3-1996

Combinatorics of Open Covers (I): Ramsey Theory

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Abstract: We study several schemas for generating from one sort of open cover of a topological space a second sort of open cover. Some of these schemas come from classical literature, others are borrowed from the theory of ultrafilters on the set of positive integers. We show that the fact that such a schema actually succeeds in producing a cover imposes strong combinatorial structure on the family of open covers of a certain sort. In particular, we show that certain analogues of Ramsey's theorem characterize some of these circumstances.
Keywords: open cover, large cover, ω–cover, γ–cover, Hurewicz, Menger, Ramsey, Rothberger, P–point, Q–point, infinite game.
AMS Subj. Class. 90D44, 04A99, 54

1 Introduction

Several important classes of topological spaces have been described by schemas of the sort where classes $\mathcal{A}$ and $\mathcal{B}$ of covers are given, as well as some procedure Π for generating from elements of $\mathcal{A}$, elements of $\mathcal{B}$. A space is said to have property $\Pi(\mathcal{A},\mathcal{B})$ if by means of the procedure Π it is possible to produce from covers in $\mathcal{A}$ a cover in $\mathcal{B}$. We study the combinatorial structure imposed on classes of open covers by the fact that a space belongs to a class described by such a schema. We emphasize the following classes of open covers:

1. $\mathcal{O}$: the collection of all open covers of X.

2. Λ: the collection of all *large* covers of X: A cover $\mathcal{C}$ is large if it is an open cover such that for each x in X the set $\{C \in \mathcal{C} : x \in C\}$ is infinite.

3. $\delta\Lambda$: the collection of all *densely large* covers of X: A cover $\mathcal{C}$ is densely large if it is an open cover of X such that there is a dense subset, say D, of X such that $\mathcal{C}$ is a large cover relative to D.

4. Ω: the collection of ω–covers of X, a notion introduced in [8]. A cover $\mathcal{C}$ of X is an ω–cover if it is an open cover of X such that no element of $\mathcal{C}$ contains X, and each finite subset of X is a subset of some element of $\mathcal{C}$.

5. Γ: the collection of γ–covers of X: An open cover $\mathcal{C}$ of X is a γ–cover if, for each x in X, the set $\{U \in \mathcal{C} : x \notin U\}$ is finite, and $\mathcal{C}$ is infinite.

We have the inclusions: $\Gamma \subseteq \Omega \subseteq \Lambda \subseteq \delta\Lambda \subseteq \mathcal{O}$.

The following a key observation about ω–covers is used often without explicit reference:

Lemma 1 *If $\mathcal{U}$ is an ω–cover of X then for every partition of $\mathcal{U}$ into finitely many classes, at least one of these classes is an ω–cover of X.*

This observation is also true of γ–covers:

Lemma 2 *If $\mathcal{U}$ is a γ–cover of X, so is every infinite subset of $\mathcal{U}$.*

In the vague description above of the schemas for generating open covers of a space, the binary operators (or procedures) Π of interest to us will all have the property of being *anti–monotonic* in the first variable, and *monotonic* in the second. This means that if $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{D}$ are nonempty subsets of $\mathcal{O}$ such that $\mathcal{A} \subset \mathcal{B}$, then we have: $\Pi(\mathcal{D},\mathcal{A}) \subseteq \Pi(\mathcal{D},\mathcal{B})$ and $\Pi(\mathcal{A},\mathcal{D}) \supseteq \Pi(\mathcal{B},\mathcal{D})$. It follows that for $\mathcal{A} \subseteq \mathcal{B} \subseteq \mathcal{O}$ one automatically has:

$$\begin{tikzcd}
\Pi(\mathcal{B},\mathcal{B}) \arrow[r] & \Pi(\mathcal{A},\mathcal{B}) \\
\Pi(\mathcal{B},\mathcal{A}) \arrow[r] \arrow[u] & \Pi(\mathcal{A},\mathcal{A}) \arrow[u]
\end{tikzcd}$$

In this diagram, as in the ones to follow, an arrow indicates that the class at the beginning point of the arrow is a subclass of the class at the terminal point. It will often turn out that the classes represented by the vertices of our diagrams are not provably equal; though this is important information which partially justifies the introduction of these classes and their study, this will not be a point of emphasis of this paper, but is postponed to [12]. Instead, we are going to study the combinatorial properties of the sets of open covers for members of our classes of spaces.

We confine ourselves to the comfortable setting of metric spaces where a variety of combinatorial tools are at our disposal. Worthy as such a pursuit may be, we didn't extract the exact topological circumstances under which our arguments will still go through.

One of the main tools in our study is the notion of an infinite game. The reader will also notice that we had a lot of guidance from two well established theories: the theory of ultrafilters on the set of positive integers, and Ramsey theory.

Quite a few new symbols are introduced throughout the paper. We have tried to keep these as suggestive as possible of the concepts they denote. Except for the new notions introduced, our notation is standard and can be gleaned from most of the more recent references in our bibliography.

Acknowledgement

I would like to thank the Department of Mathematics of Ohio University for the hospitality which I enjoyed during the Fall quarter of 1994, and also their Topology Seminar for the opportunity to present some of the material treated here. I am grateful to Winfried Just and Paul Szeptycki for their enthusiastic exploration of matters related to this paper. Conversations with them in the Yogurt Oasis greatly enhanced my understanding of the matters I discuss here.

I would also like to thank the Department of Mathematics of the University of Toronto and its Set Theory and Topology Seminar where some of this material was presented during the Spring of 1995. I must also thank Arnie Miller for very stimulating remarks which improved some of the exposition greatly, and also answered several of my questions.

An anonymous referee's report did a great deal to alleviate the reader's suffering and frustration when reading this paper. I am very grateful for this.

I received partial support for this research from NSF grant DMS 95-05375.

2 Operators.

Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty subsets of $\mathcal{O}$.

Selection from a sequence of covers.

Let Π be a procedure for building, from a sequence $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ of covers from $\mathcal{A}$, a cover from $\mathcal{B}$. We shall say that X belongs to the class $\Pi(\mathcal{A}, \mathcal{B})$ if: *for every sequence $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ of covers from $\mathcal{A}$, one can build, using procedure Π, a cover of X which is in $\mathcal{B}$.*

There are many different such processes in the literature; here we concentrate on the following three:

1. S_1: from a sequence $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ of elements of $\mathcal{A}$, select for each n a $U_n \in \mathcal{U}_n$, to obtain $\{U_n : n = 1, 2, 3, \ldots\}$, a member of $\mathcal{B}$.

2. S_{fin}: from a sequence $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ of elements of $\mathcal{A}$, select for each n a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$ to obtain a member $\cup_{n=1}^{\infty}\mathcal{V}_n$ of $\mathcal{B}$.

3. U_{fin}: from a sequence $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ of elements of $\mathcal{A}$, select for each n a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$ to obtain a member $\{\cup\mathcal{V}_n : n = 1, 2, 3, \ldots\}$ of $\mathcal{B}$.

One can check that for any collection $\mathcal{A}$ of open covers, and for $\mathcal{B}$ any one of $\mathcal{O}$, Λ, Ω or Γ the operators S_{fin} and S_1 are related as follows:

$$\mathsf{S}_1(\mathcal{A},\mathcal{B}) \subseteq \mathsf{S}_{fin}(\mathcal{A},\mathcal{B}) \tag{1}$$

Because of our peculiar requirements regarding γ–covers (that they be infinite), large covers (every point belongs to infinitely many distinct members of the cover) and ω–covers (that they don't contain an open set which covers the entire space), and since finite unions might destroy these properties, the relation of the operator $\mathsf{U}_{fin}(\cdot,\cdot)$ to the other two is a little more delicate. To allow for a smooth exposition we make the following three conventions:

Convention 1 *The symbol $\mathsf{U}_{fin}(\mathcal{A},\Gamma^*)$ denotes the class of spaces X with the following property: for every sequence $(\mathcal{U}_n : n = 1,2,3,\ldots)$ of $\mathcal{A}$–covers of X there is a sequence $(\mathcal{V}_n : n = 1,2,3,\ldots)$ such that:*

1. *for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and*
2. *either $\{\cup\mathcal{V}_n : n = 1,2,3,\ldots\}$ is a γ–cover of X, or else for all but finitely many n, $X = \cup\mathcal{V}_n$.*

Convention 2 *For Ω, the symbol $\mathsf{U}_{fin}(\mathcal{A},\Omega^{**})$ denotes the class of spaces X with the following property: for every sequence $(\mathcal{U}_n : n = 1,2,3,\ldots)$ of $\mathcal{A}$–covers of X there is a sequence $(\mathcal{V}_n : n = 1,2,3,\ldots)$ such that:*

1. *for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and*
2. *either $\{\cup\mathcal{V}_n : n = 1,2,3,\ldots\}$ is an ω–cover of X, or else for some n, $X = \cup\mathcal{V}_n$.*

Convention 3 *The symbol $\mathsf{U}_{fin}(\mathcal{A},\Lambda^{***})$ denotes the class of spaces X with the following property: for every sequence $(\mathcal{U}_n : n = 1,2,3,\ldots)$ of $\mathcal{A}$–covers of X there is a sequence $(\mathcal{V}_n : n = 1,2,3,\ldots)$ such that:*

1. *for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and*
2. *either $\{\cup\mathcal{V}_n : n = 1,2,3,\ldots\}$ is a large cover of X, or else for infinitely many n, $X = \cup\mathcal{V}_n$.*

From elementary considerations we obtain the following diagram:

$$
\begin{tikzcd}
\mathsf{U}_{fin}(\Gamma,\Gamma^*) \arrow[r] & \mathsf{U}_{fin}(\Gamma,\Omega^{**}) \arrow[r] & \mathsf{U}_{fin}(\Gamma,\Lambda^{***}) \arrow[r] & \mathsf{U}_{fin}(\Gamma,\mathcal{O}) \\
\mathsf{S}_{fin}(\Gamma,\Gamma) \arrow[r] \arrow[u] & \mathsf{S}_{fin}(\Gamma,\Omega) \arrow[r] \arrow[u] & \mathsf{S}_{fin}(\Gamma,\Lambda) \arrow[r] \arrow[u] & \mathsf{S}_{fin}(\Gamma,\mathcal{O}) \arrow[u] \\
\mathsf{S}_1(\Gamma,\Gamma) \arrow[r] \arrow[u] & \mathsf{S}_1(\Gamma,\Omega) \arrow[r] \arrow[u] & \mathsf{S}_1(\Gamma,\Lambda) \arrow[r] \arrow[u] & \mathsf{S}_1(\Gamma,\mathcal{O}) \arrow[u] \\
\mathsf{S}_1(\Omega,\Gamma) \arrow[r] \arrow[u] & \mathsf{S}_1(\Omega,\Omega) \arrow[r] \arrow[u] & \mathsf{S}_1(\Omega,\Lambda) \arrow[r] \arrow[u] & \mathsf{S}_1(\Omega,\mathcal{O}) \arrow[u] \\
 & & \mathsf{S}_1(\Lambda,\Lambda) \arrow[r] \arrow[u] & \mathsf{S}_1(\Lambda,\mathcal{O}) \arrow[u] \\
 & & & \mathsf{S}_1(\mathcal{O},\mathcal{O}) \arrow[u]
\end{tikzcd}
$$

Some of the vertices of this directed graph correspond to well–known classes:

- $\mathsf{S}_1(\Omega,\Gamma)$ denotes the γ–sets of Gerlits and Nagy – [8],
- $\mathsf{S}_1(\Omega,\mathcal{O})$ denotes the C''–sets of Rothberger – [18],
- $\mathsf{U}_{fin}(\Gamma,\Gamma^*)$ denotes the collection of sets having Hurewicz's property – [11].
- $\mathsf{U}_{fin}(\Gamma,\mathcal{O})$ denotes the collection of sets having Menger's property – [10].

One can show that even for the specific case of subsets of the real line no two of the nine classes of sets represented by the vertices of the 3×3 subdiagram made up from the first, second and fourth columns and the third, fourth and sixth (= top) rows in our diagram are provably equal (this is shown in [12]); no doubt one can in ZFC find general topological spaces which would witness that no two of these nine classes in general coincide, but we did not investigate this.

The position of the important class $\mathsf{S}_{fin}(\Omega,\Omega)$ is not indicated in the diagram. It lies between the classes $\mathsf{S}_1(\Omega,\Omega)$ and $\mathsf{S}_{fin}(\Gamma,\Omega)$. A deeper analysis of this class is taken up in [12].

2.1 Schemas based on binary relations.

Let $\mathbf{R}$ be a binary relation on $\mathcal{O}$. Let $\mathcal{A}$ and $\mathcal{B}$ be two nonempty subsets of $\mathcal{O}$. Then the symbol $\mathbf{R}(\mathcal{A},\mathcal{B})$ denotes:

for every element A of $\mathcal{A}$, there is an element B of $\mathcal{B}$ such that $(B, A) \in \mathbf{R}$.

For example, let Ref be the symbol for the "refinement" relation: An open cover $\mathcal{V}$ of X refines an open cover $\mathcal{U}$ of X if for every V in $\mathcal{V}$ there is a U in $\mathcal{U}$ such that $V \subseteq U$. We write $(\mathcal{V}, \mathcal{U}) \in \mathsf{Ref}$ for this. If we let $\mathsf{loc}\Phi$ denote the collection of locally finite open covers of the space X, then $\mathsf{Ref}(\mathcal{O}, \mathsf{loc}\Phi)$ abbreviates that (X, τ) is paracompact. Similarly, if we let $\mathsf{pt}\Phi$ denote the collection of point finite open covers of X, then $\mathsf{Ref}(\mathcal{O}, \mathsf{pt}\Phi)$ abbreviates that (X, τ) is metacompact.

Another symbol we require is: **Sub**, denoting the "subset" relation. The symbol $\mathbf{Sub}(\mathcal{A}, \mathcal{B})$ denotes: *for every element A of $\mathcal{A}$, there is an element B of $\mathcal{B}$ such that B is a subset of A.*

As an example, let K denote the set of countable open covers of X. Then $\mathsf{Sub}(\mathcal{O}, \mathsf{K})$ abbreviates that (X, τ) is Lindelöf. If Φ denotes the collection of finite open covers of a topological space, then $\mathbf{Sub}(\mathcal{O}, \Phi)$ denotes that (X, τ) is compact. Incidentally, note that in the notation just established, X has Rothberger's property C' if, and only if, X is a member of $\mathsf{S}_1(\Phi, \mathcal{O})$.

2.2 Schemas based on disjointification.

Distinct Representatives

Let κ be an infinite cardinal number. It is well known that if $\mathcal{S}$ is a family of cardinality at most κ consisting of sets, each of cardinality κ, then there is for each S in $\mathcal{S}$ a set $B_S \subseteq S$ of cardinality κ such that the family $(B_S : S \in \mathcal{S})$ is a pairwise disjoint family. This fact is of great use in infinitary combinatorics. We need an analogue of this for our topological situation. Define: *the collection $\mathcal{A}$ of open covers is countably distinctly representable by the collection $\mathcal{B}$ of open covers, relative to the binary relation* R, if: for every sequence $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ of elements of $\mathcal{A}$, there is a sequence $(\mathcal{V}_n : n = 1, 2, 3, \ldots)$ of elements of $\mathcal{B}$ such that

1. for each n, $(\mathcal{V}_n, \mathcal{U}_n) \in \mathsf{R}$, and
2. for distinct m and n, $\mathcal{V}_m \cap \mathcal{V}_n = \emptyset$.

This defines a binary relation on the collection of subsets of $\mathcal{O}$. The symbol $\mathbf{CDR}_{\mathsf{R}}$ denotes this binary relation, and $\mathbf{CDR}_{\mathsf{R}}(\mathcal{A}, \mathcal{B})$ abbreviates the assertion that $\mathcal{A}$ is countably distinctly representable by $\mathcal{B}$ relative to R. Since we shall often use such assertions as $\mathbf{CDR}_{\mathsf{Ref}}(\mathcal{O}, \mathsf{loc}\Phi)$ and $\mathbf{CDR}_{\mathsf{Ref}}(\mathcal{O}, \mathsf{pt}\Phi)$, we determine some circumstances under which these are applicable. Recall that a cover $\mathcal{U}$ of a set X is *irreducible* if for each U in $\mathcal{U}$, the set $\mathcal{U} \setminus \{U\}$ is not a cover for X. A standard argument using Zorn's Lemma shows that every point–finite cover of a set S by some of its subsets has an irreducible subcover.

Lemma 3 *Let (X,τ) be a T_2 space which has no isolated points. If the space satisfies* $\mathsf{Ref}(\mathcal{O},\mathsf{pt}\Phi)$ *(*i.e.*, the space is metacompact) then it also satisfies* $\mathsf{CDR}_{\mathsf{Ref}}(\mathcal{O},\mathsf{pt}\Phi)$.

Proof : Let $(\mathcal{U}_n : n = 1,2,3,\dots)$ be a sequence of open covers of X.

To begin, let $\mathcal{V}_1$ be an irreducible point finite open cover of X which refines $\mathcal{U}_1$. Let N be a positive integer and assume that $\mathcal{V}_1,\dots,\mathcal{V}_N$ have already been chosen such that:

1. each $\mathcal{V}_i$ is an irreducible point finite open cover of X which refines $\mathcal{U}_i$, and
2. for $i \neq j$, $\mathcal{V}_i \cap \mathcal{V}_j = \emptyset$.

Now let $\mathcal{W}$ be the set of those open subsets U of X which have the property that there are sets $A_1,\dots A_N$ and B where for each i A_i is an element of $\mathcal{V}_i$, and B is an element of $\mathcal{U}_{N+1}$, and U is a *proper* subset of $B \cap (\cap_{i=1}^N A_i)$.

For each i, since each element of $\mathcal{W}$ is a proper subset of some element of $\mathcal{V}_i$ and since $\mathcal{V}_i$ is an irreducible open cover of X, every refinement of $\mathcal{W}$ will be disjoint from $\mathcal{V}_i$ (and will be a refinement of $\mathcal{U}_{N+1}$). Now let $\mathcal{V}_{N+1}$ be an irreducible open point–finite refinement of $\mathcal{W}$.

Then $\mathcal{V}_1,\dots\mathcal{V}_{N+1}$ still satisfy the two requirements above, and the recursive selection procedure continues. $\Box$

This lemma holds in particular for the metric space setting, which we are favoring. Here is another useful fact:

Lemma 4 *Every infinite subset of a metric space is an element of the class* $\mathbf{CDR}_{\mathsf{Sub}}(\Gamma,\Gamma)$.

Several vertices in our diagram of classes arising from the selection operators are equal:

Corollary 5 $\mathsf{S}_{fin}(\Gamma,\Lambda) = \mathsf{U}_{fin}(\Gamma,\mathcal{O})$.

Proof : It is evident that the left side of the equation is contained in the right. Now let X be a space belonging to the right side. Let $(\mathcal{U}_n : n = 1,2,3,\dots)$ be a sequence of γ–covers of X. Apply Lemma 4, and find for each n a γ–cover $\mathcal{V}_n$ of X such that $\mathcal{V}_n$ is a subset of $\mathcal{U}_n$, and such that for $m \neq n$, $\mathcal{V}_m \cap \mathcal{V}_n$ is empty. Then, partition the set of positive integers into infinitely many pairwise disjoint infinite sets, $Y_1, Y_2, Y_3, \dots$. For each n, apply the fact that X is in the class $\mathsf{U}_{fin}(\Gamma,\mathcal{O})$ to the sequence $(\mathcal{V}_m : m \in Y_n)$ of γ–covers of X; let $(\mathcal{F}_m : m \in Y_n)$ be the corresponding sequence of finite sets as in the definition of $\mathsf{U}_{fin}(\Gamma,\mathcal{O})$. Then for each n, $\cup_{m\in Y_n}\mathcal{F}_m$ is a cover for X. Since for distinct values of n these covers are disjoint from each other, we see that the sequence $(\mathcal{F}_n : n = 1,2,3,\dots)$ witnesses for the original sequence of γ–covers that X belongs to $\mathsf{S}_{fin}(\Gamma,\Lambda)$. $\Box$

Using similar ideas, one proves

Corollary 6 $\mathsf{S}_1(\Gamma, \Lambda) = \mathsf{S}_1(\Gamma, \mathcal{O})$.

Also the classes $\mathsf{S}_{fin}(\Gamma, \Gamma)$ and $\mathsf{S}_1(\Gamma, \Gamma)$ are equal. This may at first glance seem obvious, but there are some subtleties involved in proving it. This is done in [12].

Splittability

The fact that an infinite set can always be partitioned into two disjoint sets, each having the same cardinality as the original set, is a consequence of the combinatorial fact stated at the introduction of the preceding paragraph.

We need a topological analogue of this – that "big" open covers for certain spaces can be partitioned into many different open covers of that space. Notice that if X belongs to $\mathbf{CDR}_{\mathsf{Sub}}(\mathcal{A}, \mathcal{B})$, then every cover of X from $\mathcal{A}$ contains countably many disjoint covers from $\mathcal{B}$. It might sometimes happen that such a strong fact is not true, but that a weaker fact, which we introduce here, is true and sufficient.

We shall say that $\mathcal{A}$ *is* $\mathcal{B}$*–splittable* if: for every element A of $\mathcal{A}$ there are two disjoint elements of $\mathcal{B}$, each a subset of A. The symbol $\mathbf{Split}(\mathcal{A}, \mathcal{B})$ denotes this.

$\mathsf{Split}(\Gamma, \Gamma)$ always holds. The operator $\mathsf{Split}(\cdot, \cdot)$ will appear from time to time in this paper, and is explored in more detail in [12].

Q–point–like schemas.

The next two schemas we describe are borrowed from the theory of ultrafilters on the set of positive integers. Recall that a free ultrafilter $\mathcal{U}$ on the set of positive integers is a Q–point if there is for each partition of the set of positive integers into disjoint finite subsets, a set in $\mathcal{U}$ which meets each of these finite sets in at most one point. Choquet calls these *rare* ultrafilters – [3], Def. 7.

The ultrafilter $\mathcal{U}$ is said to be a semi–Q–point if there is for every partition $(\mathcal{P}_n : n = 1, 2, 3, \ldots)$ of the set of positive integers into disjoint finite sets, an element F of $\mathcal{U}$ such that for each n, $\mathcal{U} \cap \mathcal{P}_n$ has at most n elements.

A set of positive integers is said to be 2*–uncrowded* if does not contain two consecutive integers.

We shall say that the space belongs to the class:

1. $\mathsf{Q}(\mathcal{A}, \mathcal{B})$ if: for every open cover $\mathcal{U}$ from $\mathcal{A}$, and for every partition of this cover into countably many disjoint nonempty finite sets $\mathcal{F}_1, \mathcal{F}_2, \mathcal{F}_3, \ldots$, there is a subset $\mathcal{V}$ of $\mathcal{U}$ which is a cover of X that belongs to $\mathcal{B}$, and which for each n has at most one element in common with $\mathcal{F}_n$.

2. $\mathsf{sQ}(\mathcal{A}, \mathcal{B})$ if: for every cover $\mathcal{U}$ from $\mathcal{A}$ and for every partition $(\mathcal{F}_n : n = 1, 2, 3, \ldots)$ of $\mathcal{U}$ into nonempty pairwise disjoint finite sets, there is a subset $\mathcal{V}$ of $\mathcal{F}$ such that $\mathcal{V}$ is a cover of X which belongs to $\mathcal{B}$, and for each n, $\mathcal{V} \cap \mathcal{F}_n$ has at most n elements.

3. $\mathsf{Uncr}(\mathcal{A},\mathcal{B})$ if : for every cover $\mathcal{F}$ from $\mathcal{A}$, and for every partition $(\mathcal{F}_n : n = 1, 2, 3, \ldots)$ of $\mathcal{F}$ into nonempty finite pairwise disjoint sets, there is a 2–uncrowded set Z of positive integers and for each n in Z there is a set F_n in $\mathcal{F}_n$ such that $\{F_n : n \in Z\}$ is in $\mathcal{B}$.

4. $\mathsf{wUncr}(\mathcal{A},\mathcal{B})$ if: for every cover $\mathcal{F}$ from $\mathcal{A}$, and for every partition $(\mathcal{F}_n : n = 1, 2, 3, \ldots)$ of $\mathcal{F}$ into nonempty finite pairwise disjoint sets, there is a 2–uncrowded set Z of positive integers such that $\cup_{n\in Z}\mathcal{F}_n$ is in $\mathcal{B}$.

There is a useful game–theoretic way in which the property $\mathsf{Q}(\Lambda,\Lambda)$ can be described. For $\mathcal{U}$ a large cover of X we define the following *accept–reject* game, $\mathsf{AR}(X,\mathcal{U})$: In the n–th inning ONE first selects a set U_n from $\mathcal{U} \setminus \{U_i : i < n\}$; then TWO responds by either accepting U_n, or by otherwise rejecting U_n. Acceptance is indicated by choosing $\epsilon_n = 1$, and rejection is indicated by choosing $\epsilon_n = 0$. Player ONE wins a play

$$U_1, \epsilon_1, \ldots, U_n, \epsilon_n, \ldots$$

if either the set $\{n : \epsilon_n = 1\}$ is finite, or else the set $\{U_n : \epsilon_n = 1\}$ is a large cover of X. Otherwise, TWO wins.

Theorem 7 *For a subset X of the real line the following statements are equivalent:*

1. *X belongs to $\mathsf{Uncr}(\Lambda,\Lambda)$.*
2. *For every large cover $\mathcal{U}$ of X, TWO does not have a winning strategy in the game $\mathsf{AR}(X,\mathcal{U})$.*
3. *X belongs to the class $\mathsf{Q}(\Lambda,\Lambda)$.*

Proof : $1 \Rightarrow 2$: Let X be a set which has the uncrowdedness property. Let $\mathcal{U}$ be an infinite large cover of X. Let $(U_n : n = 1, 2, 3, \ldots)$ be a bijective enumeration of $\mathcal{U}$. Let F be a strategy for TWO in the game $\mathsf{AR}(X)$.

We claim first that we may assume:

> *For every one–to–one finite sequence σ from $\mathcal{U}$ and for every k larger than any m for which U_m is listed in σ, there is a one–to–one sequence τ from $\mathcal{U}$ such that:*
>
> 1. *k is less than any m for which U_m is listed in τ, and*
> 2. *for each n larger than any m for which U_m is listed in τ, we have $F(\sigma \frown \tau \frown (U_n)) = 1$.*

(Otherwise, ONE defeats F by forcing TWO to accept only finitely many U_n.)

Using the italicized remark above, select positive integers

$$k_1 < k_2 < \ldots < k_n < \ldots$$

such that: For every one–to–one sequence σ of length $\leq k_n$ such that if U_m is listed in σ, then $m < k_n$, there is a one–to–one sequence τ such that if U_m is listed in τ, then $k_n \leq m < k_{n+1}$, and for all $j \geq k_{n+1}$,

$$F(\sigma \frown \tau \frown (U_j)) = 1.$$

Put $\mathcal{U}_1 = \{U_m : m < k_1\}$, and for each n put $\mathcal{U}_{n+1} = \{U_m : k_n \leq m < k_{n+1}\}$. This partitions $\mathcal{U}$ into finite non–empty sets. Since X has the uncrowdedness property, fix a 2–uncrowded infinite set Z of positive integers and for each n in Z a U_{i_n} in $\mathcal{U}_n$ such that $\{U_{i_n} : n \in Z\}$ is a large cover of X. Enumerate Z increasingly as $(n_1, n_2, \dots, n_k, \dots)$. By dropping the first few elements of Z if necessary, we may assume that $i_{n_1} > k_2$.

Choose σ_1 a one–to–one sequence from $\mathcal{U}$ of length k_1 such that each U_i listed in σ_1 has $i < k_1$. Then fix τ_1, a one–to–one sequence such that if U_i is listed in τ_1, then $k_1 \leq i < k_2$. Now i_{n_1} is larger than k_2, and so $F(\sigma_1 \frown \tau_1 \frown (U_{i_{n_1}})) = 1$.

Now $\sigma_2 = \sigma_1 \frown \tau_1 \frown (U_{i_{n_1}})$ is such that if U_i is listed in σ_1, then $i \leq i_{n_1} < k_{n_1+1}$. Choose a one–to–one τ_2 such that if U_i is listed in τ_2, then $k_{n_1+1} \leq i < k_{n_1+2}$, and for all $j \geq k_{n_1+2}$, $F(\sigma_2 \frown \tau_2 \frown (U_j)) = 1$. Since Z is 2–uncrowded, $i_{n_2} \geq k_{n_2+2}$, and so $F(\sigma_2 \frown \tau_2 \frown (U_{i_{n_2}})) = 1$. Put $\sigma_3 = \sigma_2 \frown \tau_2 \frown (U_{i_{n_2}})$, and repeat. We obtain

$$\sigma_1 \frown \tau_1 \frown (U_{i_{n_1}}) \frown \tau_2 \frown (U_{i_{n_2}}) \frown \tau_3 \frown \dots,$$

a play of the game, and TWO accepted each $U_{i_{n_j}}$.

$2 \Rightarrow 3$: Let $\mathcal{U}$ be a large cover for X and let the sequence $(\mathcal{U}_n : n = 1, 2, 3, \dots)$ be a partition of $\mathcal{U}$ into pairwise disjoint nonempty finite sets. Now assume that TWO does not have a winning strategy in the game $\mathsf{AR}(X, \mathcal{U})$.

Consider the strategy for TWO which calls on TWO to accept only the first element from each $\mathcal{U}_n$ that ONE presents, and to reject all other elements from the same $\mathcal{U}_n$. Since each $\mathcal{U}_n$ is a finite set and since ONE must present TWO with infinitely many distinct elements of $\mathcal{U}$ during a play of the game, it follows that during any play of the game TWO will accept infinitely many of the elements presented by ONE. However, since TWO does not have a winning strategy in the game, this means that there is a sequence of moves by ONE which ensures that the set of accepted elements is a large cover for X. Now this set of accepted elements contains no more than one element per $\mathcal{U}_n$. It follows that X has the property $\mathsf{Q}(\Lambda, \Lambda)$.

$3 \Rightarrow 1$: Consider a sequence $(\mathcal{U}_n : n = 1, 2, 3, \dots)$ of finite sets whose union is a large cover of X. For each n put $\mathcal{V}_n = \mathcal{U}_{3 \cdot n - 2} \cup \mathcal{U}_{3 \cdot n - 1} \cup \mathcal{U}_{3 \cdot n}$. Then $(\mathcal{V}_n : n = 1, 2, 3, \dots)$ constitutes a partition of a large cover of X into pairwise disjoint finite sets. Since X has property $\mathsf{Q}(\Lambda, \Lambda)$, select from each $\mathcal{V}_n$ a set such that the selected sets form a large cover of X. For each n let $U_{3 \cdot n - i_n}$ be the element selected from $\mathcal{V}_n$; the subscript is chosen to correspond to the subscript of the $\mathcal{U}_j$ to which the set selected from $\mathcal{V}_n$ belongs. Now it may happen that for some n we have $i_n = 0$ and $i_{n+1} = 2$, in which case the two selected elements are from adjacently indexed $\mathcal{U}_m$'s, so that the selector obtained here is not from a 2–uncrowded set of indices. This is fixed by applying property $\mathsf{Q}(\Lambda,\Lambda)$ once more to a sequence of $\mathcal{W}_n$'s, where each $\mathcal{W}_n$ consists of either a single $U_{3\cdot n-i_n}$ not adjacently indexed to any other $U_{3\cdot m-i_m}$ or else of two adjacently indexed $U_{3\cdot n-i_n}$'s. The result now follows from the observation that no selector from this sequence contains a pair of adjacently indexed elements. $\Box$

We also consider an accept–reject game for $\mathsf{Q}(\Omega,\Omega)$: An ω–covering $\mathcal{U}$ of X is given. In the n–th inning ONE first selects an element, denoted U_n, from $\mathcal{U}\setminus\{U_j : j<n\}$. TWO responds by selecting $\epsilon_n\in\{0,1\}$. ONE wins the play

$$U_1,\epsilon_1,\dots,U_n,\epsilon_n,\dots$$

if either $\{n:\epsilon_n=1\}$ is finite, or else $\{U_n:\epsilon_n=1\}$ is an ω–cover of X.

Let $\mathsf{AR}_\omega(X,\mathcal{U})$ denote this game. Using the methods above one shows:

Theorem 8 *For X a set of real numbers, the following are equivalent:*

1. *X has property $\mathsf{Q}(\Omega,\Omega)$.*
2. *X has property $\mathsf{Uncr}(\Omega,\Omega)$.*
3. *TWO does not have a winning strategy in the game $\mathsf{AR}_\omega(X,\mathcal{U})$.*

2.3 P–point–like schemas.

The next schema is based on the notion of a P–point ultrafilter on the set of positive integers. Recall that a free ultrafilter $\mathcal{U}$ on the set of positive integers is a P–point if there is for every sequence $U_1\supseteq U_2\supseteq\dots\supseteq U_n\supseteq\dots$ of elements of $\mathcal{U}$, an element U of $\mathcal{U}$ such that for every n, $U\setminus U_n$ is a finite set.

By analogy we shall say that X belongs to $\mathsf{P}(\mathcal{A},\mathcal{B})$ if: *for every sequence $(\mathcal{U}_n : n=1,2,3,\dots)$ of covers from $\mathcal{A}$ such that for each n $\mathcal{U}_{n+1}\subseteq\mathcal{U}_n$, there is a cover $\mathcal{V}$ in $\mathcal{B}$ such that $\mathcal{V}\subseteq\mathcal{U}_1$ and for each n $\mathcal{V}\setminus\mathcal{U}_n$ is finite.* As with the other operators we have introduced so far, P is anti–monotonic in the first variable, and monotonic in the second.

Theorem 9 *Let X be a set of real numbers. Then:*

1. *For any collection $\mathcal{A}$ of open covers of X, $\mathsf{S}_{fin}(\mathcal{A},\mathcal{A})$ implies $\mathsf{P}(\mathcal{A},\mathcal{A})$.*
2. *For $\mathcal{A}$ any of $\mathcal{O},\Lambda$ or Ω, $\mathsf{P}(\mathcal{A},\mathcal{A})$ implies $\mathsf{S}_{fin}(\mathcal{A},\mathcal{A})$.*

Proof : To prove 2, proceed as follows: Let $(\mathcal{G}_n : n=1,2,3,\dots)$ be a sequence of covers from class $\mathcal{A}$. We may assume that each $\mathcal{G}_n$ is countable. For each n put $\mathcal{U}_n=\cup_{m\geq n}\mathcal{G}_m$. Then the sequence $(\mathcal{U}_n : n=1,2,3,\dots)$ of open covers satisfies for each n $\mathcal{U}_{n+1}\subset\mathcal{U}_n$. Note also that each $\mathcal{U}_n$ is still a member of $\mathcal{A}$.

Now apply the property $\mathsf{P}(\mathcal{A},\mathcal{A})$: we find a countable open subcover $\mathcal{U}$ of $\mathcal{U}_1$ of X such that for each n, $\mathcal{V}\setminus\mathcal{U}_n$ is finite. Enumerate $\mathcal{V}$ bijectively as $(A_k : k = 1,2,3,\ldots)$.

For each k put $F_k = \{n : A_k \in \mathcal{G}_n\}$. If F_k is finite, let n_k be its maximum element; otherwise, let n_k be the least element of F_k which is larger than k. Then the sequence $(n_k : k = 1,2,3,\ldots)$ diverges to infinity. To see this, note that it suffices to show that if $\{k : F_k \text{ is finite}\}$ is an infinite set, then $(n_k : F_k \text{ is finite})$ diverges to infinity.

But look, if there were an m and infinitely many k such that $n_k = m$, then each of these infinitely many A_k would be an element of $\mathcal{G}_m$, and m would be the *largest* such index. This in turn implies that m is maximal such that each of these A_k is an element of $\mathcal{U}_m$. Then we have the contradiction that $\mathcal{V}\setminus\mathcal{U}_{m+1}$ is infinite.

Finally, for each n put $\mathcal{V}_n = \{A_k : n_k = n\}$. Then each $\mathcal{V}_n$ is a finite subset of the corresponding $\mathcal{G}_n$, and $\mathcal{W} = \cup_{n=1}^{\infty}\mathcal{V}_n$. $\square$

It is not true that $\mathsf{P}(\Gamma,\Gamma)$ and $\mathsf{S}_{fin}(\Gamma,\Gamma)$ are equal. The reason for this is that every space has property $\mathsf{P}(\Gamma,\Gamma)$, while only some spaces have the Menger property, and so only some spaces have property $\mathsf{S}_{fin}(\Gamma,\Gamma)$. To see that every space has property $\mathsf{P}(\Gamma,\Gamma)$, notice that from a descending chain of γ–covers we may select an element from each such that the selected elements are pairwise distinct. Since this set of selected elements is an infinite subset of the largest one of these γ–covers, it is also a γ–cover.

Simple considerations show that for Π any one of the operators Q, sQ, Uncr, wUncr or P, and for the open covers we are interested in, the relevant relations are represented in the following diagram:

$$\begin{tikzcd}
\Pi(\Omega,\Gamma) \arrow[r] & \Pi(\Omega,\Omega) \arrow[r] & \Pi(\Omega,\Lambda) \arrow[r] & \Pi(\Omega,\mathcal{O}) \\
 & & \Pi(\Lambda,\Lambda) \arrow[r] \arrow[u] & \Pi(\Lambda,\mathcal{O}) \arrow[u]
\end{tikzcd}$$

It is also clear that for nonempty families $\mathcal{A}$ and $\mathcal{B}$ of covers, one always has:

$$\begin{tikzcd}
\mathsf{sQ}(\mathcal{A},\mathcal{B}) \arrow[r] & \mathsf{Sub}(\mathcal{A},\mathcal{B}) \\
\mathsf{Q}(\mathcal{A},\mathcal{B}) \arrow[u] & \mathsf{P}(\mathcal{A},\mathcal{B}) \arrow[u]
\end{tikzcd}$$

3 Schemas motivated by Ramsey theory

Next we turn to Ramsey–theoretic ideas and we adapt some concepts from the theory of partition relations for cardinals ([4], Section 15) and partition relations for ultrafilters ([1], Section 2) to our present context. The main concept from Ramsey theory is the notion of an end–homogeneous set, and close variations of it. We adapt this for our special circumstances.

Let S be a countably infinite set and let $f : [S]^2 \to \{0, 1\}$ be a function. Then a subset R of S is said to be

1. *eventually end–homogeneous for f* if there is a finite–to–one function $\phi : R \to \{1, 2, 3, \ldots\}$ such that for all u, v and w from R, if $\phi(u) + 1 < \phi(v), \phi(w)$, then $f(\{u, v\}) = f(\{u, w\})$.

2. *end–homogeneous for f* if there is a one–to–one function $\phi : R \to \{1, 2, 3, \ldots\}$ such that for all u, v and w from R, if $\phi(u)+1 < \phi(v), \phi(w)$, then $f(\{u, v\}) = f(\{u, w\})$.

3. *eventually homogeneous for f* if there is a finite–to–one function $\phi : R \to \{1, 2, 3, \ldots\}$ and an $i \in \{0, 1\}$ such that for all u and v from R, if $\phi(u) \neq \phi(v)$, then $f(\{u, v\}) = i$.

4. *homogeneous for f* if there is an $i \in \{0, 1\}$ such that for all u and v from R, if $u \neq v$, then $f(\{u, v\}) = i$.

In our context, S and R are going to be open covers of X. It will also be necessary, in some cases, to put some restrictions on the function f. Let $\mathcal{U}$ be a cover for X which is a member of class $\mathcal{A}$. We shall say that a function

$$f : [\mathcal{U}]^2 \to \{0, 1\}$$

is an $\mathcal{A}$ *coloring* if: for each U in $\mathcal{U}$, and for every cover $\mathcal{V} \subseteq \mathcal{U}$ of X which is in $\mathcal{A}$, there is an i in $\{0, 1\}$ such that the set $\{V \in \mathcal{V} : f(\{U, V\}) = i\}$ is a cover in $\mathcal{A}$ of X. When $\mathcal{A}$ is Λ, then f is said to be a *large coloring*.

We shall say that a space X satisfies the partition relation

$$\mathcal{A} \to_\Psi \langle \mathcal{B} \rangle_2^2$$

if: Whenever $\mathcal{U}$ is a cover of X in $\mathcal{A}$, and f is a function in class Ψ from $[\mathcal{U}]^2$ to $\{0, 1\}$, then there is a subset $\mathcal{V}$ of $\mathcal{U}$ which is in $\mathcal{B}$ and which is eventually end–homogeneous with respect to f.

For a space X the symbol

$$\mathcal{A} \to_\Psi \langle (\mathcal{B}) \rangle_2^2$$

denotes that for every cover $\mathcal{U}$ of X in $\mathcal{A}$, and for every function $f : [\mathcal{U}]^2 \to \{0, 1\}$ in class Ψ there is a subset $\mathcal{V}$ of $\mathcal{U}$ which is in $\mathcal{B}$ and which is end–homogeneous for f.

We shall say that a space X satisfies the partition relation

$$\mathcal{A} \rightarrow_{\Psi} \lfloor \mathcal{B} \rfloor^2_2$$

if: Whenever $\mathcal{U}$ is a cover in $\mathcal{A}$ for X, and $f : [\mathcal{U}]^2 \rightarrow \{0,1\}$ is a function in class Ψ, then there is a subset $\mathcal{V}$ of $\mathcal{U}$ which is in the class $\mathcal{B}$, and which is eventually homogeneous for f.

The space X satisfies the partition relation

$$\mathcal{A} \rightarrow_{\Psi} (\mathcal{B})^2_2$$

if: For every cover $\mathcal{U}$ of X in the class $\mathcal{A}$, and for every function f in class Ψ from $[\mathcal{U}]^2$ to $\{0,1\}$, there is a subset $\mathcal{V}$ of $\mathcal{U}$ which is homogeneous for f.

In all the notation above, if there is no restriction on the coloring f, then we omit the subscript Ψ.

Next we discuss a fact about $\mathsf{S}_{fin}(\Omega,\Omega)$ which will be of use when we treat $\mathsf{S}_1(\Omega,\Omega)$.

Theorem 10 *If a space X satisfies $\mathsf{S}_{fin}(\Omega,\Omega)$, then it satisfies the partition relation $\Omega \rightarrow \lfloor \Omega \rfloor^2_2$.*

Proof : Let $(U_n : n = 1,2,3,\dots)$ bijectively enumerate an ω–cover $\mathcal{U}$ of the $\mathsf{S}_{fin}(\Omega,\Omega)$–set X. Let $f : [\mathcal{U}]^2 \rightarrow \{0,1\}$ be a given coloring. Then choose a sequence $((\mathcal{U}_n, i_n) : n = 1,2,3\dots)$ such that:

1. $\mathcal{U}_1$ is an ω–subcover of $\mathcal{U}$ such that for each $V \in \mathcal{U}_1$, $f(\{U_1, U\}) = i_1$.
2. $\mathcal{U}_{n+1}$ is an ω–subcover of $\mathcal{U}_n$ such that for each V in $\mathcal{U}_{n+1}$, $f(\{U_{n+1}, V\}) = i_{n+1}$.

Since X is a $\mathsf{S}_{fin}(\Omega,\Omega)$–set, select from each $\mathcal{U}_n$ a finite set $\mathcal{V}_n$ such that $\cup_{n=1}^{\infty}\mathcal{V}_n$ is an ω–cover of X. We may assume that there is a fixed $i \in \{0,1\}$ such that for each U_m in this latter union we have $i_m = i$. We may also assume that the $\mathcal{V}_n$'s are pairwise disjoint.

Define $k_1 < k_2 < \dots < k_n < \dots$ such that for each n, if U_i is in $\mathcal{V}_n$, then $i \leq k_n$. Then choose a sequence $\ell_1 < \ell_2 < \dots < \ell_n < \dots$ of positive integers such that:

1. for each $j \geq \ell_1$ $\mathcal{V}_j \subset \mathcal{U}_{k_1}$, and
2. for each $j \geq \ell_{m+1}$, $\mathcal{V}_j \subset \mathcal{U}_{k_{\ell_m}}$.

For each n put $\mathcal{P}_n = \cup_{\ell_n \leq j < \ell_{n+1}} \mathcal{V}_j$. Then the sequence $(\mathcal{P}_n : n = 1,2,3,\dots)$ is a partition of an ω–cover of X into pairwise disjoint nonempty finite sets. Then one of the sets $\cup_{n=1}^{\infty}\mathcal{P}_{2n}$ or $\cup_{n=1}^{\infty}\mathcal{P}_{2n-1}$ is an ω–cover of X. We may assume that the former is an ω–cover for X. For a set V in this cover, define $\Phi(V)$ to be n if, and only if, V is an element of $\mathcal{P}_{2n}$. Then Φ is finite–to–one and for all V and W from this cover for which $\Phi(V) \neq \Phi(W)$, we have $f(\{V, W\}) = i$. $\Box$

Problem 1 [1] *Is the converse of Theorem 10 true?*

4 The Menger property

In his 1924 paper [15], Karl Menger introduced a topological notion which is appropriately called the *Menger property* in [5]. A space X is said to have the Menger property if for every sequence $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ of open covers of X there is a sequence $(\mathcal{V}_n : n = 1, 2, 3, \ldots)$ such that for each n, $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and such that $\cup_{n=1}^{\infty}\mathcal{V}_n$ is a cover of X. Strictly speaking this is not Menger's original definition. This is a reformulation given in 1925 by Witold Hurewicz [10], who pointed out that for metric spaces (the original context of Menger's paper) this property is equivalent to the one introduced by Menger. In our notation the metric spaces having Menger's property are exactly the members of the class $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$.

If a topological space has the Menger property, then it has the well–known Lindelöf property: every open cover of the space contains a countable subset which still is a cover for the space. In our notation,

$$\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O}) \subseteq \mathbf{Sub}(\mathcal{O}, \mathsf{K}).$$

Using this and other elementary facts one sees that the following classes of metric spaces are equal:

$\mathsf{U}_{fin}(\Gamma, \mathcal{O})$, $\mathsf{U}_{fin}(\Omega, \mathcal{O})$, $\mathsf{U}_{fin}(\Lambda, \mathcal{O})$, $\mathsf{U}_{fin}(\mathcal{O}, \mathcal{O})$, $\mathsf{S}_{fin}(\Lambda, \mathcal{O})$, $\mathsf{S}_{fin}(\Omega, \mathcal{O})$, and $\mathsf{S}_{fin}(\Gamma, \mathcal{O})$.

Proposition 11 *For Lindelöf spaces,* $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O}) = \mathsf{U}_{fin}(\Gamma, \mathcal{O})$.

Proof : Let X be a set in $\mathsf{U}_{fin}(\Gamma, \mathcal{O})$. Then let $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ be a sequence of open covers of X. We may assume that X is not compact. We may also assume that each $\mathcal{U}_n$ is countably infinite and no finite subset covers X. For each n let $(U_k^n : k = 1, 2, 3, \ldots)$ enumerate $\mathcal{U}_n$ bijectively. Then let $\mathcal{W}_n$ be the collection whose m'th member is $\cup_{k=1}^{m} U_k^n$.

Then each $\mathcal{W}_n$ is a γ–cover of X. Now apply the fact that X is a member of $\mathsf{U}_{fin}(\Gamma, \mathcal{O})$ to choose from each $\mathcal{W}_n$ a finite set $\mathcal{S}_n$ such that $\cup_{n=1}^{\infty}\mathcal{S}_n$ is an open cover of X. Now disassembling the members of each $\mathcal{S}_n$, we find for each n a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$ such that $\cup_{n=1}^{\infty}\mathcal{V}_n$ is a cover of X. $\Box$

Hurewicz discovered a very useful description of spaces which have the Menger property. This description is given most economically in the language of game–theory. Let X be a topological space. Players ONE and TWO play the following infinitely long game: They play an inning for each positive integer.

[1] The answer is "yes", and is proven in [12].

In the n–th inning ONE chooses an open cover $\mathcal{U}_n$ of X; TWO responds by selecting a finite subset $\mathcal{V}_n$ of $\mathcal{U}_n$. TWO wins the play

$$\mathcal{U}_1, \mathcal{V}_1, \ldots, \mathcal{U}_n, \mathcal{V}_n, \ldots$$

of this game if $\cup_{n=1}^{\infty} \mathcal{V}_n$ is a cover of X. We shall call this game the *Menger game*; it is denoted $\mathsf{Menger}(X)$.

There is an easy observation about winning strategies for ONE in the Menger game which makes the treatment of matters concerning this game a little easier:

Lemma 12 *If ONE has a winning strategy in* $\mathsf{Menger}(X)$*, then ONE has a winning strategy F which has the property that for every finite sequence $(\mathcal{V}_1, \ldots, \mathcal{V}_n)$ of finite collections of open sets, every element of $F(\mathcal{V}_1, \ldots, \mathcal{V}_n)$ contains the union of the sets in $\mathcal{V}_1 \cup \ldots \cup \mathcal{V}_n$.*

We may further also assume that the covers played by ONE are always increasing chains of open sets; this is because TWO is allowed to pick finite subsets each inning, and not the individually chosen sets, but their unions determine the outcome of the game.

Theorem 13 *For a set X of real numbers the following are equivalent:*

1. *X has the Menger property.*
2. *ONE does not have a winning strategy in the game* $\mathsf{Menger}(X)$.
3. *X has property* $\mathsf{P}(\mathcal{O}, \mathcal{O})$.
4. *X has property* $\mathsf{P}(\Lambda, \delta\Lambda)$.
5. *X satisfies the partition relation $\Lambda \rightarrow_{\Lambda} \langle \delta\Lambda \rangle_2^2$.*

Proof : 1 $\Rightarrow$ 2: This is Theorem 10 of [10]. Since this theorem of Hurewicz is not as well-known as it deserves to be we give a fairly complete outline of Hurewicz's proof of it here.

Let F be a strategy for ONE. Then we may assume that $F(X)$, the first move of ONE according to the strategy F, is an ascending ω–chain of open sets covering X, say $F(X) = (U_{(n)} : n = 1, 2, 3, \ldots)$, listed in $\subset$–increasing order. Then, for each n, list $F(U_{(n)})$ in $\subset$–increasing order as $(U_{(n,m)} : m = 1, 2, 3, \ldots)$, and so on. Supposing that U_τ has been defined for all finite sequences τ of length at most k of positive integers, we now define for each $(n_1, \ldots, n_k)$:

$$F(U_{(n_1)}, \ldots, U_{(n_1,\ldots,n_k)}) = (U_{(n_1,\ldots,n_k,m)} : m = 1, 2, 3, \ldots).$$

Then the family

$$(U_\tau : \tau \text{ a finite sequence of positive integers})$$

has the following properties for each τ:

1. If m is less than n, then $U_{\tau\frown(m)}$ is a subset of $U_{\tau\frown(n)}$.

2. For each n, $U_\tau \subseteq U_{\tau\frown(n)}$.

3. $\{U_{\tau\frown(n)} : n \text{ a positive integer}\}$ is an open cover of X.

Now we define for each n and k:

$$U_k^n = \begin{cases} U_{(k)} & \text{if } n = 1, \\ (\cap\{U_{\tau\frown(k)} : \tau \in {}^{n-1}\omega\}) \cap U_k^{n-1} & \text{otherwise} \end{cases}$$

One then shows that for each n the set $\{U_k^n : k = 1, 2, 3, \ldots\}$, denoted $\mathcal{U}_n$, is an open cover of X. An important part of this argument is to first show (by induction) that for each $(i_1, \ldots, i_n)$ such that $\max\{i_1, \ldots, i_n\} \geq k$ one has $U_k^n \subseteq U_{(i_1,\ldots,i_n)}$. It then follows that each U_k^n is an intersection of *finitely many* open sets, and thus itself open. Next one shows (again by induction) that for each n the set $\{U_\tau : \text{length}(\tau) = n\}$ is a γ–cover of X.

Now observe that by its very definition each $\mathcal{U}_n$ is an increasing chain of open sets. Finally, before applying the fact that X has Menger's property, one verifies by induction that each $\mathcal{U}_n$ is indeed a cover of X.

Now apply the fact that X has the Menger property, $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$; we find a function f from the set of positive integers to the set of positive integers such that $\{U_{f(n)}^n : n = 1, 2, 3, \ldots\}$ is an open cover of X. But look, for each n we have

$$U_{f(n)}^n \subseteq U_{f\lceil_n\frown(f(n))}(= U_{f\lceil_{n+1}}).$$

Then the sequence $(U_{(f(1))}, \ldots, U_{(f(1),\ldots,f(n))}, \ldots)$ is a sequence of moves by TWO which defeats F.

$2 \Rightarrow 3$: Let $\mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \ldots$ be a descending sequence of open covers of X. We may assume that all the inclusions are proper. Now define a strategy F for ONE in $\mathsf{Menger}(X)$ as follows:

$F(X) = \mathcal{U}_1$ is ONE's first move. Now let $(\mathcal{V}_1, \ldots, \mathcal{V}_n)$ be a finite sequence of finite subsets of $\mathcal{U}_1$. Then let m be minimal such that $m > |\mathcal{V}_1 \cup \ldots \cup \mathcal{V}_n|$, and such that $\mathcal{U}_m \cap (\mathcal{V}_1 \cup \ldots \cup \mathcal{V}_n) = (\cap_{j=1}^\infty \mathcal{U}_j) \cap (\mathcal{V}_1 \cup \ldots \cup \mathcal{V}_n)$. Define: $F(\mathcal{V}_1, \ldots, \mathcal{V}_n) = \mathcal{U}_m$.

Then F is a legitimate strategy for ONE. But since ONE has no winning strategy in the game $\mathsf{Menger}(X)$, we see that there is a play against F which defeats it. Let

$$F(X), \mathcal{V}_1, F(\mathcal{V}_1), \ldots, \mathcal{V}_n, F(\mathcal{V}_1, \ldots, \mathcal{V}_n), \ldots$$

be such a play. From the definition of F we see that for each n there are only finitely many m such that $\mathcal{V}_m$ is not a subset of $\mathcal{U}_n$. If we let $\mathcal{U}$ be the set $\cup_{n=1}^\infty \mathcal{V}_n$, then for each n we have $\mathcal{U} \setminus \mathcal{U}_n$ is a finite set. Moreover, since TWO won this play of the game, $\mathcal{U}$ is also a cover of X.

$3 \Rightarrow 4$: Let $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ be a descending sequence of large covers of X.

Since X has the P–point property, we find a cover $\mathcal{V}$ of X such that for each n the set $\mathcal{V}\setminus\mathcal{U}_n$ is finite. If $\mathcal{V}$ is a densely large cover of X we are done. So, we may suppose that $\mathcal{V}$ is not a densely large cover of X.

Let D be a countable dense subset of X. Let E be the set of points of D at which $\mathcal{V}$ is not large – thus, for every $e\in E$ the set $\{V\in\mathcal{V}: e\in V\}$ is finite. Let $(e_n: n=1,2,3,\dots)$ enumerate the elements of E in such a way that each element is listed infinitely many times.

Let $(Y_n: n=1,2,3,\dots)$ be a partition of the set of positive integers into pairwise disjoint infinite subsets. For each m in Y_n let $\mathcal{S}_m(e_n)$ be the set of elements of $\mathcal{U}_m\setminus\mathcal{V}$ which contain e_n. Then each $\mathcal{S}_m(e_n)$ is an infinite set. Choose sets U_m from $\mathcal{S}_m(e_m)$ such that for each m we have $U_m\not\in\{U_i: i<m\}$. Then finally put $\mathcal{U}=\mathcal{V}\cup\{U_n: n=1,2,3,\dots\}$. We see that $\mathcal{U}$ is a cover of X which is large at the dense set D.

$4\Rightarrow 5$: Let X be a set having the P–point property for large covers. Let $\mathcal{U}$ be a large cover for X and let $f:[\mathcal{U}]^2\to\{0,1\}$ be a large coloring. We may assume that $\mathcal{U}$ is countably infinite. Let $(U_n: n=1,2,3,\dots)$ be a bijective enumeration of $\mathcal{U}$. Then choose a sequence $((\mathcal{U}_n, i_n): n=1,2,3,\dots)$ such that:

1. $\mathcal{U}_1$ is a large subcover of $\mathcal{U}$ such that for each $V\in\mathcal{U}_1$, $f(\{U_1,V\})=i_1$.
2. $\mathcal{U}_{n+1}$ is a large subcover of $\mathcal{U}_n$ such that for each $V\in\mathcal{U}_{n+1}$ we have $f(\{U_{n+1},V\})=i_{n+1}$.

Since X has the P–point property for large covers we select a densely large open cover $\mathcal{V}$ of X such that for each n we have: $\mathcal{V}\setminus\mathcal{U}_n$ is a finite set. For each n we define $\mathcal{V}_n=\mathcal{V}\setminus(\mathcal{U}_n\cup\{U_i: i<n\})$.

Define a sequence $k_1<k_2<\dots<k_n<\dots$ of positive integers such that for each n, if U_i is an element of $\mathcal{V}_n$, then $i\le k_n$. Then choose a sequence $\ell_1<\ell_2<\dots<\ell_n<\dots$ of positive integers such that:

1. for each $j\ge\ell_1$, $\mathcal{V}_j\subset\mathcal{U}_{k_1}$, and
2. for each $j\ge\ell_{m+1}$, $\mathcal{V}_j\subset\mathcal{U}_{k_{\ell_m}}$.

For each n put $\mathcal{P}_n=\cup_{\ell_n\le j<\ell_{n+1}}\mathcal{V}_j$. Then the sequence $(\mathcal{P}_n: n=1,2,3,\dots)$ partitions the densely large cover $\mathcal{W}=\cup_{n=1}^{\infty}\mathcal{V}_n$ of X into pairwise disjoint nonempty finite sets. Define the function $\Phi:\mathcal{W}\to\omega$ such that $\Phi(U)=n$ whenever $U\in\mathcal{P}_n$. Then Φ is finite–to–one and if W and Z are elements of $\mathcal{W}$ such that $\Phi(U)+1<\Phi(W),\Phi(Z)$, then $f(\{U,W\})=f(\{U,Z\})$.

$5\Rightarrow 1$: Let $(\mathcal{U}_n: n=1,2,3,\dots)$ be a sequence of open covers of X. By appropriately refining each we may assume that each is a locally finite cover of X and that for all distinct m and n we have $\mathcal{U}_m\cap\mathcal{U}_n=\emptyset$. Then the cover $\mathcal{U}$ which is the union of the $\mathcal{U}_n$'s, is a large cover of X. Moreover, the function $f:[\mathcal{U}]^2\to\{0,1\}$ which is defined so that $f(\{U,V\})=1$ if, and only if, U and V do not belong to the same one of the sets $\mathcal{U}_n$. Then f is a large coloring.

Let $\mathcal{V} \subset \mathcal{U}$ be a densely large cover of X which is eventually end–homogeneous for f. Let $\Psi : \mathcal{V} \rightarrow \omega$ be a finite–to–one function such that for all U, V and W from $\mathcal{V}$ such that $\Psi(U)+1 < \Psi(V), \Psi(W)$ implies that $f(\{U, V\}) = f(\{U, W\})$. Thus, for each U, for all but finitely many V and W we have $f(\{U, V\}) = f(\{U, W\})$. But a densely large set is infinite, and cannot have all its members in a locally finite family. This implies that for each U in $\mathcal{V}$, all but finitely many of the elements of $\mathcal{V}$ are in $\mathcal{U}_n$'s different from the one to which U belongs. This in turn implies that for each n, $\mathcal{V} \cap \mathcal{U}_n$ is finite. For each n put $\mathcal{V}_n = \mathcal{V} \cap \mathcal{U}_n$. Then the sequence $(\mathcal{V}_n : n = 1, 2, 3, \ldots)$ witnesses the Menger property of X for the sequence $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ of covers of X. $\square$

It is shown in [12] that the Menger property and the property $\mathsf{Split}(\Lambda, \Lambda)$ are incomparable.

5 A schema motivated by the notion of a semiselective ultrafilter.

According to K. Kunen [13], p. 387, a free ultrafilter on the set of positive integers is *semiselective* if for every sequence $(U_n : n = 1, 2, 3, \ldots)$ of elements of the ultrafilter, there is a sequence $(x_n : n = 1, 2, 3, \ldots)$ of positive integers such that for each n x_n is an element of U_n, and $\{x_n : n = 1, 2, 3, \ldots\}$ is an element of the ultrafilter. It is well–known that an ultrafilter on the positive integer is semiselective if, and only if, it is both a P–point ultrafilter and a semi–Q–point ultrafilter. The mere definition of the classes $\mathsf{S}_1(\mathcal{A}, \mathcal{A})$ shows strong analogies with that of a semiselective ultrafilter.

Closely related to the notion of a semiselective ultrafilter is the notion of a *selective* ultrafilter: given a sequence $(X_n : n < \omega)$ of pairwise disjoint nonempty subsets of ω such that none is in the ultrafilter but the union is in the ultrafilter, there exists a sequence $(x_n : n < \omega)$ of elements of ω such that $\{x_n : n < \omega\}$ is in the ultrafilter, and such that for each n, $x_n \in X_n$ (see for example Definition 1.7 of Grigorieff's paper [9], or the definition near the bottom of p. 386 of Kunen's paper [13]).

It is known that every selective ultrafilter is also semiselective, while the converse is not provable: Using the Continuum Hypothesis one can construct semiselective ultrafilters which are not selective. While there is this subtle difference of the two concepts in the context of free ultrafilters on the set of positive integers, the analogous concepts in our context coincide; we give a proof here for one of the classes of sets we consider, while for a second one this will have to wait for [12].

Rothberger's property C".

In his 1938 paper [18] Fritz Rothberger introduced the notion of a C"–set: a subset X of a space has property C" if for every sequence $(\mathcal{U}_n : n < \omega)$ of open covers of X, there is a sequence $(U_n : n < \omega)$ such that for each $n < \omega$, $U_n \in \mathcal{U}_n$, and such that $\{U_n : n < \omega\}$ is a cover for X. In our notation this says that X is a member of the class $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$. There is a clear analogy between the definitions of C" and the notion of a semiselective ultrafilter. We shall find characterizations for C"–sets which are analogous to known characterizations for selective ultrafilters, thus showing that in this context "selective" is ' 'semiselective". The following infinite game is an important tool in our study:

Two players named ONE and TWO play a game of length ω as follows: In the n–th inning ONE chooses a countable open cover $\mathcal{U}_n$ of of our metric space X, and TWO responds by choosing $U_n \in \mathcal{U}_n$. TWO wins a play

$$(\mathcal{U}_1, U_1, \ldots, \mathcal{U}_n, U_n, \ldots)$$

of the game if $\{U_n : n = 1, 2, 3, \ldots\}$ is a cover for X; otherwise ONE wins.

We shall call this game the *Rothberger game* because of its obvious connection with the Rothberger property C"; the symbol $\mathsf{Rothberger}(X)$ denotes this game on X. The game was introduced by F. Galvin in his paper [6]. In [17], Pawlikowski proved:

Theorem 14 (Pawlikowski) *For a subset X of a metric space, the following are equivalent:*

1. *X has Rothberger's property C".*
2. *ONE does not have a winning strategy in $\mathsf{Rothberger}(X)$.*

Indeed, one can show that if ONE is required to choose a large cover of X each inning, and if TWO wins only if the collection of sets chosen by TWO is a large cover of X, then the analogue of Pawlikowski's theorem holds. We shall use this fact below.

The following few facts about large covers and the Rothberger property will be useful in what follows:

Theorem 15 *Let X be a C"–set. If $\mathcal{U}_1$ and $\mathcal{U}_2$ are large covers of X, then there are large covers $\mathcal{V}_1$ and $\mathcal{V}_2$ of X such that $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$, $\mathcal{V}_1 \subset \mathcal{U}_1$ and $\mathcal{V}_2 \subset \mathcal{U}_2$.*

Proof : Let $\mathcal{U}_1$ and $\mathcal{U}_2$ be given large covers of X. Define covers $\mathcal{W}_1, \mathcal{W}_2, \ldots, \mathcal{W}_n, \ldots$ as follows for each k:

$$\mathcal{W}_{2k-1} = \{A_1 \cap \ldots \cap A_{2k-1} : A_1, \ldots, A_{2k-1} \in \mathcal{U}_1 \text{ and } |\{A_1, \ldots, A_{2k-1}\}| = 2k-1\},$$

and

$$\mathcal{W}_{2k} = \{A_1 \cap \ldots \cap A_{2k} : A_1, \ldots, A_{2k} \in \mathcal{U}_2 \text{ and } |\{A_1, \ldots, A_{2k}\}| = 2k\}.$$

Then, as X is a C" set, we choose for each k a W_k from $\mathcal{W}_k$ such that there is for each x infinitely many k such that x is in W_{2k} and there are infinitely many ℓ such that x is in $W_{2\ell-1}$.

For each k, write $W_k = A_1^k \cap \ldots \cap A_k^k$, where the sets $A_1^k, \ldots, A_k^k$ are pairwise distinct. Then choose sets $S_1, \ldots, S_k, \ldots$ and $T_1, \ldots, T_k, \ldots$ so that

1. $S_1 = A_1^1$,
2. $T_1 \in \{A_1^2, A_2^2\} \setminus \{S_1\}$,
3. $S_k \in \{A_1^{2k-1}, \ldots, A_{2k-1}^{2k-1}\} \setminus \{S_1, \ldots, S_{k-1}, T_1, \ldots, T_{k-1}\}$, and
4. $T_k \in \{A_1^{2k}, \ldots, A_{2k}^{2k}\} \setminus \{S_1, \ldots, S_k, T_1, \ldots, T_{k-1}\}$, for each k.

Then put $\mathcal{V}_1 = \{S_n : n = 1, 2, 3, \ldots\}$ and put $\mathcal{V}_2 = \{T_n : n = 1, 2, 3, \ldots\}$. We see that $\mathcal{V}_1 \cap \mathcal{V}_2 = \emptyset$ and $\mathcal{V}_1 \subset \mathcal{U}_1$, and $\mathcal{V}_2 \subset \mathcal{U}_2$ are large covers of X. $\Box$

In particular if X is a C"–set, then every large cover $\mathcal{U}$ of X can be partitioned into two disjoint large covers of X; thus every large cover of a C"–set can be partitioned into countably many disjoint large covers.

The technique of the proof of Theorem 15 can be used to show that for every sequence $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ of large covers of X, there is a sequence $(U_n : n = 1, 2, 3, \ldots)$ such that for each n, $U_n \in \mathcal{U}_n$, and such that $\{U_n : n = 1, 2, 3, \ldots\}$ is a large cover of X. This can also be deduced from the following theorem.

Theorem 16 *Let X be a C"–set. Then X belongs to the class* $\mathbf{CDR_{Sub}}(\Lambda, \Lambda)$

Proof : Let X be a set having property C" and let $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ be a sequence of large covers of X. Now define an $\omega \times \omega$ matrix $(\mathcal{V}_m^n : m, n = 1, 2, 3 \ldots)$ of large covers of X such that

1. for each n, $(\mathcal{U}_n, \mathcal{V}_1^n, \mathcal{V}_2^n, \ldots)$ is a descending sequence of large covers of X and
2. for each n, and for distinct $i, j \leq n$, $\mathcal{V}_n^i \cap \mathcal{V}_n^j = \emptyset$.

For this we use Theorem 15 repeatedly. Then, use the fact that X is a C"–set to choose for each n a sequence $(V_m^n : m = 1, 2, 3, \ldots)$ such that for each m V_m^n is an element of $\mathcal{V}_m^n$, and such that $\mathcal{V}_n^* = \{V_m^n : m = 1, 2, 3, \ldots\}$ is a large cover of X. Then the sequence $(\mathcal{V}_n^* : n = 1, 2, 3, \ldots)$ has the property that for n, m distinct, $|\mathcal{V}_n^* \cap \mathcal{V}_m^*| \leq \max\{m, n\}$. Put $\mathcal{V}_n = \mathcal{V}_n^* \setminus (\mathcal{V}_1^* \cup \ldots \cup \mathcal{V}_{n-1}^*)$. $\Box$

Theorem 17 *For a set X of real numbers, the following are equivalent:*

1. *X belongs to $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$.*
2. *X belongs to $\mathsf{S}_1(\Lambda, \Lambda)$.*
3. *X belongs to $\mathsf{S}_1(\Omega, \mathcal{O})$.*

Proof : First we see that 1 implies 2. Let $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ be a sequence of large covers of X. By Theorem 16 we select first from each $\mathcal{U}_n$ a large cover $\mathcal{V}_n \subset \mathcal{U}_n$ such that for $m \neq n$, the sets $\mathcal{V}_m$ and $\mathcal{V}_n$ are disjoint. Then let

$(Y_n : n = 1, 2, 3, \ldots)$ be a partition of the set of positive integers into infinitely many infinite pairwise disjoint subsets. Apply the fact that X is in $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ to each of the sequences $(\mathcal{V}_n : n \in Y_m)$ of open covers of X. For each m we find a sequence $(U_n : n \in Y_m)$ such that for each n in Y_m U_n is an element of $\mathcal{V}_n$, and $\{U_n : n \in Y_m\}$ is a cover for X. But then $(U_n : n = 1, 2, 3, \ldots)$ constitutes a large cover of X.
It is clear that 2 implies 3.
$3 \Rightarrow 1$: Let $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ be a sequence of covers of X. Let $(Y_n : n = 1, 2, 3, \ldots)$ be a pairwise disjoint sequence of infinite sets of positive integers whose union is the set of positive integers. For each n define $\mathcal{W}_n$ to be the set whose elements are of the form

$$U_{n_1} \cup \ldots \cup U_{n_k}$$

where k is any positive integer and $n_1 < n_2 < \ldots < n_k$ are elements of Y_n, and each U_{n_j} is an element of $\mathcal{U}_{n_j}$.

Then each $\mathcal{W}_n$ is an ω–cover of X. Now apply the fact that X is in $\mathsf{S}_1(\Omega, \mathcal{O})$, and choose for each n a W_n from $\mathcal{W}_n$ such that the set $\{W_n : n = 1, 2, 3, \ldots\}$ is a cover for X. For each n choose a sequence $i_1^n < \ldots < i_{k_n}^n$ and $U_{i_j^n}$ from $\mathcal{U}_{i_j^n}$ such that $W_n = U_{i_1^n} \cup \ldots \cup U_{i_{k_n}^n}$.

Then the sequence $(U_{i_1^1}, \ldots, U_{i_{k_1}^1}, U_{i_1^2}, \ldots, U_{i_{k_2}^2}, \ldots)$ already covers X and can be augmented to a sequence which contains one set from each $\mathcal{U}_n$. $\square$

Thus, the following classes are equal: $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$, $\mathsf{S}_1(\Lambda, \mathcal{O})$, $\mathsf{S}_1(\Lambda, \Lambda)$, $\mathsf{S}_1(\Omega, \Lambda)$, and $\mathsf{S}_1(\Omega, \mathcal{O})$.

Let $(U_n : n = 1, 2, 3, \ldots)$ be a bijective enumeration for a large cover $\mathcal{U}$ of X. A binary relation R on the set of positive integers is said to be *$\mathcal{U}$–compatible for this enumeration* if for each n the set $\{U_m : (n, m) \in R\}$ is a large cover of X.

Theorem 18 *For a set X of real numbers the following are equivalent:*

1. *X has Rothberger's property C".*
2. *For every bijective enumeration $(U_n : n = 1, 2, 3, \ldots)$ of a large cover $\mathcal{U}$ of X and for every binary relation R which is $\mathcal{U}$–compatible for this enumeration, there exists a sequence*

   $$k_1 < k_2 < \ldots < k_n < \ldots$$

   of positive integers such that for each n we have $(k_n, k_{n+1}) \in R$, and such that $\{U_{k_n} : n = 1, 2, 3, \ldots\}$ is a large cover for X.
3. *For every sequence $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ of large covers of X and for every bijective enumeration $(U_n : n = 1, 2, 3, \ldots)$ of $\cup_{n=1}^{\infty}\mathcal{U}_n$, there is an increasing function g from the set of positive integers to the set of positive*

integers such that for each n, $U_{g(n+1)} \in \mathcal{U}_{g(n)}$, and the set $\{U_{g(n)} : n = 1, 2, 3, \ldots\}$ is a large cover for X.

Proof : $1 \Rightarrow 2$: We use Pawlikowski's theorem. Let $\mathcal{U}$ be a large cover of X and let $(U_n : n = 1, 2, 3, \ldots)$ be a bijective enumeration of X. Let R be a binary relation which is $\mathcal{U}$–compatible for this enumeration. Consider the following strategy of ONE in the Rothberger game:

In the first inning ONE chooses $F(X) = \mathcal{U}$. Let U_n be selected by TWO. Then we put $F(U_n) = \{U_m : m > n \text{ and } (n, m) \in R\}$. Now let $U_{i_1}, \ldots, U_{i_n}$ be a sequence of elements of $\mathcal{U}$. Then ONE's move is

$$F(U_{i_1}, \ldots, U_{i_n}) = \{U_m : m > \max\{i_1, \ldots, i_n\} \text{ and } (i_n, m) \in R\}.$$

By 1 and Pawlikowski's theorem, this strategy is not a winning strategy for ONE. Consider and F–play which is lost by ONE, say

$$\mathcal{U}_1, U_{k_1}, \mathcal{U}_2, U_{k_2}, \ldots, \mathcal{U}_n, U_{k_n}, \ldots$$

Then by the definition of F we have 2.
$2 \Rightarrow 3$: Let $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ be a sequence of large covers of X and let $(U_n : n = 1, 2, 3, \ldots)$ be a bijective enumeration of $\mathcal{U} = \cup_{n=1}^{\infty} \mathcal{U}_n$. For each n and m, define $(n, m) \in R$ if $n < m$ and $U_m \in \mathcal{U}_n$. Then R is $\mathcal{U}$ – compatible for this enumeration of $\mathcal{U}$. Apply 2 to find an increasing sequence $k_1 < k_2 < \ldots < k_n < \ldots$ of positive integers such that for each n we have $(k_n, k_{n+1}) \in R$, and such that $\{U_{k_n} : n = 1, 2, 3, \ldots\}$ is a large cover of X. From the definition of R we see that for each n, $U_{k_{n+1}} \in \mathcal{U}_{k_n}$. Now for each n we let $g(n)$ be k_n.
$3 \Rightarrow 1$: This implication is easy. $\square$

Theorem 19 *Let X be a set of real numbers. Then the following are equivalent:*

1. *X has Rothberger's property C".*
2. *X satisfies $\Lambda \rightarrow_{\Lambda} \langle(\Lambda)\rangle_2^2$*
3. *X satisfies $\Lambda \rightarrow_{\Lambda} (\Lambda, \text{not point–finite})_2^2$*
4. *For every large cover $\mathcal{U}$ of X and for every partition $(\mathcal{V}_n : n = 1, 2, 3, \ldots)$ of $\mathcal{U}$ where each $\mathcal{V}_n$ is nonempty and point–finite, there is a sequence $(V_n : n = 1, 2, 3, \ldots)$ of elements of $\mathcal{U}$ such that for each n V_n is in $\mathcal{V}_n$, and such that $\{V_n : n = 1, 2, 3, \ldots\}$ is a large cover for X.*
5. *For every large cover $\mathcal{U}$ of X and for every function $f : \mathcal{U} \rightarrow \omega$, either there is a large cover $\mathcal{V} \subseteq \mathcal{U}$ on which f is one–to–one, or else there is a family $\mathcal{W} \subseteq \mathcal{U}$ which is not point–finite with respect to X, on which f is constant.*

Proof : $1 \Rightarrow 2$: We use Pawlikowski's theorem. Let $\mathcal{U}$ be a large cover of X and let $\Phi : [\mathcal{U}]^2 \to \{0,1\}$ be a large coloring. Let $(U_n : n = 1,2,3,\dots)$ be a bijective enumeration of $\mathcal{U}$.

Define a strategy F for ONE in the Rothberger game as follows:

$F(X) = \mathcal{U}$. For each n, define $F(U_n)$ as follows: Choose an i_n such that $\{V : \Phi(\{U_n, V\}) = i_n\}$ is a large cover for X and put $F(U_n) = \{V \in \mathcal{U} : \Phi(\{U_n, V\}) = i_n\}$. Let $(n_1,\dots,n_k)$ be given and suppose that $F(U_{n_1},\dots,U_{n_k})$ and $(i_{n_1},\dots,i_{n_1,\dots,n_k})$ have already been defined and that $F(U_{n_1},\dots,U_{n_k})$ is a large cover of X such that for every V in this large cover and for $1 \le j \le k$ we have $\Phi(\{U_{n_j}, V\}) = i_{n_1,\dots,n_j}$. Then, for every m choose $i_{n_1,\dots,n_k,m}$ such that the set $\{U_s \in F(U_{n_1},\dots,U_{n_k}) : s > m \text{ and } \Phi(\{U_m, U_s\}) = i_{n_1,\dots,n_k,m}\}$ is a large cover of X, and let $F(U_{n_1},\dots,U_{n_k},U_m)$ be this set.

Then F is a strategy for ONE in the Rothberger game. Since we are assuming that X has property C", this is not a winning strategy for ONE. Consider a play

$$F(X), U_{n_1}, F(U_{n_1}), U_{n_2}, F(U_{n_1}, U_{n_2}), \dots$$

which is lost by ONE. Then $\mathcal{V} = \{U_{n_m} : m = 1,2,3,\dots\}$ is a large cover of X. By the definition of F we also see that $n_1 < n_2 < \dots < n_k < \dots$, and that for each k, for all $m > k$, we have

$$\Phi(\{U_{n_k}, U_{n_m}\}) = i_{n_1,\dots,n_k}.$$

Define $\phi : \mathcal{V} \to \{1,2,3,\dots\}$ such that $\phi(V) = k$ if, and only if, $V = U_{n_k}$. We see that $\mathcal{V}$ is end–homogeneous with respect to Φ.

$2 \Rightarrow 3$: Let $\mathcal{U}$ be a large cover of X and let $\Phi : [\mathcal{U}]^2 \to \{0,1\}$ be a large coloring of $\mathcal{U}$. Choose a large subcover $\mathcal{V}$ of $\mathcal{U}$ which is end–homogeneous with respect to Φ.

Let $\phi : \mathcal{V} \to \omega$ be a one–to–one function and for each V in $\mathcal{V}$, let $i_V \in \{0,1\}$ be such that for each W with $\phi(V) < \phi(W)$ we have $\Phi(\{V, W\}) = i_V$. Look at the set of V in $\mathcal{V}$ for which i_V is 1. If this set is not point–finite, then we have the second option. If this set is point–finite, then its complement is a large cover of X and is a 0–homogeneous set.

$3 \Rightarrow 4$: Let $\mathcal{U}$ be a large cover of X and let $(\mathcal{V}_n : n = 1,2,3,\dots)$ be a partition of $\mathcal{U}$ into nonempty point–finite families. Define a coloring $\Phi : [\mathcal{U}]^2 \to \{0,1\}$ as follows:

$$\Phi(\{U, V\}) = \begin{cases} 0 & \text{if for each } n, \ |\{V, W\} \cap \mathcal{V}_n| \le 1 \\ 1 & \text{otherwise} \end{cases}$$

Then by 3 either there is a 1–homogeneous set which is not point–finite, or else there is a large cover $\mathcal{V} \subset \mathcal{U}$ which is 0 –homogeneous. But since each $\mathcal{V}_n$ is point–finite, the definition of Φ implies that there is a 1–homogeneous large cover of X. Now such a cover has at most one element from each $\mathcal{V}_n$.

$4 \Rightarrow 5$: Let $\mathcal{U}$ be a large cover of X and let $f : \mathcal{U} \to \omega$ be a function. For each n in the range of f, put $\mathcal{V}_n = \{V \in \mathcal{U} : f(V) = n\}$. If there is an n for which $\mathcal{V}_n$ is not point–finite with respect to X, then we have found a set on which f is

constant, and which is not point–finite with respect to X. Otherwise, each $\mathcal{V}_n$ is a point–finite family and we apply 4 to find for each n a V_n in $\mathcal{V}_n$ such that $\{V_n : n = 1, 2, 3, \ldots\}$ is a large cover for X. But then this is a large cover of X on which f is one–to–one.
$5 \Rightarrow 1$: Let $(\mathcal{U}_n : n = 1, 2, 3 \ldots)$ be a sequence of open covers of X. For each n let $\mathcal{W}_n$ be a locally finite open refinement of $\mathcal{U}_n$ such that $\mathcal{W}_m \cap \mathcal{W}_n = \emptyset$ whenever $m \neq n$.

Define $f : \mathcal{U} \rightarrow \omega$ so that $f(U) = n$ if, and only if, U is a member of $\mathcal{W}_n$. Since each $\mathcal{W}_n$ is locally finite, we see that there is a large cover of X on which f is one–to–one. But then this cover has at most one point in common with each $\mathcal{W}_n$. We find a sequence $(U_n : n = 1, 2, 3, \ldots)$ such that for each n, U_n is an element of $\mathcal{U}_n$, and $\{U_n : n = 1, 2, 3, \ldots\}$ is a cover of X. $\square$

Corollary 20 *If X has property $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$ then it has property $\mathsf{Q}(\Lambda, \Lambda)$.*

Proof : This follows from 4 of Theorem 19. $\square$

Consequently, $\mathsf{S}_1(\mathcal{O}, \mathcal{O}) = \mathsf{P}(\Lambda, \Lambda) \cap \mathsf{Q}(\Lambda, \Lambda)$.

$\mathsf{S}_1(\Omega, \Omega)$–sets.

Lemma 21 *If X has property $\mathsf{S}_1(\Omega, \Omega)$, then X has property $\mathsf{CDR}_{\mathsf{Sub}}(\Omega, \Omega)$.*

Proof : Let $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ be a sequence of ω–covers of X. For each n let $\mathcal{V}_n$ consist of sets of the form

$$\bigcap_{i=1}^{n}\left(\bigcap_{j=1}^{1+\ldots+n+(n+1)} U_j^i\right)$$

where for each n:

1. $U_j^i \neq U_{j'}^{i'}$ whenever $(i, j) \neq (i', j')$, and
2. for each i and j, U_j^i is in $\mathcal{U}_i$.

Then each $\mathcal{V}_n$ is an ω–cover of X. Since X has property $\mathsf{S}_1(\Omega, \Omega)$, we find a sequence $V_1, V_2, \ldots, V_n, \ldots$ such that for each n, V_n is in $\mathcal{V}_n$, and such that $\{V_n : n = 1, 2, 3, \ldots\}$ is an ω–cover of X.

Now we write for each n

$$V_n = \cap_{i=1}^{n}(\cap_{j=1}^{1+2+\ldots+n+(n+1)} U_j^i).$$

For each n and for each $i \leq n$ choose j_n^i in $\{1, 2, \ldots, 1 + 2 + \ldots + n + (n + 1)\}$ such that:

1. if i and t are distinct then $U^i_{j^i_n}$ and $U^t_{j^t_n}$ are distinct, and

2. if t and n are distinct then $U^m_{j^m_t}$ and $U^m_{j^m_n}$ are distinct,

This is done inductively, starting with $j^1_1 = 1$, and noting that when we are about to select j^i_{n+1}, we have already selected $1+2+\ldots+n+(i-1)$ sets $U^t_{j^t_r}$, $t \leq n$ and $t \leq r \leq n$, and sets $U^t_{j^t_{n+1}}$ for $t < i$, and we now have $1+2+\ldots+n+(n+1)$ sets, $\{U^i_j : 1 \leq j \leq 1+\ldots+n+(n+1)\}$ from which to select $U^i_{j^i_{n+1}}$. At least $(n+1)-i$ of these are distinct from every set selected so far. Let j^i_{n+1} be a subscript for one of *these* sets.

Finally we put $\mathcal{W}_n = \{U^n_{j^n_k} : k = n+1, n+2, n+3, \ldots\}$. Since for each n and for each $i \leq n$ we have $V_n \subseteq U^i_{j^i_n}$ we see that each $\mathcal{W}_n$ is an ω–cover of X, is a subset of $\mathcal{U}_n$, and is disjoint from $\mathcal{W}_k$ whenever k and n are distinct. $\Box$

Corollary 22 *Every set in* $\mathsf{S}_1(\Omega,\Omega)$ *is also in* $\mathsf{Split}(\Omega,\Omega)$.

Theorem 23 *For subsets of the real line,* $\mathsf{S}_1(\Omega,\Omega) = \mathsf{P}(\Omega,\Omega) \cap \mathsf{sQ}(\Omega,\Omega)$.

Proof : First, we show that the collection on the left of the equation is contained in the collection on the right. It is clear from Theorem **??** and the definitions that $\mathsf{S}_1(\Omega,\Omega) \subseteq \mathsf{P}(\Omega,\Omega)$. Thus, let $\mathcal{U}$ be an ω– cover for X and let $(\mathcal{P}_n : n = 1,2,3,\ldots)$ be a partition of this cover into pairwise disjoint nonempty finite sets.

Using Lemma 21 we first find a partition $(\mathcal{U}_n : n = 1,2,3,\ldots)$ of $\mathcal{U}$ into pairwise disjoint ω–covers of X. Observe that each $\mathcal{P}_k$ has nonempty intersection with at most finitely many of the sets $\mathcal{U}_n$. Thus, choose positive integers $n_1 < n_2 < \ldots < n_k < \ldots$ such that for each k we have $\mathcal{U}_{n_k} \cap (\mathcal{P}_1 \cup \ldots \cup \mathcal{P}_k) = \emptyset$. Then the sequence $(\mathcal{U}_{n_k} : k = 1,2,3,\ldots)$ is a sequence of ω–covers of X. Since X is a member of $\mathsf{S}_1(\Omega,\Omega)$, we select from each $\mathcal{U}_{n_k}$ an element V_k such that the set $\mathcal{V} = \{V_k : k = 1,2,3,\ldots\}$ is an ω–cover of X. Notice that for each k we have $|\mathcal{V} \cap \mathcal{P}_k| \leq k$. Thus, $\mathcal{V}$ is the desired ω–cover of X. This shows that also $\mathsf{S}_1(\Omega,\Omega) \subseteq \mathsf{sQ}(\Omega,\Omega)$.

Next we show that the collection on the right of the equality sign of the theorem is also contained in the collection on the left. This will complete the proof. Thus, let $(\mathcal{U}_n : n = 1,2,3,\ldots)$ be a sequence of ω–covers of X.

Let $(I_n : n = 1,2,3,\ldots)$ be a partition of the set of positive integers into disjoint finite sets such that for each n I_n has n elements. For each m let $\mathcal{W}_m$ consist of all nonempty sets of the form $U_{i_1} \cap \ldots \cap U_{i_m}$ where for each j, i_j is an element of I_j and U_{i_j} is an element of $\mathcal{U}_{i_j}$. Now as X is a member of $\mathsf{P}(\Omega,\Omega)$, fix for each n a finite set $\mathcal{V}_n \subset \mathcal{W}_n$ such that $\cup_{n=1}^{\infty}\mathcal{V}_n$ is an ω–cover of X. We may assume that the $\mathcal{V}_n$'s are pairwise disjoint, for if they were not we could replace them with sets $\mathcal{Z}_n$ where $\mathcal{Z}_1 = \mathcal{V}_1$ and for each $n > 1$, $\mathcal{Z}_n = \mathcal{V}_n \setminus (\mathcal{V}_1 \cup \ldots \cup \mathcal{V}_{n-1})$.

Thus, the sequence $(\mathcal{V}_n : n = 1,2,3,\ldots)$ is a partition of an ω–cover of X into pairwise disjoint finite sets. Since by hypothesis X also belongs to the class

$\mathsf{sQ}(\Omega,\Omega)$, we select for each n a set $\mathcal{P}_n \subseteq \mathcal{V}_n$ such that $\mathcal{P}_n$ has at most n points, and $\cup_{n=1}^{\infty}\mathcal{P}_n$ is an ω–cover for X.

Now for each n we may assume that $\mathcal{P}_n$ has n elements, say $\{S_1^n,\dots,S_n^n\}$ (when a $\mathcal{P}_n$ has fewer elements, what we are going to do will be even easier to do). Fix n and for each i write $S_i^n = U^n_{p_1^i}\cap\dots\cap U^n_{p_n^i}$ where each p_j^i is an element of I_n and $U^n_{p_j^i}$ is an element of $\mathcal{U}_{p_j^i}$. Then put $U_{p_i^i} = U^n_{p_i^i}$, an element of $\mathcal{U}_{p_i^i}$. It contains S_i.

The sequence $(U_{p_i^i} : i \in I_n, n = 1,2,3,\dots)$ is an ω–cover of X, and it can be augmented to a sequence which would still be an ω–cover for X, and which witnesses membership of X to the collection $\mathsf{S}_1(\Omega,\Omega)$ for the given sequence of ω–covers of X. $\Box$

By our preceding theorem,

$$\mathsf{P}(\Omega,\Omega)\cap\mathsf{Q}(\Omega,\Omega) = \mathsf{Q}(\Omega,\Omega)\cap\mathsf{S}_1(\Omega,\Omega).$$

In the following theorem we use this observation to characterize the property of belonging to $\mathsf{Q}(\Omega,\Omega)\cap\mathsf{S}_1(\Omega,\Omega)$ by a Ramseyan theorem. In [12] we shall show that $\mathsf{S}_1(\Omega,\Omega)\subseteq\mathsf{Q}(\Omega,\Omega)$; this means that the property $\mathsf{S}_1(\Omega,\Omega)$ is really the one characterized by the Ramseyan property below.

Theorem 24 *For a set X of real numbers the following are equivalent:*

1. *for every positive integer k, $\Omega \rightarrow (\Omega)^2_k$.*
2. *X is in $\mathsf{Q}(\Omega,\Omega)\cap\mathsf{P}(\Omega,\Omega)$.*

Proof : $1 \Rightarrow 2$: Let $\mathcal{U}$ be an ω–cover for X and let $(\mathcal{U}_n : n = 1,2,3,\dots)$ be a partition of it into nonempty finite sets. Define a coloring $\Phi_1 : [\mathcal{U}]^2 \rightarrow \{0,1\}$ such that for $\{A,B\}$ in $[\mathcal{U}]^2$ we have

$$\Phi_1(\{A,B\}) = \begin{cases} 0 & \text{if there is an } n \text{ such that } A,B\in\mathcal{U}_n \\ 1 & \text{otherwise} \end{cases}$$

Apply 1 to find an i and an ω–subcover $\mathcal{V}$ of $\mathcal{U}$ which is i–homogeneous for Φ_1. Since each $\mathcal{U}_n$ is finite, i cannot be 0. Thus i is 1 and we see that for each n $\mathcal{V}$ and $\mathcal{U}_n$ have at most one common element. This establishes that X is in $\mathsf{Q}(\Omega,\Omega)$.

Next we show that if X satisfies the partition relation, then it is in the class $\mathsf{P}(\Omega,\Omega)$. Thus, consider a descending chain $\mathcal{U}_1 \supset \mathcal{U}_2 \supset \dots \supset \mathcal{U}_n \supset \dots$ of ω–covers of X. We seek an ω–cover $\mathcal{V}$ of X such that for each n the set $\mathcal{V}\setminus\mathcal{U}_n$ is finite.

For each n put $\mathcal{R}_n = \mathcal{V}_n\setminus\mathcal{V}_{n+1}$. There are two cases to consider. The first case is that only finitely many of the $\mathcal{R}_n$'s are ω–covers of X. Then we let n_0 be the largest n for which this is the case, and we define a coloring

$$f : [\mathcal{U}_{n_0+1}]^2 \rightarrow \{0,1\}$$

so that for $\{U,V\}$ in $[\mathcal{U}_{n_0+1}]^2$, we put $f(\{U,V\})=0$ if U and V are in the same $\mathcal{R}_n$ for an $n\geq n_0+1$; else we set $f(\{U,V\})=1$. By the partition property we find an ω–cover $\mathcal{V}$ of X such that $\mathcal{V}$ is homogeneous of color 1 for f, and for every $n\geq n_0+1$, $\mathcal{V}\setminus\mathcal{U}_n$ is finite.

The second case is when for infinitely many n, $\mathcal{R}_n$ is an ω–cover of X. In this case we shall show that we can find an ω–cover of X which is in the union of these $\mathcal{R}_n$'s, and which meets each in a finite set. Thus, we may as well assume that the original sequence of ω–covers consists of covers which are disjoint from each other. Enumerate each of these bijectively such that $\mathcal{U}_n$ is listed as $(U^n_k:k=1,2,3,\ldots)$.

Define an ω–cover $\mathcal{U}$ so that its elements are of the form $U^n_k\cap U^m_\ell$ where $n+k<m$. Then define a coloring

$$f:[\mathcal{U}]^2\rightarrow\{0,1\}$$

so that for $\{U,V\}\in[\mathcal{U}]^2$ we have $f(\{U,V\})=0$ if U and V can be represented respectively as $U^{n_1}_{k_1}\cap U^{m_1}_{\ell_1}$ and $U^{n_2}_{k_2}\cap U^{m_2}_{\ell_2}$, where $n_1+k_1=n_2+k_2$.

Otherwise, we set $f(\{U,V\})=1$. By the partition hypothesis we find an ω–cover $\mathcal{V}\subset\mathcal{U}$ such that $\mathcal{V}$ is homogeneous for the coloring f. We claim that $\mathcal{V}$ is homogeneous of color 1. To see that it cannot be of color 0, consider an element of $\mathcal{V}$, say $U=U^n_k\cap U^m_\ell$. If $\mathcal{V}$ were homogeneous of color 0, then for every $V\in\mathcal{V}$, V has a representation of the form $U^{n_1}_{k_1}\cap U^{m_1}_{\ell_1}$, where $n_1+k_1=n+k$. But then these sets refine the collection $\{U^i_j:i+j=n+k\}$, a finite collection which is not an ω–cover of X, whence $\mathcal{V}$ is not an ω–cover of X.

So, let $\mathcal{V}$ be homogeneous of color 1. Enumerate $\mathcal{V}$ bijectively as $(V_n:n=1,2,3,\ldots)$; for each r, choose a representation of V_r, say

$$V_r=U^{n_r}_{k_r}\cap U^{m_r}_{\ell_r}.$$

Since $\mathcal{V}$ is homogeneous of color 1, we have that $n_r+k_r\neq n_s+k_s$ whenever $r\neq s$. This implies that the sequence $(m_r:r=1,2,3,\ldots)$ diverges to infinity. But then the ω-cover $\{U^{m_r}_{\ell_r}:r=1,2,3,\ldots\}$ meets each $\mathcal{U}_n$ in a finite set (recall that the $\mathcal{U}_n$'s are disjoint from each other!).

$2\Rightarrow 1$: Now assume that X is in $\mathsf{P}(\Omega,\Omega)\cap\mathsf{Q}(\Omega,\Omega)$, that $\mathcal{U}$ is an ω cover for X and that $\Phi:[\mathcal{U}]^2\rightarrow\{0,1\}$ is a coloring. Enumerate $\mathcal{U}$ bijectively as $(U_n:n=1,2,3,\ldots)$.

Choose i_1 such that $\mathcal{U}_1=\{U\in\mathcal{U}:\Phi(\{U_1,U\})=i_1\}$ is an ω–cover of X. Then, at stage $n+1$ choose i_{n+1} such that $\mathcal{U}_{n+1}=\{U\in\mathcal{U}_n:\Phi(\{U_{n+1},U\})=i_{n+1}\}$ is an ω cover for X. Applying the fact that X is in $\mathsf{S}_1(\Omega,\Omega)$ to the sequence $(\mathcal{U}_n:n=1,2,3,\ldots)$ of ω–covers of X we find for each n a $U_{k_n}\in\mathcal{U}_n$ such that the set $\{U_{k_n}:n=1,2,3,\ldots\}$ is an ω cover of X. Partitioning this set into two according to whether i_{k_n} is zero or one, we see that we obtain an ω–subcover for which the i_{k_n} all have the same value.

We may assume that for all n, $i_{k_n}=1$. Now we choose positive integers $m_1<m_2<\ldots<m_\ell<\ldots$ such that: For each i, for all $r\leq m_i$ and for all $j\geq m_{i+1}$, $U_{k_j}\in\mathcal{U}_{k_r}$.

For each i put $\mathcal{V}_i = \{U_{n_k} : m_i \leq k < m_{i+1}\}$. Then the sequence $(\mathcal{V}_i : i = 1, 2, 3, \ldots)$ is a partition of an ω–cover of X into nonempty finite subsets. By property $\mathsf{Q}(\Omega, \Omega)$ we find for each i a V_i in $\mathcal{V}_i$ such that $\{V_i : i = 1, 2, 3, \ldots\}$ is an ω cover of X. But each of the sets $\{V_{2i} : i = 1, 2, 3, \ldots\}$ and $\{V_{2i+1} : i = 1, 2, 3, \ldots\}$ is homogeneous of color 1 for Φ, and at least one of them is an ω–cover of X. $\square$

.

Theorem 25 *For a set X of real numbers, the following are equivalent:*

1. $\Omega \rightarrow (\Omega)^2_2$.
2. *for each k,* $\Omega \rightarrow (\Omega)^2_k$.
3. *For each k and each n,* $\Omega \rightarrow (\Omega)^n_k$.
4. $\Omega \rightarrow (\Omega, 4)^3$.

Proof : The implication $3 \Rightarrow 4$ is clearly true. The proof of the implication $4 \Rightarrow 1$ is like that of Theorem 2.1 of [1]. We must show that $1 \Rightarrow 2$, and $2 \Rightarrow 3$.
$2 \Rightarrow 3$: For the sake of simplicity we work this out for $n = 3$; the proof for each n is similar, using induction. By 2 and by Theorem 24 we know that X is in $\mathsf{Q}(\Omega, \Omega) \cap \mathsf{S}_1(\Omega, \Omega)$.

Thus, let k be a positive integer and let $\mathcal{U}$ be an ω–cover of X. Let $f : [\mathcal{U}]^3 \rightarrow \{0, 1, \ldots, k\}$ be a given coloring. Let $(U_n : n = 1, 2, 3, \ldots)$ enumerate $\mathcal{U}$ bijectively.

Inductively define a sequence $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ of ω–covers of X and a sequence $(i_n : n = 1, 2, 3, \ldots)$ of elements of $\{0, 1, \ldots, k\}$ as follows: Let $\Phi_1 : [\mathcal{U} \setminus \{U_1\}]^2 \rightarrow \{0, 1, \ldots, k\}$ be the coloring defined by $\Phi_1(\mathcal{V}) = f(\{U_1\} \cup \mathcal{V})$. Apply 1 to find an i_1 and an ω–cover $\mathcal{U}_1$ which is i_1–homogeneous for Φ_1.

Assume that $\mathcal{U}_1 \supseteq \mathcal{U}_2 \supseteq \ldots \supseteq \mathcal{U}_n$, ω–covers of X, as well as $i_1, \ldots, i_n$ have been selected such that for each $j \leq n$ and for each $\mathcal{V} \in [\mathcal{U}_j]^2$ we have $f(\{U_j\} \cup \mathcal{V}) = i_j$.

Define $\Phi_{n+1} : [\mathcal{U}_n \setminus \{U_{n+1}\}]^2 \rightarrow \{0, 1, \ldots, k\}$ so that for each $\mathcal{V}$ in $[\mathcal{U}_n]^2$ we have $\Phi_{n+1}(\mathcal{V}) = f(\{U_{n+1}\} \cup \mathcal{V})$. Then apply 1 to find an i_{n+1} and an ω–cover $\mathcal{U}_{n+1} \subseteq \mathcal{U}_n$ which is i_{n+1}–homogeneous for Φ_{n+1}.

Since X is in $\mathsf{S}_1(\Omega, \Omega)$, there is a sequence $(V_n : n = 1, 2, 3, \ldots)$ such that for each n V_n is an element of $\mathcal{U}_n$, and such that $\{V_n : n = 1, 2, 3, \ldots\}$ is an ω–cover of X. For each k we may choose n_k such that $V_k = U_{n_k}$. Since one of the classes in a partition of an ω–cover into finitely many classes is again and ω–cover, we may assume that there is a fixed i such that for all k we have $i_{n_k} = i$.

Choose $1 < k_1 < k_2 < \ldots < k_m < \ldots$ such that for all $j \geq k_1$ we have $U_{n_j} \in \mathcal{U}_{n_1}$, and for all ℓ, for all $j \geq k_{\ell+1}$ and for all $i \leq k_\ell$, we have $U_{n_j} \in \mathcal{U}_{n_i}$. Put $\mathcal{P}_1 = \{U_{n_i} : i \leq k_1\}$ and $\mathcal{P}_{n+1} = \{U_{n_i} : k_n \leq i < k_{n+1}\}$. Then the sequence $(\mathcal{P}_n : n = 1, 2, 3, \ldots)$ is a partition of an ω–cover of X into pairwise disjoint nonempty finite sets. Since X has the 2–uncrowdedness property for ω–covers,

we find a 2–uncrowded set Z of positive integers and for each k in Z a W_k in $\mathcal{P}_k$ such that the set $\{W_k : k = 1, 2, 3, \ldots\}$ is an ω–cover of X. But then this ω–cover is homogeneous of color i for f.
The proof of $1 \Rightarrow 2$ is standard. $\Box$

$\mathsf{S}_1(\Gamma, \Gamma)$.

As could be gleaned from the earlier sections, the game–theoretic tool is powerful in analysing the classes of sets we are studying here. Here is a natural game associated with the class $\mathsf{S}_1(\Gamma, \Gamma)$: Players ONE and TWO play an inning per positive integer. In the n'th inning, ONE chooses a γ–cover $\mathcal{U}_n$ of X, and TWO responds by selecting a set $U_n \in \mathcal{U}_n$. TWO wins a play

$$\mathcal{U}_1, U_1, \ldots, \mathcal{U}_n, U_n, \ldots$$

if $\{U_n : n = 1, 2, 3, \ldots\}$ is a γ–cover of X; otherwise, ONE wins. The symbol $\Gamma_1(X)$ denotes this game.

Theorem 26 *For a set X of real numbers, the following are equivalent:*

1. *X belongs to the class $\mathsf{S}_1(\Gamma, \Gamma)$.*
2. *ONE has no winning strategy in the game $\Gamma_1(X)$.*

Proof : The proof of the implication $2 \Rightarrow 1$ uses a standard argument. We show that $1 \Rightarrow 2$. Thus, let F be a strategy for ONE. Then use F to define open subsets U_τ, $\tau \in {}^{<\omega}\omega$, of X as follows.

The first move of ONE, $F(X)$, is enumerated bijectively as $(U_{(n)} : n < \omega)$. Then for each n, $F(U_{(n)}) \setminus \{U_{(n)}\}$ is enumerated bijectively as $(U_{(n,m)} : m < \omega)$. Assume that for each τ of length at most k we have already defined U_τ. Then we define $(U_{(n_1,\ldots,n_k,m)} : m = 1, 2, 3, \ldots)$ to be

$$F(U_{(n_1)}, \ldots, U_{(n_1,\ldots,n_k)}) \setminus \{U_{(n_1)}, \ldots, U_{(n_1,\ldots,n_k)}\}$$

where the enumeration $(U_{(n_1,\ldots,n_k,m)} : m = 1, 2, 3, \ldots)$ is bijective.

By the rules of the game we see that for every finite sequence σ the set $\{U_{\sigma\frown(m)} : m = 1, 2, 3, \ldots\}$ is a γ–cover of X. Applying the fact that X is in $\mathsf{S}_1(\Gamma, \Gamma)$, we find for each σ an n_σ such that the set $\{U_{\sigma\frown(n_\sigma)} : \sigma \text{ a finite sequence}\}$ is a γ cover of X.

Recursively define $n_1, n_2, \ldots$ so that $n_1 = n_\emptyset$ and $n_{k+1} = n_{(n_1,\ldots,n_k)}$. Then the sequence

$$U_{(n_1)}, U_{(n_1,n_2)}, \ldots, U_{(n_1,\ldots,n_k)}, \ldots$$

is the sequence of moves of TWO during a play of the game in which ONE used F. If this sequence has infinitely many distinct terms, it constitutes a γ–cover of X, thus defeating F. The fact that this sequence of moves by TWO indeed has infinitely many distinct terms follows from the way we have modified the moves of ONE before letting TWO respond. $\Box$
.

5.1 The Hurewicz property.

According to Hurewicz [11] X has the *Hurewicz property* if for every sequence $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ of open covers of X there is a sequence $(\mathcal{V}_n : n = 1, 2, 3, \ldots)$ such that for each n $\mathcal{V}_n$ is a finite subset of $\mathcal{U}_n$, and such that

$$X \subseteq \bigcup_{n=1}^{\infty} \bigcap_{m \geq n} (\cup \mathcal{V}_m).$$

A set X has the property $\mathsf{U}_{fin}(\mathcal{O}, \Gamma^*)$ if, and only if, it has the Hurewicz property.

The following game is naturally associated with this covering property: In the n–th inning ONE selects a large cover $\mathcal{U}_n$ of X; TWO responds by selecting a finite set $\mathcal{V}_n \subset \mathcal{U}_n$. TWO wins the play

$$\mathcal{U}_1, \mathcal{V}_1, \ldots, \mathcal{U}_n, \mathcal{V}_n, \ldots$$

if for each x for all but finitely many n, $x \in \cup \mathcal{V}_n$; otherwise, ONE wins. This game is denoted $\mathsf{Hurewicz}(X)$.

Theorem 27 *For a set X of real numbers the following are equivalent:*

1. *X has the Hurewicz property.*
2. *ONE does not have a winning strategy in the game* $\mathsf{Hurewicz}(X)$.

Proof : A standard argument shows that if ONE does not have a winning strategy in the game $\mathsf{Hurewicz}(X)$, then X has the Hurewicz property. We prove the other implication. Let F be a strategy for ONE. ONE's first move according to strategy F is denoted $F(X)$. Define large covers of X has follows:

1. $F(X) = (U_{(n)} : n = 1, 2, 3, \ldots)$, and
2. Assuming that U_σ has been defined for each finite sequence σ of positive integers of length at most m, we define:

$$(U_{(n_1,\ldots,n_m,k)} : k = 1, 2, 3, \ldots) = F(\{U_{(n)} : n \leq n_1\}, \ldots, \{U_{(n_1,\ldots,n_{m-1},k)} : k \leq n_m\}).$$

Since X has the Hurewicz property, we find for every finite sequence σ of positive integers a positive integer n_σ such that, setting $\mathcal{V}_\sigma = \{U_{\sigma \frown (n)} : n \leq n_\sigma\}$, each element of X belongs to all but finitely many of the sets $\cup \mathcal{V}_\sigma$.

Now recusively select a sequence $n_1, n_2, \ldots, n_k, \ldots$ of positive integers such that

$$n_{k+1} = \begin{cases} n_\emptyset & \text{if } k = 0, \text{ and} \\ n_{(n_1,\ldots,n_k)} & \text{otherwise} \end{cases}$$

Then the sequence $\mathcal{V}_{(n_1)}, \ldots, \mathcal{V}_{(n_1,\ldots,n_k)}, \ldots$ is a sequence of moves by TWO against the strategy F of ONE which defeats F. Consequently F is not a winning strategy for ONE. $\square$

Corollary 28 *Every set of real numbers which has the Hurewicz property satisfies* $\mathsf{CDR}_{\mathsf{Sub}}(\Lambda,\Lambda)$.

Proof : Let X be a set with the Hurewicz property and let $(\mathcal{U}_n : n = 1, 2, 3, \ldots)$ be a sequence of large covers of X. Let $(Y_n : n = 1, 2, 3, \ldots)$ be an infinite sequence of pairwise disjoint infinite subsets of the set of positive integers whose union is the set of positive integers. Define a strategy F for ONE in the game $\mathsf{Hurewicz}(X)$ as follows: The first move of ONE is $F(X) = \mathcal{U}_1$ and for every finite sequence $(\mathcal{V}_1, \ldots, \mathcal{V}_n)$ of finite sets of open sets, define

$$F(\mathcal{V}_1, \ldots, \mathcal{V}_n) = \mathcal{U}_m \setminus (\mathcal{V}_1 \cup \ldots \cup \mathcal{V}_n)$$

whenever n is in Y_m. This is a legitimate strategy for ONE but is not a winning strategy. Accordingly there is an F–play which is lost by ONE. Let

$$F(X), \mathcal{V}_1, F(\mathcal{V}_1), \ldots, \mathcal{V}_n, F(\mathcal{V}_1, \ldots, \mathcal{V}_n), \ldots$$

be a play lost by ONE. Then each element of X is in all but finitely many of the sets $\cup\mathcal{V}_n$. Observe that by the definition of F the sequence of sets $(\mathcal{V}_n : n = 1, 2, 3 \ldots)$ is pairwise disjoint. For each n put $\mathcal{R}_n = \cup_{m \in Y_n} \mathcal{V}_m$. Then each $\mathcal{R}_n$ is a large cover of X, and these covers are pairwise disjoint. $\square$

Corollary 29 *If X has the Hurewicz property then it satisfies* $\mathsf{Split}(\Lambda,\Lambda)$.

5.2 $\mathsf{S}_1(\Omega, \Gamma)$.

According to Gerlits and Nagy [8], a set X of real numbers has the γ–property if there is for every sequence $(\mathcal{U}_n : n < \omega)$ of ω–covers of X a sequence $(U_n : n < \omega)$ such that for each n U_n is an element of $\mathcal{U}_n$ and $\{U_n : n < \omega\}$ is a γ–cover for X.

There is a natural game, denoted $\mathsf{Gamma}(X)$, associated with this property: In the n–th inning ONE selects an ω–cover of X and TWO responds by selecting an element of this ω–cover. They play an inning per positive integer. TWO wins a play if the collection of sets selected by TWO is a γ–cover for X; ONE wins otherwise.

Theorem 30 *For a set X of real numbers the following are equivalent:*

1. *X is in* $\mathsf{Sub}(\Omega, \Gamma)$.
2. *X is in* $\mathsf{S}_1(\Omega, \Gamma)$.
3. *ONE has no winning strategy in* $\mathsf{Gamma}(X)$.
4. *For all positive integers n and k, $\Omega \to (\Gamma)^n_k$.*

Proof : The implication $1 \Rightarrow 2$ is due to Gerlits and Nagy –[8].
$2 \Rightarrow 3$: Let F be a strategy for ONE in the game $\mathsf{Gamma}(X)$. Define ω–covers for X as follows:

1. $(U_{(n)} : n = 1, 2, 3, \ldots)$ enumerates $F(X)$, the first move of ONE, and
2. assuming that U_σ is already defined for every sequence of length at most m of positive integers, we define $(U_{(n_1,\ldots,n_m,k)} : k = 1, 2, 3, \ldots)$ to be:

$$F(U_{(n_1)}, \ldots, U_{(n_1,\ldots,n_m)}) \setminus \{U_{(n_1)}, \ldots, U_{(n_1,\ldots,n_m)}.$$

For every finite sequence σ of positive integers the set $\{U_{\sigma\frown(n)} : n = 1, 2, 3, \ldots\}$ is an ω–cover of X. Now apply 2 to select for each σ a positive integer n_σ such that the selection

$$\{U_{\sigma\frown(n_\sigma)} : \sigma \text{ finite sequence of positive integers}\}$$

is a γ–cover of X. Then define a sequence $n_1, n_2, n_3, \ldots$ of positive integers such that $n_1 = n_\emptyset$, and for each k larger than 1, $n_{k+1} = n_{(n_1,\ldots,n_k)}$. Then the sequence

$$U_{(n_1)}, \ldots, U_{(n_1,\ldots,n_k)}, \ldots$$

is a sequence of moves by TWO during a play in which ONE used the strategy F, and this sequence constitutes a γ–cover for X. Thus, F is not a winning strategy for ONE.
$3 \Rightarrow 4$. We show that $\Omega \rightarrow (\Gamma)^2_2$. The proof for higher exponents and more colors then uses this fact and the usual methods for proving Ramsey's theorem for higher exponents and more colors.

Thus, let $\mathcal{U}$ be an ω–cover of X and let $f : [\mathcal{U}]^2 \rightarrow \{0, 1\}$ be a given coloring. Enumerate $\mathcal{U}$ bijectively as $(U_{(n)} : n = 1, 2, 3, \ldots)$. We shall now define a strategy F for ONE in the game $\mathsf{Gamma}(X)$.

The first move by ONE according to F is $F(X) = \mathcal{U}$. For each n we choose an i_n in $\{0, 1\}$ such that the set $F(U_{(n)} = \{V \in \mathcal{U} : f(\{U_{(n)}, V\}) = i_{(n)}\}$ is an ω–cover of X; for convenience we enumerate it bijectively as $(U_{(n,m)} : m = 1, 2, 3, \ldots)$. Assume that for every finite sequence σ of length at most m of positive integers we have defined U_σ and $i_{\sigma\lceil_{length(\sigma)-1}}$ such that

1. i_σ is in $\{0, 1\}$, and
2. with σ equal to $(n_1, \ldots, n_k)$, we have: $(U_{(n_1,\ldots,n_k,m)} : m = 1, 2, 3, \ldots)$ enumerates the ω–cover

   $$\{V \in F(U_{(n_1)}, \ldots, U_{(n_1,\ldots,n_k)}) : f(\{U_{(n_1,\ldots,n_k)}, V\}) = i_{(n_1,\ldots,n_k)}$$

   of X.

Let $(n_1, \dots, n_m)$ be given. Then $U_{(n_1,\dots,n_m)}$ is an element of $F(U_{(n_1)}, \dots, U_{(n_1,\dots,n_{m-1})})$. Choose $i_{(n_1,\dots,n_m)} \in \{0, 1\}$ so that

$$F(U_{(n_1)}, \dots, U_{(n_1,\dots,n_m)}) = (U_{(n_1,\dots,n_m,k)} : k = 1, 2, 3, \dots)$$

bijectively enumerates the ω–cover

$$\{V \in F(U_{(n_1)}, \dots, U_{(n_1,\dots,n_{m-1})}) : F(\{U_{(n_1,\dots,n_m)}, V\}) = i_{(n_1,\dots,n_m)}\}$$

of X.

This defines a strategy for ONE. By hypothesis ONE has no winning strategy. Thus, choose a play against this strategy which defeats it, say

$$U_{(n_1)}, \dots, U_{(n_1,\dots,n_k)}, \dots$$

Then this sequence of sets has the property that it constitutes a γ–cover for X, and for all $k < \ell$ we also have

$$f(\{U_{(n_1,\dots,n_k)}, U_{(n_1,\dots,n_\ell)}\}) = i_{(n_1,\dots,n_k)}.$$

Then choose $i \in \{0, 1\}$ such that for infinitely many k we have $i_{(n_1,\dots,n_k)} = i$. Put $\mathcal{V} = \{U_{(n_1,\dots,n_k)} : i_{(n_1,\dots,n_k)} = i\}$. Then $\mathcal{V}$ is a γ–cover of X which is homogeneous for the coloring f, and which is a subset of the ω–cover $\mathcal{U}$.
$4 \Rightarrow 1$: This implication is easy. $\square$

Theorem 31 *The following classes are equal:* $\mathsf{S}_1(\Omega, \Gamma)$, $\mathsf{Uncr}(\Omega, \Gamma)$, $\mathsf{Ramsey}(\Omega, \Gamma)$, $\mathsf{Q}(\Omega, \Gamma)$, *and* $\mathsf{sQ}(\Omega, \Gamma)$.

At this point the second diagram in the article has been simplified to:

$$\begin{tikzcd}
\mathsf{Hurewicz} \arrow[r] & \mathsf{U}_{fin}(\Gamma, \Omega^{**}) \arrow[r] & \mathsf{Menger} \\
\mathsf{S}_{fin}(\Gamma, \Gamma) \arrow[u] \arrow[r] & \mathsf{S}_{fin}(\Gamma, \Omega) \arrow[u] & \\
\mathsf{S}_1(\Gamma, \Gamma) \arrow[u] \arrow[r] & \mathsf{S}_1(\Gamma, \Omega) \arrow[u] \arrow[r] & \mathsf{S}_1(\Gamma, \Lambda) \arrow[uu] \\
\gamma - \text{sets} \arrow[u] \arrow[r] & \mathsf{S}_1(\Omega, \Omega) \arrow[u] \arrow[r] & \mathsf{C}'' \arrow[u]
\end{tikzcd}$$

For typographical reasons the class $\mathsf{S}_{fin}(\Omega, \Omega)$ which lies between $\mathsf{S}_1(\Omega, \Omega)$ and $\mathsf{S}_{fin}(\Gamma, \Omega)$ has been left out. In [12] it is shown that for subspaces of the real line, of these twelve classes only $\mathsf{S}_1(\Gamma, \Gamma)$ and $\mathsf{S}_{fin}(\Gamma, \Gamma)$ provably coincide.

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