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# A Glimpse into Secondary Students' Understanding of Functions

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**Abstract**

In this article we examine how secondary school students think about functional relationships. More specifically, we examined seven students' intuitive knowledge in regards to representing two real-world situations with functions. We found students do not tend to represent functional relationships with coordinate graphs even though they are able to do so. Instead, these students tend to represent the physical characteristics of the situation. In addition, we discovered that middle-school students had sophisticated ideas of dependency and covariance. All the students were able to use their models of the situation to generalize and make predictions. These findings suggest that secondary students have the ability to describe covariant and dependent relations and that their models of functions tend to be more intuitive than mathematical – even for the students in algebra II and calculus. Our work suggests a possible framework that begins describing a way of analyzing students' understanding of functions.

**Keywords:** *Functions; Students' Thinking, Mathematics*

## Introduction

Reform efforts in mathematics education have broadened the focus of teaching and learning to include not only how students perform, but also how students come to *understand* important ideas in mathematics (NCTM, 1991, 1995, 2000; NGA, 2010). The concept of a functional relationship is foundational to understanding mathematics throughout the K-12 curriculum (Smith, 2003, 1996). In the U.S., understanding functions is a key component in the adoption of the Common Core State Standards for Mathematics (CCSSM) and in the assessment of students' ability to model real-world situations (Darling-Hammond, Burkhardt, & Schoenfeld, 2011).

Much of the earlier research about the learning and teaching of functions has focused on describing how student understandings differ from the formal notion of function. Researchers of concept images have documented secondary and undergraduate student misconceptions of the formal definition of function and how these misconceptions reveal student concept images that differ from the formal definition in stable ways (Vinner, 1992; Vinner & Dreyfus, 1989). Other researchers have documented students' misunderstandings of coordinate graphing conventions (Schoenfeld, Smith, & Arcavi, 1993) and of student difficulties with translating among various

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formal representations of functional relationships (Dreyfus & Eisenberg, 1982; Moschkovich, Schoenfeld, & Arcavi, 1993). Indeed, these difficulties with the formal definition and representations of function are displayed not only by secondary and undergraduate students, but also by many mathematics teachers (Even, 1993; Even & Tirosh, 2002; Norman, 1992; Stein, 1990).

Although this significant body of research has detailed students' understandings of the formal definition and representations of function at the secondary level, only recently have researchers investigated students' informal and pre-formal understandings of functions at the elementary and middle school levels (Ainley, Pratt, & Hansen, 2006; Blanton & Kaput, 2011; Doorman & Gravemeijer, 2009; Ellis, 2007; Gravemeijer, 1999; Rivera & Becker, 2008; Warren, Cooper, & Lamb, 2006). Much of this research has demonstrated that when young students are provided with specific mathematical models, students are able to demonstrate some functional relationships. This research provides a window into students' implicit understandings of functions and their abilities to construct very specific representations: tables and some symbolic representations. However, if students begin to develop the means to model and discuss functional relationships without formal instruction, then is it the case that middle and high school students can bridge between this informal or formal knowledge to new functional situations?

We propose that knowledge of students' intuitive understanding of functional relationships, as represented by their modeling schemes and verbal communication, is crucial in helping teachers build from students' informal thinking toward the formal concept of function. In addition, we hold that students should learn mathematics "so as to be useful" (Freudenthal, 1968); we expect students will employ the mathematics they learn in school in responding to novel mathematical situations. For example, one would imagine secondary students at different grade levels to respond differently to realistic situations based on their experience and instruction in function over time. The purpose of this article is to describe how secondary students from across grade levels demonstrate understandings of functional relationships in two physical contexts, outside of formal function instruction.

The central concern of this work is to better understand student thinking about contextualized situations that can be modeled by functions in order to promote understanding and reasoning in algebra. We use the term student thinking to refer to what students say or do, which is a proxy for their understanding. In more detail, understanding is a complex, dynamic state in which a student is able to connect a piece of knowledge to other, related pieces of knowledge (Carpenter & Lehrer, 1999; Hiebert & Carpenter, 1992; Hiebert et al., 1997). In this view, understanding means a student's knowledge is organized into networks of mental representations in which the strength of the network is related to the number of connections linking the common conceptual ideas (Carpenter & Lehrer, 1999).

Most work in the past literature on student knowledge of functions focuses upon students' acquisition of *formal* knowledge of functions and the procedural skills necessary to construct and manipulate conventional representations of functions (Dreyfus & Eisenberg, 1982; Schoenfeld et al., 1993; Vinner & Dreyfus, 1989). This research often describes students' knowledge of functions as fragmented and inflexible in utilizing different representations of functions (Confrey, 1988; Dreyfus & Eisenberg, 1982; Rizzuti & Confrey, 1988); it focuses on students' concept image of functions and the extent to which that concept image is aligned, or misaligned, with a formal definition of functions (Vinner & Dreyfus, 1989); it debates students' inabilities to perceive functions as higher-level, abstract objects (Sfard, 1992; Slavit, 1997). These findings have implications for teaching because teachers have to understand the limits of students' formal procedural and conceptual thinking about functions.

In order to better understand how students develop and formalize the concept of function, we here examine students' *informal* or initial ideas regarding functional relationships. To focus on students' informal understandings' of function, we rely on the notion of quantitative reasoning (Smith & Thompson, 1996; Thompson & Thompson, 1995), which is grounded in problems that are "verbal descriptions of situations constituted by interrelated quantities" (Smith & Thompson, 2007, p. 102). These problems enable students to conceptualize the problem elements in ways that are related to the problem situation rather than only as quantities and operations – thus encouraging a focus on students' notions of the functional relationships arising from the problem situation. Second, requiring students to focus on the relationships among quantities instead of procedural operations or formal notations can provide rich descriptions of students' intuitive understandings of the functional relationships inherent in the problem situation and students' abilities to communicate their thinking. As Confrey and Smith (1994) point out, when students are encouraged to "generate functional relationships by acting within contextual situations and by using multiple representations in both creating and representing their solution processes, legitimate and diverse ways of thinking about functions are created" (p. 32).

To focus on students' informal understandings and representations of functional relationships, rather than the ways they fall short in formal function analysis, we draw upon research that interprets students' informal reasoning within contextual domains. First, covariance, or covariational reasoning, accounts for the development of students' ability to coordinate the changes in two varying quantities (Carlson, Jacobs, Coe, Larsen, & Hsu, 2002). In Carlson et al.'s framework for covariational reasoning, students begin to model dynamic events by recognizing that there are two quantities changing in a given situation. In order to observe or see a *quantity* as changing, rather than just a real-world object as changing, the student must recognize how some quality of the object can be measured with numerical values (including a tacit recognition of a unit of measurement) and how these values vary in the situation (Thompson, 1994). As students display higher levels of covariational reasoning, they not only recognize the two covarying quantities, but they start to coordinate the changes in one quantity with the changes in the other, including the direction of change and the amount of change (Carlson et al., 2002).

A second aspect of students' informal reasoning about functions is the recognition of dependency – a relation between two quantities in which the values of one quantity depend on the values of the other. This reasoning entails viewing one quantity as an independent variable. In this way it differs from covariation, in which changes in two quantities can be coordinated without designating some priority of one quantity over the other. However, it is not a focus of this article to examine the way a person decides which variable to treat as independent. For instance, in a real-world situation there may be a particular reason to treat a particular quantity as independent because it is perceived to have a causal relationship to the other quantity, such as a person's height depending on her age rather than vice versa. But in purely mathematical situations where there is no causation, this assigning of an independent variable may be arbitrary or conventional, such as treating a circle's area as depending on its radius rather than vice versa.

Students are aware of the difference when they can describe or demonstrate when two variables covary or when the change is one variable is coordinated by a change in the other versus when one is dependent on the other. This knowledge does not necessarily address a students' understanding that two variables might be covarying and both dependent on another third variable. In any case, as the student develops dependency reasoning, they can come not only to view one quantity's values as depending on the other's, but also to determine a stable process by which the quantity's values can be determined from the other's. This process conception of the functional relationship between the two quantities is ultimately crucial to the development of the formal understanding of function (Dubinsky & Harel, 1992; Oehrtman, Carlson, & Thompson, 2008). Only when this process conception of function is developed can students later reify functions into abstract objects in a way that is necessary for advanced undergraduate mathematics (Sfard, 1992; Sfard & Thompson, 1994; Slavit, 1995, 1997).

Our goal was to learn how students demonstrate their understanding about functional relationships within contextual situations (and when not currently being taught about functions). The research questions were: 1) how might students from ages 12 – 17 demonstrate knowledge about functional relationships within a context, whether intuitively or based on their prior knowledge of mathematics and 2) how might students represent these relationships and use them to illustrate their understandings?

With these questions in mind, we designed tasks and student interviews to encourage students to model and talk about their thinking. Our intent was to encourage them to represent the features of a given functional situation they considered important, but to do so without necessarily relying on prior formal knowledge of mathematics. We created tasks that would allow students to represent their thinking to us freely, yet allow them to quantify (Thompson & Thompson, 1995) the given situations as much or as little as they saw fit. Our intent was not to assess the extent of these students' formal knowledge of functions and skills in constructing conventional representations, but to probe their ideas about functional relationships and to make note of their choice of salient features and means of representation. In short, we attempted to gain understanding of how these students make sense of functional relationships.

One goal of the study was to describe these representations and discussions. A second goal was to develop a broad general coding scheme for levels of student understanding of functional relationships. The idea was that such a coding scheme could be used in a larger study, to evaluate the effectiveness of an instructional program or to compare different populations. We discuss this general scheme at the end of the article.

We chose a small sample of students at the secondary level at different ages in varied courses to examine understandings of functional relationships. These seven students were from the U.S. in a Midwest urban area and volunteered to participate in a one-hour interview in which each student completed two tasks independently. The students were from two schools – a junior high and a high school. Interview transcripts of these students were analyzed for the purposes of this paper. These seven students varied in age (12 to 17), grade level ($7^{th}$ to $12^{th}$), mathematics course enrollment (pre-Algebra to pre-Calculus), and sex (male and female). Their characteristics are shown in Table 1 and their names have been changed to pseudonyms.

In order to address our research questions, interview data were collected and analyzed using a qualitative methodology. All students were interviewed once, two months into the beginning of the school year. Each student was interviewed by one of the authors. These interviews consisted of two different tasks and were audio- and video-taped. The protocols for the two semi-structured interviews are included in appendices A and B. Each student was asked to think about, describe, and represent the elements they recognized as changing in the situations. Although interviewers followed the protocol for each task, they also asked additional questions to further probe students' thinking in relevant areas.

**Table 1. Student characteristics**

| **Participants** | | | | | |
|---|---|---|---|---|---|
| Student | Age | Grade | Course | Sex | Ethnicity |
| William | 12 | 7 | Pre-Algebra | M | African American |
| Arlis | 13 | 8 | Pre-Algebra | F | Caucasian |
| Rachel | 14 | 8 | Algebra | F | Hispanic |
| Jennie | 14 | 9 | Geometry | F | Caucasian |
| Movae | 15 | 10 | Algebra II | F | African American |
| Ki | 16 | 11 | Algebra II | M | Caucasian |
| Carlos | 17 | 11 | Pre-Calculus | M | Hispanic |

We chose two tasks for the interviews. The Ball Drop task was used so students could conduct the experiment more than once and see the action. It consisted of an experiment in which a Ping-Pong ball was dropped from the height of a yardstick. Students watched the experiment and then described the elements they observed changing. They were asked to represent this situation on paper and to answer questions about their representations. The task also allowed students to examine different related variables. The Growth task was chosen as a realistic situation that allowed students to discuss the relationship of height and age and build and use a graph to make predictions beyond the teenage years. It included a description of a girl's growth in height over several years of her life: 17 inches when she was born, 35 inches at age 2, and 45 inches at age 5. Students were asked to represent her growth, make generalizations, and explain their thinking about the relationship between age and height.

To analyze the interview data we used qualitative techniques (Erickson, 1986; Schatzman & Strauss, 1973; Wolcott, 1994). Our process included a first read of the data in which initial and general coding took place. To ensure reliability of the codes, two readers discussed their results until consensus was reached. Through this process, general themes were identified. These themes emerged from both the literature review and this initial reading and are described below.

The general themes were utilized to frame a second and more intense reading of the data. During this reading, categories within each of the themes were identified. Two readers met, discussed and refined the coding until consensus was reached. At this point, general assertions, largely through induction, were made for each category. These data were read a third time to search for confirming and disconfirming evidence to justify the assertions. Classes of data were organized, and relationships were drawn, where possible, between these classes. These relationships allowed further analysis.

Initially, the following general themes were found based on the literature and the reading: representation, covariance, dependency, prediction, and pattern. Through a second read of the data, categories within each theme were derived. These themes and their categories are described below. Near the end of the article we also present the general scheme for functional reasoning that emerged from these categories.

***Representations.*** We examined both student representations and explanatory comments in search of clues to student understanding of the functional relationships described or observed during the interviews. We made note of the physical characteristics of these representations, whether graphic, pictorial, or numeric. We, then, reviewed student representations and transcripts for evidence of students' tendencies to rely upon accuracy and detail in conveying information, as compared to their tendencies to rely upon trends or make estimates in creating their representations. Next, we examined student representations and their transcripts in search of evidence of their interest in, awareness of, and ability to illustrate observed change or movement in their representations. For instance, a student might use arrows between rows to denote a consistent change. And, finally, we made note of student comments and explanations characterizing what they seemed to value as effective conveyance of information about functional relationships. From a second read of the data the following categories emerged: *type, salient features,* and *use*.

***Covariance.*** Our second and third themes – covariance and dependency – were separated to understand how students think about these interconnected but different constructs. Covariance was defined as knowledge that two elements were changing in the problem in a coordinated manner. As we read each transcript, we searched for evidence that either confirmed or contradicted this notion of covariance. Once this initial analysis was completed, we discovered students' responses could be categorized as *explicit, implicit, indefinite,* or *no evidence*. Students' responses were categorized as *explicit* when they clearly expressed that two elements were changing. Responses were labeled *implicit* if there was some evidence that covariance was understood even though it was not directly stated. Both of these categories indicate what Carlson, et al. (2002) might describe as the first level of covariational reasoning, in which a student does recognize that there are two varying quantities, but does not necessarily coordinate these two quantities or their simultaneous variation (Carlson et al., 2002). On the other hand, if there was evidence of a higher level of covariational reasoning, in which the student recognizes simultaneous change between the two varying quantities, and coordinates these simultaneous changes in some way, then this was classified separately as dynamic change. If there was some evidence that seemed to imply that students might understand covariance, but this evidence was unclear, their notion of covariance was labeled *indefinite*. In some cases it was noted that students never mentioned or made reference to the idea that two elements in the problem were changing. Their notion of covariance was categorized as *no evidence*.

***Dependency***. Dependency is the understanding that the values of one quantity depend on the values of the other. Evidence of students' understandings of dependency was in a similar way to covariance categorized as *explicit, implicit, indefinite,* or *no evidence*. The understanding of dependency does not necessarily involve the quantification of how one quantity can be determined from the other.

***Prediction.*** For both tasks students were asked to make predictions at different points in the interview. Students did this on the basis of their *model*, the *data* given or collected, or *intuition*. These categories were not mutually exclusive. For example, at different points in the interview students used prior knowledge to make a prediction and at other times used their models for the same purpose.

***Pattern.*** This category refers to students' attempts to describe the overall pattern of the functional relationship. Some students attempted to form *rules or generalizations* to describe the pattern. These written rules often were quantified in some way, but did not necessarily employ formal or conventional notations. Other students tended to describe the pattern *verbally* or qualitatively. Students' verbal descriptions were labeled as either conceptually correct or conceptually flawed. In some cases students made no attempt to describe the overall pattern.

Sometimes a student may describe a rule quantifying how one quantity changes as the other changes; this indicates a higher level of covariational reasoning than simply noticing that two quantities are changing (Carlson et al., 2002). Sometimes a student may describe an explicit rule for the value of one quantity in terms of the other; this indicates a higher level of relational reasoning than simply noticing that one quantity's values depend on the other quantity's values. The categories for this theme were no pattern, verbally stated, and rule stated.

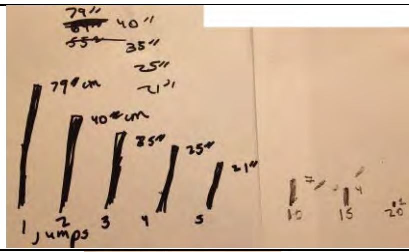
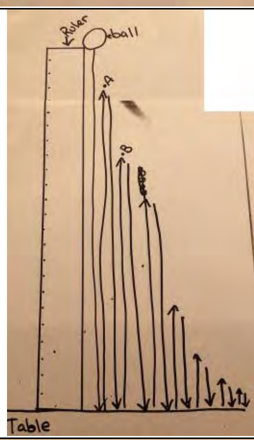
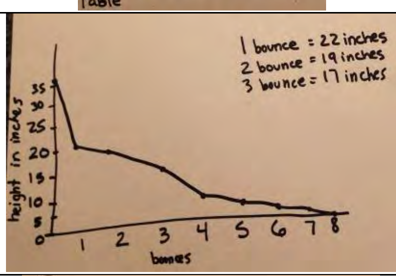
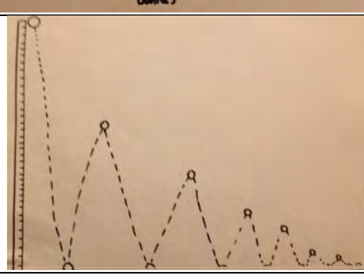
The seven students interviewed each completed the same two tasks. Table 2 lists the various types and characteristics of representations that students constructed to illustrate the ball drop and growth situations during the interviews. Every student for both tasks chose to graph the situations as one of their representations. Five of the seven chose it as the first representation for the growth task, but only three for the ball drop task. Four students, Arlis, Jennie, Movae, and Carlos, constructed two representations for each of the two tasks; and three students, William, Rachel, and Ki, completed one representation, only, for each task. The 22 total representations included 11 coordinate graphs, five other numerical representations (one vertical scale, three bar graphs, and one table of data), and six pictorial representations (stick figures for the growth task or the complete pathway of the bouncing ball for the ball drop task).

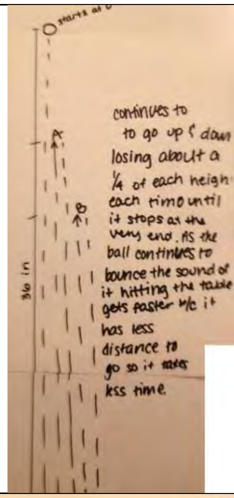
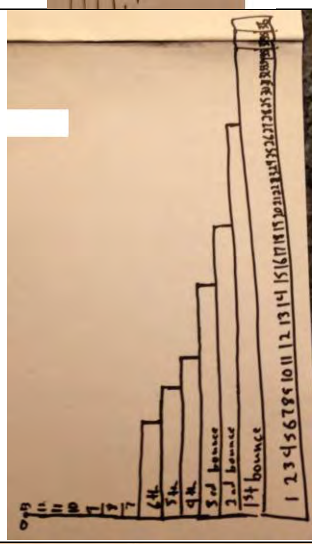
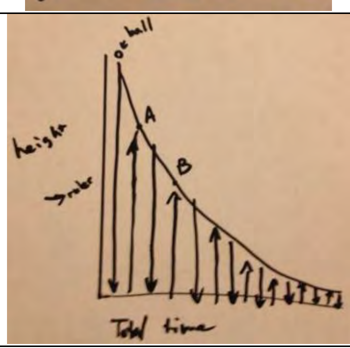
**Table 2.**
**Students' first and second representation of each task**

| Participants | Ball Drop Task | Growth Task |
|---|---|---|
| William | Graph (bar) | Graph (bar) |
| | Table | |
| Arlis | Picture (ball path) | Table |
| | Graph (linear) | Graph (linear) |
| Rachel | Graph (linear) | Graph (linear) |
| Jennie | Picture (ball path) | Vertical scale |
| | Graph (linear) | Graph (linear) |
| Movae | Picture (ball path) | Graph (linear) |
| | Graph (linear) | Picture (stick figures) |
| Ki | Graph (bar) | Graph (linear) |
| Carlos | Picture (ball path) | Graph (linear) |
| | Graph (linear) | Picture (stick figures) |

Six of the seven students interviewed, all except William, demonstrated ability to construct coordinate graphs to illustrate either or both of the functional relationships inherent in the task situations. Only one student, Rachel, constructed coordinate graphs for her first representation for each task. Of the six students who constructed coordinate graphs, only five of their 12 first representations were coordinate graphs. The remaining seven first representations by these students were alternative representations, including the vertical scale, the table of data, one bar graph, and four pictorial drawings of the pathway of the bouncing ball Table 3a below highlights students' first choice of representing the data.

**Table 3a.**
***Student representations and response analysis for Ball Drop task***

| | Ball Drop - Representation | Covariation | Dependency | Prediction | Pattern |
|---|---|---|---|---|---|
| William<br>Age 12 |  | Implicit | Implicit | Model<br>Data | Verbal<br>Correct |
| Arlis<br>Age 13 |  | Explicit | Implicit | Model | Verbal<br>Correct |
| Rachel<br>Age 14 |  | Explicit | Explicit | Data | Rule<br><br>Verbal<br>Correct |
| Jennie<br>Age 14 |  | Implicit | Implicit | Data | Verbal<br>Correct |

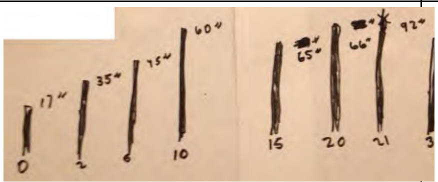
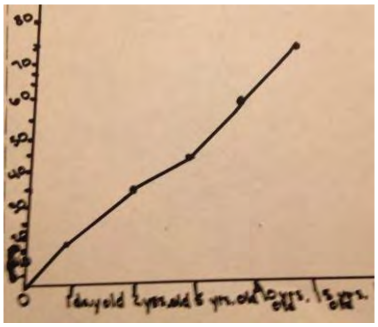
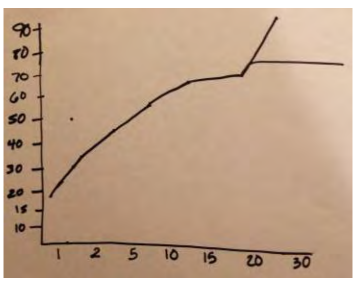
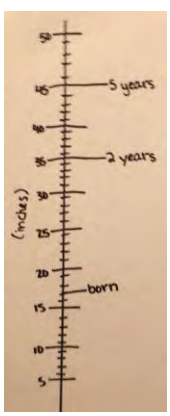
| | | | | | |
|---|---|---|---|---|---|
| Movae<br>Age 15 |   | Explicit | Implicit | Model | Rule |
| Ki<br>Age 16 |   | Implicit | Implicit | Model | Rule |
| Carlos<br>Age 17 |   | Implicit | Implicit | Model | Verbal Correct |

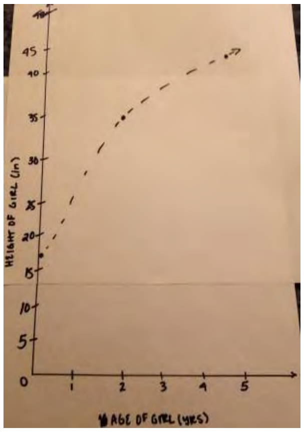
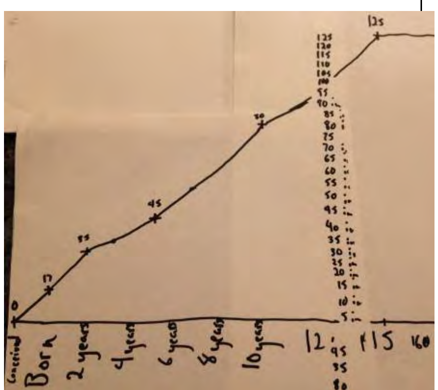
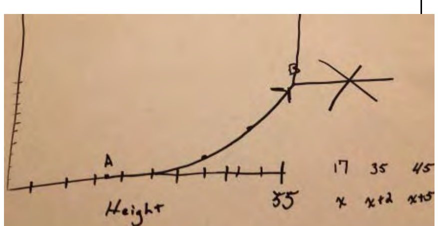
Here is a description of the students' tendency to use exact measures or detail to convey information in their representations, as compared to any tendencies to abstract and model only essential information from the task situations. Seventeen instances of this tendency were observed, based on student comments and/or interviewer observations. Only one student, William, referred to his representation as an abstraction of the situation, when he commented that although numbers would provide exact information, his graph of the ball drop task, in comparison, provided a "sense" of the situation. In contrast, most other instances can be interpreted as a preference to use exact measures or detail to convey information. For example, Rachel commented that in her graph of the growth task, the three given heights were the most important information, while the other heights shown on the graph were merely estimates. Jennie on two occasions added labels to her models in order to include additional information or to clarify her work. Jennie also commented that she did not show all of the actual bounces of the ball in her model because she could not count fast enough to track them all. Movae commented she was concerned that both her drawing and graph of the ball task omitted information she was unable to include. Arlis drew a very detailed drawing of the ball drop task, including the entire path of the ball as it bounced up and down the table upon which the yardstick was placed, as well as the fully detailed yardstick.

Next, we summarize in Table 3b the manner in which students were able to connect observed changes in height for the growth task, or the movement of the ball or the change in its height for the ball drop task, to the manner in which they constructed their representations for these situations. Out of the tasks completed, five students, Arlis, Rachel, Jennie, Movae, and Carlos, were able to articulate these explanations, whether partial or complete, for six of the seven tasks. For example, both Rachel and Jennie commented they had used lines to connect the top of each bounce in order to highlight the decrease in height with each bounce. According to Jennie, "unconnected dots" in lieu of her connecting line would not convey this same information effectively. Other students were less successful in incorporating observed change in a representation. Carlos, for example, commented that he could not illustrate the "speed" (or the frequency, presumably) of the bounces in his graph of the ball drop task. Similarly, when Movae could not include the increasing frequency of the bounces in her drawing of the ball drop situation, she merely added a verbal description to convey this information.

This section summarizes students' comments and explanations regarding features of their graphs and other representations of the growth and ball situations, indicating their perceptions of necessary features of an effective representation. Four students, Arlis, Rachel, Movae, and Carlos, indicated that a representation should include all given data or observed measures, that it not omit any important details, and that its size should be adequate to accommodate all of these important facts. Jennie seemed concerned that her drawing for the growth task and that her graph for the ball drop task include sufficient labels. Rachel and Jennie both wanted their models to be easy for an observer to understand, and in addition, Jennie did not want an observer to have to estimate any measures on her graph of the growth situation. Most students indicated that visual conveyance of information was important to them in making their models. For example, Jennie wanted her coordinate graph to "look" like the growth situation, which, as she told us, was the reason she chose the vertical axis to represent the girl's height. Movae drew the entire pathway of the bouncing ball because it illustrated what she "sees happening." Carlos drew a coordinate graph to represent the growth situation because, in his words, that design was the easiest for him to see. Arlis, who drew a pictorial model of the bouncing ball for the ball task, was reluctant to construct a second representation of the situation because doing so would require an observer also to "watch the experiment to be able to understand what things are doing."

**Table 3b.**
**Student representations of Growth task and analysis of student thinking**

| | Ball Drop - Representation | Covariation | Dependency | Prediction | Pattern |
|---|---|---|---|---|---|
| William<br>Age 12 |  | Implicit | Implicit | Data Intuitive | Verbal Correct |
| Arlis<br>Age 13 |  | Implicit | Indefinite | Data | Verbal |
| Rachel<br>Age 14 |  | Explicit | Explicit | Data Intuitive | Rule State |
| Jennie<br>Age 14 |   | No Evidence | No Evidence | Model Intuitive | No Pattern |

| | | | | | |
|---|---|---|---|---|---|
| Movae<br>Age 15 |   | Implicit | Indefinite | Intuitive | No Pattern |
| Ki<br>Age 16 |  | Explicit | No Evidence | Data | Verbal |
| Carlos<br>Age 17 |   | Implicit | Indefinite | Data | Verbal |

Students' notions of covariance and dependency were intertwined in their comments about elements changing in the problem situation. Typically, students tended to talk about the relationship of these elements at three different points in the interview: when asked to describe in general what was changing in the situation, when asked to reflect on what their model represented, and when describing the pattern of the data. And from these probes students' responses were described as explicitly, implicitly, indefinitely, or not demonstrating a notion of covariance and dependency. Students' responses will be used in the following discussion.

The responses were broken down into these four categories to distinguish students' attention to the relationship between the elements they found changing in the situation. To understand students' notions of covariance and dependency, how these two notions at times differed and at other times were woven together in students' comments, and how their responses were categorized, we will examine three individual cases – Rachel, William, and Movae – that depict these key features, but that also share common elements with the other four cases.

For both tasks, Rachel clearly explained that two elements were changing in a coordinated manner, and that one was dependent on the other. Early in the interview when asked what she saw happening in the ball drop experiment she explained, "As you bounce the ball, the height after each bounce started to like decrease." From this comment it appears she noticed that two elements— bounce number and the height of the ball— were changing. Her use of the word 'after' demonstrated an understanding that the height was dependent on each consecutive bounce. There were two other instances in the interview in which Rachel again focused on the dependent relationship. When asked about the important features of her graph she stated, "The number of bounces and the height that they reach." And when she was asked to describe the ball drop experiment to someone else she said, "After every bounce, the height that the ball will reach will decrease." Rachel's understanding of covariance and dependency were not limited to one task. She also described in the growth task how two elements changed and how one was dependent on the other. When asked what she would tell her parents about this situation she stated, "We had to show incrementation on how a girl's height changed and we made a graph and showed how much height she increased by at different points in her life." Rachel's language described the interconnectedness of covariance and dependency. There were no instances in which she described the two elements changing but did not mention the dependent relationship.

William's comments were similar to Rachel's in that his language indicated ideas for both covariance and dependency consistently across both tasks. However, his comments were categorized as implicit. Initially, when discussing what he saw occurring in the ball drop experiment he stated, "Well the jumps start out big and then each, the next jump is smaller than the jump before and eventually it's not bouncing anymore." This is what Carlson et al. (2002) describes as type 1 covariation. Clearly, his language depicts a dependent relationship; however, the two elements changing are implied. This might be reflected in the notion that time, which is continually changing, does not need to be explicitly pointed out. He used the word "jump" to refer to both the bounce and the height simultaneously. Similarly, in the growth task he used ambiguous language to refer to the changing elements and their relationship. He stated, "Each year the girl shows a good amount of growth;" and, "Well basically they start, every year they grow a couple more inches." Again his language focused on the dependent relationship using words such as "each" in the first quote and "every" in the second quote. He never directly stated that age and height are the two elements changing or that height is necessarily dependent on age.

The third case (Movae) is similar to the first two cases in that her language for one of the tasks is explicit although not as consistent. For the other task, her responses were similar to other cases in that she used language that did not focus on how two elements changed. For example, in the ball drop task she did not always refer to the dependent relationship of the ball's height to its bounce. When describing her representation she stated, "the information is like the height each one bounced and like for each bounce how high it went approximately." Here, she stated that two elements were changing, an indication of the notion of covariance, but not of dependency. However, later she stated, "As the ball bounced more times it bounced less high." Here she referred to the dependency of the ball's height to further bounces. There was even less evidence of understanding of covariance and dependency in her responses to the growth task. There was only one response in which she described the relationship: "You see something getting bigger like a person (you) usually think, oh they're getting older." In this instance, her notion of covariance was implicit in that bigger refers to height and older to age. The notion of dependency was reversed; her statement depicts age as dependent on height. Movae's case demonstrates instances in which students' notions of covariance and dependency were at times indefinite and unrelated. Students' understanding of functions: predictions

For both tasks, students were asked to use the data given to create a representation of the situation. They were then asked to explain their thinking, make predictions and if possible to make generalizations. Through these interchanges, data on how students made predictions and generalizations were collected. After an initial reading and coding of the data, it was evident that students tended to make predictions and generalize based on the following categories: their model, (a mathematical or verbal description of how one variable depends on the other including diagrams or graphs), the given data, and/or prior knowledge or intuition. In some cases students used more than one method for a single task.

For the growth task, three of the students used a model as one way to estimate different heights for the girl in the problem. In all three cases, the students had drawn a continuous line graph to represent the girl's growth. Each responded in a typical way explaining that the line was simply an estimate of the girl's height and not her exact height. For example, Jennie responded, "Well, it is to connect the two years to five years and also because between the two and five years she did grow increasingly, but you don't know exactly how much." Another one of the students explained that the line represented her average height between known points.

Four of the seven students used a model to generalize and extend the pattern in the ball drop task. These students tended to find the first four or five data points and then extended the pattern based on the pattern they observed in their representation. It may be argued, however, that the students simply extended the pattern from the experiment they just observed. This is an obvious effect but does not necessarily account for how students extended their graphs. Each of these four students drew the first few data points and then, when asked to extend the pattern to when the ball stopped, they noticed that the ball's height decreased at a decreasing rate of change and drew the line accordingly.

A fifth student used his model to generalize the pattern that would occur if the ball was dropped from twice the height. William stated, "Well I think these might be doubled. They might go like twice as high. So like this might be 2, 8, 14 (instead of 1, 4, and 7). . . . although that (the numbers) may not be as important as just the pattern." This student focused on the overall pattern generated from his graph to generalize the pattern for the new situation.

In the growth task four of the students used specific data points, rather than a holistic model of the situation, to make predictions. In each case the students determined how many inches the girl grew between ages 0 and 2, and 2 and 5 to create a rule to estimate her height at ages 10, 15, and 20. For example, Arlis used the following reasoning to determine that the girl would grow 15 inches for each five-year period between the ages 5, 10, 15, and 20:

Another student used a similar strategy. Ki determined, incorrectly, that the girl grew 45 inches in the first five years and therefore would grow another 45 inches each subsequent five-year span. So by age 10, by his calculations, the girl would be 80 inches (by adding incorrectly) and then 125 inches at age 15. Even though he described generally that a person's growth slows as his or her age increases, this information seemed irrelevant or isolated from his model of the growth of this girl. The other two students used both common sense and the data to estimate the girl's height at different ages. For example, Rachel stated, "Well every like five years the girl grew about ten inches taller and, but then when she like got to be like 16 or 17, she probably stayed just the same and didn't grow too many more inches taller."

For the ball task, three students used the data to extend the pattern. For example, when asked how he was going to extend the graph, William stated, "Well it'd probably be, after 21 (the height of the fifth bounce), it'd go down to about maybe 10 and then 8, 7, 5, 3, 2, and then it bounces like one a couple times and then basically dies." The other two students examined the difference in heights between consecutive bounces and used this information to find points to extend the graph. Rachel explained how she determined the heights of the bounces that she had not observed, "It seems like it isn't the same amount each time. You could take the average. For these three (bounces 1, 2, and 3), it'd probably decrease between two and three inches each time, like 2.5." She disregarded the decrease in height between the starting point of the ball and its first bounce. She continued by saying, "And between the next two bounces that was about three inches. And then the next one was about 2.5 inches and then maybe about one (at the end)." For these three students it is evident that they examined the height of the ball for different bounce numbers or the differences in heights for the first few occurrences and then attempted to use this information to extend their pattern.

There were four instances, all arising from the growth task, in which students relied on prior knowledge either partially or totally to make predictions. William and Rachel used the data to find a rule to estimate the girl's growth, but used knowledge of how tall a girl probably would be for different ages to check whether their estimates made sense. The other two students, Jennie and Movae, both used an informal rule to determine that the girl's adult height would be 70 inches. For example, Jennie stated, "I think she would be 70 inches tall. Because I heard that when you are two years old, how tall you are at two years old if you double that, that is how tall you will actually be when you grow up." Movae's response was similar. These instances describe two different ways students used prior knowledge. Two used it to determine the reasonableness of their answers while two other students used prior information as a steadfast rule discounting any data given in the problem.

There are four ways in which students typically represent functional relationships: graphs, tables, verbal descriptions, and equations. This study focused particularly on students' preferences in representing realistic functional situations for the two tasks. We chose to use two situations that could not be described by easily recognized rules or simple equations in order to encourage students to describe verbally the behavior of the relationship between the changing elements in each. This section focuses on how students described the overall pattern of the functional situation. We found that although the predominant choice of representation was a graph, some students nevertheless attempted to describe the functional relationship verbally and at times with rules. Below is a description of how students attempted to describe this pattern for the two tasks.

During the growth task only one student (Rachel) tried to use a rule to describe the pattern. Four described it verbally, but only one did so correctly; the other three described the pattern incorrectly or ambiguously. The final two students did not describe the pattern at all. Rachel's rule was "Well every like five years, she grew about 10 inches taller and, but then when she like got to be like 16 or 17, she probably just stayed the same and didn't grow too many more inches taller." The rule does not fit the graph of the girl's growth. In fact, the only description that closely modeled the girl's growth was a statement made by William: "She kind of starts out growing fast and then she slowly slows down and by the time she's 21, she's really slowed down. She's not really growing." This was the only description that depicted the decrease in the rate of change. This is a higher level of Carlson et al.'s (2002) covariational reasoning: not just noticing the two covarying quantities (height and time), and not just coordinating the direction of change, but recognizing the change in the rate of change. Two of the students described the girl's growth as being linear. For example, Carlos stated, "I figured it would be like a steady increase until she gets to a certain point and pretty much levels off and she won't grow anymore." Similarly, Ki stated, "It obviously goes up. It kind of is a little consistent as your grow. I mean not exactly, but close."

Arlis's description of her growth was ambiguous, "She is just getting older and the inches and keeps getting taller." One possibility is that the pattern was too complex for students to describe.

All seven of the students were able to depict correctly the pattern in the ball drop task verbally or to generate a rule. Three students used a rule to describe the overall pattern of this functional relationship. Rachel stated that "the ball's height lowered at a constant rate." Movae described the pattern as, "I have just been seeing it the whole time, that maybe it loses about a quarter of its height each time until it has no height to go." And finally Ki described it as, "Well it's losing at a ratio to how much it, to where it started from." Each of these rules has some merit. The other students described the pattern in a similar manner. For example, William stated, "The bounces are getting smaller and smaller until the very end;" and Jennie said, "It kept getting lower and lower." Perhaps because the functional relationship inherent in this situation appeared to them to exhibit a consistent decrease in the rate of change, students were able to make sense of the pattern, describe it verbally, and even attempt to impose a mathematical rule.

To begin the discussion section, we want to make it clear to the reader that students' understanding of functional relationships along with how they represent them are influenced by their current and previous mathematics instruction (Ainley et al., 2006). We are examining how students respond to functional situations because of and in spite of these experiences. We might expect that all the students who have previously taken algebra 1 to be able to explicitly represent and describe both covariational and dependent relationships and students who have not to only do so implicitly or not at all. Even with our small sample, this was not the case.

Our findings indicate that these students did not always choose to construct a coordinate graph to represent the functional relationships, even though they indicated they were capable of doing so. Instead, they often chose a representation resembling some of the physical characteristics of the situation, even when the representation contained some degree of abstraction. The students are not engaging in vertically mathematizing, (Gravemeijer, 1999) or in other words, are still "mathematically impoverished" (Ainley et al., 2006, p.24). This fact is evidenced, for example, by Jennie's choice of a one-dimensional vertical axis to represent the girl's height over time in the growth task, and by Jennie's, Movae's, and Arlis's drawings of the entire pathway of the bouncing ball for the ball task. In general, these students, in constructing their representations, were very interested in detail and showed preference for using labels and even verbal description to add additional information to their representations.

This detail work parallels the emphasis that is put on the procedure of graphing in traditional curricular resources (such as labeling axes with capitalized words), but not what is state mandates and assessment specifications (Darling-Hammond et al., 2011) that focus more on interpretation. It might be that as Klahr and Simon (1999) suggest, students tend to use weak or informal methods to solve realistic or novel situations instead of more formalized or strong methods. This is similar to what we found in our study; although all the students have had experiences with formal graphing, their representations for examining an applied problem lack formality and instead focus on students' perceptions of the problem mechanics. The implication for teachers and curriculum writers is to create a culture of practice where students are able to use strong or formalized methods (such as the coordinate plane) to solve problems that are more realistic in nature. One such framework is provided by Ainley et al. (2006), where they describe the importance and balance of placing students in realistic situations by creating tasks with both purpose and utility. "Whilst engaged in a purposeful task, learners may learn to use a particular mathematical idea in ways that allow them to understand how and why that idea is useful, by applying it in that purposeful context." (Ainley et al., 2006, p. 30)

Our findings also indicate that what constitutes a good representation for these students is one that contains complete information, and that includes the specific data described in the given situation (as in the case of the growth task) or measured by the student during the task (as in the case of the ball task). We found these students were typically interested in precision and were not comfortable in dealing with unknowns or estimates in the process of initially constructing their representations. From these students' perspectives, a good representation is one that presents information visually, or that does not require an observer to interpret symbolic or abstract notation. Rather, the way to present information most effectively is through inclusion of necessary, explanatory detail. Interestingly, after introducing the concept of function formally, many teachers find it difficult to get students to notate their graphs. Perhaps if the concept of function were allowed to grow naturally out of students' informal understanding, students would more aptly notate and label graphs.

These students were not universally capable of incorporating the dynamic aspect of any change or movement inherent in the given situation – whether or not they recognized and articulated this change. The tendencies we observed among these students seem to indicate that they did not perceive their graphs or other representations as tools that possess some utilitarian value by virtue of the manner in which they were constructed, or by any inherent scheme of representation. We surmise that because the student was the one who created the representation, from his or her perspective, the creation held no more power or utility than had been granted by the creator; all of its features were subject to the control of the student-creator, which is similar to how Gravemeijer uses the ideas of emergent modeling and horizontal mathematizing (Doorman & Gravemeijer, 2009; Gravemeijer, 1999). Students first use these informal or emergent models to solve the problem. They must be able to do this a number of times before they are able to abstract the model and use it as an object to be manipulated or to generalize it to other similar sets of problems – the ability to vertically mathematize.

Our findings suggest students have notions of covariance and dependency. By aggregating explicit and implicit comments, we found every student referenced the covariant relationship in the situations and all but one referenced the dependent relationship. In fact, all but one student made reference to covariance for both tasks. There was some evidence but less consistency for the notion of dependency. All students commented on the dependent element in the ball task, but only two of them referenced the relationship in the growth task. More empirical work is needed to determine the reason for this, but one explanation might be that the growth task focused on the common element of time or age, either of which from the students' perspective was simply assumed as one element of change and therefore not worthy of comment. This evidence that dependency relies on covariational reasoning is similar as to what Carlson et al. found (2002).

It is also apparent from our data that students across grade levels tend makes estimates, and predictions based on their model, the data, or intuition. It is reassuring that students tend to look for patterns graphically and numerically to generalize and make predictions. However, our data also demonstrate that students do not do so efficiently with data from realistic situations. Furthermore, students who had taken advanced mathematics courses did not show a noticeable difference in this skill.

Although the interviews did not focus on students' ability to represent the situations using formulas or rules, a few students attempted to do so. Neither situation, however, lent itself to an easily recognizable rule. Even so, three students (not dependent on age) constructed a rule that could have been tested if given enough time. The majority of the students (five out of seven) correctly described the overall pattern of the ball task while one student correctly described the girl's growth. It might be that the ball's pattern was more predictable with a consistent rate of change whereas the girl's growth was not, therefore leading students to correctly identify the ball's pattern but not the girl's growth.

An emergent functional reasoning scheme

With such a small data set, it is impossible to make any significant generalizations using the current coding/representation analysis. However, by looking so closely at that small sample it may allow us to begin analyzing larger sample sizes. To this end, we have merged our categories into a functional reasoning scheme that would be easier to use and could be valuable in further studies with larger populations (see Table 4 for a summary description of the levels).

We notice, for instance, that in order to see dependence, one must be able to recognize covariance. If you treat explicit, implicit, and indefinite as scores of 3, 2, 1, no student should have a dependency score higher than a covariance score. This is true for this sample of students. Likewise, our sample supports the hypothesis that a student cannot generate a rule for the situation if they do not have some sense of dependency. As demonstrated in Table 4, this allows us to merge our scales for our categories of covariance, dependence, and pattern for our categories into the following scheme, where a student:

Level 1: Only attends to a single variable.
Level 2: Implicitly (2a) or explicitly (2b) indicates some understanding of *covariance*, but not dependence.
Level 3: Implicitly or explicitly indicates an understanding of *dependence* (i.e. that one variable's value is dependent on the other, but is unable to articulate a rule that might be used to generalize the situation).
Level 4: Generates a (verbal) *rule* that generalizes the situation that only attends to *order* (e.g. the ball's height decreases with the number of bounces (or time)).
Level 5: Generates a (mathematical) rule that could be used to generalize the situation *quantitatively* (e.g. at each bounce the ball half as high as it was on the previous bounce).

**Table 4.**
**Students' levels within overall scheme of functions understanding**

| | **Students** | | | | | | | **Totals** | |
|---|---|---|---|---|---|---|---|---|---|
| **Level** | William | Arlis | Rachel | Jennie | Movae | Ki | Carlos | Ball | Growth |
| **1 – One variable** | | | | G | | | | 0 | 1 |
| **2a – Covariance Only (implicit)** | | G | | | | | G | 0 | 2 |
| **2b – Covariance Only (explicit)** | | | | | | G | | 0 | 1 |
| **3 - Dependence** | | | | | G | | | 0 | 1 |
| **4 – Verbal Rule** | BG | B | | B | | | B | 4 | 1 |
| **5 – Quantitative Rule** | | | BG | | B | B | | 3 | 1 |

Key: B – Ball Drop Task
G – Growth Task

## Implications

Our findings provide a small glimpse into secondary students' thinking about functions. We found the students we interviewed did not always choose first to make coordinate graphs to represent their understanding of a given functional relationship and that this held true regardless of their ability to construct graphs. In addition, these students, regardless of their preferences for one type of representation over another, did not always use these representations to make predictions about the data. At the same time, all seven students were able to make some sort of extension or prediction about the data in one or both of the task situations. In contrast, only three students were able to state a quantitative rule to describe either or both of the task situations. This might, in part, be explained by the tasks, which themselves are tough to describe quantitatively. However, six of the seven students indicated they designed their representations so that they visually resembled the task situations. This may mean students are not accustomed to describing realistic situations this way or that the situations lend themselves to visually representing the data. Either way, if one goal is to have students describe situations with a rule, students are not doing this without being asked.

We have extracted one possible explanation for these seemingly contradictory findings: Perhaps these students were able to use their representations to make predictions about the data because the predictions or extensions seemed to fit *visually* into the representational scheme comprising their models (e.g., the predicted measure fell along an existing line in the graph, or the extension merely required lengthening an existing line in an already-established direction), but *not* because of mathematical insight.

If the above is true, it holds important implications for teaching. Students need to understand the latent power of a representational scheme such as the coordinate graphing system. Instruction should be designed to help them make connections between mathematical features (a descriptive rule or pattern) of the functional relationship being represented, and the method by which the representation is constructed. In addition, instruction should not be restricted to the procedural aspects of constructing coordinate graphs or to the technicalities of creating a mathematical rule that can accommodate a functional relationship. We need to bridge the gap between what students are thinking (i.e., visual resemblance) and what we want them ultimately to know and do.

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