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Students' Reasoning Around the Functional Relationship

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Proportional reasoning is related to flexible use of the scalar and functional relationships that exist in proportional situations. More specifically, in regard to the functional relationship, students' understanding of the multiplicative comparison that exists between two quantities in a ratio is a key concept. We conducted student interviews with 12 high performing students to examine their conception of the functional relationship. Analyses provided initial evidence that the majority of students did not conceive of the multiplicative comparison when solving problems designed to press the functional relationship, indicating students' written work that makes use of the functional relationship should not imply understanding of the multiplicative comparison.

Keywords: Algebra and Algebraic Thinking, Cognition, Middle School Education

Introduction and Purpose

Students' ability to understand and apply proportional reasoning is critical to their future success in mathematics and science. Yet, there is evidence students are not developing proportional reasoning during their school experiences (Brahmia, Boudreaux, & Kanim, 2016: Cohen, Anat Ben, & Chayoth, 1999). One way to address this issue is for instruction to focus on developing understandings regarding mathematical relationships present in proportional situations, allowing for connections across topical borders into more sophisticated mathematics (e.g., rate of change, slope, and covariation) and other content areas (e.g., physics and chemistry) (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002; Teuscher & Reys, 2010).

One important aspect of the development of proportional reasoning is the understanding of the multiplicative comparison relationship that exists between two quantities in a ratio (Lobato, Ellis, & Charles, 2010; Tourniaire, & Pulos, 1985). The research focuses on students' conception(s) of the multiplicative comparison relationship (Lo, Watanabe, & Cai, 2004) and how they articulate their conception(s) of that relationship in a problem solving situation.

Theoretical Framework

We argue it is important for students to demonstrate the ability to flexibly and fluently make use of the scalar and functional relationships to solve problems (mathematics perspective), and at the same time they must possess understandings of these relationships through both a composed unit and multiplicative comparison conception (student cognition perspective).

Mathematics Perspective: Scalar and Functional Proportional Relationships

From a mathematics perspective proportional situations involve an equivalent relationship between ratios, such that $a/b = c/d$. There are two multiplicative relationships that can be found within proportions – scalar and functional (Tourniaire & Pulos, 1985). Imagine the situation "*Callie bought 7 cookies for $3. How many cookies can Callie buy for $12?*" as represented in the first row of Table 1. One can solve this problem by scaling up both elements of the original ratio by a factor of 4 to find 28 cookies for $12. This is referred to as the *scalar relationship* because we can scale up both components of the ratio by a common scale factor to create a new equivalent ratio. Alternatively, imagine the situation "*Callie bought 6 cookies for $2. How many cookies can Callie buy for $13?*" as represented in the second row of Table 1. Rather than scaling the original ratio by a factor of 6.5, one can use a simpler mathematical relationship expressing the number of cookies in

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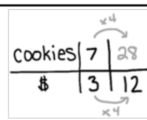
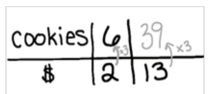
terms of the dollars (cookies are 3 times the dollars). This is the *functional relationship* because one variable is defined as a function of the other.

Table 1: Examples of contexts and student work around the scalar and functional relationships and students' related proportional reasoning conceptions

Mathematical Relationship		Students' Proportional Reasoning Conceptions	
Scalar Relationship		Composed Unit	Students view the quantities in the ratio relationship as separate entities requiring coordination in conjunction with one another. For example, *"If I can buy 7 cookies for $3, then if I multiply the number of cookies times 4 to get 28, I have to multiply the number of dollars times 4 and get $12."*
Functional Relationship		Multiplicative Comparison	Students view one quantity in the ratio relationship in terms of the other quantity. For example, *"If I can buy 6 cookies for $2, then my cookies are always 3 times my number of dollars. Therefore, if I have $13, I know my cookies are 3 times more or 39 cookies."*

Missing-value proportion-based problems can be solved using both relationships. How one chooses to solve the problem typically depends on two factors. First, the context and the numbers therein may make the use of one relationship more efficient (Steinthorsdottir & Sriraman, 2009). Second, the choice is influenced by that particular student's proportional reasoning conception.

Student Conception: Composed Unit and Multiplicative Comparison

Students' conceptions of proportional relationships may parallel the scalar and functional relationships, but their use of a relationship does not necessarily mean a particular conception is present. Students may tend to see a given ratio as a *composed unit* involving the joining of two quantities in a ratio relationship (Lobato et al., 2010) which can then be scaled up or down to create equivalent ratios (see first row of table 1). Alternatively, students may tend to see one component of the ratio as a *multiplicative comparison* of the other component (see second row of table 1). Ideally students will possess both conceptions in order to flexibly operate with ratios in different contexts and situations.

Given the parallel nature of these relationships and conceptions, one may expect students to explain solution processes for scalar items through composed unit thinking and functional items through multiplicative comparison thinking. However, researchers have cautioned that students' solution processes that make use of the functional relationship may actually involve a composed unit conception (Lamon, 1993; Simon & Placa, 2012). For example, given the problem *Callie bought 6 cookies for $2. How many cookies can Callie buy for $13?*, a student may divide 6 cookies by $2 with a result of 3. How a student then expresses the meaning of '3' may indicate a focus on the composed unit or multiplicative comparison conception of the ratio relationship.

- Composed unit example: "Well 6 divided by 2 equals 3, so I knew $1 could buy 3 cookies. I had to multiply by the 3 cookies for a dollar by $13 to get $39." In this example, the student expresses the meaning of the '3' as 3 cookies and part of a composed unit, $1 for 3 cookies.
- Multiplicative comparison example: "Well 6 divided by 2 equals 3 that means the cookies are always 3 times the dollars. If I have $13, I multiply this by 3 to get the number of cookies." In this example, the student expresses the meaning of the '3' as a multiplicative comparison between cookies and dollars, 3 times more.

Students need both conceptions to fluently and flexibly proportionally reason. Yet there is little empirical evidence related to how students who make use of the functional relationship express their conception of this relationship. This study addresses the question: how do students articulate their

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conception of the functional relationship when presented with problems that press the functional relationship?

Methods

We conducted interviews with 12 grade six students who exhibited high performance on an assessment of proportional reasoning to examine their conceptions of the functional relationship. The individual student interviews were conducted at one school immediately following the administration of the assessment. Three items that pressed the functional relationship were selected. Students were presented with their worked solution strategies for the three problems (bulleted below) and asked to describe their process. The interviews were video recorded and then transcribed for analysis.

- Marta found brownie deal with 3 brownies for $9. How many brownies can she buy with $12?
- Tomas found a hamburger deal with 4 hamburgers for $28. How much will 5 hamburgers cost?
- Mark found a hamburger deal with 8 hamburgers for $32. How much will it cost to buy 5 hamburgers?

Analysis

Thirty solution strategies from 12 students were available for analysis as not all students received all items in the interview process. The first round involved coding the written strategies as explicit, implicit, or indeterminate in regards to written evidence for use of a *composed unit understanding*. We opted to code based on evidence of composed unit understanding due to its relatively high frequency in our cursory examination of the strategies. An explicit code typically was applied to written use of the words 'per' or 'each'. An implicit code involved evidence of adding or multiplicative scaling ratios. All other work was coded as indeterminate. The second round of coding involved examining students' interview responses as confirms, indeterminate, or multiplicative comparison in regards to evidence for composed unit understanding. A code of confirms indicated verbal evidence of use of a composed unit strategy, typically involving the use of the words 'per' or 'each'. The indeterminate response could not be clearly coded as a scalar or functional strategy. The multiplicative comparison code involved evidence of one quantity being defined in terms of the other quantity multiplicatively, using the word 'times'.

Results

Of the 30 solution strategies, 27 were correct. The following categories (and counts) resulted from the coding process for the correct written work: explicit (7), implicit (6), and indeterminate (14) indication of composed unit understanding. We followed this by examining the type of thinking indicated in the interview response. The following categories emerged from the interviews: confirmation of a composed unit conception (23), indeterminate (3), and multiplicative comparison (1).

The number relationships were designed to press students to use multiplicative comparison understanding. While all students made use of the functional relationship, only one response out of 27 provided clear articulation of the multiplicative comparison between the quantities in the ratio and 23 of the 27 responses provided evidence of composed unit thinking. This indicates either the problems did not encourage multiplicative comparison conceptions or that the majority of the students did not possess this understanding. A second finding is written solution strategies often do not provide an explicit indication of whether students used composed unit or multiplicative comparison thinking.

Discussion

The results provide evidence that students' use of the functional relationship in their solution strategy does not indicate an understanding of the multiplicative comparison. Students typically made use of the functional relationship through division to generate a unit rate. These results provide empirical evidence to support previous statements in the literature (Lamon, 2005; Simon & Placa, 2012) regarding students' difficulty in understanding the functional relationship as a multiplicative comparison.

A small sample and the use of a discrete, easy to visualize context and missing value problem types may have impacted the findings. It is possible the context and/or problem type increased students' use of per-one or composed unit thinking; a different one may promote more multiplicative comparison thinking.

The primary implication is the need for intentional intervention to make explicit the functional relationship as a multiplicative comparison. Students' use of the functional relationship to generate a unit rate is often assumed to indicate students' understanding of the functional relationship as a multiplicative comparison. It is imperative that we build teachers' knowledge around these relationships and students' conceptions if we want students to develop a multi-faceted ability to proportionally reason.

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