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Abstract

In the present research, we evaluated the effectiveness of a multi-year professional development program in mathematics for elementary teachers. Each year the program focused on a different domain of mathematics. We found the program increased teachers' knowledge of (a) number and operations, (b) measurement and geometry, and (c) probability and statistics. We also examined the relation between mathematical knowledge and teaching practices. Across the three domains neither pretest nor posttest mathematical knowledge were related to classroom teaching practices. However, change in knowledge was positively related to six different dimensions of teaching practice for number and operations, and for measurement and geometry; and was positively related to four or six dimensions for probability and statistics. That is, those teachers with greater changes in knowledge demonstrated more effective instruction.

Keywords: *mathematics professional development; teacher knowledge; teacher practices*

1. Introduction

"Effective mathematics teaching requires understanding what students know and need to learn and then challenging and supporting them to learn it well" (NCTM, 2000, 16).

Research on learning has demonstrated that students should learn with understanding (Cohen, McLaughlin, & Talbert, 1993; Hiebert, 1986). Students need to understand topics in greater detail, become better problem solvers, and learn mathematics as an interconnected web of knowledge as opposed to isolated topics divorced from real-world or complex events (Romberg, 1992; Hiebert & Carpenter, 1992; Tate, 1994). More specifically and as the beginning quote highlights, national organizations such as NCTM (1989, 1991, 2000) and MSEB (1990, 1991, 1993) recommend students learn to communicate and reason with mathematics, make connections within and outside of mathematics, and become problem solvers.

Implementing instruction based on these recommendations for learning requires new types of mathematical and pedagogical knowledge for elementary and middle school teachers. Unfortunately, after more than two decades of the mathematics reform movement, teachers are still entering the teaching field unprepared to teach mathematics in the way envisioned by these standards (Ball, 2000; Frykholm, 1996; Lloyd & Behm, 2005; Zeichner, 1993). In addition, most elementary teachers have not experienced mathematics in the manner in which they are being asked to teach (Cohen & Ball, 1990; Knapp & Peterson, 1995). They need an understanding of how mathematical knowledge is constructed among topics and how students might think (informally and formally) about the mathematics.

Unfortunately, most teachers lack exposure to the necessary rich mathematical experiences that could develop these various types of knowledge and it is difficult for teachers to acquire the knowledge, skills, and dispositions to teach in these ways while in the classroom (Cuban, 1990; Frykholm, 1996, 2004; Lloyd & Behm, 2005; Richardson, 1990). Therefore, it is critical for teachers to be in situations where they can experience mathematics in the aforementioned ways through rigorous professional development.

It is within the context of this reform movement in mathematics that this professional development framework was conceptualized. As will be discussed below, it is important that the mathematics education community closely examine the relationships between the process of knowledge construction and instructional practices with respect to mathematics if teachers are to gain the types knowledge and experiences that will assist them in the creation of intellectually rich learning environments for their students.

This paper highlights the theoretical foundation for a professional development project over two consecutive three-year periods, spanning six years in all, and consisting of a week-long intensive institute followed by continued support throughout the academic year. The goal of the professional development was to transform teachers' knowledge of mathematics, mathematics pedagogy, and students' mathematical ideas and to transform their instructional practices.

1.1 Transforming Knowledge

To accomplish our initial goals, teachers need to possess knowledge of mathematics knowledge about how students learn mathematics, as well as instructional practices that support quality mathematics teaching (Ball & Cohen, 1999; Borko, 2004; Wilson & Berne, 1999). Furthermore, according to recent developments in teacher learning theory, mathematics content and pedagogical content knowledge play a greater role in professional development (Loucks-Horsley, et al, 2003).

While knowledge of mathematics content may be at the root of some of the challenges elementary school teachers face, secondary and middle school teachers are more likely to face a different, but related, kind of deficit. Secondary teachers may be competent with common mathematical knowledge, however, reformed teaching requires a special type of knowledge that is grounded in the specific acts of teaching (Ball, Hill, & Bass, 2005). Common content knowledge is defined as the procedural and conceptual understandings of mathematics we use to solve mathematical problems and recognize incorrect answers and definitions. Specialized knowledge extends beyond common content knowledge and is the specialized knowledge of mathematics that is particular to mathematics teachers (Ball, Hill, & Bass, 2005). This type of knowledge allows mathematics teachers to analyze and use multiple solution strategies and representations, provide mathematical explanations, and identify misconceptions.

Knowledge of student thinking relates to the knowledge that allows teachers to predict possible student solution strategies, anticipate likely misconceptions, and to interpret students' ideas. Knowledge of instructional practices is the knowledge teachers use when creating instructional sequences to facilitate student learning such as choosing curricular materials, assessing students, asking questions and reflecting on how to improve their practice (Ball, Thames & Phelps, 2005; Hill & Ball, 2004; Hill et al., 2004).

Ball and colleagues' expansions of the types of teacher knowledge provide a useful framework for teacher professional development and research and are aligned with the principles and major features of teaching for understanding described above. In designing activities to be used in all three of the core classes, we have carefully considered teachers' existing knowledge and the knowledge required for teaching, as described above, for early and upper elementary teachers and middle school teachers.

1.2 Transforming Practice

While developing knowledge about instruction is crucial, we also consider the need for teachers to engage in the professional activities of teaching, or what Loucks-Horsley et al. (2003) refer to as "building a professional culture." In fact, recent developments in the field of professional development design suggest that collaboration, community building, and participation in communities of learners are key elements in sustaining the impact of high quality professional development (Ball & Cohen, 1999; Borko, 2004; Loucks-Horsley, et al, 2003; Wilson & Berne, 1999).

We believe one key element in learning to participate in a professional community is developing what refer to as a "stance of inquiry" (Ball & Cohen, 1999). Effective standards-based instruction is grounded in teachers' ongoing wondering and inquiries about mathematics, student learning and instructional practices. "One way to put the aim here is to help teachers learn the intellectual and professional stance of inquiry – … that would support their generation of multiple conjectures about an issue in practice, their production of alternative explanations, and their efforts to weigh them reactionally" (Ball & Cohen, 1999, p. 27). In other words, teachers should learn to engage in productive discussions and collaborations about their everyday work in classrooms. This type of work should include selecting materials and designing lessons, analyzing student work, wondering about student thinking, and thinking critically about what type of tasks to pose and questions to ask students next. The knowledge and dispositions required in order to develop a stance of inquiry and to become a productive member of a professional community will be discussed next.

1.3 Theoretical Framework for the DMT Professional Development

Many conceptual models for learning and teaching refer to *hypothetical learning trajectories* (HLTs) or *learning progressions* (Baroody, et al, 2004; Clements & Sarama, 2004; Gravemeijer, 1999; Hiebert, et al, 1997; Simon, 1995). Generally, an HLT describes the mathematical path that teachers envision their students taking as they explore specific mathematical domains. HLTs include models of children's initial ideas, a sequence of instructional tasks, and descriptions of children's progressions of learning and thinking.

We attempt to prepare teachers to explore the terrain of teaching mathematics in new ways. Through the professional development, teachers learn to examine the types of situations or tasks they provide students. These situations focus on encouraging students to think about the best possible solution strategy and which mathematical models might help communicate the solution and possibly the process. At this point, it is crucial that teachers begin to question their own (typically traditional in nature and limited in scope) mathematical knowledge and the knowledge (based on real experiences) that their students bring to the classroom. We do this by helping them learn to develop an understanding of possible *learning trajectories or learning progressions* (Gravemeijer & van Galen, 2003; Simon, 1995) for various mathematical topics.

According to Simon (1995, 2004), HLT's are made up of several components: the learning that defines the direction, the learning activities and the hypothetical learning process – a prediction of how the students' thinking and understanding will evolve in the context of the learning activities. Moreover, Baroody, et al (2004) emphasize that HLT's should focus on the "big ideas" and should incorporate both linear, or "ladder-like" trajectories, as well as multiple path or "branching tree" trajectories (p. 254). Finally, Hiebert, et al (1997) describe hypothetical learning trajectories as "…the teacher's vision of the mathematical path that the students might take, and its hypothetical nature comes from the fact that it is based on the teacher's guess about how learning might proceed along the path. The trajectory guides the teacher's task selection, but feedback from students and the teacher's assessments of the residues that are being formed lead to revisions in the trajectory. Tasks are selected purposefully, but the sequence can be revised" (p. 34).

As Gravemeijer (2004) points out, HLTs are tailored to a specific classroom and teacher at a given time. (For this reason, developing HLTs for general use may be a misuse of the HLT concept.) Gravemeijer (1999) develops another concept, *local instructional theory*, to refer to more general instructional sequences that may be useful on a larger scale. "The idea is that teachers use their insight in the local instruction theory to choose instructional activities and to design HLT for their own students. In my view, LITs [Local Instructional Theories] can never free the teachers from having to design HLTs for their own classrooms. Nevertheless, I would argue that using a local instruction theory as a framework of reference could enhance the quality of the learning trajectories" (Gravemeijer, 2004, 107-108). This is ultimately the goal of the DMT professional development: to help teachers learn to utilize LITs in the creation of learning-teaching progressions that fit their students and classrooms (van den Heuvel-Panhuizen, 2001).

1.4 A Model for Teaching for Understanding: DMT Instructional Theory

Our model for teaching for understanding stems from the notions of "guided reinvention" and "mathematizing" (Freudenthal; 1973, 1991; Treffers, 1987). As Gravemeijer and van Galen (2003) describe, guided reinvention is a process of first allowing students to develop informal strategies for solving problems, and then, by critically examining those strategies, encouraging students to develop more sophisticated, formal, conventional and abstract strategies and algorithms. By comparing invented solution strategies, students learn which manipulations make sense for given contexts and are encouraged to develop more general procedures. Eventually, the contexts fade into the background and the "manipulations themselves … acquire meaning of their own" (Gravemeijer & van Galen, 2003, p. 116). This is the essence of what can be described as mathematical abstraction (Simon et al., 2010).

Through solving novel problems and examining multiple approaches, students make connections between existing knowledge (informal ideas) and new knowledge (more formal mathematical ideas). By critically examining their own and others' strategies, students are encouraged to build functional understanding which exemplifies the importance of social interactions in classrooms. "By thinking and talking about similarities and differences between arithmetic procedures, students can construct relationships between them. … the instructional goal is not necessarily to inform one procedure by the other but, rather, to help students build a coherent mental network in which all pieces are joined to others with multiple links" (Hiebert & Carpenter, 1992, p. 68).

Closely related to guided reinvention, our model also incorporates Treffer's (1987) notions of horizontal and vertical mathematization. Horizontal mathematization occurs when students represent a contextualized problem mathematically in order to find a solution strategy. Vertical mathematization involves taking the mathematical matter to a higher level, and is evident when students no longer adhere to the isolated attributes of specific problem solving contexts but instead view their solution processes and representations as objects of mathematical examination—a process of reification that places the focus not on the problem at hand but mathematics in general. Mathematizing covers such activities as generalizing, justifying, formalizing, and curtailing – including, but not limited to, developing an abstract algorithm (Gravemeijer & van Galen, 2003). By focusing on both types of mathematizing in their classrooms, teachers must maintain a focus on the inherent structure of the mathematical ideas that are emerging. In addition, they must address students' misconceptions as they arise so these misconceptions do not hinder the mathematizing progression. One of the results of mathematizing is that teachers connect students' informal strategies, many of which may be developed outside of school, with more formal mathematical ideas. "One would predict that if children possessed internal networks constructed both in and out of school, and if they recognized the connections between them, their understanding and performance in both settings would improve" (Hiebert & Carpenter, 1992, p. 79).

Such a process starts with carefully chosen problems or situations to provide students. To solve these problems, students must model the situation to some degree. Rather than beginning with the standard algorithms and attempts to concretize them, teaching begins with students' common sense solutions to contextual problems that are sensible for them. By reflecting on the solution procedures they have used, students develop more sophisticated models and procedures that they can also use in other situations (Gravemeijer, 1999; Gravemeijer & van Galen, 2003). In other words, teaching starts with considering students' ideas about solving real world problems. Teaching practices are not limited to only what students derive independently and naturally, but acknowledging and building upon students' inventions is viewed as a necessity in this instructional model.

Mathematical knowledge originates from students' attempts to model contextual situations. Models then become the basis for solving related problems and as a means for support for more formal mathematical reasoning (Doorman & Gravemeijer, 2009; Gravemeijer & van Galen, 2003). As Cobb (2000) described, this use of modeling "…implies a shift in classroom mathematical practices such that ways of symbolizing developed to initially express informal mathematical activity take on a life of their own and are used subsequently to support more formal mathematical activity in a range of situations" (p. 319). In this way, modeling is a fundamental process in learning mathematics. However, this view of models and modeling contrasts with current practices in mathematics instruction in which models are used to "concretize expert knowledge" (Gravemeijer & van Galen, 2003, p. 118) and contextual problems are presented only after students have mastered traditional ways of solving problems. In this way, guided reinvention and mathematizing via the use of models, turn the focus toward students' ways of using models rather than on teacher or curriculum created ways of using models.

Through enacting aspects of "guided reinvention" and "mathematizing" teachers develop a classroom practice that is based on the tenets of teaching for understanding. To help frame their instructional practices, we used Hiebert and colleagues five features: tasks, the role of the teacher, the culture of the classroom, mathematical tools, and equity. Learning tasks are seen as opportunities for students to explore mathematics. They should enable students to "problemtize" the mathematics and walk away from the activity with a greater understanding of the mathematics (Hiebert, Carpenter, Fennema, Fuson, Human, Murray, Olivier, & Wearne, 1996). The role of the teacher is that of a facilitator. Teachers should choose worthwhile tasks and ask students to share their work and reflect on necessary elements within the mathematics. The culture of the classroom is one of a community in which students interact with each other about the subject. The environment should promote socio-mathematical values (Kazemi & Stipek, 2001), where students find it safe to share ideas. The role of mathematical tools is to deepen students' understanding. It is important to build understanding by having students use tools or mathematical models to make appropriate connections. Finally, the mathematics should be accessible and equitable for all students with expectations that they will learn mathematics with understanding and ample opportunities to do so.

2. Method and Data Sources

There are two questions framing this study. The first question is, "How does elementary teachers' knowledge of mathematics change over a six month period after participating in a professional development project?" The second question is, "Is change in knowledge related to teaching practice?"

Each year the professional development covered a different content area. During the summer the professional development institutes focused on the following topics: year one - number and algebra, year two - measurement and geometry, and year three - data analysis, probability, and statistics.

The first year focused on number, number operation, and algebra. Professional development participants developed an understanding of students' learning of these topics with significant time spent developing their own knowledge of the topics by means of investigating rich problem-solving situations. Participants learned to create classroom environments that encourage mathematizing and to utilize various models and notational systems that provide teachers with the means to both model and extend students' own mathematical development and reasoning. Connections to algebraic topics as well as foundational rational number concepts were also addressed with special emphasis given to reforming many participants' previously held misconceptions of these more sophisticated topics.

The second year focused on measurement and geometry. Professional development activities included not only developing participants' mathematical knowledge of measurement and geometry but also experiences with understanding the complexities of student reasoning and development in these domains. The third year focused on probability and statistics with special emphasis given to extending and deepening teachers' limited conceptualizations of the nature of probability and statistics. Whether due to a lack of experience as a learner of mathematics or a pre-dominance of poorly designed curricular materials, many teachers (and indeed U.S. citizens as a whole) often lack a fundamental knowledge-based relative to probability and statistics (Shaughnessy, 2007). The data were analyzed and are reported separately for the different content areas (i.e., number and operation, measurement and geometry, and probability and statistics).

2.1 Data Collection

To examine teachers' understanding of mathematics and students and mathematics and teaching in the areas of (a) number and operation, (b) measurement and geometry, and (c) probability and statistics, as well as teaching practice, various sources of data were collected and analyzed using the principles of both quantitative and qualitative research; however, our focus here is on the quantitative data.

2.1.1 Content Knowledge Inventory

The content area for the first year of professional development focused on number. To measure teachers' knowledge in this area we created a 16 item knowledge inventory based predominantly on items found in the literature (Ball, 1989; Ball, Lubienski, & Mewborn, 2001; Ball & Wilson, 1990; Carpenter, Franke, & Levi, 2003; Empson & Junk, 2004; Kennedy, Ball, McDiarmid, 1993; Lamon, 1999; Ma 1999; Schifter, 1998, 2001; Sowder, Philipp, Armstrong, & Schappelle, 1998). The items were aimed at uncovering both the teachers' knowledge of number and their pedagogical knowledge. The knowledge inventory was given to all participating teachers before the professional development began and was given again six months later. The Cronbach's alpha for the inventory was .72.

During the second year, we focused on measurement and geometry. As with number and operation topics, we reviewed the body of literature focusing on the mathematical knowledge required for teaching. Much of the research on teaching and learning of measurement and geometry implies that teachers' knowledge can be assessed when they are asked to respond to situated teaching scenarios. We created a 9 item knowledge inventory based on the literature. The items placed teachers in a hypothetical situation in which they had to respond to students' conjectures and misconceptions on measurement or geometry tasks (Barrett, Jones, Thornton & Dickson, 2003; Battista, 1999; Kribs-Zaleta & Bradshaw, 2003; Lehrer, 2003; Ma, 1999; Renee, 2004). Just as we had with the first year's inventory, participants took the second year's knowledge inventory prior to the professional development and again approximately six months later. Cronbach's alpha for the inventory was .64.

The third knowledge inventory focused on data analysis, probability and statistics. Relevant literature was used in a similar fashion to the previous years to find items for the inventory. Special emphasis was given to items emphasizing both content knowledge of probability and statistics, but also knowledge of mathematics for teaching probabilistic and statistical topics and more specifically, responding to common student misconceptions of these domains (Konold & Higgins, 2003; Jones, Langrall, Thornton & Mogill, 1999; Shaughnessy, 1992). The 16 item knowledge inventory had a Cronbach's alpha of .71.

2.1.2 Observation Instrument

During the school year, classroom observations were conducted in each of the classrooms. To provide a more accurate portrayal of instructional practices, two to three classroom observations were conducted in each of the teachers' classrooms. The Classroom Observation Scoring Sheet (see Appendix A) was based on our conceptual framework for teaching for understanding. In particular, we used Hiebert et al.'s (1997) five core features and a sixth feature of classroom discourse. These features are (1) learning tasks are seen as opportunities for students to explore mathematics (Cronbach's alpha = .85), (2) the role of the teacher is that of a facilitator (Cronbach's alpha = .90), (3) the culture of the classroom is one of a community in which students interact with each other about the subject

(Cronbach’s alpha = .87), (4) mathematical tools are used to deepen students’ understanding (Cronbach’s alpha = .94), (5) the mathematics is accessible and equitable for all students (Cronbach’s alpha = .83), and (6) the discourse encourages dialogic conversations (Cronbach’s alpha = .93).

Over the course of the project, DMT staff observed 268 lessons. At the beginning of each year, we co-observed teachers and independently scored the observations to verify that we were using the observation instrument consistently. When there were differences, the observation scripts were examined and discussed until raters came to an agreement on how to score the observation and how to use the instrument for similar cases in the future. The inter-rater reliability for the approximately 10 percent of observations was high (Kappa = .92).

2.2 Participants and Situation

To answer the research questions, we partnered with local Title I schools whose administrators were interested in participating in the study. We asked the entire school’s faculty to participate. We did not want to work with a select group of teachers who volunteered or self-selected to be part of the study. Thus, all teachers in each of the seven schools were a part of the professional development.

We chose seven schools where there was a need for mathematics improvement (based on standardized test scores) and all the staff was, at least, willing to participate in mathematics professional development. The seven elementary schools were in 3 districts: 2 urban and 1 rural (see Table 1). Each of the schools had a large percentage of students who were considered disadvantaged. For instance, between 58% and 89% of their students came from homes of low social-economic-status (SES) and each school had a relatively high number of English-Language-Learners, special education, and/or Title 1 students when compared to district or state averages.

Table 1: School Demographics

School	A (K-6)	B (K-5)	C (K-3)	D (4-6)	E (K-5)	F (K-5)	G (K-6)
Enrollment	369	531	509	409	490	570	213
Teachers participating	13	15	12	8	18	20	8
Racial/Ethnicity							
American Indian	2%	0%	1%		1%	1%	0%
African American	1%	1%	1%		1%	0%	1%
Asian/Pacific Islander	1%	0%	1%		1%	1%	1%
Hispanic	7%	45%	10%		70%	70%	56%
White, non-Hispanic	89%	54%	87%		28%	28%	42%
ELL	2%	16%	10%		30%	30%	14%
Migrant	0%	3%	0%		3%	1%	2%
Free/reduced lunch	88%	58%	62%		85%	89%	79%
Special Education	20%	18%	11%		7%	7%	8%

As part of the negotiations with school administrators, 100% of the teachers were involved in the study. However, because of summer plans, (e.g., weddings, vacations, surgeries, etc.) we were not able to collect pretest data on all the teachers. There were a total of 94 teachers in the study, but we report on 70 teachers with complete data sets (i.e., pretest on knowledge inventory, posttest on knowledge inventory, and instructional practice scores) for the professional development focused on number and operation, 68 teachers with complete data sets for the professional development focused on measurement and geometry, and 64 teachers with complete data sets for the professional development focused on probability and statistics. We will report our findings by content area.

3. Results

The primary purpose of this investigation was to evaluate the effectiveness of our professional development, in terms of teachers’ knowledge of mathematics and their classroom practice. We will begin by reporting descriptive statistics for content knowledge and observational data of teaching practice. We will then report on the relation between knowledge and teaching practice.

3.1 Changes in Teachers' Content Knowledge

The pretest scores on the knowledge inventories show that teachers' mathematical knowledge across the three domains is limited (see the leftmost column in Table 2). These results are similar to the results found in past research (e.g., Empson & Junk, 2004; Ma, 1999; Schifter, 2001).

Table 2: Mean Percentage of Teachers' Correct Responses on Knowledge Inventories

Domain	Pretest	Posttest
Number and Operation	.44 (.02)	.65 (.02)
Measurement and Geometry	.40 (.02)	.59 (.02)
Probability and Statistics	.52 (.02)	.75 (.02)

Note. Numbers in parentheses are the standard error of the mean.

In the spring (six months after the initial training), the knowledge inventory was administered again to teachers. Posttest scores, presented in the rightmost column of Table 2, were significantly higher than pretest scores for (a) number and operations, $t(69) = 9.24$, $p < .001$, (b) measurement and geometry, $t(67) = 9.88$, $p < .001$, and (c) probability and statistics, $t(63) = 11.17$, $p < .001$. The professional development significantly increased teachers' overall mathematical knowledge.

3.2 Observation of Teachers' Practice

For each teacher, we computed an average score across the items making up the six features of the teacher observation instrument (see Table 3). These scores are on a scale where 0 indicates none of the behavior was observed and a 3 indicates the teacher always showed the behavior.

Table 3: Mean Score of Teachers' Observed Practice by Feature

Feature	**Mean Score**
Nature of Classroom Tasks	1.89 (.06)
Role of the Teacher	1.63 (.05)
Social Culture of the Classroom	1.59 (.05)
Mathematical Tools as Learning Supports	1.56 (.05)
Equity and Accessibility	1.58 (.04)
Classroom Discourse	1.41 (.06)

Note. Numbers in parentheses are the standard error of the mean. Scores are the mean of individual items on a subscale. Each item ranged from 0 to 3.

3.3 Teacher Content Knowledge in Relation to Instruction

We used a Pearson correlation to examine the relation between teachers' content knowledge and their instructional practices (see Table 4). We found that, regardless of the domain, initial content knowledge (i.e., pretest score) was not correlated with teachers' instructional practices based on our measures. The professional development increased knowledge; however, content knowledge after the professional development (i.e., posttest score) was not correlated with teachers' instructional practices. We did find significant relations between teachers' content knowledge gains and their instructional practices. The teachers whose content knowledge scores increased the most over the six months of professional development demonstrated better instructional practices.

Interestingly, the number of years a teacher had taught was not related to the content knowledge inventory scores or the gains they made during the professional development. Across the domains, there was a slight inverse relation between years of experience and instructional practice (the correlations ranged from -.16 to -.01 across the six features of instruction); the greater the number of years one taught was negatively correlated with the scores on the instrument measuring instructional practices; however, these correlations were not significant..

Table 4: Correlations among Content Knowledge and Features of Instruction by Domain

Domain			
Number and Operation	*Pretest*	*Posttest*	*Change*
Feature			
Nature of Classroom Tasks	-.20	.15	.32*
Role of the Teacher	-.20	.12	.30*
Social Culture of the Classroom	-.22	.16	.35**
Mathematical Tools as Learning Supports	-.22	.14	.33**
Equity and Accessibility	-.17	.19	.35**
Classroom Discourse	-.14	.21	.33**
Measurement and Geometry			
Feature			
Nature of Classroom Tasks	-.01	.16	.29*
Role of the Teacher	-.15	.03	.29*
Social Culture of the Classroom	-.14	.09	.37**
Mathematical Tools as Learning Supports	-.18	-.02	.27*
Equity and Accessibility	-.20	.00	.34**
Classroom Discourse	-.06	.09	.25*
Probability and Statistics			
Feature			
Nature of Classroom Tasks	.05	.24	.21
Role of the Teacher	-.13	.12	.25*
Social Culture of the Classroom	-.04	.11	.16
Mathematical Tools as Learning Supports	-.21	.17	.39**
Equity and Accessibility	-.07	.24	.33**
Classroom Discourse	-.07	.20	.29*

***Correlation is significant at the 0.01 level (2-tailed).*

**Correlation is significant at the 0.05 level (2-tailed).*

4. Discussion, Conclusions and Implications

This study examined teachers' knowledge related to the teaching of mathematics and its relationship to instructional practices. Over two grant cycles and six years of data, we demonstrated that professional development focusing on developing students' mathematical thinking had a positive effect on teachers' knowledge. Most importantly, we found that when teachers engaged in reforming their mathematics teaching, the degree of knowledge change was significant to the increased change in practice.

This relation is somewhat explained by Ball and Cohen's (1999) conjecture that when teachers have the opportunity to try these ideas out in practice, both their knowledge and practice begin to change. However, in our case, all teachers were given the opportunities to try out DMT ideas in their classrooms. Their principals, colleagues, and mathematics coaches all supported the professional development and its tenets. Some teachers still remained resistant to change their practice. These teachers' knowledge did not increase as did the teachers' who did attempt to put these ideas into practice.

More specifically, the results of this study revealed a strong relationship between teachers' knowledge gains on all three inventories – measuring number and algebra concepts, measurement and geometry concepts, and probability and statistics concepts and their instructional practices regarding the teaching for understanding. This might mean that the teachers who participated in the week-long summer institute attempted to implement these ideas consistently in their classroom increased their knowledge of mathematics and students and their knowledge of mathematics and teaching to a larger degree than teachers who did not attempt to implement these ideas in their classroom.

We found that teachers who were observed trying the DMT methods in their classrooms also had greater content knowledge gains. These teachers encouraged their students to solve, for instance, addition problems using decomposition and compensation strategies and observed first hand that their students could not only solve the problems correctly, but were also able to describe the process in ways that indicated they were developing mathematical knowledge with understanding (Hiebert, et al., 1997). These findings are consistent with conclusions found in the research literature (Borko, Eisenhart, Brown, Underhill, Jones, & Agard, 1992; Fennema et. al., 1996; Mosenthal, 1995; Onosko, 1990, 1991; Thompson, 1992; Wilson, 1994).

Something of specific interest was the teachers whose scores on the content knowledge inventories that were generally higher, but whose instructional practices did not focus on the attributes of teaching for understanding. This might suggest that content knowledge alone may not be the most robust measure for efforts to improve teachers' instructional practices and eventually achievement.

Another finding is that the more the teachers' content knowledge scores improved the more their teaching practices reflected characteristics of teaching for understanding. One reason for this may be that the professional development model focused on aspects of teaching for understanding (Cohen, McLaughlin, & Talbert, 1993; Hiebert & Carpenter, 1992).

Sherin (2002) describes three changes typically seen when teachers try to shift from the traditional teaching of mathematics to an approach that falls more in line with teaching for understanding. The first change is seen in the instructional materials teachers use to teach mathematics. The focus in most textbooks is on procedural knowledge of the mathematics. Recently there have been numerous curricular programs that have been developed to be more in line with NCTM (2000) standards (Encyclopedia Britannica, 1998; Lappan, Fey, Fitzgerald, Friel, & Phillips, 1997; TERC, 1998). However, we found teachers do not necessarily need these programs in order to teach for understanding. Much of the DMT professional development focused on enabling teachers to create worthwhile tasks to help students explore mathematical relationships (Doerr, 2006; Gravemeijer, 2004). Teachers in our program created these types of problems on their own or together with other teachers. Similar to what Cohen (1990) and Wilson (1990) found, teachers will teach traditionally until they are put in situations where they are shown how to change their teaching practices and, then, attempt it themselves.

Sherin (2002) describes the second change as their style of teaching. Teaching for understanding is difficult because it means teachers cannot rely on the textbook or provide students with just one solution strategy. Instead, teachers must understand the structural components of the mathematics, how students come to understand the topic, and what different teaching approaches and ideas might work with students were are excelling or struggling. Because the DMT professional development focused clearly on these aspects, teachers were given the vision and impetus to make these changes in the classroom. Teachers began providing tasks that encouraged students to think and solve problems. The students were given the opportunity to share their ideas and discuss which ones were mathematically more efficient and/or which ones made more sense based on the structure of the task and what was being asked of them. This type of teaching is difficult and demands more of the teacher in areas of classroom management as well as a deeper knowledge of mathematics and students' thinking.

The final change is an increased focus on classroom discourse. By posing appropriate questions, discourse can be used to elicit and analyze students' thinking (Brendefur & Frykholm, 2000; Kazemi, 1998; Kazemi & Stipek, 2001). We found that discourse was significantly related to gains teachers made on the content knowledge inventory. When teachers were observed having dialogic conversations, we noted that more students were involved in making mathematical connections and defending their ideas mathematically to other students.

Our study suggests that elementary and middle school teachers' knowledge of mathematics, mathematics pedagogy, and learning progressions of students can be increased through participating in an ongoing professional development experience that focuses on developing mathematical thinking. As teachers participate in this type of rich professional development and as teachers attempt to put these ideas into practice, it is evident that both their content knowledge increases and their instructional practices move toward teaching for understanding.

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APPENDIX A

DMT Observation Scoring Sheet

Teacher: **Grade Level:** **Date:**

Attribute	**Description**	**Scoring**
		None Little Most All **0 --------1--------2--------3**
Nature of Classroom Tasks	Connect with where students are	
	Leave behind something of mathematical value	
	Requires students to use HOT when addressing the concept or problem	
	Provides opportunities for students to consider alternative solutions and strategies	
Role of the Teacher	Select tasks with goals in mind	
	Share essential information	
	Uses combination of conceptual and procedural knowledge	
	Provides opportunities for students to consider alternative solutions and strategies	
Social Culture of the Classroom	Ideas and methods are valued	
	Students choose and share their methods	
	Mistakes and misconceptions are learning opportunities for everyone	
	Correctness resides in mathematical argument	
	Safe learning environment	
Mathematical Tools as Learning Supports	Meaning for tools must be constructed by each user	
	Used with purpose—to solve problems	
	Used for recording, communicating, and thinking	
Equity and Accessibility	Tasks are accessible to all students	
	Every student is heard	
	All contribute	
Classroom Discourse	Elaborate explanations	
	Justify solution strategies – (logical argument)	
	Conceptual Press (univocal – dialogic)	