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TOPOLOGICAL CRITERIA FOR SCHLICHTNESS

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Abstract We give two sufficient criteria for schlichtness of envelopes of holomorphy in terms of topology. These are weakened converses of results of Kerner and Royden. Our first criterion generalizes a result of Hammond in dimension 2. Along the way, we also prove a generalization of Royden’s theorem.

Keywords: envelope of holomorphy; schlichtness; covering space

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Let \( \Omega \subseteq \mathbb{C}^n \) be a domain. The envelope of holomorphy of \( \Omega \) is a pair \((\widetilde{\Omega}, \pi)\) consisting of a connected Stein manifold \( \widetilde{\Omega} \) and a locally biholomorphic map \( \pi: \widetilde{\Omega} \to \mathbb{C}^n \), together with a holomorphic inclusion \( \alpha: \Omega \to \widetilde{\Omega} \), characterized by the following properties: \( \pi \circ \alpha \) is the identity, and each holomorphic function \( f \) on \( \Omega \) has a unique holomorphic extension \( F_f \) on \( \widetilde{\Omega} \), with \( f = F_f \circ \alpha \). Let \( \Omega' = \pi(\widetilde{\Omega}) \) and let \( i = \pi \circ \alpha: \Omega \to \Omega' \). The envelope of holomorphy \((\widetilde{\Omega}, \pi)\) is schlicht if \( \pi: \widetilde{\Omega} \to \Omega' \) is biholomorphic. One would like to give conditions on \( \Omega \) to have a schlicht envelope of holomorphy.

Two results of Kerner and Royden lead to necessary conditions. Kerner [5] has shown that \( \alpha^*: \pi_1(\Omega) \to \pi_1(\widetilde{\Omega}) \) is surjective. Royden [8] has shown that \( \alpha^*: H^1(\widetilde{\Omega}; \mathbb{Z}) \to H^1(\Omega; \mathbb{Z}) \) is injective. It follows trivially that if \((\widetilde{\Omega}, \pi)\) is schlicht, so \( \widetilde{\Omega} = \Omega' \), then \( i^*: \pi_1(\Omega) \to \pi_1(\Omega') \) is surjective and \( i^*: H^1(\Omega'; \mathbb{Z}) \to H^1(\Omega; \mathbb{Z}) \) is injective.

Neither of these conditions is sufficient, by a result of Fornæss and Zame [1] (see [2, §3]). Following an idea of Hammond [2], one may seek sufficient conditions by adjoining to the surjectivity of \( i^* \) (or injectivity of \( i^* \)) the assumption that \( \pi: \widetilde{\Omega} \to \Omega' \) is a covering space. This strong assumption is still reasonable, as covering maps certainly occur among envelopes of holomorphy; indeed, Fornæss and Zame show in [1] that for any covering map \( \pi: \widetilde{\Omega} \to \Omega' \) there is a domain \( \Omega \subseteq \Omega' \) with envelope of holomorphy \((\widetilde{\Omega}, \pi)\).

Specifically, Hammond has shown that, in dimension \( n = 2 \), if \( i^*: \pi_1(\Omega) \to \pi_1(\Omega') \) is surjective and \( \pi: \widetilde{\Omega} \to \Omega' \) is a covering map, then \((\widetilde{\Omega}, \pi)\) is schlicht. We give an elementary proof of Hammond’s theorem in all dimensions \( n \geq 2 \). In addition, we give a sufficient condition for schlichtness in terms of the injectivity of \( i^* \) on cohomology, again assuming \( \pi \) is a covering map. Along the way, we give an alternative proof of Royden’s theorem, which also extends it to coefficient groups other than \( \mathbb{Z} \).

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**Theorem 1.** If \( \pi \) is a covering map and \( i_* : \pi_1(\Omega) \to \pi_1(\Omega') \) is surjective, then \((\tilde{\Omega}, \pi)\) is schlicht.

This extends the theorem of Hammond for dimension \( n = 2 \). Hammond’s proof relies on a result of Jupiter \([4]\), which is special to dimension 2.

**Proof.** The number of sheets of the covering map \( \pi \) is equal to the index of \( \pi_* (\pi_1(\tilde{\Omega})) \) in \( \pi_1(\Omega') \) (see, for example, \([3, \text{Proposition 1.32}]\)). The surjectivity of \( i_* = \pi_* \circ \alpha_* \) implies that \( \pi_* \) is surjective. Hence, the index of the image subgroup is 1, so \( \pi : \tilde{\Omega} \to \Omega' \) is 1-sheeted, i.e. a homeomorphism. Since \( \pi \) is a holomorphic homeomorphism, it is biholomorphic and so \( \tilde{\Omega} \) is schlicht. \( \square \)

Compare with the more technical proof in \([2]\).

The cohomology in Royden’s result is Čech cohomology with coefficients in the sheaf of locally constant \( \mathbb{Z} \)-valued functions. Since our spaces are manifolds, Čech cohomology coincides with singular cohomology (with coefficients in \( \mathbb{Z} \)); see, for example, \([6, \text{Theorem 73.2}]\). Recall also that by the universal coefficient theorem, \( H^1(X; G) = \text{Hom}(\pi_1(X), G) \) for a path-connected space \( X \) and abelian coefficient group \( G \) \([3, \text{p. 198}]\).

Before we go on, observe that this proves Royden’s theorem as a consequence of Kerner’s theorem and extends it to other coefficient groups.

**Theorem 2 (Royden).** For any abelian group \( G \), \( \alpha^* : H^1(\Omega; G) \to H^1(\tilde{\Omega}; G) \) is injective.

**Proof.** Since \( \alpha_* : \pi_1(\Omega) \to \pi_1(\tilde{\Omega}) \) is surjective,

\[
\alpha^* : \text{Hom}(\pi_1(\Omega), G) \to \text{Hom}(\pi_1(\tilde{\Omega}), G)
\]

is injective and these Hom groups coincide with \( H^1(\Omega; G) \), \( H^1(\tilde{\Omega}; G) \). \( \square \)

Royden proves this for \( G = \mathbb{Z} \) using Čech cohomology, in particular the exponential short exact sequence (hence the restriction to \( G = \mathbb{Z} \)). No such result holds for higher cohomology groups \([1, \text{Theorem 4}]\).

Now, we aim to give a sufficient criterion for schlichtness by assuming \( i^* : H^1(\Omega'; G) \to H^1(\Omega; G) \) is injective for every abelian group \( G \), and that \( \pi \) is a covering map. Our strategy is to deduce that \( \pi_* : \pi_1(\tilde{\Omega}) \to \pi_1(\Omega') \) is surjective, as in the proof of Theorem 1. This would follow if we could deduce that \( i_* : \pi_1(\Omega) \to \pi_1(\Omega') \) is surjective, but, in general, injectivity of \( \text{Hom}(A, G) \to \text{Hom}(B, G) \) does not imply surjectivity of \( B \to A \). The problem is that if the image of \( B \) is a proper subgroup which is not contained in any proper normal subgroup, then there is no non-zero \( f : A \to G \) vanishing on the image of \( B \). For example, let \( \mathfrak{S}_3 \) be the symmetric group on three letters and let \( B = \mathbb{Z}/2\mathbb{Z} \) be the subgroup generated by a transposition. If \( f : \mathfrak{S}_3 \to G \) is any group homomorphism such that the restriction \( f \mid B \) is zero, then \( f \) itself is zero.

We must solve this problem by adjoining a hypothesis to ensure that every proper subgroup of \( \pi_1(\Omega') \) is contained in a proper normal subgroup. However, this alone is not enough. For, suppose that \( B \subset \pi_1(\Omega') \) is a proper subgroup, contained in a proper normal
subgroup $N$. We get a non-zero homomorphism $f: \pi_1(\Omega') \to G = \pi_1(\Omega)/N$, namely the quotient map, whose restriction to $B \subseteq N$ is zero, so $\text{Hom}(\pi_1(\Omega'),G) \to \text{Hom}(B,G)$ is not injective. This will prove the theorem we want, but only if $G$ is abelian, so we can identify these Hom groups with singular cohomology.

So we need to know that every proper subgroup of $\pi_1(\Omega')$ is not only contained in a proper normal subgroup, but in one such subgroup $N$ whose quotient $G = \pi_1(\Omega)/N$ is abelian.

Fortunately, this condition is more natural than it sounds. It holds if $\pi_1(\Omega')$ is nilpotent, as in that case every maximal proper subgroup is normal and has prime index (see [7, Theorem 5.40]).

We get the following.

**Theorem 3.** If $\pi$ is a covering map, $\pi_1(\Omega')$ is nilpotent and $i^*: H^1(\Omega';G) \to H^1(\Omega;G)$ is injective for every abelian group $G$, then $(\tilde{\Omega}, \pi)$ is schlicht.

**Proof.** Since $i^* = \alpha^* \circ \pi^*$ is injective, $\pi^*$ is injective as well. Via $\pi_*$, we regard $\pi_1(\tilde{\Omega})$ as a subgroup of $\pi_1(\Omega')$. Recall that if $H$ is any nilpotent group, then every maximal proper subgroup $N$ of $H$ is normal and has prime index, and, in particular, $H/N$ is abelian. If $\pi_1(\tilde{\Omega}) \not\subseteq \pi_1(\Omega')$, there exists a maximal subgroup $\pi_1(\tilde{\Omega}) \subseteq N \not\subseteq \pi_1(\Omega')$ and hence a surjection $\pi_1(\Omega') \to G = \pi_1(\Omega')/N$ to an abelian group with $\pi_1(\tilde{\Omega})$ mapping to zero. This surjection is non-zero and lies in the kernel of

$$\pi^*: H^1(\Omega';G) = \text{Hom}(\pi_1(\Omega'),G) \to \text{Hom}(\pi_1(\tilde{\Omega}),G) = H^1(\tilde{\Omega};G)$$

for the abelian group $G = \pi_1(\Omega')/N$, contradicting the injectivity of $\pi^*$.

It follows that $\pi_1(\tilde{\Omega}) = \pi_1(\Omega')$. As before, this implies that $\pi$ is a degree 1 covering map, and hence a biholomorphism. □

Solvability would not be enough, as shown by the example of $\mathbb{Z}/2\mathbb{Z} \subset S_3$. This would not only obstruct the proof given above, but would actually lead to a counter-example to the version of the statement, with solvable in place of nilpotent.

**Example 4.** Recall that Artin’s braid group on three strands, denoted $B_3$, is the fundamental group of the complement of the braid arrangement $A_2$ in $\mathbb{C}^3$, the union of the three hyperplanes defined by $(y-x)(z-x)(z-y) = 0$. Quotienting by the small diagonal, the line $x = y = z = 0$, $B_3$ is the fundamental group of $\Omega' \subset \mathbb{C}^2$, the complement of the union of three lines through the origin in $\mathbb{C}^2$. Let $B_2 \subset B_3$ be a subgroup corresponding to two of the strands, so $B_2 \cong \mathbb{Z}$ has index 3 in $B_3$ and is not normal. There exists a covering space $\tilde{\Omega} \to \Omega'$ such that $\pi_1(\tilde{\Omega}) = B_2 \subset B_3$. Since $U$ is a Stein manifold, so is $\tilde{\Omega}$ [9]. By [1, Theorem 5], there exists a domain $\Omega \subset \Omega'$ with envelope of holomorphy $\tilde{\Omega}$. This is not schlicht, but for every abelian group $G$, $\text{Hom}(B_3,G) \to \text{Hom}(B_2,G)$ is injective.

More generally, let $H$ be any finitely presented group. $H$ is the fundamental group of a 2-complex, which may be embedded in $\mathbb{R}^5$, or, for that matter, $\mathbb{C}^3$; then, a tubular neighbourhood $\Omega'$ of this complex (in $\mathbb{C}^3$) still has $\pi_1(\Omega') = H$. Any subgroup $K \subset H$
occurs as the fundamental group of a covering space $\hat{\Omega} \to \Omega'$. Again, $\hat{\Omega}$ is Stein since $\Omega'$ is, and there exists a domain $\Omega \subset \Omega'$ with envelope of holomorphy $\hat{\Omega}$.

It is not necessary to assume that $i^*$ is injective when coefficients are taken in any abelian group $G$. It would be enough to assume that $i^*$ is injective when coefficients are taken in any finite cyclic group, in any abelian quotient $G$ of $\pi_1(\Omega')$ or even just in a single abelian quotient $G = \pi_1(\Omega')/N$ for some proper normal subgroup $N$ containing $\pi_1(\hat{\Omega})$.

If, in addition, $\pi: \hat{\Omega} \to \Omega'$ is a normal covering space, then $\pi_1(\hat{\Omega}) \subseteq \pi_1(\Omega')$ is a normal subgroup and we can take $G$ to be an abelian quotient of $\pi_1(\Omega')/\pi_1(\hat{\Omega})$, which is the group of deck transformations.

**Corollary 5.** Suppose $\pi$ is a normal covering map with deck transformation group $H$. If there exists a non-zero abelian quotient $G$ of $H$ such that $i^*: H^1(\Omega'; G) \to H^1(\Omega; G)$ is injective, then $(\hat{\Omega}, \pi)$ is schlicht.

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**References**