1-1-1997

Combinatorics of Open Covers (III): Games, $C_p(X)$

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FUNDAMENTA
MATHEMATICAE
152 (1997)

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Abstract. Some of the covering properties of spaces as defined in Parts I and II are here characterized by games. These results, applied to function spaces $\mathsf{C}_{\mathrm{p}}(X)$ of countable tightness, give new characterizations of countable fan tightness and countable strong fan tightness. In particular, each of these properties is characterized by a Ramseyan theorem.

Let $\mathbb{N}$ denote the set of positive integers. As in [11] and [20] the following two selection hypotheses and their associated games will be our main concern: Let $\mathcal{A}$ and $\mathcal{B}$ be collections of subsets of an infinite set S. Then $\mathsf{S}_1(\mathcal{A},\mathcal{B})$ denotes the hypothesis that for every sequence $(O_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(T_n : n \in \mathbb{N})$ such that for each n, $T_n \in O_n$, and $\{T_n : n \in \mathbb{N}\}$ is an element of $\mathcal{B}$. The associated game is denoted by $\mathsf{G}_1(\mathcal{A},\mathcal{B})$, and is played as follows: The players, ONE and TWO, play an inning per positive integer. In the nth inning ONE first selects a set $O_n \in \mathcal{A}$, after which TWO selects an element $T_n \in O_n$. A play $(O_1, T_1, O_2, T_2, \ldots)$ is won by TWO if $\{T_n : n \in \mathbb{N}\}$ is in $\mathcal{B}$; otherwise ONE wins. If ONE has no winning strategy in the game $\mathsf{G}_1(\mathcal{A},\mathcal{B})$, then the families $\mathcal{A}$ and $\mathcal{B}$ satisfy hypothesis $\mathsf{S}_1(\mathcal{A},\mathcal{B})$. In this sense the game is a *sufficient* test whether $\mathcal{A}$ and $\mathcal{B}$ satisfy $\mathsf{S}_1(\mathcal{A},\mathcal{B})$; the main interest in such games stems from the fact that often they are a *necessary* test for the validity of the selection hypothesis. When they are, they are a powerful tool to prove theorems about the combinatorial structure of $\mathcal{A}$ and $\mathcal{B}$.

The symbol $\mathsf{S}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$ denotes the second selection hypothesis of interest to us: for every sequence $(O_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$ there is a sequence $(T_n : n \in \mathbb{N})$ such that for each n, T_n is a finite subset of O_n, and $\bigcup_{n=1}^{\infty} T_n$ is an element of $\mathcal{B}$. The associated game is denoted by $\mathsf{G}_{\mathrm{fin}}(\mathcal{A},\mathcal{B})$, and is

1991 *Mathematics Subject Classification*: 90D44, 54D20, 03E99, 04A99.

Key words and phrases: Rothberger property, Menger property, ω-cover, $\mathsf{S}_1(\Omega,\Omega)$, $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$, $\mathsf{C}_{\mathrm{p}}(X)$, countable fan tightness, countable strong fan tightness, infinite games.

Funded in part by NSF grant DMS 95-05375.

played as follows: Two players, ONE and TWO, play an inning per positive integer. In the nth inning ONE first selects a set $O_n \in \mathcal{A}$, after which TWO selects a finite subset T_n of O_n. A play $(O_1, T_1, O_2, T_2, \ldots)$ is won by TWO if $\bigcup_{n=1}^{\infty} T_n$ is in $\mathcal{B}$; otherwise ONE wins. If ONE has no winning strategy in the game $\mathsf{G}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$, then the families $\mathcal{A}$ and $\mathcal{B}$ satisfy hypothesis $\mathsf{S}_{\mathrm{fin}}(\mathcal{A}, \mathcal{B})$.

For the duration of this paper X is an infinite Tikhonov space and $\mathcal{O}$ denotes the collection of all open covers of X. The open cover $\mathcal{U}$ of X is an *ω-cover* if X is not a member of it and every finite subset of X is contained in an element of $\mathcal{U}$. The symbol Ω denotes the collection of all ω-covers of X.

In the first two sections of the paper we study the two selection hypotheses and their associated games for the case when $\mathcal{A}$ and $\mathcal{B}$ are both Ω. We show that the nonexistence of a winning strategy for ONE in the associated game is a necessary (and sufficient) condition for the validity of the corresponding selection hypothesis (Theorems 2 and 5).

In the third section we turn our attention to spaces of countable tightness and $\mathsf{C}_{\mathrm{p}}(X)$: $\mathbb{R}^X$ denotes the Cartesian product of X copies of the real line $\mathbb{R}$, endowed with the Tikhonov product topology. The subset of *continuous* functions from X to $\mathbb{R}$ with the topology it inherits from $\mathbb{R}^X$ is denoted by $\mathsf{C}_{\mathrm{p}}(X)$; this is the *topology of pointwise convergence*. Theorems of Arkhangel'skiĭ, Arkhangel'skiĭ and Pytkeev, Gerlits and Nagy, and Sakai expose a duality between the closure properties of $\mathsf{C}_{\mathrm{p}}(X)$ and the combinatorics of open covers of X. We use the results from the first two sections to give Ramsey-theoretic characterizations of "classical" closure properties of $\mathsf{C}_{\mathrm{p}}(X)$, as well as a number of other characterizations normally associated with ultrafilters on the set of positive integers (Theorem 11 in Section 4 and Theorem 13 in Section 5). Directly after each of these two theorems we discuss to what extent they are theorems about the special spaces $\mathsf{C}_{\mathrm{p}}(X)$, and give two companion results (Theorems 11B and Theorem 13B), which give a connection with two cardinal numbers associated with combinatorics of the real line.

1. Games and $\mathsf{S}_1(\Omega, \Omega)$. Fritz Rothberger introduced the property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ in [18], Fred Galvin introduced the game $\mathsf{G}_1(\mathcal{O}, \mathcal{O})$ in [7] and Janusz Pawlikowski proved in [17]:

THEOREM 1 (Pawlikowski). *X has property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ if, and only if,* ONE *does not have a winning strategy in the game $\mathsf{G}_1(\mathcal{O}, \mathcal{O})$.*

Masaki Sakai introduced the property $\mathsf{S}_1(\Omega, \Omega)$ in [19]. The next theorem can be used to give unified proofs of some of the results of [11] and [20]. Here we shall use it in our analysis of $\mathsf{C}_{\mathrm{p}}(X)$.

THEOREM 2. *X has property $\mathsf{S}_1(\Omega, \Omega)$ if, and only if,* ONE *does not have a winning strategy in $\mathsf{G}_1(\Omega, \Omega)$.*

The significant implication is that if X has property $\mathsf{S}_1(\Omega,\Omega)$, then ONE does not have a winning strategy in the corresponding game. The following fact from [19] is key to our proof of this:

THEOREM 3. *X has property $\mathsf{S}_1(\Omega,\Omega)$ if, and only if, every finite power of X has property $\mathsf{S}_1(\mathcal{O},\mathcal{O})$.*

Towards proving Theorem 2, let X be a space with property $\mathsf{S}_1(\Omega,\Omega)$. We may assume that for $m\neq n$, X^m and X^n are disjoint. For each n, X^n has property $\mathsf{S}_1(\mathcal{O},\mathcal{O})$, whence so does $Y:=\sum_{n=1}^{\infty}X^n$.

Let F be a strategy for ONE of $\mathsf{G}_1(\Omega,\Omega)$ on X and define a strategy G for ONE of $\mathsf{G}_1(\mathcal{O},\mathcal{O})$ on Y as follows: With $F(X)=(U_n:n\in\mathbb{N})$ ONE's first move in $\mathsf{G}_1(\Omega,\Omega)$ on X, define ONE's first move in $\mathsf{G}_1(\mathcal{O},\mathcal{O})$ on Y by $G(Y)=(U_m^n:m,n\in\mathbb{N})$. TWO responds with a set $U_{m(1)}^{n(1)}$ from $G(Y)$. Then $U_{m(1)}$ is a response of TWO to $F(X)$. Apply F to find $F(U_{m(1)})=(U_{m(1),m}:m\in\mathbb{N})$, an ω-cover of X. Then define $G(U_{m(1)}^{n(1)})=(U_{m(1),m}^n:n,m\in\mathbb{N})$. TWO responds with $U_{m(1),m(2)}^{n(2)}$ from $G(U_{m(1)}^{n(1)})$; $U_{m(1),m(2)}$ is a response of TWO to $F(U_{m(1)})$. First compute $F(U_{m(1)},U_{m(1),m(2)})=(U_{m(1),m(2),m}:m\in\mathbb{N})$ and then define $G(U_{m(1)}^{n(1)},U_{m(1),m(2)}^{n(2)})=(U_{m(1),m(2),m}^n:n,m\in\mathbb{N})$, and so on.

As Y has property $\mathsf{S}_1(\mathcal{O},\mathcal{O})$, choose a G-play lost by ONE of $\mathsf{G}_1(\mathcal{O},\mathcal{O})$. It is of the form $G(Y),U_{m(1)}^{n(1)},G(U_{m(1)}^{n(1)}),U_{m(1),m(2)}^{n(2)},\ldots$, where

$$F(X),U_{m(1)},F(U_{m(1)}),U_{m(1),m(2)},\ldots$$

is an F-play of $\mathsf{G}_1(\Omega,\Omega)$. Since $U_{m(1)}^{n(1)},U_{m(1),m(2)}^{n(2)},\ldots$ is an open cover of Y, the sequence $U_{m(1)},U_{m(1),m(2)},\ldots$ is an ω-cover of X, and F is defeated. ■

2. $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$ and games. Witold Hurewicz showed in [10] that $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ is equivalent to a property which was introduced in [16] by Karl Menger. To distinguish it from another covering property also introduced by Hurewicz, $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ is called *Menger's property*. In that same paper Hurewicz implicitly studied the game $\mathsf{G}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$. Rastislav Telgársky later made this game explicit in [23]. Hurewicz proved in Theorem 10 of [10]:

THEOREM 4 (Hurewicz). *The space X has property $\mathsf{S}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$ if, and only if, ONE does not have a winning strategy in $\mathsf{G}_{\mathrm{fin}}(\mathcal{O},\mathcal{O})$.*

In Theorem 3.9 of [11] it was shown that a topological space has property $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$ if, and only if, every finite power of X has the Menger property. Using this fact, the method of Theorem 2 and Hurewicz's Theorem, one proves:

THEOREM 5. *A space X has property $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$ if, and only if,* ONE *has no winning strategy in $\mathsf{G}_{\mathrm{fin}}(\Omega,\Omega)$.*

By results of [11] and of [20], $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$ is also characterized by a partition relation reminiscent of the one introduced in [3] (Theorem 2.3(iii)) for P-point ultrafilters on the set of positive integers. This and several other characterizations of the property $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$ have been proved by *ad hoc* methods. The game $\mathsf{G}_{\mathrm{fin}}(\Omega,\Omega)$ can be used to give a fairly unified treatment of the theory of $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$. As with $\mathsf{G}_1(\Omega,\Omega)$, we shall here use it to analyse $\mathsf{C}_{\mathrm{p}}(X)$.

3. Countable tightness. We use the following notation for a free ideal J of subsets of a set S:

$$J^* = \{S\setminus X : X\in J\},\qquad J^+ = \{X\subseteq S : X\notin J\}.$$

Then J^* is said to be the *dual* of J, and is a filter. Moreover, J^+ is $\{X\subseteq S : (\forall Y\in J^*)(X\cap Y\neq\emptyset)\}$. It is also customary to define these two notions for free filters in the obvious way.

If a space is not first countable then convergence of sequences does not describe its closure operator. The following notion is central to the several weakened forms of the sequential description of closures that have been considered: Since the main difficulties arise at points which are not isolated, we define this notion only for such points. For a space Y and for a nonisolated point $y\in Y$, the symbol Ω_y denotes the set $\{A\subset Y : y\notin A \text{ and } y\in\overline{A}\}$.

If Y is a T_1-space, then for $A\in\Omega_y$ those subsets of A which are not in Ω_y is a free ideal on A, denoted by $I_{y,A}$. Then we have $I^+_{y,A}=\{B\in\Omega_y : B\subseteq A\}$. The dual $I^*_{y,A}$ in A of this ideal is also denoted by $\mathcal{F}_{y,A}$ and is the trace on A of subsets of Y having y in their interior.

A space has *countable tightness* if for any subset A and any $x\in\overline{A}$, there is a *countable* $B\subseteq A$ with $x\in\overline{B}$. If Y has countable tightness we may at a nonisolated point y restrict our attention to the countable sets in Ω_y. There is a standard way to obtain from a free ideal J on a countable set S a countably tight space: Let ∞ be a point not in S, and define a topology τ_J on $Y := S\cup\{\infty\}$ as follows: Every point of S will be isolated, while the open neighborhoods of ∞ are sets of the form $\{\infty\}\cup S\setminus X$, $X\in J$. One can show that (Y,τ_J) is a T_4-space; since it is countable, it also has countable tightness. Moreover, Ω_∞ is J^+ and J^* is the filter of neighborhoods of ∞ relativized to S.

In view of the preceding remarks, the study of Ω_y at nonisolated points of a countably tight T_1-space encompasses the study on countable sets of free ideals I, their dual filters I^*, and their complements I^+. A lot is known about these objects, especially in the case where the ideal is maximal. It sometimes happens that combinatorial properties for filters are equivalent to each other when considered for "small" filters, but different from each other for maximal filters, and sometimes it happens that properties that are equivalent to each other for maximal filters are for general filters not equivalent to each other. Examples of these phenomena abound in the literature (see for example [3], [4] and [9]). It is natural to ask for which filters some of these combinatorial properties are equivalent. We shall show that for appropriate X, $\mathsf{C}_{\mathrm{p}}(X)$ is a rich source of such filters.

If X is uncountable then $\mathsf{C}_{\mathrm{p}}(X)$ is not first countable but it could have countable tightness. The following consequence of a theorem of Arkhangel'skiĭ and Pytkeev is a key tool in the study of countably tight function spaces. (A proof can be found in [2], Theorem II.1.1.)

Theorem 6 (Arkhangel'skiĭ–Pytkeev). *For a Tikhonov space X, every finite power of X has the Lindelöf property if, and only if, $\mathsf{C}_{\mathrm{p}}(X)$ has countable tightness.*

Gerlits and Nagy added a further important characterization in terms of X of the countable tightness of $\mathsf{C}_{\mathrm{p}}(X)$ to this list—see [8]:

Theorem 7 (Gerlits–Nagy). *For a Tikhonov space X, every finite power of X has the Lindelöf property if, and only if, every (open) ω-cover of X contains a countable subset which is an ω-cover.*

We now introduce for topological spaces a series of properties which are usually studied in connection with ultrafilters and relate these to two important strengthenings of countable tightness. Let Y be a countably tight T_1-space and let y be a nonisolated point of Y.

Y has *property* $\mathsf{K}(\Omega_y, \Omega_y)$ if for every first countable compact Hausdorff space Z, for each $A \in \Omega_y$, and for every function $f : A \to Z$, if there is an $a \in Z$ such that for every neighborhood U of a the set $\{t \in A : f(t) \in U\} \in \Omega_y$, then there is a $B \subset A$ such that $B \in \Omega_y$ and a is the unique limit point of the set $\{f(x) : x \in B\}$. An analogue of this property was introduced near the bottom of page 386 of [13] as a characterization of P-point ultrafilters on $\mathbb{N}$.

Following the standard combinatorial definition of a P-point ultrafilter on $\mathbb{N}$ we say that Y has property $\mathsf{P}(\Omega_y, \Omega_y)$ if there is, for each descending sequence $A_1 \supseteq \ldots \supseteq A_n \supseteq \ldots$ in Ω_y, an A in Ω_y such that for each n, $A \setminus A_n$ is finite.

Y has *property* $\mathsf{Q}(\Omega_y, \Omega_y)$ if for each countable $A \in \Omega_y$, for each partition of A into pairwise disjoint finite sets, there is an element of Ω_y which is a subset of A and meets each of the blocks of the partition in a single point.

The following definitions are inspired by Booth's characterization of P-point ultrafilters in Theorem 4.7(iv) and of Ramsey ultrafilters in Theorem 4.9(iii) of [5]: Y has *property* $\mathsf{B}_{\mathrm{linear}}(\Omega_y, \Omega_y)$ if for every $A \in \Omega_y$ and for every linear order R of A there is a $B \subseteq A$ such that $B \in \Omega_y$, and the order type of B relative to R is ω or ω^*. The space has *property* $\mathsf{B}_{\mathrm{tree}}(\Omega_y, \Omega_y)$ if for each $A \in \Omega_y$ and for each tree order R of A there is a $B \in \Omega_y$ which is a chain or an antichain of the tree (A, R).

The following two properties feature in [5], Theorem 4.9(v), in [9], Definition 1.11 and Corollary 1.15 and the definition after Proposition 6.4, and in [15], Proposition 0.8: Y has *property* $\mathsf{Ind}_{\mathrm{fin}}(\Omega_y, \Omega_y)$ if there is, for every descending sequence $(A_n : n \in \mathbb{N})$ in Ω_y and each bijective enumeration $(a_m : m \in \mathbb{N})$ of A_1, a function $H : \mathbb{N} \to [\mathbb{N}]^{<\aleph_0}$ such that: if $m < n$ then $\sup H(m) < \sup H(n)$ and $|H(m)| < |H(n)|$; $\bigcup_{n=1}^{\infty}\{a_j : j \in H(n)\} \in \Omega_y$; for each n, $\{a_j : j \in H(n+1)\} \subseteq A_{\sup H(n)}$. The space has *property* $\mathsf{Ind}_1(\Omega_y, \Omega_y)$ if for every descending sequence $(A_n : n \in \mathbb{N})$ in Ω_y and for every bijective enumeration $(a_m : m \in \mathbb{N})$ of A_1, there is a strictly increasing function $g : \mathbb{N} \to \mathbb{N}$ such that $\{a_{g(n)} : n \in \mathbb{N}\} \in \Omega_y$, and for each n, $a_{g(n+1)} \in A_{g(n)}$.

The next two definitions are also inspired by characterizations of P-point or Ramsey ultrafilters by various authors—for example in [3] and [4]. Y has property $\mathsf{C}_{\mathrm{fin}}(\Omega_y, \Omega_y)$ if, for each $A \in \Omega_y$ and for every function $f : A \to \omega$, either there is a subset B of A such that $B \in \Omega_y$ and f is finite-to-one on B, or f is constant on B. The following equivalent form of this assertion is often used: for each A in Ω_y and for each partition $A = \bigcup_{n=1}^{\infty} A_n$ such that no A_n is in Ω_y, there is a $B \subseteq A$ such that $B \in \Omega_y$ and for each n, $B \cap A_n$ is finite.

The space has property $\mathsf{C}_1(\Omega_y, \Omega_y)$ if, for each $A \in \Omega_y$ and for every function $f : A \to \omega$, either there is a subset B of A such that $B \in \Omega_y$ and f is one-to-one on B, or there is a subset B of A such that $B \in \Omega_y$ and f is constant on B. This statement in turn is equivalent to the following: for each A in Ω_y and for each partition $A = \bigcup_{n=1}^{\infty} A_n$ such that no A_n is in Ω_y, there is a $B \subseteq A$ such that $B \in \Omega_y$ and for each n, $B \cap A_n$ has at most one element.

4. Countable fan tightness. A topological space Y has *countable fan tightness at y* if the selection hypothesis $\mathsf{S}_{\mathrm{fin}}(\Omega_y, \Omega_y)$ holds. The game $\mathsf{G}_{\mathrm{fin}}(\Omega_y, \Omega_y)$ is the *countable fan tightness game at y*. A space has *countable fan tightness* if it has countable fan tightness at each element. Countable fan tightness implies countable tightness, but not conversely. Moreover, for a T_1-space Y of countable tightness and for a nonisolated point $y \in Y$, countable fan tightness at y is equivalent to saying that for every countable $A \in \Omega_y$ the filter $\mathcal{F}_{y,A}$ satisfies the selection property $\mathsf{S}_{\mathrm{fin}}(\mathcal{F}^+_{y,A}, \mathcal{F}^+_{y,A})$. Accordingly, let us say that a free filter $\mathcal{F}$ on a countable set S is a *fan tight* filter if it has the selection property $\mathsf{S}_{\mathrm{fin}}(\mathcal{F}^+, \mathcal{F}^+)$.

According to A. Mathias a family $\mathcal{A}$ of subsets of a countable set A is a *moderately happy family* if there is a free filter $\mathcal{F}$ on A such that $\mathcal{A} = \mathcal{F}^+$, and for every *descending* sequence $(A_n : n \in \mathbb{N})$ of elements of $\mathcal{A}$, there is an element X of $\mathcal{A}$ such that for each n, $X \setminus A_n$ is finite. He calls the dual ideal $\mathcal{F}^* := \{A \setminus X : X \in \mathcal{F}\}$ a *moderately happy ideal* ([15], Definitions 9.0 and 9.1); S. Grigorieff calls $\mathcal{F}^*$ a P-*point ideal* ([9], Definition 1.8 and Proposition 1.9) or a *weak p-T-ideal* ([9], Definition 6.3 and Proposition 6.4). Thus, if $\mathcal{F}$ is a fan tight filter, then $\mathcal{F}^+$ is a moderately happy family and $\mathcal{F}^*$ is a P-point ideal.

Let $\mathcal{P}$ and $\mathcal{Q}$ be collections of subsets of the set S and let n and k be in $\mathbb{N}$. The symbol $\mathcal{P} \to \lceil \mathcal{Q} \rceil^2_2$ means that for each element A of $\mathcal{P}$ and for each function $f : [A]^2 \to \{0, 1\}$ there is a $B \subseteq A$, a finite-to-one function g with domain B and an $i \in \{0, 1\}$ such that $B \in \mathcal{Q}$, and for all b and c in B, $f(\{b, c\}) = i$ whenever $g(b) \neq g(c)$. We say that B is *eventually homogeneous for* f. The symbol $\mathcal{P} \to (\mathcal{Q})^n_k$ means that for each element A of $\mathcal{P}$ and for each function $f : [A]^n \to \{1, \ldots, k\}$ there is a $B \subseteq A$ and an $i \in \{1, \ldots, k\}$ such that $B \in \mathcal{P}$, and f has the value i on $[B]^n$.

4.1. *Countable fan tightness for* $\mathsf{C}_{\mathrm{p}}(X)$**.** In view of the work in [11] the following theorem of Arkhangel'skiĭ [1] connects countable fan tightness in topological function spaces with the combinatorial property $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$ of open covers:

THEOREM 8 (Arkhangel'skiĭ). *For a Tikhonov space X, every finite power of X has Menger's property if, and only if, $\mathsf{C}_{\mathrm{p}}(X)$ has countable fan tightness.*

It is well known that the set of irrational numbers does not have Menger's property, and that Menger's property is preserved by continuous images. The fact that addition is a continuous function from $\mathbb{R}^2$ to $\mathbb{R}$ implies that if X is a set of real numbers such that $X + X$ is the set of irrational numbers, then all finite powers of X have the Lindelöf property but X^2 does not have the Menger property. Then $\mathsf{C}_{\mathrm{p}}(X)$ has countable tightness, but does not have countable fan tightness. Such a set of real numbers exists. To see this, let $(x_\alpha : \alpha < 2^{\aleph_0})$ bijectively enumerate the set of irrational numbers. If y is a real number and $Y \subseteq \mathbb{R}$ is a set of cardinality less than $2^{\aleph_0}$, then the set $\{t :$ for some u in Y, $y - t + u$ or $t + u$ is rational or $t + y$ is rational$\}$ has cardinality less than $2^{\aleph_0}$. Thus, letting x_α play the role of y and letting Y be the already selected z's and t's, we can recursively choose z_α and t_α, $\alpha < 2^{\aleph_0}$, such that:

1. $x_0 + t_0$ is irrational, and put $z_0 = x_0 - t_0$;
2. For $0 < \beta < 2^{\aleph_0}$ and for all δ less than β each of $t_\beta + z_\delta$, $t_\beta + t_\delta$, $x_\beta - t_\beta + z_\delta$ and $x_\beta - t_\beta + t_\delta$ is irrational, and put $z_\beta = x_\beta - t_\beta$.

Then $X = \{z_\beta : \beta < 2^{\aleph_0}\} \cup \{t_\gamma : \gamma < 2^{\aleph_0}\}$ is the required set.

We use the operation $\mathsf{Coz}(f,\delta) := \{x \in X : |f(x)| < \delta\}$ which associates open subsets of X with $f \in \mathsf{C}_{\mathrm{p}}(X)$ and positive real numbers δ, to translate back-and-forth between closure properties of $\mathsf{C}_{\mathrm{p}}(X)$ and cover properties of X. We use $\mathbf{o}$ to denote the constant function with value 0 from X to $\mathbb{R}$. Since $\mathsf{C}_{\mathrm{p}}(X)$ is a topological vector space, it is homogeneous. Thus determining if a point belongs to the closure of a set reduces to determining if $\mathbf{o}$ belongs to the closure of a corresponding set. Similarly, when playing the countable fan tightness game on $\mathsf{C}_{\mathrm{p}}(X)$, we may assume that the point at which it is played is $\mathbf{o}$. We shall use the following lemmas about $\Omega_{\mathbf{o}}$ heavily:

LEMMA 9. *If X is an infinite Tikhonov space, then there is a sequence $(g_n : n < \omega)$ in $\Omega_{\mathbf{o}}$ such that for each n, g_n is nonnegative and there is an x such that $g_n(x) = 1$, and for all m and n, if $m \neq n$, then for all x, $g_m(x) \cdot g_n(x) = 0$.*

LEMMA 10. *Let A and B be elements of $\Omega_{\mathbf{o}}$, let C and D be subsets of $\mathsf{C}_{\mathrm{p}}(X)\setminus\{\mathbf{o}\}$, let h be an element of $\mathsf{C}_{\mathrm{p}}(X)$ and let $(g_n : n < \omega)$ be a sequence as in Lemma 9. Then:*

(1) $\{|f| : f \in C\} \in \Omega_{\mathbf{o}}$ *if, and only if*, $C \in \Omega_{\mathbf{o}}$.
(2) $\{f + g : f \in A$ *and* $g \in B\} \in \Omega_{\mathbf{o}}$.
(3) $\{|f| + h : f \in A\} \in \Omega_h$.
(4) *If* $\{a \in A : (\exists c \in C)(|c| \le |a|)\} \in \Omega_{\mathbf{o}}$, *then* $C \in \Omega_{\mathbf{o}}$.
(5) *If* $\{|f| + |g| : f \in C$ *and* $g \in D\} \in \Omega_{\mathbf{o}}$, *then C and D are in* $\Omega_{\mathbf{o}}$.
(6) *If* $(c_n : n < \omega)$ *bijectively enumerates C and* $\{|c_n| + g_n : n < \omega\} \in \Omega_{\mathbf{o}}$, *then* $C \in \Omega_{\mathbf{o}}$.

THEOREM 11. *Let X be a Tikhonov space such that $\mathsf{C}_{\mathrm{p}}(X)$ has countable tightness. Then the following are equivalent:*

(a) $\mathsf{C}_{\mathrm{p}}(X)$ *has countable fan tightness.*
(b) ONE *does not have a winning strategy in the game* $\mathsf{G}_{\mathrm{fin}}(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$.
(c) $\mathsf{C}_{\mathrm{p}}(X)$ *has property* $\mathsf{Ind}_{\mathrm{fin}}(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$.
(d) $\mathsf{C}_{\mathrm{p}}(X)$ *has property* $\mathsf{K}(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$.
(e) $\mathsf{C}_{\mathrm{p}}(X)$ *has property* $\mathsf{P}(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$.
(f) $\Omega_{\mathbf{o}} \to \lceil \Omega_{\mathbf{o}} \rceil^2_2$.
(g) $\mathsf{C}_{\mathrm{p}}(X)$ *has property* $\mathsf{B}_{\mathrm{linear}}(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$.
(h) $\mathsf{C}_{\mathrm{p}}(X)$ *has property* $\mathsf{C}_{\mathrm{fin}}(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$.

Proof. (a)⇒(b). Since $\mathsf{C}_{\mathrm{p}}(X)$ has countable fan tightness, X has property $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$. Then ONE has no winning strategy in $\mathsf{G}_{\mathrm{fin}}(\Omega, \Omega)$, played on X. We use this information to prove (b).

Fix a well-ordering $\prec$ of the finite subsets of $\mathsf{C}_{\mathrm{p}}(X)$. Let σ be a strategy for ONE of $\mathsf{G}_{\mathrm{fin}}(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$. Since $\mathsf{C}_{\mathrm{p}}(X)$ has countable tightness we may assume that all ONE's moves are countable sets. By (1) of Lemma 10 we may assume that each element of each move of ONE is nonnegative. Let $(g_n : n < \omega)$ be a sequence as in Lemma 9. Use σ as follows to define a strategy τ for ONE of $\mathsf{G}_{\rm fin}(\Omega,\Omega)$ on X:

Let $(f_n : n<\omega)$ bijectively enumerate $\sigma(\mathsf{C}_{\rm p}(X))$, the first move of ONE of $\mathsf{G}_{\rm fin}(\Omega_{\mathbf{o}},\Omega_{\mathbf{o}})$. For each n the set $U_n := \mathsf{Coz}(f_n+g_n;1/2)$ is an open proper (because at some x, $f_n(x)+g_n(x)\geq 1$) subset of X. Since $\{f_n+g_n : n<\omega\}$ is in $\Omega_{\mathbf{o}}$, $\tau(X)=\{U_n : n<\omega\}$ is a legitimate move of ONE of $\mathsf{G}_{\rm fin}(\Omega,\Omega)$ played on X: For let F be a finite nonempty subset of X. All but finitely many of the g_n's are zero on F. Choose n so large that at each $x\in F$, $f_n(x)<1/2$. Then $F\subseteq U_n$. For $S_1\subseteq\tau(X)$ a move of TWO of $\mathsf{G}_{\rm fin}(\Omega,\Omega)$, let T_1 be the $\prec$-least finite subset of $\sigma(\mathsf{C}_{\rm p}(X))$ with $S_1=\{\mathsf{Coz}(f_j+g_j,1/2) : f_j\in T_1\}$. Then T_1 is a legitimate move of TWO of $\mathsf{G}_{\rm fin}(\Omega_{\mathbf{o}},\Omega_{\mathbf{o}})$. Let $F_1\subseteq\omega$ be the finite set with $T_1=\{f_n : n\in F_1\}$.

To determine $\tau(S_1)$ look at the response $\sigma(T_1)=(f_{F_1,n} : n<\omega)$, enumerated bijectively, of ONE of $\mathsf{G}_{\rm fin}(\Omega_{\mathbf{o}},\Omega_{\mathbf{o}})$. Then for each n, $U_{F_1,n}:=\mathsf{Coz}(f_{F_1,n}+g_n,(1/2)^2)$ is an open proper subset of X and $\tau(S_1)=(U_{F_1,n} : n<\omega)$ a valid move of ONE of $\mathsf{G}_{\rm fin}(\Omega,\Omega)$ on X. TWO of this game responds with a finite subset S_2 of $\tau(S_1)$. Let T_2 be the $\prec$-least finite subset of $\sigma(T_1)$ such that $S_2=\{U_{F_1,n} : f_{F_1,n}\in T_1\}$, and let $F_2\subset\omega$ be the finite set with $T_2=\{f_{F_1,n} : n\in F_2\}$.

To determine $\tau(S_1,S_2)$ look at the response $\sigma(T_1,T_2)=(f_{F_1,F_2,n} : n<\omega)$, enumerated bijectively, of ONE of $\mathsf{G}_{\rm fin}(\Omega_{\mathbf{o}},\Omega_{\mathbf{o}})$. For each n, $U_{F_1,F_2,n}=\mathsf{Coz}(f_{F_1,F_2,n}+g_n,(1/2)^3)$ is an open proper subset of X and $\tau(S_1,S_2)=(U_{F_1,F_2,n} : n<\omega)$ is a valid move of ONE of $\mathsf{G}_{\rm fin}(\Omega,\Omega)$.

Continuing like this we define a strategy τ for ONE of $\mathsf{G}_{\rm fin}(\Omega,\Omega)$. But τ is not a winning strategy. Look at a τ-play $\tau(X),S_1,\tau(S_1),S_2,\tau(S_1,S_2),\ldots$ which was lost by ONE of $\mathsf{G}_{\rm fin}(\Omega,\Omega)$. Corresponding to it we have a σ-play $\sigma(\mathsf{C}_{\rm p}(X)),T_1,\sigma(T_1),T_2,\sigma(T_1,T_2),\ldots$ of $\mathsf{G}_{\rm fin}(\Omega_{\mathbf{o}},\Omega_{\mathbf{o}})$. The correspondence between these two plays is such that for a sequence $F_1,\ldots,F_n,\ldots$ of finite subsets of ω we have, for each n, $T_n=\{f_{F_1,\ldots,F_{n-1},m} : m\in F_n\}$ and $S_n=\{\mathsf{Coz}(f_{F_1,\ldots,F_{n-1},m}+g_m,(1/2)^n) : f_{F_1,\ldots,F_{n-1},m}\in T_n\}$.

Since $\bigcup_{n=1}^\infty S_n$ is an ω-cover of X it follows that $\mathbf{o}$ is in $\overline{\bigcup_{n=1}^\infty T_n}$. But then ONE of $\mathsf{G}_{\rm fin}(\Omega_{\mathbf{o}},\Omega_{\mathbf{o}})$ lost this play despite following the strategy σ.

(b)$\Rightarrow$(c). Let $(A_n : n\in\mathbb{N})$ be a descending sequence from $\Omega_{\mathbf{o}}$ and enumerate A_1 bijectively as $(a_n : n\in\mathbb{N})$. Define a strategy σ for ONE in the game $\mathsf{G}_{\rm fin}(\Omega_{\mathbf{o}},\Omega_{\mathbf{o}})$ as follows: ONE's first move is $\sigma(\mathsf{C}_{\rm p}(X))=A_1$. If TWO responds with the finite subset T_1 then ONE computes $x_1:=1+\sup\{j : a_j\in T_1\}$, and plays $\sigma(T_1)=\{a_j\in A_{x_1} : j>x_1\}$. If TWO now responds with a finite subset $T_2\subset\sigma(T_1)$, then ONE first computes $x_2\geq 1+\sup\{j : a_j\in T_2\}$ so large that $|\{j : a_j\in A_{x_1}$ and $j\leq x_2\}|>x_1$ and $a_{x_2}\in A_{x_1}$. Then ONE plays $\sigma(T_1,T_2):=\{a_j\in A_{x_2} : j>x_2\}$. If TWO responds with the finite set $T_3\subset\sigma(T_1,T_2)$ then ONE first computes $x_3\geq 1+\sup\{j : a_j\in T_3\}$ so large that $|\{j : a_j \in A_{x_2} \text{ and } j \leq x_3\}| > x_1 + x_2$, and $a_{x_3} \in A_{x_2}$. Then ONE plays $\sigma(T_1, T_2, T_3) = \{a_j \in A_{x_3} : j > x_3\}$, and so on.

By (b) there is a σ-play lost by ONE, say $\sigma(\mathsf{C}_{\mathrm{p}}(X)), T_1, \sigma(T_1), T_2, \sigma(T_1, T_2), T_3, \ldots$ Using the associated sequence $x_1, x_2, x_3, \ldots$, define $H : \mathbb{N} \to [\mathbb{N}]^{<\aleph_0}$ as follows: $H(1) = \{1, \ldots, x_1\}$, $H(2) = \{j : a_j \in A_{x_1} \text{ and } j \leq x_2\}$, $H(3) = \{j : a_j \in A_{x_2} \text{ and } j \leq x_3\}$, and so on. Observe that for each n, $\sup H(n) = x_n$, $|H(1)| = x_1$ and for $n > 1$, $|H(n)| > x_1 + \ldots + x_{n-1}$. Moreover, for each n, $T_{n+1} \subseteq \{a_j : j \in H(n+1)\} \subset A_{\sup H(n)}$. Since $\bigcup_{n=1}^{\infty} T_n \in \Omega_{\mathbf{o}}$, H is as required.

(c)$\Rightarrow$(d). Let Z be a first countable compact Hausdorff space and let $A \in \Omega_{\mathbf{o}}$ and a function $f : A \to Z$ be given such that for an element a of Z, $\{x \in A : f(x) \in U\} \in \Omega_{\mathbf{o}}$ for each neighborhood U of a. Choose a sequence $(V_n : n \in \mathbb{N})$ of nonempty open subsets of Z such that: a is an element of each, $\overline{V}_n \subset V_m$ whenever $m < n$, for each open set U containing a there is an n with $V_n \subset U$, and $\{a\} = \bigcap_{n<\infty} V_n$. Since $\mathsf{C}_{\mathrm{p}}(X)$ has countable tightness we may assume that A is countable.

Define a descending sequence of elements of $\Omega_{\mathbf{o}}$ as follows: $A_1 = \{x \in A : f(x) \in V_1\}$. For each n put $A_{n+1} = \{x \in A_n : f(x) \in V_{n+1}\}$. Enumerate A_1 bijectively as $(a_n : n \in \mathbb{N})$. By (c) let $H : \mathbb{N} \to [\mathbb{N}]^{<\aleph_0}$ be a function such that: for each n, $\{a_j : j \in H(n+1)\} \subset A_{\sup H(n)}$; for $m < n$, $\sup H(m) < \sup H(n)$ and $|H(m)| < |H(n)|$; $B = \bigcup_{n=1}^{\infty} \{a_j : j \in H(n)\} \in \Omega_{\mathbf{o}}$. Thus for each n, for all but finitely many points x from B, $f(x) \in V_n$. So a is the unique limit point of the values of f on B.

(d)$\Rightarrow$(e). Let $(A_n : n \in \mathbb{N})$ be a descending sequence of subsets of X such that for each n, $\mathbf{o}$ is in $\overline{A}_n \setminus A_n$. Define a function f from A_1 to the compact Hausdorff space ${}^{\omega}2$ so that for each $x \in A_1$,

$$f(x)(n-1) = \begin{cases} 1 & \text{if } x \in A_n, \\ 0 & \text{otherwise.} \end{cases}$$

Consider the element h of ${}^{\omega}2$ which is equal to 1 everywhere. Then for each $m > 0$ the set $\{x \in A_1 : i \leq m \Rightarrow f(x)(i) = 1\}$ contains the set A_m, an element of $\Omega_{\mathbf{o}}$; it follows that for each neighborhood U of h the set $\{x \in A_1 : f(x) \in U\}$ is in $\Omega_{\mathbf{o}}$. Apply (c) to find a subset B of A_1 such that h is the unique limit point of $\{f(x) : x \in B\}$. Then for each n the set $B \setminus A_n$ is finite.

(e)$\Rightarrow$(f). Let A be an element of $\Omega_{\mathbf{o}}$ and let $f : [A]^2 \to \{0,1\}$ be given. Since we are assuming that $\mathsf{C}_{\mathrm{p}}(X)$ has countable tightness we may assume that A is countable. Enumerate A bijectively as $(a_n : n < \omega)$.

Recursively choose $i_0, i_1, \ldots, i_n, \ldots \in \{0,1\}$ and a descending sequence $A_0 \supset A_1 \supset \ldots \supset A_n \supset \ldots$ of subsets of A which are in $\Omega_{\mathbf{o}}$ such that

1. $A_0 = \{a_n : n > 1 \text{ and } f(\{a_0, a_n\}) = i_0\}$, and for each n,
2. $A_{n+1} = \{a_m \in A_n : m > n+2 \text{ and } f(\{a_{n+1}, a_m\}) = i_{n+1}\}$.

Then for each n, $a_{n+1} \notin A_n$. Apply $\mathsf{P}(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$ to the sequence of A_n's to find a $B \in \Omega_{\mathbf{o}}$ such that $B \subset A_0$ and for each n, $B \setminus A_n$ is finite.

Write $B = B_0 \cup B_1$, where for $j \in \{0,1\}$ we have $B_j = \{a_n \in B : i_n = j\}$. For one of the two values of j we have $B_j \in \Omega_{\mathbf{o}}$; we may assume that $B_0 \in \Omega_{\mathbf{o}}$.

List B_0 as $(a_{n_j} : j < \omega)$ using the earlier enumeration of A. We may assume that for all $k < m$, the largest n_j with $a_{n_j} \notin A_k$ is less than the largest n_i with $a_{n_i} \notin A_m$. Define, recursively, sequences $j_1 < \ldots < j_k < \ldots$ of positive integers and $C_0, C_1, \ldots, C_k, \ldots$ of finite subsets of B so that: $C_0 = \{a_{n_0}\}$, $C_1 = \{a_{n_1}, \ldots, a_{n_{j_1}}\}$, where $j_1 \geq 0$ is maximal with $a_{n_{j_1}} \notin A_{n_0}$, and $C_{k+1} = \{a_{n_{j_k+1}}, \ldots, a_{n_{j_{k+1}}}\}$, where $j_{k+1} > j_k$ is maximal with $a_{n_{j_{k+1}}} \notin A_{n_{j_k}}$.

Then the sequence $(C_k : k < \omega)$ is a partition of B into disjoint finite sets, and at least one of $\bigcup_{k=1}^{\infty} C_{2k}$ or $\bigcup_{k=1}^{\infty} C_{2k-1}$ is in $\Omega_{\mathbf{o}}$. Whichever of these it is, verifies the claimed partition property.

(f)⇒(g). Let A be an element of $\Omega_{\mathbf{o}}$ and let R be a linear ordering of A. Since $\mathsf{C}_{\mathrm{p}}(X)$ has countable tightness we may assume that A is countable. Enumerate A bijectively as $(a_n : n < \omega)$. Then define $\phi : [A]^2 \to \{0,1\}$ so that

$$\phi(\{a_m, a_n\}) = \begin{cases} 0 & \text{if } m < n \text{ and } a_m \, R \, a_n, \\ 1 & \text{otherwise.} \end{cases}$$

By (c) we find a set $B \subset A$ and an $i \in \{0,1\}$, and a finite-to-one function $g : B \to \omega$ such that $B \in \Omega_{\mathbf{o}}$ and for all $b, c \in B$ we have $\phi(\{b,c\}) = i$ whenever $g(b) \neq g(c)$. If $i = 1$, then it follows that B has order type ω relative to R; if $i = 0$, then B has order type ω^* relative to R.

(g)⇒(h). Let $A \in \Omega_{\mathbf{o}}$ and a function $f : A \to \omega$ be given. We may assume that A is countable. For each n put $A_n = \{x \in A : f(x) = n\}$. If there is an n such that $A_n \in \Omega_{\mathbf{o}}$, then we are done. Thus we may assume that no A_n is in $\Omega_{\mathbf{o}}$. We may also assume that each A_n is infinite.

Define a linear order R on A so that each A_n has order type ω^* relative to R, and if $a \in A_m$ and $b \in A_n$, and $m < n$, then $a \, R \, b$. Apply g to find a subset B of A which is an element of $\Omega_{\mathbf{o}}$, and which has order type ω or ω^* relative to R. By the definition of R and the fact that no A_n is in $\Omega_{\mathbf{o}}$ while B is, we see that B has order type ω—as such B meets each A_n in a finite set, meaning that f is finite-to-one on B.

(h)⇒(a). Let $(A_n : n < \omega)$ be a sequence of elements of $\Omega_{\mathbf{o}}$. Choose a sequence $(g_n : n < \omega)$ as in Lemma 9. For each n put $B_n = \{g_n + |a| : a \in A_n\}$. Then $\bigcup_{n=0}^{\infty} B_n$ is an element of $\Omega_{\mathbf{o}}$, but no B_n is an element of $\Omega_{\mathbf{o}}$. We may assume that the B_n's are pairwise disjoint (else replace each B_{n+1} by $B_{n+1} \setminus (B_0 \cup \ldots \cup B_n)$).

Apply h to choose for each n a finite subset $F_n \subset B_n$ such that $\bigcup_{n=0}^{\infty} F_n \in \Omega_{\mathbf{o}}$. For each n choose a finite set $C_n \subset A_n$ such that $F_n = \{g_n + |g| :$

$g \in C_n\}$. By (1) of Lemma 10 and the properties of the g_n's, $\bigcup_{n=0}^{\infty} C_n$ is in $\Omega_{\mathbf{o}}$. This is seen as follows: Suppose on the contrary that $\bigcup_{n=0}^{\infty} C_n$ is not in $\Omega_{\mathbf{o}}$, and let U be a neighborhood of $\mathbf{o}$ disjoint from $\bigcup_{n=0}^{\infty} C_n$. We may assume that G is a finite subset of X and ε is a positive real such that U is $\{f \in \mathsf{C}_{\mathrm{p}}(X) : \text{for each } x \text{ in } G, |f(x)| < \varepsilon\}$. Thus, for each n, for each $x \in G$, and for each $g \in C_n$, $|g(x)| \geq \varepsilon$. But each g_n is nonnegative, so that for each $g \in C_n$, $g_n(x) + |g(x)| \geq \varepsilon$. Thus U also witnesses that $\bigcup_{n=0}^{\infty} F_n$ is not in $\Omega_{\mathbf{o}}$, a contradiction. ■

4.2. *Countable fan tightness for* T_1*-spaces***.** The proofs above show that if Y has countable tightness, then the following seven implications hold at any nonisolated $y \in Y$:

1. ONE has no winning strategy in $\mathsf{G}_{\mathrm{fin}}(\Omega_y, \Omega_y) \Rightarrow \mathsf{S}_{\mathrm{fin}}(\Omega_y, \Omega_y)$;
2. $\mathsf{S}_{\mathrm{fin}}(\Omega_y, \Omega_y) \Rightarrow \mathsf{Ind}_{\mathrm{fin}}(\Omega_y, \Omega_y)$;
3. $\mathsf{Ind}_{\mathrm{fin}}(\Omega_y, \Omega_y) \Rightarrow \mathsf{K}(\Omega_y, \Omega_y)$;
4. $\mathsf{K}(\Omega_y, \Omega_y) \Rightarrow \mathsf{P}(\Omega_y, \Omega_y)$;
5. $\mathsf{P}(\Omega_y, \Omega_y) \Rightarrow \Omega_y \to \lceil \Omega_y \rceil_2^2$;
6. $\Omega_y \to \lceil \Omega_y \rceil_2^2 \Rightarrow \mathsf{B}_{\mathrm{linear}}(\Omega_y, \Omega_y)$;
7. $\mathsf{B}_{\mathrm{linear}}(\Omega_y, \Omega_y) \Rightarrow \mathsf{C}_{\mathrm{fin}}(\Omega_y, \Omega_y)$.

One can also show that for countably tight spaces the converses of the implications in 2, 3 and 4 hold while the implications in 1 (*), 5 and 7 are *not* reversible.

For 5: We give an example of the form (Y, τ_{J_1}), where J_1 is a free ideal on $\mathbb{N}$. First, choose a partition $(S_n : n \in \mathbb{N})$ of $\mathbb{N}$ such that each S_n is infinite. Define $J_1 = \{A \subset \mathbb{N} : (\forall n)(A \cap S_n \text{ is finite})\}$. As the sequence $(A_n : n \in \mathbb{N})$, where for each n, $A_n = \bigcup_{m \geq n} S_m$, shows, $\mathsf{P}(\Omega_\infty, \Omega_\infty)$ fails. Ramsey's theorem implies that $\Omega_\infty \to (\Omega_\infty)_2^2$ holds. (This example was given in Proposition 9 of the Appendix of [9].)

For 7: Let S be $\mathbb{Q}$ and let J_2 be $\{A \subset \mathbb{Q} : A \text{ nowhere dense}\}$. Then every element of Ω_∞ is somewhere dense in $\mathbb{Q}$, whence $\mathsf{B}(\Omega_\infty, \Omega_\infty)$ fails. But $\mathsf{C}_{\mathrm{fin}}(\Omega_\infty, \Omega_\infty)$ holds. To see this, let A be an element of Ω_∞. We may assume that A is a dense subset of the interval (a, b). Write $A = \bigcup_{n=1}^{\infty} S_n$, where no S_n is in Ω_∞. Also, let $(I_n : n \in \mathbb{N})$ bijectively enumerate a basis for the inherited topology of (a, b). Then each $S(I_n) = \{m : S_m \cap (I_n \setminus \bigcup_{j<m} S_j) \neq \emptyset\}$ is infinite. For each n choose an infinite set $B_n \subset S(I_n)$ such that the B_n's are pairwise disjoint. Choose an increasing sequence of k_n's such that for each n, $k_n \in B_n$. Then for each n, select a point $x_{k_n} \in S_{k_n} \cap (I_n \setminus \bigcup_{j<k_n} S_j)$. The set $\{x_{k_n} : n \in \mathbb{N}\}$ is in Ω_∞ and has at most finitely many points in common with each S_n.

(*) An example for 1 will be discussed below when we treat countable strong fan tightness.

PROBLEM 1. *Find a space of countable tightness which illustrates that the implication in* 6 *is not reversible.*

It is not clear for which spaces Y of countable tightness a property from the list

$$\{\Omega_y \to \lceil \Omega_y \rceil^2_2;\ \mathsf{B}_{\mathrm{linear}}(\Omega_y,\Omega_y);\ \mathsf{C}_{\mathrm{fin}}(\Omega_y,\Omega_y);\ \mathsf{S}_{\mathrm{fin}}(\Omega_y,\Omega_y)\}$$

determines that ONE has no winning strategy in $\mathsf{G}_{\mathrm{fin}}(\Omega_y,\Omega_y)$. Some notation is needed to formulate a partial result in this regard. The minimal cardinality of a neighborhood base for the point y of Y is denoted by $\chi(Y,y)$. For functions f and g from $\mathbb{N}$ to $\mathbb{N}$, $f \prec g$ denotes that $\lim_{n\to\infty}(g(n)-f(n)) = \infty$. Then $\prec$ defines a partial ordering. The symbol $\mathfrak{d}$ denotes the cofinality of this partially ordered set.

THEOREM 11B. *For an infinite cardinal number* κ *the following are equivalent*:

(1) $\kappa < \mathfrak{d}$.
(2) *For each* T_1*-space* X *of countable tightness and for each* $y \in X$ *such that* $\chi(X,y)=\kappa$, ONE *has no winning strategy in* $\mathsf{G}_{\mathrm{fin}}(\Omega_y,\Omega_y)$.
(3) *For each* T_1*-space* X *of countable tightness*, *if* y *is an element of* X *such that* $\chi(X,y)=\kappa$, *then* X *has countable fan tightness at* y.
(4) *For each* T_1*-space* X *of countable tightness*, *if* y *is an element of* X *such that* $\chi(X,y)=\kappa$, *then* X *has property* $\mathsf{Ind}_{\mathrm{fin}}(\Omega_y,\Omega_y)$ *at* y.
(5) *For each* T_1*-space* X *of countable tightness*, *if* y *is an element of* X *such that* $\chi(X,y)=\kappa$, *then* X *has property* $\mathsf{K}(\Omega_y,\Omega_y)$ *at* y.
(6) *For each* T_1*-space* X *of countable tightness and for each* $y \in X$ *with* $\chi(X,y)=\kappa$, X *has property* $\mathsf{P}(\Omega_y,\Omega_y)$.
(7) *For each* T_1*-space* X *of countable tightness and for each* $y \in X$ *such that* $\chi(X,y)=\kappa$, $\Omega_y \to \lceil \Omega_y \rceil^2_2$ *holds*.
(8) *For each* T_1*-space* X *of countable tightness and for each* $y \in X$ *such that* $\chi(X,y)=\kappa$, X *has property* $\mathsf{B}_{\mathrm{linear}}(\Omega_y,\Omega_y)$.
(9) *For each* T_1*-space* X *of countable tightness and for each* $y \in X$ *with* $\chi(X,y)=\kappa$, X *has property* $\mathsf{C}_{\mathrm{fin}}(\Omega_y,\Omega_y)$.

Proof. We prove (1)⇒(2) and (9)⇒(1).

(1)⇒(2). Let X be a T_1-space of countable tightness and let $y \in X$ be a point with $\chi(X,y)=\kappa$. Let $\mathcal{B}$ be a neighborhood basis of cardinality κ for y. Let σ be a strategy for ONE in $\mathsf{G}_{\mathrm{fin}}(\Omega_y,\Omega_y)$. Since X has countable tightness we may assume that in each inning σ calls on ONE to play a countable set.

Define for each finite sequence τ of positive integers a point a_τ in X as follows: $(a_n : n \in \mathbb{N})$ bijectively enumerates ONE's first move $\sigma(\emptyset)$. For the response $\{a_j : j \le n_1\}$ of TWO, $(a_{n_1,n} : n \in \mathbb{N})$ bijectively enumerates ONE's move $\sigma(\{a_j : j \le n_1\})$. For the response $\{a_{n_1,j} : j \le n_2\}$ of TWO,

$(a_{n_1,n_2,n} : n \in \mathbb{N})$ bijectively enumerates ONE's move $\sigma(\{a_j : j \leq n_1\}, \{a_{n_1,j} : j \leq n_2\})$, and so on.

Define for each B in $\mathcal{B}$ an increasing function f_B recursively as follows:

1. $f_B(1) = \min\{n : n > 1 \text{ and } a_n \in B\}$;
2. $f_B(n+1)$ is the least $m > f_B(n)$ such that for each finite sequence τ of length at most $f_B(n)$ of positive integers not exceeding $f_B(n)$, there is a $j \leq m$ such that $a_{\tau\frown j} \in B$.

For a function f from $\mathbb{N}$ to $\mathbb{N}$ let $f^1(k)$ denote $f(k)$, and for $m \in \mathbb{N}$ let $f^{m+1}(k)$ denote $f(f^m(k))$.

From the properties of the f_B's we see that for all n, $f_B(n) \leq f_B^n(1)$. For each B define g_B so that $g_B(n) = f_B^n(1)$. The family $\{g_B : B \in \mathcal{B}\}$ is by cardinality considerations not cofinal in the order $\prec$. Thus choose an increasing function g such that $g(1) > 2$ and for each B the set $\{n : g_B(n) < g(n)\}$ is infinite. For notational convenience let T_1 denote the set $\{a_j : j \leq g(1)\}$ and for $k > 1$ let T_k denote the set $\{a_{g^1(1),\ldots,g^{k-1}(1),j} : j \leq g^k(1)\}$. Then

$$\sigma(\emptyset), T_1, \sigma(T_1), T_2, \sigma(T_1, T_2), \ldots$$

is a play of the game $\mathsf{G}_{\mathrm{fin}}(\Omega_y, \Omega_y)$ during which ONE used the strategy σ. To see that TWO won this play, it suffices to see that for each $B \in \mathcal{B}$ there is an m with $B \cap T_m$ nonempty.

Let $B \in \mathcal{B}$ be given. Since for each n, $g(n) < g^n(1)$, the set $\{n : g_B(n) < g^n(1)\}$ is infinite. Let m be the least element of this set. If m is 1, then $f_B(1) < g(1)$ and we see that $B \cap T_1$ is nonempty. So, assume that m is larger than 1. Then we have:

$$g_B(m) < g^m(1) \quad \text{but} \quad g^{m-1}(1) \leq g_B(m-1) = f_B^{m-1}(1).$$

Since $m - 1 < f_B^{m-1}(1)$ the sequence $(g^1(1), \ldots, g^{m-1}(1))$ is one of the sequences considered in the definition of $f_B(f_B^{m-2}(1) + 1)$. Consequently, we have a $j \leq f_B(f_B^{m-2}(1) + 1)$ such that $a_{g^1(1),\ldots,g^{m-1}(1),j} \in B$. Then $B \cap T_m \neq \emptyset$, since

$$f_B(f_B^{m-2}(1) + 1) \leq f_B(f_B^{m-1}(1)) = f_B^m(1) < g^m(1).$$

(9)⇒(1). Let X be a set of real numbers of cardinality κ. Then $\chi(\mathsf{C}_{\mathrm{p}}(X), \mathbf{o}) = |X| = \kappa$. By (9), $\mathsf{C}_{\mathrm{p}}(X)$ has property $\mathsf{C}_{\mathrm{fin}}(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$. By Theorem 11, $\mathsf{C}_{\mathrm{p}}(X)$ has countable fan tightness. By Arkhangel'skiĭ's theorem and Theorem 3.9 of [11], X has property $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$. We have shown that each set of real numbers of cardinality κ has property $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$. But then by Theorem 4.6 of [11], $\kappa < \mathfrak{d}$. ■

The position of the universal quantifier in the clauses of Theorem 11B is important: One cannot show that if at a point y of a space Y one of the conclusions of a clause of Theorem 11B is true, then $\chi(Y, y) < \mathfrak{d}$. To see this, consider the closed unit interval, denoted by X. Since it has property $\mathsf{S}_{\rm fin}(\Omega,\Omega)$, it follows that $\mathsf{C}_{\rm p}(X)$ has countable fan tightness. By Theorem 11 each of the conclusions of Theorem 11B holds for $\Omega_{\mathbf{o}}$. However, $\chi(\mathsf{C}_{\rm p}(X),\mathbf{o}) = |X| = 2^{\aleph_0}$.

The analogies between the properties of countably fan tight $\mathsf{C}_{\rm p}(X)$ and of P-point ultrafilters raise the question if the existence of an uncountable set X of real numbers for which $\mathsf{C}_{\rm p}(X)$ is countably fan tight implies the existence of a P-point ultrafilter. This is not so: There is always an uncountable set X of real numbers with property $\mathsf{S}_{\rm fin}(\Omega,\Omega)$ (even one that is not σ-compact, by Section 5 of [11]): Thus by Arkhangel'skiĭ's theorem, there is always an uncountable set of real numbers for which $\mathsf{C}_{\rm p}(X)$ is countably fan tight. But by Theorem VI.4.8 of [22] it is consistent that there is no P-point ultrafilter.

5. Countable strong fan tightness. A topological space has *countable strong fan tightness at the point y* if $\mathsf{S}_1(\Omega_y,\Omega_y)$ holds. The game $\mathsf{G}_1(\Omega_y,\Omega_y)$ is also called the *strong fan tightness game at y*. A space has *countable strong fan tightness* if it has this property at each point. Every space which has countable strong fan tightness has countable fan tightness.

For a T_1-space Y of countable tightness and for a nonisolated point y, strong countable fan tightness at y is equivalent to saying that for every countable $A \in \Omega_y$ the filter $\mathcal{F}_{y,A}$ satisfies the selection hypothesis $\mathsf{S}_1(\mathcal{F}^+_{y,A},\mathcal{F}^+_{y,A})$. In general, let us say that a free filter $\mathcal{F}$ on $\mathbb{N}$ is a *strong fan tight filter* if it satisfies $\mathsf{S}_1(\mathcal{F}^+,\mathcal{F}^+)$.

According to Mathias a family $\mathcal{A}$ of infinite subsets of $\mathbb{N}$ is a *happy family* if there is a free filter $\mathcal{F}$ such that $\mathcal{A} = \mathcal{F}^+$, and for every *descending* sequence $(A_n : n \in \mathbb{N})$ of elements of A there is an $X \in \mathcal{A}$ such that for each n, $X \setminus \{1,\ldots,n\} \subseteq A_n$ ([15], Definition 0.1). Grigorieff calls the dual ideal $\mathcal{F}^*$ of $\mathcal{F}$ a *weak T-ideal* ([9], Definition 1.2), a *selective ideal* (Definition 1.7) or an *inductive ideal* (Definition 1.11)—see Corollary 1.15 of [9] and Proposition 0.8 of [15]. Thus, if $\mathcal{F}$ is a strong fan tight filter, then $\mathcal{F}^+$ is a happy family and $\mathcal{F}^*$ is an inductive ideal.

In the following section we shall also include a result about the *square bracket partition relation* which is defined as follows: Let $\mathcal{A}$ and $\mathcal{B}$ be families of subsets of S, and let k, l and m be in $\mathbb{N}$. Then $\mathcal{A} \to [\mathcal{B}]^n_{k/\le l}$ denotes the statement that for each $A \in \mathcal{A}$ and for each function $f : [A]^n \to \{1,\ldots,k\}$ there is a subset B of A in $\mathcal{B}$ and a subset J of $\{1,\ldots,k\}$ of cardinality at most l such that all values of f on $[B]^n$ lie in J. The negation of this statement is denoted by $\mathcal{A} \not\to [\mathcal{B}]^n_{k/\le l}$. When l is $k-1$ it is customary to leave off all reference to l and to simply write $\mathcal{A} \to [\mathcal{B}]^n_k$ and $\mathcal{A} \not\to [\mathcal{B}]^n_k$ for these two statements. This partition relation was introduced and extensively studied for cardinal numbers in [6].

5.1. *Countable strong fan tightness for* $\mathsf{C}_{\mathrm{p}}(X)$**.** In [19] Sakai proved:

THEOREM 12 (Sakai). *For each Tikhonov space* X *the following are equivalent*:

(1) $\mathsf{C}_{\mathrm{p}}(X)$ *has countable strong fan tightness.*
(2) *Each finite power of* X *has Rothberger's property* C''.
(3) X *has property* $\mathsf{S}_1(\Omega,\Omega)$.

By Sakai's theorem, Arkhangel'skiĭ's theorem and Theorem 2.3 from [11] there is a set of real numbers (for example, the Cantor set) which has property $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$ but not property $\mathsf{S}_1(\Omega,\Omega)$. Consequently, there is a set X of real numbers such that $\mathsf{C}_{\mathrm{p}}(X)$ has countable fan tightness, but does not have countable strong fan tightness. The following characterizations of countable strong fan tightness give *mutatis mutandis* characterizations of property $\mathsf{S}_1(\Omega,\Omega)$.

THEOREM 13. *Let* X *be a Tikhonov space such that* $\mathsf{C}_{\mathrm{p}}(X)$ *has countable tightness. Then the following are equivalent*:

(a) $\mathsf{C}_{\mathrm{p}}(X)$ *has countable strong fan tightness.*
(b) ONE *does not have a winning strategy in* $\mathsf{G}_1(\Omega_{\mathbf{o}},\Omega_{\mathbf{o}})$.
(c) $\mathsf{C}_{\mathrm{p}}(X)$ *has property* $\mathsf{Ind}_1(\Omega_y,\Omega_y)$.
(d) $\mathsf{C}_{\mathrm{p}}(X)$ *has both properties* $\mathsf{P}(\Omega_{\mathbf{o}},\Omega_{\mathbf{o}})$ *and* $\mathsf{Q}(\Omega_{\mathbf{o}},\Omega_{\mathbf{o}})$.
(e) *For all* n *and* k *in* $\mathbb{N}$, $\mathsf{C}_{\mathrm{p}}(X)$ *satisfies* $\Omega_{\mathbf{o}}\to(\Omega_{\mathbf{o}})^n_k$.
(f) $\mathsf{C}_{\mathrm{p}}(X)$ *satisfies* $\mathsf{B}_{\mathrm{tree}}(\Omega_{\mathbf{o}},\Omega_{\mathbf{o}})$.
(g) $\mathsf{C}_{\mathrm{p}}(X)$ *satisfies* $\mathsf{C}_1(\Omega_{\mathbf{o}},\Omega_{\mathbf{o}})$.
(h) $\mathsf{C}_{\mathrm{p}}(X)$ *satisfies* $\Omega_{\mathbf{o}}\to[\Omega_{\mathbf{o}}]^2_3$.

Proof. (a)⇒(b). Fix a sequence $(g_n : n<\omega)$ as in Lemma 9. Let σ be a strategy for ONE. Since $\mathsf{C}_{\mathrm{p}}(X)$ has countable tightness, we may assume that in each inning ONE chooses a countable set.

We define a strategy τ for ONE in $\mathsf{G}_1(\Omega,\Omega)$ played on X: Look at ONE's first move, $\sigma(\mathsf{C}_{\mathrm{p}}(X))$, and enumerate it as $(f_{(n)} : n<\omega)$. For each n define $U_{(n)}=\mathsf{Coz}(|f_{(n)}|+g_n,1/2)$. Then define $\tau(X)=\{U_{(n)} : n<\omega\}$. This is an ω-cover of X. TWO of $\mathsf{G}_1(\Omega,\Omega)$ responds by selecting a $U_{(n_1)}$. This is translated back as the move $f_{(n_1)}$ for TWO of $\mathsf{G}_1(\Omega_{\mathbf{o}},\Omega_{\mathbf{o}})$. Apply σ to find $\sigma(f_{(n_1)})=(f_{(n_1,m)} : m<\omega)$. Then define $\tau(U_{(n_1)})$ to be $(U_{(n_1,m)} : m<\omega)$, where $U_{(n_1,m)}=\mathsf{Coz}(|f_{(n_1,m)}|+|g_m|,(1/2)^2)$. To this TWO of the $\mathsf{G}_1(\Omega,\Omega)$-game responds with a set $U_{(n_1,n_2)}$, which is translated back as the move $f_{(n_1,n_2)}$ for TWO of the game $\mathsf{G}_1(\Omega_{\mathbf{o}},\Omega_{\mathbf{o}})$, and so on.

By Theorem 2, τ is not a winning strategy for ONE of $\mathsf{G}_1(\Omega,\Omega)$ on X. Let $\tau(X),U_{(n_1)},\tau(U_{(n_1)}),U_{(n_1,n_2)},\ldots$ be a τ-play which is lost by ONE. Look at the corresponding σ-play $\sigma(\mathsf{C}_{\mathrm{p}}(X)),f_{(n_1)},\sigma(f_{(n_1)}),f_{(n_1,n_2)},\ldots$ Since the set $\{U_{(n_1,\ldots,n_k)} : k\in\mathbb{N}\}$ is an ω-cover, $\mathbf{o}$ is in the closure of the sequence of $f_{(n_1,\ldots,n_k)}$'s. Thus ONE of $\mathsf{G}_1(\Omega_{\mathbf{o}},\Omega_{\mathbf{o}})$ lost this σ-play.

(b)⇒(c). Let $(A_n : n \in \mathbb{N})$ be a descending sequence of elements of $\Omega_{\mathbf{o}}$, and let $(a_n : n \in \mathbb{N})$ be a bijective enumeration of A_1. Define a strategy σ for ONE in the game $\mathsf{G}_1(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$ as follows: ONE's first move is $\sigma(\mathsf{C}_{\mathrm{p}}(X)) = A_1$. If TWO responds with a_{n_1}, then ONE plays $\sigma(a_{n_1}) = A_{n_1} \setminus \{a_j : j \le n_1\}$. If TWO now responds with $a_{n_2} \in \sigma(a_{n_1})$, then we know that $n_1 < n_2$; ONE's next move is $\sigma(a_{n_1}, a_{n_2}) = A_{n_2} \setminus \{a_j : j \le n_1 + n_2\}$, and so on.

By (b) we find a play $\sigma(\mathsf{C}_{\mathrm{p}}(X)), a_{n_1}, \sigma(a_{n_1}), a_{n_2}, \sigma(a_{n_1}, a_{n_2}), \ldots$ lost by ONE. For each k put $g(k) = a_{n_{k+1}}$. Then g is as required.

(c)⇒(d). It is clear that $\mathsf{Ind}_1(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$ implies property $\mathsf{P}(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$. To see that it implies property $\mathsf{Q}(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$, let A be an element of $\Omega_{\mathbf{o}}$ which is partitioned into the pairwise disjoint nonempty finite subsets $(F_n : n \in \mathbb{N})$. Define an enumeration $(a_n : n \in \mathbb{N})$ of A such that if $a_i \in F_n$ and $a_j \in F_m$, and if $n < m$, then $i < j$. It follows that if $a_n \in F_m$, then $m \le n$.

For each m set $A_m = \bigcup_{n \ge m} F_n$. Now apply (c) to find a strictly increasing function g such that $B := \{a_{g(n)} : n \in \mathbb{N}\} \in \Omega_{\mathbf{o}}$, and for each n, $a_{g(n+1)} \in A_{g(n)}$. For each n choose $f(n)$ such that $a_{g(n)} \in F_{f(n)}$. By the way we set things up we have, for each n, $g(n) < f(n) \le g(n+1)$. But then B meets each F_n in at most one point.

(d)⇒(e). We give an argument for $\Omega_{\mathbf{o}} \to (\Omega_{\mathbf{o}})^2_2$; the proof for higher values of the superscript and the subscript then follows a standard induction argument.

Let $A \in \Omega_{\mathbf{o}}$ and a coloring $f : [A]^2 \to \{0, 1\}$ be given. We may assume that A is countable, and enumerate it bijectively as $(a_n : n < \omega)$. Put $A_{-1} = A$ and define a descending sequence $A_0, A_1, \ldots$ of elements of $\Omega_{\mathbf{o}}$ and a sequence $i_0, i_1, \ldots$ of elements of $\{0, 1\}$ as follows:

Choose i_0 so that the set $\{a_m : f(\{a_0, a_m\}) = i_0\}$ is in $\Omega_{\mathbf{o}}$. Suppose that $n \ge 0$ is given and that $A_0, \ldots, A_n$ and $i_0, \ldots, i_n$ have been defined such that for each $j \le n$, $A_j = \{a_m \in A_{j-1} : m > j \text{ and } f(\{a_j, a_m\}) = i_j\}$ and $A_j \in \Omega_{\mathbf{o}}$.

Then choose i_{n+1} such that $A_{n+1} = \{a_m \in A_n : m > n + 1 \text{ and } f(\{a_{n+1}, a_m\}) = i_{m+1}\}$ is in $\Omega_{\mathbf{o}}$. First apply property $\mathsf{P}(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$ to the sequence $(A_n : n < \omega)$ to find a set $B \in \Omega_{\mathbf{o}}$ such that $B \subseteq A$ and for each n, $B \setminus A_n$ is finite. For each n put $B_n = B \cap (A_n \setminus A_{n+1})$. Then each B_n is a finite set. Choose a sequence $m_0 < m_1 < \ldots < m_k < \ldots$ of positive integers such that for each k, if $j \ge m_{k+1}$, then for each $a_n \in \bigcup_{t \le m_k} B_t$, $B_j \subset A_n$.

For each k put $S_k = \bigcup_{m_k \le j < m_{k+1}} B_j$. Then each S_k is a finite set and the union of the S_k's is in $\Omega_{\mathbf{o}}$. At least one of the two sets $\bigcup_{k \text{ even}} S_k$ or $\bigcup_{k \text{ odd}} S_k$ is in $\Omega_{\mathbf{o}}$. We may assume it is the former. Apply property $\mathsf{Q}(\Omega_{\mathbf{o}}, \Omega_{\mathbf{o}})$: We find a set $B \subset A$ such that B is in $\Omega_{\mathbf{o}}$, $B \cap S_k \ne \emptyset$ implies k is even, and for each k, $|B \cap S_k| \le 1$. By the construction of the S_k's, B is end-homogeneous for f (in the order imposed by the original enumeration of A). Thus, for each $a_n \in B$, if $a_m \in B$ and $n < m$, then $f(\{a_n, a_m\}) = i_n$. Now partition

B into two sets B_0 and B_1, where we put $a_n \in B_j$ if $i_n = j$. Each of these two sets is homogeneous for the coloring f, and one of these is in $\Omega_{\mathbf{o}}$.

(e)⇒(f). Let R be a tree relation on $A \in \Omega_{\mathbf{o}}$. Define $f : [A]^2 \to \{0,1\}$ so that

$$f(\{a,b\}) = \begin{cases} 0 & \text{if } \{a,b\} \text{ is an antichain,} \\ 1 & \text{otherwise.} \end{cases}$$

From (e) we find a subset B of A such that f is constant on $[B]^2$, and $B \in \Omega_{\mathbf{o}}$. If f is constant of value 0, then B is an antichain; in the other case it is a chain.

(f)⇒(g). Let an $A \in \Omega_{\mathbf{o}}$ as well as a function $f : A \to \omega$ be given. We may assume that A is countable. For each n set $A_n = \{a \in A : f(a) = n\}$. If for some n, A_n is in $\Omega_{\mathbf{o}}$, we are done. Otherwise we define a tree ordering R on A such that the only branches are the A_n's. No subset of A which is in Ω_y is a chain, so that by (f) there is a subset B of A which is an antichain and in Ω_y. Then f is one-to-one on B.

(g)⇒(h). The proof starts like that of (d)⇒(e), with one small innovation near the end. Let $A \in \Omega_{\mathbf{o}}$ be given, as well as $f : [A]^2 \to \{1,2,3\}$. We may assume that each element of A is nonnegative and that A is countable and enumerate it bijectively as $(a_n : n \in \mathbb{N})$.

First select a descending sequence $A_1 \supset A_2 \supset \ldots$ in $\Omega_{\mathbf{o}}$ along with a sequence $l_1, l_2, \ldots$ with terms in $\{1,2,3\}$ such that each $A_n \in \Omega_{\mathbf{o}}$, and for $a_m \in A_n$, $m > n$ and $f(\{a_n, a_m\}) = l_n$. Let $(g_n : n \in \mathbb{N})$ be a sequence as in Lemma 9. For each n put $B_n = \{a_m + g_n : a_m \in A_n \setminus A_{n+1}\}$. Then $\bigcup_{n=1}^{\infty} B_n \in \Omega_{\mathbf{o}}$, while no B_n is in $\Omega_{\mathbf{o}}$. Applying g we choose for each n an $a_{m_n} + g_n \in B_n$ such that the set $\{a_{m_n} + g_n : n \in \mathbb{N}\}$ is in $\Omega_{\mathbf{o}}$. Consequently, the set $\{a_{m_n} : n \in \mathbb{N}\}$ is in $\Omega_{\mathbf{o}}$. Next select $n_1 < \ldots < n_k < \ldots$ such that for all $i \le n_1$ we have $m_i \le m_{n_1}$ and for all $j \ge n_2$ we have $a_{m_j} \in A_{m_{n_1}}$, and for all $i \le n_k$ we have $m_i \le m_{n_k}$ and for all $j \ge n_{k+1}$, $a_{m_j} \in A_{m_{n_k}}$. Put $S_1 = \{a_{m_i} : i \le n_1\}$, and for all k, $S_{k+1} = \{a_{m_i} : n_k < i \le n_{k+1}\} \setminus (S_1 \cup \ldots \cup S_k)$. One (or both) of $\bigcup_{i=1}^{\infty} S_{2i-1}$ or $\bigcup_{i=1}^{\infty} S_{2i}$ is in $\Omega_{\mathbf{o}}$; we may assume it is the latter.

Since each S_{2i} is finite, it is not in $\Omega_{\mathbf{o}}$. Yet, $\bigcup_{i=1}^{\infty} S_{2i} \in \Omega_{\mathbf{o}}$. Apply g to the sequence of S_{2i}'s: in each we pick an element $a_{m_{j_i}}$ such that $\{a_{m_{j_i}} : i \in \mathbb{N}\} \in \Omega_{\mathbf{o}}$. Since now for $i < r$ we have $m_{j_i} < m_{j_r}$ and $a_{m_{j_r}} \in A_{m_{j_i}}$, we see that the sequence $(a_{m_{j_i}} : i \in \mathbb{N})$ is end-homogeneous for the partition f. Partitioning this sequence into three classes according to the values of the $l_{m_{j_i}}$'s, we find that one (or more) of these three classes is in $\Omega_{\mathbf{o}}$—whichever it is, is in fact a homogeneous set for the coloring f.

(h)⇒(a). One could argue as follows: Let $(A_n : n \in \mathbb{N})$ be a sequence from $\Omega_{\mathbf{o}}$. Enumerate each A_n bijectively as $(a_m^n : m \in \mathbb{N})$. Let $(g_m : m \in \mathbb{N})$ be a sequence of functions as in Lemma 9. By Lemma 10(1) we may assume that each a_m^n is a nonnegative function. Define B to be the set $(g_m + a_n^m + a_k^n : m <$

$n < k \in \mathbb{N}$) listed with the triples (m, n, k) in lexicographically increasing order, and without repetitions. Then B is in $\Omega_{\mathbf{o}}$. It is well known that if $\mathcal{A} \to [\mathcal{A}]^2_3$, then for all $k \in \mathbb{N}$, $\mathcal{A} \to [\mathcal{A}]^2_{k/\leq 2}$.

Define a partition $f : [B]^2 \to \{0, 1, 2, 3, 4\}$ as follows (now (m_1, n_1, k_1) lexicographically precedes (m_2, n_2, k_2)):

$$f(\{g_{m_1} + a^{m_1}_{n_1} + a^{n_1}_{k_1}, g_{m_2} + a^{m_2}_{n_2} + a^{n_2}_{k_2}\}) = \begin{cases} 0 & \text{if } m_1 = m_2 \text{ and } n_1 = n_2, \\ 1 & \text{if } m_1 = m_2 \text{ and } n_1 < n_2, \\ 2 & \text{if } m_1 < m_2 \text{ and } n_1 = n_2, \\ 3 & \text{if } m_1 < m_2 \text{ and } n_1 < n_2, \\ 4 & \text{if } m_1 < m_2 \text{ and } n_1 > n_2. \end{cases}$$

Apply (h) to find a subset C of B which is in $\Omega_{\mathbf{o}}$ such that on $[C]^2$, f has at most two values. Since for each x and $m < n < k$ we have $\max\{g_m(x), a^m_n(x), a^n_k(x)\} \leq g_m(x) + a^m_n(x) + a^n_k(x)$, we see that each of the sets $\{g_m : (\exists k)(\exists n)(k > n > m \text{ and } g_m + a^m_n + a^n_k \in C)\}$, $\{a^m_n : (\exists k > n)(g_m + a^m_n + a^n_k \in C)\}$ and $\{a^n_k : (\exists m)(g_m + a^m_n + a^n_k \in C)\}$ is in $\Omega_{\mathbf{o}}$.

Because of the properties of the g_m's there are infinitely many different values of m for which $g_m + a^m_n + a^n_k$ is in C. This means that f's value-set is not contained in $\{0, 1\}$. Moreover, C cannot contain an infinite path whose consecutive terms are 4-colored, since this would mean there is an infinite descending sequence in $\mathbb{N}$. This rules out value-sets $\{0, 4\}$ and $\{1, 4\}$ for f on C. In each of the remaining cases (the case $\{1, 2\}$ needs some caution) one finds for each A_j an $x_j \in A_j$ such that $\{x_j : j \in \mathbb{N}\}$ is in $\Omega_{\mathbf{o}}$ by augmenting an appropriate subset of the set of middle terms of elements of C, or the set of last terms. ■

An analogue in the $\mathsf{S}_1(\Omega, \Omega)$-context of (a)⇔(h) is proved in [21]. Analogies between the properties of countably strong fan tight $\mathsf{C}_{\mathrm{p}}(X)$ and of Ramsey ultrafilters might suggest that the existence of an uncountable set X of real numbers for which $\mathsf{C}_{\mathrm{p}}(X)$ is countably fan tight implies the existence of a Ramsey ultrafilter. This is not true. It was shown in Section 5 of [11] that an assumption weaker than $\mathfrak{d} = \aleph_1$ implies there is an uncountable set X of real numbers with property $\mathsf{S}_1(\Omega, \Omega)$: Applying Sakai's theorem, there is then an uncountable set of real numbers for which $\mathsf{C}_{\mathrm{p}}(X)$ is countably strong fan tight. But by Theorem 5.1 of [13] it is consistent that $\mathfrak{d} = \aleph_1$ and there are no Ramsey ultrafilters. The model given for Theorem VI.4.8 of [22] which shows that it is consistent that there is no P-point ultrafilter satisfies $2^{\aleph_0} = \aleph_2$. But any model with no P-point ultrafilters and with $2^{\aleph_0} = \aleph_2$ must have $\mathfrak{d} = \aleph_1$, because in [12] Ketonen showed that if $\mathfrak{d} = 2^{\aleph_0}$, then there are P-point ultrafilters. Thus the existence of an uncountable set of real numbers with property $\mathsf{S}_1(\Omega, \Omega)$ does not even imply the existence of P-point ultrafilters.

The fact that the partition relation in (h) characterizes countable strong fan tightness for $\mathsf{C}_{\mathrm{p}}(X)$, but does not characterize Ramseyness of ultrafilters (a result of Blass' from [4]), is another indication that the relationship between countable strong fan tightness of spaces and Ramseyness of ultrafilters is not very close. It also indicates, as we shall see below, that Theorem 13 is more a theorem about the special countably tight spaces $\mathsf{C}_{\mathrm{p}}(X)$ rather than about general countably tight spaces than one might at first expect.

5.2. *Countable strong fan tightness in* T_1*-spaces***.** If Y has countable tightness, then the following six implications hold at the nonisolated point y:

1. ONE has no winning strategy in $\mathsf{G}_1(\Omega_y, \Omega_y) \Rightarrow Y$ has countable strong fan tightness at y.
2. Y has countable strong fan tightness at $y \Rightarrow \mathsf{Ind}_1(\Omega_y, \Omega_y)$ ([9], Proposition 1.12).
3. $\mathsf{Ind}_1(\Omega_y, \Omega_y) \Rightarrow \mathsf{P}(\Omega_y, \Omega_y)$ and $\mathsf{Q}(\Omega_y, \Omega_y)$.
4. $\mathsf{P}(\Omega_y, \Omega_y)$ and $\mathsf{Q}(\Omega_y, \Omega_y) \Rightarrow$ for all n and k, $\Omega_y \to (\Omega_y)^n_k$.
5. For each n and k $\Omega_y \to (\Omega_y)^n_k \Rightarrow \mathsf{C}_1(\Omega_y, \Omega_y)$.
6. For each n and k $\Omega_y \to (\Omega_y)^n_k \Rightarrow \Omega_y \to [\Omega_y]^2_3$.

One can show that for countably tight spaces the converses of the implications in 2 and 3 hold while the implications in 1, 4 and 6 are *not* reversible. Moreover, for countably tight spaces the implication $\Omega_y \to [\Omega_y]^2_3 \to \mathsf{C}_1(\Omega_y, \Omega_y)$ is not in general provable.

For 1: The referee found the following example, and kindly permitted me to include it in this paper. The example is of the form (Y, τ_{J_3}). Moreover, the example is a countably *strong* fan tight space where ONE does not even have a winning strategy in the game $\mathsf{G}_{\mathrm{fin}}(\Omega_\infty, \Omega_\infty)$. This illustrates for both the fan tightness game and the strong fan tightness game that in general the game does not characterize the selection property. Incidentally, this also illustrates that fan tight filters on $\mathbb{N}$ or strong fan tight filters on $\mathbb{N}$ are not P^+-filters or $+$-Ramsey filters respectively, as introduced by Laflamme in [14].

The idea is to first define a strategy for ONE, from it define the appropriate ideal which makes this strategy a winning strategy, and then to show that the resulting space is countably strong fan tight at ∞. First, let $(Y_n : n \in \mathbb{N})$ be a partition of $\mathbb{N}$ into pairwise disjoint infinite sets. The strategy F is constructed such that:

1. For each finite sequence $(T_1, \ldots, T_k)$ of finite nonempty subsets of $\mathbb{N}$, each a subset of a different Y_t, there is an i with $F(T_1, \ldots, T_k) = Y_i$.
2. For each i there is a unique sequence $(T_1, \ldots, T_k)$ of finite subsets of $\mathbb{N}$ with:

(a) each T_j a nonempty subset of some Y_{k_j}, $k_j \neq i$,
(b) if $j \neq m$, then $k_j \neq k_m$, and
(c) $F(T_1, \ldots, T_k) = Y_i$.

Since ONE may always replace a move (such as the empty set) of TWO by a set properly containing it and pretend that this was actually TWO's move, there is no loss in generality when we require that the domain of F be finite sequences whose terms are nonempty. ONE's first move will be $F(\{1\})$, and if by inning n, TWO has played $T_1, \ldots, T_n$, then ONE's $(n+1)$th move will be $F(\{1\}, T_1, \ldots, T_n)$.

For every play $P := O_1, T_1, O_2, T_2, \ldots$ during which ONE used F, we put $S(P) = \bigcup_{n\in\mathbb{N}} T_n$. Let J_3 be the collection of $X \subseteq \mathbb{N}$ such that there are finitely many plays $P_1, \ldots, P_k$ during which ONE used F, such that $X \subseteq \bigcup_{j\leq n} S(P_j)$. One must check that J_3 is indeed a proper free ideal. The effect of 2 is that during any play, TWO never gets to choose a finite subset from any Y_i from which a finite subset was chosen during an earlier inning; this implies properness of J_3. Freeness is also easily checked.

By the definition of J_3, F is a winning strategy for ONE in $\mathsf{G}_{\mathrm{fin}}(\Omega_\infty, \Omega_\infty)$. To see that the space (Y, τ_{J_3}) has countable strong fan tightness, consider a sequence $(A_n : n \in \mathbb{N})$ from Ω_∞. Then no A_n is contained in J_3. To find a selector for the A_n's, proceed as follows: First, for each $j \in Y_1$ choose an $a_j \in A_j$. If the set $\{a_j : j \in Y_1\}$ is not in J_3, we are done. Else, there are finitely many F-plays $P_1, \ldots, P_n$ such that $\{a_j : j \in Y_1\} \subseteq S(P_1) \cup \ldots \cup S(P_n)$. Since $S(P_1) \cup \ldots \cup S(P_n)$ meets each Y_j in a finite set, we then proceed to choose for each $j \in Y_2$ an $a_j \in A_j$ but outside $S(P_1) \cup \ldots \cup S(P_n)$, and so on.

For 4 the example (Y, τ_{J_1}) given after Theorem 11 illustrates the point.

For 6, Blass has used the Continuum Hypothesis in [4] to construct a weakly Ramsey ultrafilter which is not a Ramsey ultrafilter, i.e., an ultrafilter $\mathcal{U}$ on $\mathbb{N}$ such that $\mathcal{U} \to [\mathcal{U}]^2_3$, but $\mathcal{U}$ does not satisfy $\mathcal{U} \to (\mathcal{U})^2_3$. Let J be the set $\{\mathbb{N} \setminus X : X \in \mathcal{U}\}$. Then the topology τ_J on $Y = \mathbb{N} \cup \{\infty\}$ is such that (Y, τ_J) satisfies the following: for all k, $\Omega_\infty \to [\Omega_\infty]^2_{k/\leq 2}$, but $\Omega_\infty \not\to (\Omega_\infty)^2_3$. Blass showed that a weakly Ramsey ultrafilter is necessarily a P-point one. Thus, if $\mathsf{C}_1(\Omega_\infty, \Omega_\infty)$ were true, this would imply that $\mathcal{U}$ is also a Q-point ultrafilter, hence a Ramsey ultrafilter, contrary to its being only a weakly Ramsey ultrafilter.

PROBLEM 2. *Find a countably tight space which illustrates that the implication in* 5 *is not reversible.*

PROBLEM 3. *Find a countably tight space which has property* $\mathsf{C}_1(\Omega_y, \Omega_y)$, *but not property* $\Omega_y \to [\Omega_y]^2_3$.

It is not clear for which spaces Y of countable tightness a property from the list $\{\Omega_y \to (\Omega_y)^2_2, \mathsf{C}_1(\Omega_y,\Omega_y), \mathsf{S}_1(\Omega_y,\Omega_y), \Omega_y \to [\Omega_y]^2_3\}$ implies that One has no winning strategy in $\mathsf{G}_1(\Omega_y,\Omega_y)$. Here is a partial result in this direction. Let $\mathsf{cov}(\mathcal{M})$ denote the minimal number of first category sets needed to cover $\mathbb{R}$.

Theorem 13B. *For an infinite cardinal κ the following are equivalent*:

(1) $\kappa < \mathsf{cov}(\mathcal{M})$.
(2) *For each T_1-space X of countable tightness and for each $y \in X$ such that $\chi(X,y)=\kappa$, One has no winning strategy in $\mathsf{G}_1(\Omega_y,\Omega_y)$.*
(3) *For each T_1-space X of countable tightness, if y is an element of X such that $\chi(X,y)=\kappa$, then X has countable strong fan tightness at y.*
(4) *For each T_1-space X of countable tightness, if y is an element of X such that $\chi(X,y)=\kappa$, then X has property $\mathsf{Ind}_1(\Omega_y,\Omega_y)$.*
(5) *For each T_1-space X of countable tightness and for each $y \in X$ with $\chi(X,y)=\kappa$, X has properties $\mathsf{P}(\Omega_y,\Omega_y)$ and $\mathsf{Q}(\Omega_y,\Omega_y)$.*
(6) *For each T_1-space X of countable tightness and for each $y \in X$ such that $\chi(X,y)=\kappa$, for all k and n, $\Omega_y \to (\Omega_y)^n_k$ holds.*
(7) *For each T_1-space X of countable tightness and for each $y \in X$ with $\chi(X,y)=\kappa$, X has the property $\mathsf{C}_1(\Omega_y,\Omega_y)$.*
(8) *For each T_1-space X of countable tightness and for each $y \in X$ with $\chi(X,y)=\kappa$, X has the property $\Omega_y \to [\Omega_y]^2_3$.*

Proof. We prove (1)⇒(2), (7)⇒(1) and (8)⇒(1).

(1)⇒(2). Let X be a T_1-space of countable tightness and let y be an element of X such that $\chi(X,y)=\kappa$. Fix a neighborhood basis $\mathcal{B}$ of y of minimal cardinality and let σ be a strategy for One in $\mathsf{G}_1(\Omega_y,\Omega_y)$.

Define a family of points a_τ, τ a finite sequence of positive integers, as follows: $(a_n : n \in \mathbb{N})$ is a bijective enumeration of One's first move, $\sigma(\emptyset)$. For the move a_{n_1} by Two, $(a_{n_1,n} : n \in \mathbb{N})$ bijectively enumerates One's move $\sigma(a_{n_1})$. For the move a_{n_1,n_2} by Two, $(a_{n_1,n_2,n} : n \in \mathbb{N})$ bijectively enumerates One's move $\sigma(a_{n_1}, a_{n_1,n_2})$, and so on.

For B in $\mathcal{B}$, define $S_B = \{f : \text{for each } n,\ a_{f(1),\dots,f(n)} \notin B\}$. Then each S_B is closed and nowhere dense. By (1) there is an f not in any S_B; fix one, say f. Then One loses the play $\sigma(\emptyset), a_{f(1)}, \sigma(a_{f(1)}), a_{f(1),f(2)}, \sigma(a_{f(1)}, a_{f(1),f(2)}), \dots$

(7)⇒(1). Let X be a set of real numbers of cardinality κ. Then $\chi(\mathsf{C}_\mathrm{p}(X),\mathbf{o}) = |X| = \kappa$, and so by (7), $\mathsf{C}_\mathrm{p}(X)$ has property $\mathsf{C}_1(\Omega_\mathbf{o},\Omega_\mathbf{o})$. By Theorem 13 this implies that $\mathsf{C}_\mathrm{p}(X)$ has countable strong fan tightness. By Sakai's theorem X has property $\mathsf{S}_1(\Omega,\Omega)$. We have shown that (7) implies that every set of real numbers of cardinality κ has property $\mathsf{S}_1(\Omega,\Omega)$. By Theorem 4.8 of [11], $\kappa < \mathsf{cov}(\mathcal{M})$.

(8)⇒(1). Let X be a set of real numbers of cardinality κ. Then in the T_1-space $\mathsf{C}_{\mathrm{p}}(X)$ we have $\chi(\mathsf{C}_{\mathrm{p}}(X), \mathbf{o}) = \kappa$. By (8), $\mathsf{C}_{\mathrm{p}}(X)$ satisfies $\Omega_{\mathbf{o}} \to [\Omega_{\mathbf{o}}]^2_3$. By Theorem 13 this implies that $\mathsf{C}_{\mathrm{p}}(X)$ has countable strong fan tightness. By Sakai's theorem X has property $\mathsf{S}_1(\Omega, \Omega)$. We have shown that (8) implies that every set of real numbers of cardinality κ has property $\mathsf{S}_1(\Omega, \Omega)$. By Theorem 4.8 of [11], $\kappa < \mathsf{cov}(\mathcal{M})$. ■

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Received 10 January 1996;
in revised form 30 October 1996