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WARING DECOMPOSITIONS OF MONOMIALS

WERONIKA BUCZYŃSKA, JAROSŁAW BUCZYŃSKI, AND ZACH TEITLER

To Tony Geramita on the occasion of his 70th birthday.

Abstract. A Waring decomposition of a polynomial is an expression of the polynomial as a sum of powers of linear forms, where the number of summands is minimal possible. We prove that any Waring decomposition of a monomial is obtained from a complete intersection ideal, determine the dimension of the set of Waring decompositions, and give the conditions under which the Waring decomposition is unique up to scaling the variables.

1. Introduction

Let $F \in \mathbb{C}[x_0, \dots, x_n]$ be a complex homogeneous polynomial of degree d. The **Waring rank** of F, denoted $R(F)$, is the least r such that $F = {\ell_1}^d + \dots + {\ell_r}^d$ for some linear forms $\ell_1, \dots, \ell_r$. Any expression $F = {\ell_1}^d + \dots + {\ell_r}^d$ with $r = R(F)$ is a **Waring decomposition**.

For example, $xy = \frac{1}{4}(x+y)^2 - \frac{1}{4}(x-y)^2$ and $xyz = \frac{1}{24}(x+y+z)^3 - \frac{1}{24}(x+y-z)^3 - \frac{1}{24}(x-y+z)^3 + \frac{1}{24}(x-y-z)^3$. Similar decompositions may be found for any monomial (see §2). These decompositions are not unique, as one may permute the variables and scale them: for xy, replace (x, y) with $(sx, \frac{1}{s}y)$ and for xyz, replace (x, y, z) with $(sx, ty, \frac{1}{st}z)$. For any monomial $F = x_0^{d_0} \cdots x_n^{d_n}$, one may scale the variables x_i by scalars λ_i such that $\prod \lambda_i^{d_i} = 1$, leaving the monomial fixed but changing the Waring decompositions. It is natural to ask if all Waring decompositions are obtained in this way; that is, whether Waring decompositions of monomials are unique up to scaling the variables.

Earlier studies of Waring decompositions have considered the question of uniqueness, going back to classical results such as Sylvester's Pentahedral Theorem and more recent work such as [RS00, Mel09], mostly concentrating on actual uniqueness (not up to scaling) of Waring decompositions of general forms. More generally, for $F \in \mathbb{C}[x_0, \dots, x_n]$ homogeneous of degree d and $r = R(F)$, the *variety of sums of powers* $\mathrm{VSP}(F)$ is the closure in $\mathrm{Hilb}_r(\mathbb{P}^n)$ of the set VSP° of reduced tuples $\{[\ell_1], \dots, [\ell_r]\}$ such that $F = \ell_1^d + \dots + \ell_r^d$ (see §5). These varieties have proved to be of interest; see [Muk92, RS00, IR01].

We describe $\mathrm{VSP}(F)$ and determine its dimension when $F = x_0^{d_0} \cdots x_n^{d_n}$ is a monomial. We answer the question of uniqueness of Waring decomposition up to scaling the variables, which amounts to determining whether a torus action on $\mathrm{VSP}^\circ(F)$ is transitive.

Both [RS11] and [CCG12] noted that a Waring decomposition $F = \ell_1^d + \dots + \ell_r^d$ may be obtained with $\{[\ell_1], \dots, [\ell_r]\}$ a complete intersection. We show that in fact every Waring decomposition of F is a complete intersection of a certain form.

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Theorem 1. *Suppose $F \in \mathbb{C}[x_0, \dots, x_n]$ is a monomial $F = x_0^{d_0} \cdots x_n^{d_n}$ with $0 < d_0 \leq \cdots \leq d_n$, $d = d_0 + \cdots + d_n$, and $F = {\ell_1}^d + \cdots + {\ell_r}^d$ for $r = R(F)$. Let $\mathcal{I} \subset \mathbb{C}[\alpha_0, \dots, \alpha_n]$ be the homogeneous ideal of functions vanishing on $Q = \{[\ell_1], \dots, [\ell_r]\} \subset \mathbb{P}^n$. Then $\mathcal{I}$ is a complete intersection of degrees $d_1 + 1, \dots, d_n + 1$, generated by:*

$$ {\alpha_1}^{d_1+1} - \phi_1 {\alpha_0}^{d_0+1}, \dots, {\alpha_n}^{d_n+1} - \phi_n {\alpha_0}^{d_0+1} $$

for some homogeneous polynomials $\phi_i \in \mathbb{C}[\alpha_0, \dots, \alpha_n]$ of degree $d_i - d_0$.

As a consequence of this and some additional restrictions on the polynomials ϕ_i we compute the dimension of the variety of sums of powers of a monomial.

Theorem 2. *Suppose $F \in \mathbb{C}[x_0, \dots, x_n]$ is a monomial $F = x_0^{d_0} \cdots x_n^{d_n}$ with $0 < d_0 \leq \cdots \leq d_n$. Let h be the Hilbert function of $\mathbb{C}[x_0, \dots, x_n]/(x_1^{d_1+1}, \dots, x_n^{d_n+1})$. Then $\mathrm{VSP}(F)$ is irreducible and $\dim \mathrm{VSP}(F) = h(d_1 - d_0) + \cdots + h(d_n - d_0)$.*

Corollary 3. $\dim \mathrm{VSP}(F) \geq n$*, with equality if and only if $F = (x_0 \cdots x_n)^k$.*

Finally we answer the uniqueness question.

Theorem 4. *Suppose $F \in \mathbb{C}[x_0, \dots, x_n]$ is a monomial $F = x_0^{d_0} \cdots x_n^{d_n}$ with $0 < d_0 \leq \cdots \leq d_n$. Let $(\mathbb{C}^*)^{n+1}$ act on $\mathbb{C}[x_0, \dots, x_n]$ by scaling the variables. The action of the n-dimensional subtorus $\mathcal{T} = \{(\lambda_0, \dots, \lambda_n) \mid \prod \lambda_i^{d_i} = 1\}$ on $\mathrm{VSP}^\circ(F)$ is transitive if and only if $d_0 = \cdots = d_n$.*

This uniqueness had been shown for $F = xyz$ by Bruce Reznick (2008, personal communication). Despite the very classical nature of the subject, it was not possible to address these questions in greater generality until very recently, as Waring ranks of monomials were only determined in 2011. This is remarkable when one considers that Waring ranks have been studied for at least 160 years. Indeed, the ranks and decompositions of quadratic forms were understood classically. The ranks and decompositions of polynomials in two variables are completely understood, following work by Sylvester in 1851 [Syl51] and recent work by Comas and Seiguer [CS11]. Beyond these cases it is a difficult problem to determine or even to give decent bounds on $R(F)$, even for seemingly simple polynomials such as monomials. For much more on this topic, including history, see [IK99] and more recently [Rez10]. Regarding monomials, the paper [LT10] found $R(xyz) = 4$, $R(xyzw) = 8$, and $R(xyz^2) = 6$. A recent paper of Ranestad and Schreyer [RS11] gives a lower bound for rank (of any homogeneous polynomial) which, for monomials, has the following consequence: If $F = {x_0}^{d_0} \cdots {x_n}^{d_n}$ is a monomial, $d_0 \leq \cdots \leq d_n$, then

$$ R(F) \geq (d_0 + 1)(d_1 + 1) \cdots (d_{n-1} + 1). \tag{5} $$

And conversely, it was communicated to us by K. Ranestad [Ran11, p. 15] that

$$ R(F) \leq (d_1 + 1) \cdots (d_{n-1} + 1)(d_n + 1). \tag{6} $$

As a special case, when $d_0 = \cdots = d_n$, this determines the rank $R((x_0 \cdots x_n)^d) = (d+1)^n$, see [RS11]. The proof of this upper bound described in [RS11] uses the Bertini theorem; another proof is given in [CCG12]. We give a third, elementary proof below.

Finally, in 2011 the Waring rank problem for monomials was solved by Carlini, Catalisano, and Geramita [CCG12].

Theorem 7 (Carlini, Catalisano, Geramita)**.** *The rank of a monomial* $x_0^{d_0} \cdots x_n^{d_n}$ *with* $d_0 \leq \cdots \leq d_n$ *is equal to* $(d_1+1)\cdots(d_n+1)$.

That is, the rank is equal to the upper bound found by Ranestad and Schreyer.

It is this result of Carlini, Catalisano, and Geramita which opens the possibility of studying VSP of monomials. We give an alternative proof along the way to proving our theorems. Note, however, that their proof is remarkably direct.

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Notation. Except in §2 we work over $\mathbb{C}$.

We recall a few well-known facts about the apolarity pairing; see, for example, [IK99, §1.1] for details. Let $T = \mathbb{C}[x_0, \ldots, x_n]$ and $S = \mathbb{C}[\alpha_0, \ldots, \alpha_n]$. Elements $D \in S$ act on polynomials $F \in T$ by letting α_i act as the differentiation operator $\partial/\partial x_i$. The induced pairing $S_a \otimes T_d \to T_{d-a}$ (where $T_k = 0$ for $k < 0$) is called the apolarity pairing; it is a perfect pairing when $d = a$. For a fixed $F \in T$, the set $F^\perp = \{D \in S : DF = 0\}$ is an ideal in S, called the annihilator of F. Suppose F is a homogeneous polynomial of degree d and $F = \ell_1^d + \cdots + \ell_r^d$ for linear forms $\ell_1, \ldots, \ell_r$. Consider the set of points $\{[\ell_1], \ldots, [\ell_r]\}$ in the projective space $\mathbb{P}^n$. The defining ideal $\mathcal{I} = \mathcal{I}(\{[\ell_1], \ldots, [\ell_r]\})$ of this set of points is a saturated radical homogeneous ideal with $\mathcal{I} \subseteq F^\perp$.

Remark. Suppose $F \in \mathbb{C}[x_0, \ldots, x_n, y_1, \ldots, y_m]$ is a homogeneous polynomial that depends only on the first $n+1$ variables $x_0, \ldots, x_n$, but considered as a polynomial in $n+m+1$ variables. If $F = \ell_1^d + \cdots + \ell_r^d$ is a Waring decomposition (i.e. $r = R(F)$) with ℓ_i linear forms apriori in $n+m+1$ variables, then in fact each ℓ_i depends only on $x_0, \ldots, x_n$, see [Lan11, Ex. 3.2.2.2]. Thus one can easily generalize Theorems 1 and 2 to any monomials, i.e., without the assumption $d_i > 0$.

2. Explicit expression for monomials as sums of powers

Let $F = x^{\mathbf{d}} = {x_0}^{d_0} \cdots {x_n}^{d_n} \in \Bbbk[x_0, \ldots, x_n]$ with $0 < d_0 \leq \cdots \leq d_n$ and let $d = d_0 + \cdots + d_n$. For $\Bbbk = \mathbb{C}$, Corollary 3.8 of [CCG12] shows that there is a power sum decomposition

$$x_0^{d_0} \cdots x_n^{d_n} = \sum_{\substack{0 \leq a_i \leq d_i \\ i=1,\ldots,n}} (x_0 + \zeta_1^{a_1} x_1 + \cdots + \zeta_n^{a_n} x_n)^d \gamma_{a_1,\ldots,a_n},$$

where ζ_i is a primitive (d_i+1)-th root of unity for $i = 1, \ldots, n$, for some coefficients γ.

In this section, we give the γ explicitly. For this section only, we do not work over $\mathbb{C}$, as we wish to consider the most general field possible.

While we reserve $R(F)$ for the Waring rank of a complex polynomial, one may ask similar questions over other fields. Given $F \in \Bbbk[x_0, \ldots, x_n]$, the Waring rank of F with respect to $\Bbbk$, denoted $R_\Bbbk(F)$, is the least r such that $F = c_1\ell_1^d + \cdots + c_r\ell_r^d$

for some constants $c_i \in \Bbbk$ and linear forms ℓ_i with coefficients in $\Bbbk$. (When $\Bbbk = \mathbb{C}$, or more generally if $\Bbbk$ is algebraically closed, the coefficients are unnecessary, as $c_i\ell_i^d$ can be replaced by $(c_i^{1/d}\ell_i)^d$.)

In this generality rank may not always be finite, because some polynomials are not sums of powers. For example, xy is not a sum of squares in characteristic 2 and xyz is not a sum of cubes in characteristics 2 or 3.

If F is a polynomial over $\Bbbk$ and $\Bbbk \subseteq \mathbb{K}$ then $R_{\Bbbk}(F) \geq R_{\mathbb{K}}(F)$. In general it may be strictly greater; see, for example, [Old40, Rez10].

Suppose $\Bbbk$ is a field such that (d_i+1)-th roots of unity exist in $\Bbbk$ for $i = 1, \dots, n$ and C is invertible, where

$$C = \binom{d}{d_0, d_1, \dots, d_n}(d_1+1)\cdots(d_n+1),$$

where $\binom{d}{d_0,d_1,\dots,d_n}$ is the multinomial coefficient $d!/(d_0!\cdots d_n!)$. For $1 \leq i \leq n$ let $\zeta_i \in \Bbbk$ be a primitive (d_i+1)-th root of unity. We claim that

$$x_0^{d_0}\cdots x_n^{d_n} = \frac{1}{C}\sum_{\substack{0\leq a_i\leq d_i\\ i=1,\dots,n}} (x_0 + \zeta_1^{a_1}x_1 + \cdots + \zeta_n^{a_n}x_n)^d(\zeta_1^{a_1}\cdots\zeta_n^{a_n}). \tag{8}$$

In particular this expression has $(d_1+1)\cdots(d_n+1)$ summands, implying $R_{\Bbbk}(F) \leq (d_1+1)\cdots(d_n+1)$, which is (6) when $\Bbbk = \mathbb{C}$.

Consider the monomial $x^{\mathbf{m}} = x_0^{m_0}\cdots x_n^{m_n}$ for some $\mathbf{m} = (m_0, \dots, m_n)$ such that $m_0 + \cdots + m_n = d$. The coefficient of this monomial in the right hand side of the above equation is

$$C_{\mathbf{m}} = \frac{1}{C}\binom{d}{m_0, \dots, m_n}\sum_{\substack{0\leq a_i\leq d_i\\ i=1,\dots,n}} \zeta_1^{a_1(m_1+1)}\cdots\zeta_n^{a_n(m_n+1)}.$$

This factors as

$$\frac{1}{C}\binom{d}{m_0, \dots, m_n}\left(\sum_{a_1=0}^{d_1}(\zeta_1^{m_1+1})^{a_1}\right)\cdots\left(\sum_{a_n=0}^{d_n}(\zeta_n^{m_n+1})^{a_n}\right).$$

Since each $\zeta_i^{m_i+1}$ is still a (d_i+1)-th root of unity (not necessarily primitive), we have

$$\sum_{a_i=0}^{d_i}(\zeta_i^{m_i+1})^{a_i} = \begin{cases} d_i+1, & \zeta_i^{m_i+1} = 1,\\ 0 & \text{otherwise}\end{cases}$$

Hence,

$$C_{\mathbf{m}} = \begin{cases} \frac{1}{C}\binom{d}{m_0,\dots,m_n}(d_1+1)\cdots(d_n+1), & \zeta_1^{m_1+1} = \cdots = \zeta_n^{m_n+1} = 1\\ 0 & \text{otherwise}\end{cases}$$

In particular $C_{\mathbf{d}} = 1$, that is, $x^{\mathbf{d}} = x_0^{d_0}\cdots x_n^{d_n}$ appears on the right hand side of (8) with coefficient 1.

Suppose $x^{\mathbf{m}}$ has nonzero coefficient in (8). Then by above $\zeta_i^{m_i+1} = 1$ for $i = 1, \dots, n$, and we see each m_i+1 is a multiple of d_i+1 for $i > 0$. Together with $m_0 + m_1 + \cdots + m_n = d_0 + \cdots + d_n$ we get $m_0 \leq d_0$. If for some $i > 0$ we have $m_i + 1 > d_i + 1$, then $m_i + 1 \geq 2(d_i+1)$ and so $m_0 \leq d_0 - (d_i+1) < 0$, since $d_0 \leq \cdots \leq d_n$. This contradiction shows that the only term with a nonzero coefficient in (8) is the term having each $m_i + 1 = d_i + 1$, i.e., the term $x^{\mathbf{d}}$.

3. Hilbert function

Now we begin working toward a proof of Theorem 1, along the way giving an alternative proof of Theorem 7. Henceforth $F = x_0^{d_0} \cdots x_n^{d_n}$ and $\mathcal{I}$ is as described in Theorem 1. We start by computing the Hilbert function of $\mathcal{I}$.

From this section onwards we work over the field $\Bbbk = \mathbb{C}$.

Let $\mathcal{J} \subset S$ be a complete intersection ideal generated in degrees $d_1+1, \ldots, d_n+1$. Note that $\dim(S/\mathcal{J})_t = (d_1+1)\cdots(d_n+1)$ for $t \gg 0$.

Proposition 9. *Suppose $F = x_0^{d_0} \cdots x_n^{d_n} = \ell_1^d + \cdots + \ell_r^d$ with r minimal possible, i.e., $r = R(F)$. Let $\mathcal{I} = \mathcal{I}(\{[\ell_1], \ldots, [\ell_r]\})$. Then the Hilbert function of $\mathcal{I}$ is equal to the Hilbert function of $\mathcal{J}$.*

The difference between our treatment of the proposition and the proof of Theorem 7 in [CCG12] is that we compare the Hilbert functions of $\mathcal{I}$ and $\mathcal{I} \cap (\alpha_0)$, whereas [CCG12] compare $\mathcal{I}$ and $\mathcal{I} : \alpha_0 + (\alpha_0)$, which leads to a remarkably quick proof of Theorem 7. (An earlier version of [CCG12] compared $\mathcal{I}$ and $\mathcal{I} + (\alpha_0)$.)

The remainder of this section gives a proof of this proposition, via some lemmas. First, a linear algebra lemma will be helpful.

Lemma 10. *If A, B, and C are finite dimensional vector spaces such that*

$$\begin{array}{ccc} A & \supset & A \cap B \\ \cap & & \cap \\ C & \supset & B \end{array}$$

then

$$\dim(A \cap B) \geq \dim A + \dim B - \dim C.$$

We denote

$$q_t = \dim\Big(\mathcal{I}_t \cap (\alpha_0 \cdot S_{t-1})\Big).$$

Lemma 11. *With notation as above, for all $t \in \mathbb{Z}$, $q_t \leq \dim \mathcal{J}_{t-1}$. Moreover, if for some t we have $q_t < \dim \mathcal{J}_{t-1}$, then $q_{t+kd_0} < \dim \mathcal{J}_{t+kd_0-1}$ for all $k > 0$.*

Proof. For convenience, let $(b_0, b_1, b_2, \ldots, b_n) = (d_0, d_1+1, d_2+1, \ldots, d_n+1)$. Note that

$$F^\perp \cap (\alpha_0) = \alpha_0 \cdot (\alpha_0^{b_0}, \alpha_1^{b_1}, \ldots, \alpha_n^{b_n}).$$

It is convenient also to write

$$(F^\perp)_t \cap \alpha_0 \cdot S_{t-1} = \alpha_0 \cdot (\alpha_0^{b_0} S_{t-1-b_0} + \cdots + \alpha_n^{b_n} S_{t-1-b_n}).$$

Then by inclusion-exclusion we may write the dimension of this space as

$$\dim\big((F^\perp)_t \cap \alpha_0 \cdot S_{t-1}\big) = \sum_{j=1}^{n+1} (-1)^{j-1} \sum_{0 \leq i_1 < \cdots < i_j \leq n} \dim S_{t-1-b_{i_1}-\cdots-b_{i_j}}.$$

Separating terms which omit or include b_0,

$$\begin{aligned}\dim((F^\perp)_t \cap \alpha_0 \cdot S_{t-1}) &= \left[\sum_{j=1}^{n}(-1)^{j-1}\sum_{1\le i_1<\cdots<i_j\le n}\dim S_{t-1-b_{i_1}-\cdots-b_{i_j}}\right]\\ &\quad+\left[\sum_{j=1}^{n+1}(-1)^{j-1}\sum_{0=i_1<\cdots<i_j\le n}\dim S_{t-1-b_{i_1}-\cdots-b_{i_j}}\right]\\ &=\left[\sum_{j=1}^{n}(-1)^{j-1}\sum_{1\le i_1<\cdots<i_j\le n}\dim S_{t-1-b_{i_1}-\cdots-b_{i_j}}\right]\\ &\quad-\left[\sum_{j=0}^{n}(-1)^{j-1}\sum_{1\le i_1<\cdots<i_j\le n}\dim S_{t-b_0-1-b_{i_1}-\cdots-b_{i_j}}\right]\\ &=\dim \mathcal{J}_{t-1}-\dim \mathcal{J}_{t-b_0-1}+\dim S_{t-b_0-1}.\end{aligned}$$

Thus

$$\dim((F^\perp)_t \cap \alpha_0 \cdot S_{t-1}) - \dim S_{t-d_0-1} = \dim \mathcal{J}_{t-1} - \dim \mathcal{J}_{t-d_0-1}. \tag{12}$$

We apply this to the diagram below:

$$\begin{array}{ccc}\mathcal{I}_t \cap \alpha_0 \cdot S_{t-1} & \supset & \mathcal{I}_t \cap ({\alpha_0}^{d_0+1} \cdot S_{t-d_0-1})\\ \cap & & \cap \\ (F^\perp)_t \cap (\alpha_0 \cdot S_{t-1}) & \supset & {\alpha_0}^{d_0+1} \cdot S_{t-d_0-1}.\end{array}$$

By Lemma 10,

$$q_t - \dim(\mathcal{I}_t \cap (\alpha_0^{d_0+1} \cdot S_{t-d_0-1})) \le \dim \mathcal{J}_{t-1} - \dim \mathcal{J}_{t-d_0-1}.$$

Note that, since $\mathcal{I}$ is radical,

$$\mathcal{I}_t \cap (\alpha_0^{d_0+1} \cdot S_{t-d_0-1}) = \alpha_0^{d_0} \cdot (\mathcal{I}_{t-d_0} \cap (\alpha_0 \cdot S_{t-d_0-1})),$$

which has dimension q_{t-d_0}, giving

$$q_t - q_{t-d_0} \le \dim \mathcal{J}_{t-1} - \dim \mathcal{J}_{t-d_0-1}. \tag{13}$$

Now for any t,

$$\begin{aligned} q_t &= (q_t - q_{t-d_0}) + (q_{t-d_0} - q_{t-2d_0}) + \cdots\\ &\le (\dim \mathcal{J}_{t-1} - \dim \mathcal{J}_{t-d_0-1}) + (\dim \mathcal{J}_{t-d_0-1} - \dim \mathcal{J}_{t-2d_0-1}) + \cdots\\ &= \dim \mathcal{J}_{t-1},\end{aligned}$$

as claimed.

If for some t we have $q_t < \dim \mathcal{J}_{t-1}$, then by (13) with t replaced by $t + d_0$, we get $q_{t+d_0} \le \dim \mathcal{J}_{t+d_0-1} - \dim \mathcal{J}_{t-1} + q_t < \dim \mathcal{J}_{t+d_0-1}$. Inductively we obtain the second claim of the lemma. $\square$

We may avoid the inclusion-exclusion and alternating sums in the above proof with the following argument.

Alternative proof of Equation (12). As above, suppose $F = x_0^{d_0} \cdots x_n^{d_n}$ so that $F^\perp = (\alpha_0^{d_0+1}, \dots, \alpha_n^{d_n+1})$.

Let $\mathcal{J} = (\alpha_1^{d_1+1}, \dots, \alpha_n^{d_n+1})$, so $\mathcal{J}$ is a complete intersection ideal generated in degrees $d_1 + 1, \dots, d_n + 1$.

Note that

$$F^{\perp} \cap (\alpha_0) = \alpha_0 \cdot (\alpha_0^{d_0}, \alpha_1^{d_1+1}, \ldots, \alpha_n^{d_n+1}) = \alpha_0 \cdot (\mathcal{J} + (\alpha_0^{d_0})).$$

Consider the graded S-module $\mathcal{J}/\alpha_0^{d_0}\mathcal{J}$. We write two short exact sequences of graded S-modules as follows:

$$0 \to \mathcal{J}(-d_0) \xrightarrow{\alpha_0^{d_0}} \mathcal{J} \to \frac{\mathcal{J}}{\alpha_0^{d_0}\mathcal{J}} \to 0,$$
$$0 \to S(-d_0) \xrightarrow{\alpha_0^{d_0}} \mathcal{J} + (\alpha_0^{d_0}) \to \frac{\mathcal{J}}{\alpha_0^{d_0}\mathcal{J}} \to 0.$$

One may check easily that these are exact (for the second one this is a consequence of $(\alpha_0^{d_0}) \cap \mathcal{J} = \alpha_0^{d_0}\mathcal{J}$). Counting dimensions of the $(t-1)$-th graded piece,

$$\dim \mathcal{J}_{t-1} - \dim \mathcal{J}_{t-d_0-1} = \dim \left(\frac{\mathcal{J}}{\alpha_0^{d_0}\mathcal{J}}\right)_{t-1} = \dim(\mathcal{J} + (\alpha_0^{d_0}))_{t-1} - \dim S_{t-d_0-1},$$

which recovers (12) as above and the rest of the proof follows as before. □

Lemma 14. *With notation as above, for all integers t we have*

$$\dim(S_t/\mathcal{I}_t) \geq \dim S_{t-1} - q_t.$$

Proof. Consider:

$$\begin{array}{ccc} \alpha_0 S_{t-1} & \supset & \mathcal{I}_t \cap \alpha_0 \cdot S_{t-1} \\ \cap & & \cap \\ S_t & \supset & \mathcal{I}_t. \end{array}$$

By Lemma 10,

$$\dim(S_t/\mathcal{I}_t) = \dim S_t - \dim \mathcal{I}_t \geq \dim S_{t-1} - \dim(\mathcal{I}_t \cap \alpha_0 \cdot S_{t-1}) = \dim S_{t-1} - q_t,$$

which proves the lemma. □

From the two lemmas, we recover the asymptotic version of Proposition 9. That is, for sufficiently large t, we have:

$$\begin{aligned} R(F) = r &= \dim(S_t/\mathcal{I}_t) && \text{for } t \gg 0 \\ &\geq \dim S_{t-1} - q_t && \text{by Lemma 14} \\ &\geq \dim S_{t-1} - \dim \mathcal{J}_{t-1} && \text{by Lemma 11} \\ &= (d_1+1)\cdots(d_n+1) && \text{for } t \gg 0. \end{aligned} \tag{15}$$

Since $r \leq (d_1+1)\cdots(d_n+1)$ by the explicit expression in Section 2, all inequalities must be equalities. This gives an alternative proof of Theorem 7. In particular, for sufficiently large t:

$$\dim S_t - \dim \mathcal{I}_t = \dim S_{t+1} - \dim \mathcal{I}_{t+1} = \dim S_t - q_{t+1}, \text{ i.e., } \dim \mathcal{I}_t = q_{t+1}. \tag{16}$$

Lemma 17. *We have $\mathcal{I} : \alpha_0 = \mathcal{I}$.*

Proof. Multiplication by α_0 gives a one-to-one map $\mathcal{I}_t \to \mathcal{I}_{t+1} \cap \alpha_0 \cdot S_t$. For $t \gg 0$, since $\dim \mathcal{I}_t = q_{t+1}$, this multiplication map is onto, hence $(\mathcal{I} : \alpha_0)_t = \mathcal{I}_t$ in sufficiently high degree. Thus $\mathcal{I}$ and $\mathcal{I} : \alpha_0$ agree up to saturation. But both ideals are saturated, so $\mathcal{I} = \mathcal{I} : \alpha_0$.

Explicitly, let $\beta \in (\mathcal{I} : \alpha_0)_t$. Then $\beta^N \in (\mathcal{I} : \alpha_0)_{tN}$ and $\alpha_0\beta^N \in \mathcal{I}_{tN+1}$. For $N \gg 0$, $\dim \mathcal{I}_{tN} = \dim(\mathcal{I} \cap (\alpha_0))_{tN+1}$, so $\beta^N \in \mathcal{I}_{tN}$, and since $\mathcal{I}$ is radical, $\beta \in \mathcal{I}$. □

Lemma 18. *For all t, we have* $\dim \mathcal{I}_t = q_{t+1}$.

Proof. Lemma 17 shows that multiplication by α_0 gives a bijection $\mathcal{I}_t = (\mathcal{I} : \alpha_0)_t \to (\mathcal{I} \cap (\alpha_0))_{t+1}$ in every degree t. □

Proof of Proposition 9. By Lemmas 11 and 18,

$$\dim \mathcal{I}_t = q_{t+1} \leq \dim \mathcal{J}_t.$$

By the "moreover" part of Lemma 11, if for some t we have a strict inequality $\dim \mathcal{I}_t < \dim \mathcal{J}_t$, then for all nonnegative k,

$$\dim(S_{t+kd_0}/\mathcal{I}_{t+kd_0}) > \dim S_{t+kd_0} - \dim \mathcal{J}_{t+kd_0},$$

a contradiction with (15) for sufficiently large k. □

Corollary 19. *For $F = x_0^{d_0} \cdots x_n^{d_n} = \ell_1^d + \cdots + \ell_r^d$ with $d_0 \leq \cdots \leq d_n$ and $r = R(F)$, each of the linear forms ℓ_i has x_0 appearing with nonzero coefficient.*

Proof. The coordinate α_0 on a point $[\ell] = [\alpha_0 x_0 + \cdots + \alpha_n x_n] \in \mathbb{P}T_1$ gives the coefficient of x_0 in the linear form ℓ. As above, let $\mathcal{I}$ be the defining ideal of the set of points $Q = \{[\ell_1], \ldots, [\ell_r]\}$. Then $\mathcal{I} : \alpha_0$ is the defining ideal of the subset of points in Q not lying on the hyperplane $\{\alpha_0 = 0\}$. By Lemma 17, $\mathcal{I} : \alpha_0 = \mathcal{I}$, so all the points $[\ell_i]$ are not contained in this hyperplane, and hence have nonzero coefficients of x_0. □

4. Complete intersections

Fix $k \in \{0, \ldots, n\}$, and $\bar{\phi} = (\phi_1, \ldots, \phi_k)$, with $\phi_i \in S_{d_i - d_0}$. We define the following homogeneous ideal:

$$\mathcal{I}(k, \bar{\phi}) = (\alpha_i^{d_i+1} - {\alpha_0}^{d_0+1}\phi_i \mid i \in \{1, \ldots, k\}).$$

Every such ideal is a complete intersection ideal generated in degrees $d_1+1, \ldots, d_k+1$.

Example 20. The explicit expression in Section 2 corresponds to

$$\bar{\phi} = (\alpha_0^{d_1-d_0}, \ldots, \alpha_0^{d_n-d_0}).$$

The following Proposition proves Theorem 1.

Proposition 21. *Any ideal $\mathcal{I}$ as in Theorem 1 is of the form $\mathcal{I}(n, \bar{\phi})$, for some $\bar{\phi} = (\phi_1, \ldots, \phi_n)$.*

Proof. Let $\mathcal{I}_{\leq t}$ denote the ideal generated by the homogeneous elements in $\mathcal{I}$ of degree at most t. We will prove by induction that $\mathcal{I}_{\leq t} = \mathcal{I}(k, \bar{\phi})$ for some $k = k(t)$ and $\bar{\phi} = \bar{\phi}(t)$. More precisely, we claim that $k(t) = \#\{i : d_i + 1 \leq t\}$. For $t = 0$, $\mathcal{I}_{\leq 0} = \mathcal{I}(0, \emptyset) = 0$. Suppose $\mathcal{I}_{\leq t-1} = \mathcal{I}(k, \bar{\phi})$ for $k = \#\{i : d_i + 1 \leq t - 1\}$ and some $\bar{\phi} = \bar{\phi}(t-1)$. If $t \neq d_i + 1$ for any $i \in \{1, \ldots, n\}$, then

$$\dim \mathcal{I}_t = \dim \mathcal{J}_t = \dim(\mathcal{I}_{\leq t-1})_t.$$

The first equality follows from Proposition 9 and the second from the fact that $\mathcal{I}(k, \bar{\phi})$ is a complete intersection of the same degrees as $\mathcal{J}$, up to degree t. So there is no new generator of $\mathcal{I}$ in degree t, and $\mathcal{I}_{\leq t} = \mathcal{I}(k, \bar{\phi})$.

Now suppose $t = d_{k+1} + 1 = \cdots = d_l + 1 < d_{l+1} + 1$ for some $l > k$. Then by the same argument, there must be exactly $l - k$ new linearly independent generators $(\rho_{k+1}, \ldots, \rho_l)$ of $\mathcal{I}$ in degree t. Each ρ_i must be in $(F^{\perp})_t$. Using the generators of the lower degree, we may eliminate from ρ_i the summands divisible by $\alpha_j^{d_j+1}$ for each $1 \le j \le k$. That is, we can assume that each of the new generators is of the form

$$\rho_i = \sum_{j=k+1}^{l} c_{ij}{\alpha_j}^{d_j+1} - \psi_i{\alpha_0}^{d_0+1}$$

for some $\psi_i \in S_{t-d_0-1}$ and $c_{ij} \in \mathbb{C}$. We claim the matrix $C := (c_{ij})_{i,j=k+1}^{l}$ is invertible. Suppose on contrary that there exists a linear combination ρ of the ρ_i's such that $\rho = \psi{\alpha_0}^{d_0+1}$ (possibly $\psi = 0$). Since $\mathcal{I}$ is radical, $\psi\alpha_0 \in \mathcal{I}_{t-d_0}$. But then $\rho \in \mathcal{I}_{\le t-1}$ and one of the generators ρ_i is redundant, a contradiction.

So C is invertible, and by replacing $(\rho_{k+1}, \dots, \rho_l)$ with the linear combinations $C^{-1}(\rho_{k+1}, \dots, \rho_l)^t$ (and analogously for ψ_i), we may assume:

$$\rho_i = {\alpha_i}^{d_i+1} - \psi_i{\alpha_0}^{d_0+1}.$$

Set $\bar{\phi}' = (\phi_1, \dots, \phi_k, \psi_{k+1}, \dots, \psi_l)$. By the above, we have $\mathcal{I}_{\le t} = \mathcal{I}(l, \bar{\phi}')$. □

We have just seen that if $\mathcal{I} \subset F^\perp$ is a radical one-dimensional ideal, then $\mathcal{I} = \mathcal{I}(n, \bar{\phi})$ for some $\bar{\phi}$. One may ask which $\bar{\phi} = (\phi_1, \dots, \phi_n)$ can occur. For any $\bar{\phi}$, $\mathcal{I}(n, \bar{\phi})$ is a one-dimensional complete intersection ideal. So the question is, for which $\bar{\phi}$ is $\mathcal{I}(n, \bar{\phi})$ radical? An obvious necessary condition is that each ϕ_i should not be divisible by α_i^2.

In addition, if $\mathcal{I} = \mathcal{I}(n, \bar{\phi})$ is radical, then each $\phi_i \notin \mathcal{I}(i-1, (\phi_1, \dots, \phi_{i-1}))$, the subideal generated by the first $i-1$ generators of $\mathcal{I}$. Otherwise the generator $\alpha_i^{d_i+1} - \phi_i\alpha_0^{d_0+1}$ of $\mathcal{I}$ can be replaced with $\alpha_i^{d_i+1}$, so the ideal is not radical. The following example shows that this necessary condition is not sufficient, even combined with $\alpha_i^2 \nmid \phi_i$.

Example 22. For $F = xy^2z^3$, $\mathcal{I}$ must have the form $\mathcal{I} = (\beta^3 - L\alpha^2, \gamma^4 - Q\alpha^2)$ for some linear form $L = L(\alpha, \beta, \gamma)$ and quadratic form $Q = Q(\alpha, \beta, \gamma)$. Consider the example $\mathcal{I} = (\beta^3 - \alpha^2\gamma, \gamma^4 - \alpha^2\beta^2)$. Then $L = \gamma$ is not divisible by β^2, $Q = \beta^2$ is not divisible by γ^2, and $Q \notin (\beta^3 - \alpha^2\gamma)$, nevertheless this ideal $\mathcal{I}$ is not radical. One can check easily that $P = \alpha^4\beta - \beta^2\gamma^3 \notin \mathcal{I}$ but $P^2 \in \mathcal{I}$. More concretely, $\mathcal{I}$ defines a scheme of length 2 at $[1:0:0] \in \mathbb{P}^2$.

Proposition 23. *Let $\bar{\phi}$ be general, i.e., each $\phi_i \in S_{d_i-d_0}$ is general. Then $\mathcal{I}(n, \bar{\phi})$ is radical.*

Proof. Let $\mathcal{B} := \prod_{i=1}^{n} S_{d_i-d_0}$. In $\mathcal{B} \times \mathbb{P}^n$ consider a variety $\mathcal{Q}$ defined by $\mathcal{I}(n, \bar{\phi})$, where $\bar{\phi} = (\phi_1, \dots, \phi_n)$, $\mathcal{I}(n, \bar{\phi}) = (\alpha_i^{d_i+1} - {\alpha_0}^{d_0+1}\phi_i \mid i \in \{1, \dots, n\})$, and $\phi_i = \sum_{|J|=d_i-d_0} f_J\alpha^J$, where f_J are the coordinates on the $S_{d_i-d_0}$ component of $\mathcal{B}$. Then $\mathcal{Q}$ is a complete intersection and each fiber $\mathcal{Q}_b$ of $\mathcal{Q} \to \mathcal{B}$ is a complete intersection. In particular, $\mathcal{Q}$ is Cohen-Macauley [Eis95, Prop. 18.13] and the map $\mathcal{Q} \to \mathcal{B}$ is equidimensional, thus flat [Eis95, Thm 18.16]. Some of the fibers are reduced, and since being reduced is an open property in a flat family, it follows that a general fiber is reduced. □

5. Dimension of variety of sums of powers

For a homogeneous form $F \in T$ of degree d, and $r > 0$, let

$$\mathrm{VSP}^{\mathrm{aff},\circ}(F, r) = \left\{ \hat{Q} = \{\ell_1, \dots, \ell_r\} \mid \hat{Q} \in \mathrm{Hilb}_r(T_1) \text{ is reduced and } F = \ell_1^d + \dots + \ell_r^d \right\}$$

and
$$\mathrm{VSP}^{\mathrm{aff}}(F,r) = \overline{\mathrm{VSP}^{\mathrm{aff},\circ}(F,r)} \subset \mathrm{Hilb}_r(T_1).$$
Also, let
$$\mathrm{VSP}^{\circ}(F,r) = \{Q = \{[\ell_1],\dots,[\ell_r]\} \mid \\ Q \in \mathrm{Hilb}_r(\mathbb{P}(T_1)) \text{ is reduced and } F = \ell_1^d + \dots + \ell_r^d\}$$
and
$$\mathrm{VSP}(F,r) = \overline{\mathrm{VSP}^{\circ}(F,r)} \subset \mathrm{Hilb}_r(\mathbb{P}(T_1)).$$
When $r = R(F)$ we omit it from the notation. $\mathrm{VSP}(F,r)$ is called the *variety of sums of powers*, see [RS00]. $\mathrm{VSP}^{\mathrm{aff},\circ}(F)$ parametrizes Waring decompositions of F. We make the general remark that $\dim \mathrm{VSP}^{\mathrm{aff}}(F) = \dim \mathrm{VSP}(F)$. First, observe the following easy lemma.

Lemma 24. *Suppose $F = {\ell_1}^d + \dots + {\ell_r}^d$ is a Waring decomposition of a homogeneous polynomial F (i.e., $r = R(F)$). If $F = c_1{\ell_1}^d + \dots + c_r{\ell_r}^d$ for some $c_i \in \mathbb{C}$, then $c_1 = \dots = c_r = 1$.*

Proof. Suppose on the contrary that for instance $c_r \neq 1$. Then combining the two decompositions we obtain:
$$\begin{aligned}(c_r - 1)F &= c_r({\ell_1}^d + \dots + {\ell_r}^d) - (c_1{\ell_1}^d + \dots + c_r{\ell_r}^d)\\ &= (c_r - c_1){\ell_1}^d + \dots + (c_r - c_{r-1}){\ell_{r-1}}^d.\end{aligned}$$
This contradicts the minimality of r. □

By this lemma, the obvious projectivization map $\mathrm{VSP}^{\mathrm{aff},\circ}(F) \to \mathrm{VSP}^{\circ}(F)$ is finite of degree d^r ($r = R(F)$). Indeed, each ℓ_i can only be replaced by a scalar multiple $\lambda\ell_i$ when λ is a d-th root of unity.

We now turn to monomials. As before, let $F = x_0^{d_0}\cdots x_n^{d_n}$, $0 < d_0 \leq \dots \leq d_n$, $d = d_0 + \dots + d_n$, and $r = R(F) = (d_1+1)\cdots(d_n+1)$. Also, let $\mathcal{J} = (\alpha_1^{d_1+1},\dots,\alpha_n^{d_n+1})$. We get a formula for the dimension of the space of solutions to the Waring decomposition problem.

Proposition 25. *Suppose F is a monomial as above. Let h be the Hilbert function of $S/\mathcal{J}$. Then $\mathrm{VSP}(F)$ is irreducible and $\dim \mathrm{VSP}(F) = h(d_1 - d_0) + h(d_2 - d_0) + \dots + h(d_n - d_0)$.*

To prove this, we first describe the space parametrizing the radical one-dimensional ideals $\mathcal{I} \subset F^{\perp}$, corresponding to points in $\mathrm{VSP}^{\circ}(F)$.

Proposition 26. *Any ideal $\mathcal{I}$ as in Proposition 9 is of the form $\mathcal{I}(n,\bar{\phi})$ for a unique $\bar{\phi} = (\phi_1,\dots,\phi_n)$ such that no term of any ϕ_i lies in $\mathcal{J}$.*

Proof. Existence of such a $\bar{\phi}$ is obvious: if a term of ϕ_i is divisible by some $\alpha_j^{d_j+1}$, it can be eliminated by subtracting an appropriate multiple of the generator $\alpha_j^{d_j+1} - \phi_j\alpha_0^{d_0+1}$ of $\mathcal{I}$. Suppose ϕ_i can be replaced by ϕ_i', leaving $\mathcal{I}$ the same, with no term of either ϕ_i or ϕ_i' lying in $\mathcal{J}$. Then the generators $\alpha_i^{d_i+1} - \phi_i\alpha_0^{d_0+1}$ and $\alpha_i^{d_i+1} - \phi_i'\alpha_0^{d_0+1}$ must differ by a combination of previous generators,
$$(\phi_i - \phi_i')\alpha_0^{d_0+1} = \sum_{j=1}^{i-1} \psi_j(\alpha_j^{d_j+1} - \phi_j\alpha_0^{d_0+1}), \tag{27}$$

for some ψ_j. The right hand side is a combination of generators of $\mathcal{I}$ with no term in $\mathcal{J}$. From the above equation it is clearly divisible by $\alpha_0^{d_0+1}$. We claim more generally that any element of $\mathcal{I}$ with no term in $\mathcal{J}$ is necessarily divisible by $\alpha_0^{d_0+1}$. Indeed, in a combination as in (27), each term of each product $\psi_j \alpha_j^{d_j+1}$ must cancel; the only surviving terms arise from the products $\psi_j \phi_j \alpha_0^{d_0+1}$.

Now, suppose $\beta \in \mathcal{I}$ has no term in $\mathcal{J}$, so β is divisible by $\alpha_0^{d_0+1}$. Then $\gamma = \beta/\alpha_0^{d_0+1}$ lies in $\mathcal{I} : \alpha_0^{d_0+1}$, which is $\mathcal{I}$, by Lemma 17. And it is still true that no term of γ is in $\mathcal{J}$, so again γ is divisible by $\alpha_0^{d_0+1}$. Continuing in this way, β is divisible by $(\alpha_0^{d_0+1})^k$ for every $k \geq 0$. Hence $\beta = 0$.

Applying this to (27), $\phi_i - \phi_i' = 0$, as desired. □

We have the following converse.

Proposition 28. *Let $\bar{\phi} = (\phi_1, \ldots, \phi_n)$ be a general tuple of homogeneous polynomials of degree* $\deg \phi_i = d_i - d_0$ *with no terms in $\mathcal{J}$. Then $\mathcal{I}(n, \bar{\phi})$ is radical.*

Proof. For each i, let $B_i' \subset S_{d_i - d_0}$ be the set of monomials of degree $d_i - d_0$ which are not in $\mathcal{J}$, so B_i' gives a basis for $(S/\mathcal{J})_{d_i - d_0}$. For each i, let $\langle B_i' \rangle \subset S_{d_i - d_0}$ be the linear span of B_i'. The hypothesis means each ϕ_i is general in $\langle B_i' \rangle$.

Let $\mathcal{B}' = \prod_{i=1}^n \langle B_i' \rangle$. Let $\mathcal{Q}' \subset \mathcal{B}' \times \mathbb{P}^n$ be defined by $\mathcal{I}(n, \bar{\phi}) = (\alpha_i^{d_i+1} - \phi_i \alpha_0^{d_0+1} \mid i \in \{1, \ldots, n\})$ where $\bar{\phi} \in \mathcal{B}'$ and $(\alpha_0, \ldots, \alpha_n)$ are coordinates on $\mathbb{P}^n$. The rest of the proof is the same as the proof of Proposition 23. For the statement that "some of the fibers are reduced", a reduced fiber is provided by the explicit expression given in Section 2. □

Now the proof of Proposition 25 is immediate.

Proof of Proposition 25. The map $\mathcal{B}' \dashrightarrow \mathrm{VSP}^\circ(F)$, given by $\bar{\phi} \mapsto \mathcal{I}(n, \bar{\phi})$ (the defining ideal of the fiber $\mathcal{Q}'_{\bar{\phi}} \subset \mathbb{P}^n$) whenever this is radical, is defined on an open subset of $\mathcal{B}'$ by Proposition 28, and (on this open subset) is one-to-one and onto by Proposition 26. Since $\mathcal{B}'$ is irreducible, so is $\mathrm{VSP}(F)$, and

$$\dim \mathrm{VSP}(F) = \dim \mathrm{VSP}^\circ(F) = \dim \mathcal{B}' = \sum \dim \langle B_i' \rangle = \sum h(d_i - d_0)$$

as claimed. □

Proof of Corollary 3. Since $d_i \geq d_0$, each $h(d_i - d_0) \geq 1$. Since $\mathcal{J}$ has no generators in degree 1, $h(1) = \dim S_1 \geq 2$, and $h(t) \geq h(1)$ for $t \geq 1$, so $h(d_i - d_0) = 1$ if and only if $d_i - d_0 = 0$. □

Example 29. For $F = x^2 y^2 z^2$, $\mathcal{I}$ must have the form $\mathcal{I} = (\beta^3 - a\alpha^3, \gamma^3 - b\alpha^3)$ for some constants a, b, so $\dim \mathrm{VSP}(F) = 2$, and this is equal to $h(d_1 - d_0) + h(d_2 - d_0) = 2h(0)$.

This ideal is a radical complete intersection if and only if $ab \neq 0$. In that case, rescaling the variables takes $\mathcal{I}$ to $(\beta^3 - \alpha^3, \gamma^3 - \alpha^3)$.

Finally, we consider the natural group operation on $\mathrm{VSP}(F)$ mentioned in the introduction. Let $(\mathbb{C}^*)^{n+1}$ act on $S = \mathbb{C}[x_0, \ldots, x_n]$ by scaling the variables. The n-dimensional subtorus $\mathcal{T} = \{(\lambda_0, \ldots, \lambda_n) \mid \prod \lambda_i^{d_i} = 1\}$ leaves $F = x_0^{d_0} \cdots x_n^{d_n}$ fixed, and so acts on $\mathrm{VSP}(F)$. Also, let $\mathcal{S}_F$ be the group of permutations of $\{0, \ldots, n\}$ respecting the degree tuple $(d_0, \ldots, d_n)$, in the sense that a permutation π lies in $\mathcal{S}_F$ if and only if, for each i, $d_{\pi(i)} = d_i$. When $\mathcal{S}_F$ acts on S by permuting variables it leaves F fixed, so again acts on $\mathrm{VSP}(F)$. The actions of $\mathcal{T}$ and $\mathcal{S}_F$ commute, so $\mathrm{VSP}(F)$ carries an action of the n-dimensional algebraic group $\mathcal{T} \times \mathcal{S}_F$.

Theorem 4 states that this action is transitive (in fact the $\mathcal{T}$-action alone is already transitive) if and only if $F = (x_0 \cdots x_n)^k$.

Proof of Theorem 4. Let $F = x_0^{d_0} \cdots x_n^{d_n}$ with $d_0 \leq \cdots \leq d_n$. If $d_0 < d_n$ then $\dim \mathrm{VSP}^\circ(F) > n = \dim \mathcal{T} = \dim(\mathcal{S}_F \times \mathcal{T})$, so the action can not be transitive. If $d_0 = \cdots = d_n = k$ then by Theorem 1 every Waring decomposition of F is cut out by an ideal of the form $(\alpha_1^{k+1} - \phi_1 \alpha_0^{k+1}, \ldots, \alpha_n^{k+1} - \phi_n \alpha_0^{k+1})$, where the ϕ_i are scalars. Since the ideal of a Waring decomposition is a radical ideal, the ϕ_i are in fact nonzero scalars. Then scaling each x_i by $\phi_i^{-1/k}$ takes this Waring decomposition to the one cut out by $(\alpha_1^{k+1} - \alpha_0^{k+1}, \ldots, \alpha_n^{k+1} - \alpha_0^{k+1})$, showing that $\mathcal{T}$ acts transitively on $\mathrm{VSP}^\circ(F)$. □

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