not necessarily the ¼ piece. 2. The winner 'uncovers' the game board first. Players cannot uncover more than they have left and must skip a turn if they roll more than they have on their game board.
Questions to Elicit Student Understanding	"How many equal sized pieces make the '1'?" "How many pieces to cover?" The smaller the piece (or unit fraction), the more of these unit fractions it will take to make 1. The larger the piece (or unit fraction) the fewer of these pieces it will take to make 1. "Can you rename this as an equivalent fraction or rename it as a decimal." "How many of the unit fraction will it take to reach 0, ½ or 1?" "How do we record the fraction or decimal if it's greater than 1?" "Draw a diagram or write a number sentence that reflects your thinking and describes which landmark number you think its closest to." "What are your reasons for thinking this way? Use models/diagrams to justify responses."
Notes	• Students have many ways to think about 'filling up' the 1 or whole. They can also rename the fraction after it fills up the whole.

	• Challenge problems provide students the opportunity to grapple with equivalence. • In every task, students should be using equivalent bar models and double number lines to discuss and diagram equivalent fractions. This will be helpful with uncommon denominators. Students will understand the need to change the size of the pieces to their equivalent to make addition and subtraction easier. • Students should be involved in a discussion about the size of fractions created, and how they know by written justification with diagrams, models, or number sentences.

Day 15-16Task: Making Trail Mix: Extending Addition and subtraction of fractions

Concepts	**Extending Addition and Subtraction of Fractions**
Key Developments and Understandings	-Estimating and knowing the magnitude of the fractions being added or subtracted together is crucial! **Units and Unitizing** -When adding decimals the rules of whole number place-value still apply. As you 'fill' one place value unit with 10 of those units, you compose 1 of the next larger place value unit. **Partitioning and Iterating:** -Just as with whole numbers, the ability to partition (decompose) fractions and decimals is important when adding or subtracting, so that parts of the fraction can be joined or separated using strategies students use with whole numbers. **Equivalence and Relationships:** -When adding or subtracting fractions, the need for a common denominator should be explained as converting to the same 'measure' or 'sized piece' so that the sum represented by adding the numerators represents a quantity of the same unit. - Adding and subtracting with like and unlike denominators: If students are familiar with adding of fractions with like denominators and finding equivalence, students should be able to add fractions with unlike denominators easily. -Fractions can be compared by means of common denominators, common numerators, or land mark numbers

	such as 0, ½, and 1. -The ‘size’ of the denominator must be considered when converting to equivalent fractions, specifically how this affects the numerator. For example, 3/6 = 6/12 because sixths are twice the size of twelfths (or twelfths are half the size of sixths). Therefore, it should take twice the number of twelfths to create a fraction equivalent to 3/6. -Equivalence doesn’t change the size of the whole or 1 or the part, the pieces get smaller=more parts. -The ‘size’ of the denominator must be considered to gain understanding of how close the fraction is to 0, ½ 1. How many of the unit fractions to get close to the fraction landmarks? -If the numerator of two fractions are the same, the students need to conceptually understand which denominator, or the measuring piece is the largest. -The referent whole needs to be known before being able to compare fractions. -Fractions can be compared easily if the size of 1 is equal. -
Materials Needed	Trail Mix recipe black line master(included), ingredients to make the trail mix (if desired), each ingredient written on a note card, math notebooks, poster paper to notate final copies of student strategies to share and post in the class, fraction rods, paper strips, 10x10 grids as needed.
Lesson Duration	1 day for students to work on strategies for expanding their ingredient for the whole class 1 day for presenting and mixing the trail mix, and providing extension to multiplication of fractions
Task: Making Trail Mix: Increasing a recipe to serve all students in the class	**Day 15:** **Process:** **Warm up** Ask students, “how would I know how much water 3 people drank altogether if they each drank 2/3 of a gallon?”

	Students might say, "I could add 2/3 +2/3+2/3, which equals 6/3. " They might recognize with a drawing filling up each cup, and the water needing to 'overflow' into another cup, that it fills up 3 cups. 6/3= 3 cups. "How would I know how much pizza was eaten by 4 people if they each ate 1/5 of a pizza?" Students might say, "If I added all of those pieces I would get four, 1/5 pieces, which equals 4/5. They didn't eat the whole thing." -Students should recognize that the fractional quantity can be iterated, or repeated to find the total. Also, some students might see that the fractional quantity can be multiplied by the number of people. Conceptually, students need to understand that this is repeated addition, so the denominator won't change when the piece is iterated. This results in an improper fraction. Teacher can press for how many one-thirds in '1', or how many one-fifths in '1', to make sense of the concept of mixed numbers (e.g. 6/3 is the same as 2.) **Task:** Students are able to look at a recipe for trail mix that serves 6 people. Students will need to decide how to expand the recipe and each ingredient to serve the number of students and extra adults in their class. The teacher can explain what a trail mix is used for. Students may not be familiar with the use. Teacher discusses how to make the number of servings fit the number of people in the class. Some students have tried repeated addition, although a faster way should be suggested. For example, the teacher could start a ratio table with '1 recipe will feed 6 people, '2 recipes (or doubled) would be enough for 12 people." Some students may use derived facts to help them reach the number of servings needed. Groups of 3 to 4 students are given an ingredient from the recipe written on a 3x5 card. If needed, the same ingredient can be given to different groups. Students begin working in groups and sharing their

	strategies for expanding. Groups can decide on one strategy or use every strategy to notate for the final copy poster. Extension: Once each group has given their strategy for expanding the ingredients, students may want to figure out how much total trail mix there will be when all of the ingredients have been put in. Can students figure out how much each student will get from the base recipe using proportional thinking? Can they use this information to find out if the serving size changes with more ingredient and more people being served? **Day 16:Process** Students are able to present their posters to explain all of the ways they thought of expanding their ingredient. If students found out how much each person receives from the whole mix, they can present their justification. The teacher may also use the following tasks for students to practice and discuss: **The use of a ratio table would be helpful for students to note their thinking and show proportionality.** 1) If a recipe for punch calls for 1/2 cup of juice to serve 4 people, how many ½ cups would be needed to serve 8 people? 16 people? 10 people? (Students can double each ½ cup portion for 8 and 16 people. For 10 people, students could use the 8 people portion added to half of the 4 people portion: 8 people =1 cup. ¼ of a cup (2 people) + 1 cup(8 people)= 1 ¼ cup. 2) If a recipe for cookies calls for 1/3 of a cup of butter for a recipe for 24 cookies (or 2 dozen), how many 1/3 cups of butter would I need for double the amount of cookies? How about triple? (For double the amount, students could add two, 1/3 cups to make 2/3, for triple the amount they could cut the 1/3 in ½ to make 1/6, and add 1/6 to 1/3= ½ cup.)
Questions to Elicit Student Understanding	"Can you find a notation that helps us organize your thinking?" "What happens when you add (iterate) the numerator and denominator to expand the recipe?" "Is there an equivalent fraction or a different number name for the numerator being larger than the

	denominator?" "Explain and show how you know this will serve the right amount of people." "Can you find a decimal equivalent to the fraction?" "Can you show me the decimal equivalent on the 10x10 grid?"
Notes:	

Day 17-Post test or Extension:Dividing Whole Numbers and Fractions

Concepts	Finding the fractional part of whole numbers and fractions
Key Developments and Understandings	-Consider whole number multiplication: the first factor tells how much of the second factor you have or want. -Finding the fractional part of a whole number (such as 12), is not unlike the task of finding a fractional part of a whole. Multiplying by the fraction involves partitioning the whole number into the number of parts that is named by the denominator. -A task involving finding how many fractional parts are in a whole number involves putting the fractional parts together thus making wholes or counting all of the fractional parts. The operator notion of rational numbers is about shrinking and enlarging, contracting and expanding, enlarging and reducing, or multiplying and dividing. Operators transform numbers, and are a set of instructions for carrying out a process. If the denominator of the operator fraction is larger than the numerator, the result will be smaller than the whole number we began with. If the numerator is larger than the denominator in the operator fraction, the result will be bigger than the whole number. -The 'size' of the denominator must be considered when converting to equivalent fractions, specifically how this affects the numerator. For

	example, 3/6 = 6/12 because sixths are twice the size of twelfths (or twelfths are half the size of sixths). Therefore, it should take twice the number of twelfths to create a fraction equivalent to 3/6.
Materials Needed:	Paper strips: 3-4 per student, math journals or notebook paper, chart paper for class notes. Have available: 10x10 grid paper (4 on a papge) 2 sheets per student or plastic pages and expo markers for less copies made, overhead of 10x10 grid
Lesson Duration	1 day
Task: <u>Dividing</u> **<u>Whole Numbers and</u>** **<u>Fractions</u>**	**Process;** **Fractional amount of a whole number:** **Task1:** Teacher writes on the board: "We had 6 cans of peaches. If we ate 2/3 of those cans of peaches, how many did we eat?" (4) Students may draw out the cans of peaches or represent the cans by folding the paper into 6^{th}. If they divide the parts and put them in 3 equal groups, each group would have 2 cans or 2 sections of paper, representing 1/3 of the cans. To find 2/3, they would need to add the 2 cans from each 1/3 portion together to make 4 cans. **A ratio table is a notation that can help with fractional parts as well as increasing the amount of cans of peaches.** **Task2:** **We had 8 yards of of ribbon. Each decoration needs 2/5 of a yard of ribbon. How many decorations can we make? (20)** Students can draw 8 rectangles to represent the yards of ribbon. Each rectangle is cut or folded into fifths. Students might circle 2/5 of a ribbon for the decoration. Each yard would have 2 decorations each, making 16 decorations and 1/5 left over. The remaining 1/5 pieces will make the four remaining decorations, which equals 20 decorations. **Fractional amount of a fraction:** **Task 1:** Students receive a contextual task to investigate why multiplying a fraction by its reciprocal works by

	using paper folding. **James had 2/3 piece of a foot of string. How much more string would he need to equal one foot of string? (1/2 of 2/3)** Students use a paper strip to fold into thirds, or three, 1/3 pieces. Studens can model '2/3' by folding the last one-third piece behind the 2/3 piece. There is ½ of the 2/3 needed to make 1 foot. Students and teacher can discuss that three, one-half pieces (of 2/3) will make 1 foot, if 2/3 is the amount of string to begin with. **Task 2:** **Tori has ¾ of a yard of dirt. How much more dirt will she need to equal one yard of dirt?** This task may be demonstrated by using a piece a paper strip and folding to show three, ¼ pieces. Students can model '3/4' by folding the last one-fourth piece behind. There is 1/3 of the ¾ needed to make 1 yard of dirt. Students and teacher can discuss that four, one-third pieces will make 1 yard of dirt, if ¾ is the amount of dirt to begin with.

APPENDIX B

Pre/Post Test

Pre/Post Test

Coding: IP=Iterating/Partitioning, E/R=Equivalence/Relationships, U=Units/Unitizing, R/S=Representations/Situations

Fair Share

IP 1.1 There were three pizzas ordered for the study group. Four people were studying. How much pizza will each person get, if they shared the pizzas equally? Draw a picture or use the sentence starter to show your understanding:

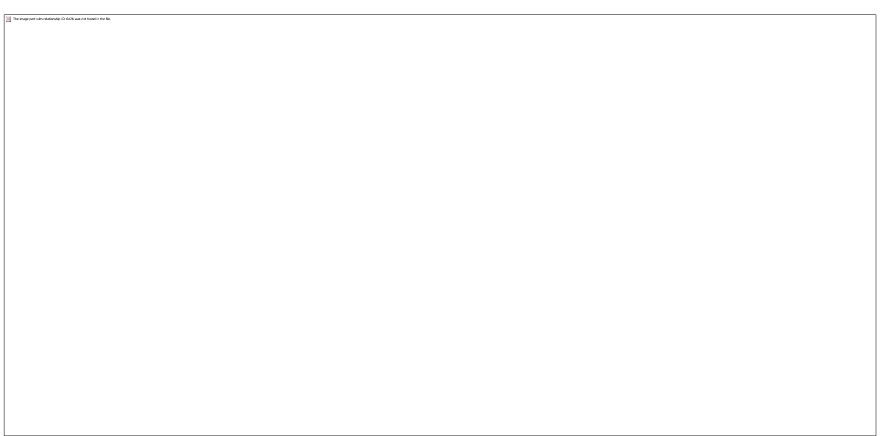
Each person will get_____, I know this because:

Fair share (build on previous, now, more difficult)

IP 1.2. Pick the diagram which best shows the amount of pizza 5 people would get if they shared 4 pizzas

A

B.



Use the sentence starter to describe how you know this is the correct answer and diagram. I picked this diagram because:

E/R and I/P 1.1 Are these fractions close to zero, ½ or 1? Draw a diagram such as a bar model, number line or number sentence to show how you know. Add a sentence to explain your reasoning.

5/8 is close to:

E.R 1.2

11/10 is close to:

E/R , I/P, U 2.1

Use a bar model or number line to explain your answer:

$$\frac{1}{2} = \frac{?}{6}$$

I/P 1.3

Two pizzas were delivered. Each was cut into 4 pieces or four, $\frac{1}{4}$ pieces. Use a bar model to show the pizza you would choose to have the largest $\frac{1}{4}$ $piece$. Why did you choose this pizza?

I/P, U, E/R 2.1

How many $\frac{1}{10}$ pieces are there in $\frac{14}{10}$? _______ How many one whole and left over $\frac{1}{10}$ pieces are there? _________

Use a bar model or number line to show how you know this.

Using real life situations to compare unit fractions (pizza cut into six pieces, pizza cut into eight pieces

E/R, I/P 2.1

If you wanted the largest piece of pizza, would you want 1 piece of pizza from a pizza cut into $\frac{1}{6}$ pieces, or from a pizza cut into $\frac{1}{8}$ pieces (should this be phrased as 'cut into 6 pieces and cut into 8 pieces'?)

Which is larger (numerator comparison, then denominator comparison)? Ordering, greater than, less than or equal to with visual representations, and justification, ** should this be unit fractions first?

E/R, I/P 2.2

Use a bar model or number line to prove which fraction is larger $\frac{3}{4}$ or $\frac{3}{6}$?

E/R, I/P 2.3 Which fraction is larger $\frac{4}{5}$ or $\frac{8}{9}$? Explain your thinking without a diagram.

More, less or equal to 1-**I think I already have this in task 3?

Addition and subtraction of fractions, mixed numbers

I/P 3. 1

Show or write how you know the following answers:

What is the sum of $\frac{2}{10}+\frac{2}{10}$=

What is the sum of $1\frac{4}{10}+2\ \frac{6}{10}$

Multiplication of fractions by a whole number- word problems, between what 2 numbers does your lie?

I/P 4.1

U 1.1

Joe filled 6 containers with $\frac{1}{3}$ cup of raisins for snack. How many cups of raisins does Joe have? Use a diagram and/or number sentence to prove how you know

Is Joe's total amount between….write how you know.

0----------1 cups

$1\frac{1}{3}$---------2 cups

$2\frac{1}{3}$---------3 cups

E/R 3.1

U 2.1

E/R, U Rename $\frac{25}{100}$ as an equal fraction and decimal

Renaming fractions as decimals (.62=62/100) where on the number line?

U, I/P 2.2

Where would you place $\frac{62}{100}$ on the number line? Use mathematical reasons and explain how you know: