11-1-2010

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 DOI: 10.1109/tcst.2009.2039796

A Study of the Lyapunov Stability of an Open-Loop Induction Machine

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***Abstract*—The induction motor is widely utilized in industry and exists in a plethora of applications. Until the last 20 years or so, it was primarily used in an open-loop fashion (i.e., balanced sinusoidal voltages, constant load torque and viscous friction) with its inherent stability counted on to allow operation over a wide range of operating conditions. Unlike classical arguments based on the steady-state torque-slip curve, a rigorous analytical stability argument using the full nonlinear dynamical model is presented. In particular, conditions for global asymptotic stability of the induction motor in the sense of Lyapunov are given in terms of the motor parameters, operating slip, and synchronous frequency.**

***Index Terms*—Global asymptotic stability, induction motor, Lyapunov stability, open-loop stability, power balance equation.**

I. INTRODUCTION

A CLASSICAL way of depicting the steady-state operation of the induction motor is the torque versus slip curve as shown in Fig. 1 (see [1] and [2]). We denote the stator electrical frequency as ω_S, the steady-state rotor speed as ω_{R0}, the number of pole-pairs as n_p, and the *normalized slip* S as

$$S \triangleq \frac{\omega_S - n_p\omega_{R0}}{\omega_S}. \quad (1)$$

In Fig. 1, τ is the steady-state output torque of the induction motor, τ_p is the peak torque, and S_p is the *pull-out slip* which corresponds to the peak torque τ_p.

The torque versus slip curve indicates the stability of the induction motor about steady-state operating points, but does not ensure it. For example, the stable steady-state operating points for motoring must satisfy $0 < S < S_p$. To explain, suppose the motor is operating at slip S_1 producing the torque τ_0 as shown in Fig. 1. Then an increase in the load torque on the machine would slow the motor down *decreasing* the steady-state speed ω_{R0}. As (1) indicates there is a consequent *increase* in the steady-state slip S (i.e., a shift to the right from the operating slip S_1 in Fig. 1). The increased slip gives an *increase* in the steady-state output torque accommodating the increase in load torque. On the other hand, consider the motor operating at the slip $S_2 > S_p$ in Fig. 1 producing the same steady-state output torque τ_0 as when operating at S_1. Now an *increase* in the load torque again results in a *decrease* in ω_R and thus an increase in the steady-state slip [see (1)], i.e., to the right of S_2 in Fig. 1. However, as Fig. 1 shows, a lower output torque is now produced which cannot meet the increased load demand. Hence the motor will stall. Note that this argument is based on

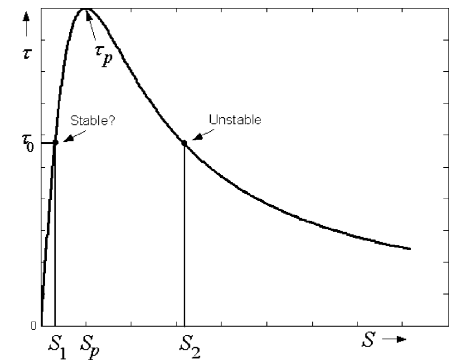

Fig. 1. Torque versus normalized slip curve. The linearized system model is unstable for $S \geq S_p$ and may or may not be stable for $S < S_p$.

steady-state conditions and does not account for transients. In fact it can be shown analytically (see the Appendix) that with no viscous friction, the operating points for which $S \geq S_p$ have a corresponding linearized system model that is unstable.[1] On the other hand, operating points with $S < S_p$ may or may not be stable (even with viscous friction) as was shown in [3] and [4] where a linearized analysis was employed. A fundamental observation here is that stability *cannot* be ascertained from the torque versus slip curve. Consequently, we argue that a rigorous stability analysis is needed.

In this work, we analyze the *open-loop* stability of induction machines, i.e., the input is a fixed set of sinusoidal steady-state voltages, the load consists of a constant load-torque τ_{L0} and a viscous friction load $f\omega_R$, $f \geq 0$. In particular, using Lyapunov theory we give sufficient conditions for global asymptotic stability (GAS) of the machine. Roughly speaking, these conditions are satisfied for a lightly loaded machine when $R_S R_R - (M n_p \omega_{R0}/2)^2 > 0$, where R_S, R_R are the stator and rotor resistance values, respectively, and M is the mutual inductance. That is, we analytically prove that under such conditions, the machine can be started from rest up to its operating speed running in open-loop. The detailed analysis presented here and the latter result are an extension to an earlier version given in [5].

To understand why starting an induction machine requires a light load, consider the machine at rest, so $\omega_R = 0$ and $S = 1$. Then usually $S_p \ll 1$ so that at the startup of the motor, the (instantaneous) slip $S \gg S_p$. As Fig. 1 shows, the torque produced by the motor is then low and thus the motor must be lightly loaded so that it can come up to full speed under open-loop conditions. After getting up to full speed, the motor can then be loaded and run stably.[2] A contribution of this work is to show this rigorously using analytical techniques.

Manuscript received January 14, 2009. Manuscript received in final form December 25, 2009. First published February 18, 2010; current version published October 22, 2010. Recommended by Associate Editor L. Dessaint.

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Digital Object Identifier 10.1109/TCST.2009.2039796

[1]See also [2, p. 175].

[2]Consequently, one cannot expect to obtain globally asymptotically stable results for a fully loaded machine.

We begin in Section II by deriving a nonlinear error model of the induction motor (about an arbitrary operating point) in the stator field coordinate system. In Section III a power balance equation of the motor is developed that is then rewritten in terms of error state variables. The results of Section III are then utilized in Section IV to develop a Lyapunov function that is used to obtain sufficient conditions for the global asymptotic stability of the induction motor. Section V provides a numerical example to illustrate the theoretical results. Finally, concluding remarks are presented in Section VI.

To our knowledge there has not been much work reported in the literature applying Lyapunov theory to study the *open-loop* stability of the induction motor using the full differential equation model of the machine. One exception is the work of Ahmed-Zaid in [6] where a Lyapunov approach (integral manifold) was used to study the stability of an induction generator connected to the grid whose stator resistance was zero and where some of the transients were neglected. On the other hand, for closed-loop control there has been extensive work using Lyapunov (passivity)-based ideas for induction machines most notably in [7] and the references therein.

II. Stator Field Model of the Induction Motor

The starting point for the analysis is the two-phase equivalent model of the motor (see [1] and [2]). The parameters of the two phase induction motor are the stator-side inductance L_S and resistance R_S, the rotor-side inductance L_R and resistance R_R, the mutual inductance M, the number of rotor pole pairs n_p, the moment of inertia of the rotor J, and the viscous friction coefficient f.

The variables consist of the angular position of the rotor θ_R, the angular speed $\omega_R = d\theta_R/dt$, the load torque τ_L, the stator currents i_{Sa} and i_{Sb}, the stator voltages u_{Sa} and u_{Sb}, and the rotor currents i_{Ra} and i_{Rb}, where a and b denote the equivalent two phases of the motor.

A. Space Vector Model

A space vector model of the induction motor is [1], [2]

$$\begin{aligned} R_S\underline{i}_S + L_S\frac{d}{dt}\underline{i}_S + M\frac{d}{dt}\left(\underline{i}_R e^{jn_p\theta_R}\right) &= \underline{u}_S \\ R_R\underline{i}_R + L_R\frac{d}{dt}\underline{i}_R + M\frac{d}{dt}\left(\underline{i}_S e^{-jn_p\theta_R}\right) &= 0 \\ n_pM\,\mathrm{Im}\left\{\underline{i}_S\left(\underline{i}_R e^{jn_p\theta_R}\right)^*\right\} - \tau_L &= J\frac{d\omega_R}{dt} \end{aligned} \tag{2}$$

where the state vector's (complex) stator current, rotor current and stator voltage are defined as

$$\underline{i}_S \triangleq i_{Sa} + ji_{Sb}, \quad \underline{i}_R \triangleq i_{Ra} + ji_{Rb}, \quad \underline{u}_S \triangleq u_{Sa} + ju_{Sb}.$$

The total load torque on the motor τ_L is defined as

$$\tau_L \triangleq f\omega_R + \tau_{L0}$$

where τ_{L0} denotes the external load torque exerted on the rotor, and is henceforth assumed to be constant.

B. Stator Field Coordinate System Model

Next, the model (2) is transformed into a stator field coordinate system. The transformation is defined as

$$\begin{aligned} \underline{i}_{Sdq} &\triangleq i_{Sd} + ji_{Sq} \triangleq \underline{i}_S e^{-j\omega_S t} \\ \underline{i}_{Rdq} &\triangleq i_{Rd} + ji_{Rq} \triangleq \underline{i}_R e^{jn_p\theta_R} e^{-j\omega_S t} \\ \underline{u}_{Sdq} &\triangleq u_{Sd} + ju_{Sq} \triangleq \underline{u}_S e^{-j\omega_S t} \end{aligned} \tag{3}$$

or

$$\begin{aligned} \underline{i}_S &= \underline{i}_{Sdq} e^{j\omega_S t} \\ \underline{i}_R &= \underline{i}_{Rdq} e^{-jn_p\theta_R} e^{j\omega_S t} \\ \underline{u}_S &= \underline{u}_{Sdq} e^{j\omega_S t} \end{aligned} \tag{4}$$

where ω_S is the electrical frequency of the voltage source applied to the stator and is assumed to be constant. Substituting (4) into the space vector model (2) and simplifying results in

$$\begin{aligned} & R_S\underline{i}_{Sdq} + L_S\frac{d\underline{i}_{Sdq}}{dt} + j\omega_S L_S\underline{i}_{Sdq} + M\frac{d\underline{i}_{Rdq}}{dt} \\ & \quad + j\omega_S M\underline{i}_{Rdq} = \underline{u}_{Sdq} \\ & R_R\underline{i}_{Rdq} + L_R\frac{d\underline{i}_{Rdq}}{dt} + j\left(\omega_S - n_p\omega_R\right)L_R\underline{i}_{Rdq} \\ & \quad + M\frac{d\underline{i}_{Sdq}}{dt} + j\left(\omega_S - n_p\omega_R\right)M\underline{i}_{Sdq} = 0 \\ & n_pM\,\mathrm{Im}\{\underline{i}_{Sdq}\left(\underline{i}_{Rdq}\right)^*\} - \left(f\omega_R + \tau_{L0}\right) = J\frac{d\omega_R}{dt}. \end{aligned} \tag{5}$$

Expanding into real and imaginary parts, we obtain the state space representation

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}) + B\mathbf{u} \tag{6}$$

where

$$\begin{aligned} \mathbf{x} &\triangleq \begin{bmatrix}\omega_R & i_{Sd} & i_{Sq} & i_{Rd} & i_{Rq}\end{bmatrix}^T \\ \mathbf{u} &\triangleq \begin{bmatrix}u_{Sd} & u_{Sq} & \tau_{L0}\end{bmatrix}^T \\ \mathbf{f}(\mathbf{x}) &\triangleq \begin{bmatrix} \frac{n_pM}{J}\left(i_{Sq}i_{Rd} - i_{Sd}i_{Rq}\right) - \frac{f}{J}\omega_R \\ \frac{R_R(1-\sigma)}{\sigma M}i_{Rd} + \frac{n_pM}{\sigma L_S}\omega_R i_{Rq} - \frac{R_S}{\sigma L_S}i_{Sd} \\ \quad + \frac{n_p(1-\sigma)}{\sigma}\omega_R i_{Sq} + \omega_S i_{Sq} \\ \frac{R_R(1-\sigma)}{\sigma M}i_{Rq} - \frac{n_pM}{\sigma L_S}\omega_R i_{Rd} - \frac{R_S}{\sigma L_S}i_{Sq} \\ \quad - \frac{n_p(1-\sigma)}{\sigma}\omega_R i_{Sd} - \omega_S i_{Sd} \\ \frac{R_S(1-\sigma)}{\sigma M}i_{Sd} - \frac{n_pM}{\sigma L_R}\omega_R i_{Sq} - \frac{R_R}{\sigma L_R}i_{Rd} \\ \quad - \frac{n_p}{\sigma}\omega_R i_{Rq} + \omega_S i_{Rq} \\ \frac{R_S(1-\sigma)}{\sigma M}i_{Sq} + \frac{n_pM}{\sigma L_R}\omega_R i_{Sd} - \frac{R_R}{\sigma L_R}i_{Rq} \\ \quad + \frac{n_p}{\sigma}\omega_R i_{Rd} - \omega_S i_{Rd} \end{bmatrix} \\ B &\triangleq \begin{bmatrix} 0 & 0 & -\frac{1}{J} \\ \frac{1}{\sigma L_S} & 0 & 0 \\ 0 & \frac{1}{\sigma L_S} & 0 \\ \frac{\sigma-1}{\sigma M} & 0 & 0 \\ 0 & \frac{\sigma-1}{\sigma M} & 0 \end{bmatrix} \end{aligned}$$

and σ is the leakage factor defined as

$$\sigma \triangleq 1 - \frac{M^2}{L_S L_R}.$$

The equilibrium conditions are obtained by setting the derivatives in the stator field model (5) to zero and then equating the real and imaginary parts to obtain

$$
\begin{aligned}
R_S i_{Sd0} - \omega_S L_S i_{Sq0} - \omega_S M i_{Rq0} &= u_{Sd0} \\
R_S i_{Sq0} + \omega_S L_S i_{Sd0} + \omega_S M i_{Rd0} &= u_{Sq0} \\
R_R i_{Rd0} - \omega_S L_R i_{Rq0} + n_p \omega_{R0} L_R i_{Rq0} - \omega_S M i_{Sq0} & \\
+ n_p \omega_{R0} M i_{Sq0} &= 0 \\
R_R i_{Rq0} + \omega_S L_R i_{Rd0} - n_p \omega_{R0} L_R i_{Rd0} + \omega_S M i_{Sd0} & \\
- n_p \omega_{R0} M i_{Sd0} &= 0 \\
n_p M \left(i_{Sq0} i_{Rd0} - i_{Sd0} i_{Rq0} \right) - \left(f \omega_{R0} + \tau_{L0} \right) &= 0. \qquad (7)
\end{aligned}
$$

C. Error Model

Next, to facilitate the Lyapunov analysis of the induction motor, we translate the origin of the system (6) to an arbitrary equilibrium point $\mathbf{x}_0$ given by

$$
\mathbf{x}_0 = \left[\omega_{R0} \quad i_{Sd0} \quad i_{Sq0} \quad i_{Rd0} \quad i_{Rq0} \right]^T
$$

which is a solution to (7). The set of error state variables about this equilibrium point are defined as

$$
\begin{aligned}
e_1 &\triangleq \omega_R - \omega_{R0} \\
e_2 &\triangleq i_{Sd} - i_{Sd0} \\
e_3 &\triangleq i_{Sq} - i_{Sq0} \\
e_4 &\triangleq i_{Rd} - i_{Rd0} \\
e_5 &\triangleq i_{Rq} - i_{Rq0}. \qquad (8)
\end{aligned}
$$

Eliminating the state variables in the model (6) using the error variables given by (8) we obtain the error model of the induction motor

$$
\dot{\mathbf{e}} = A(\mathbf{x}_0)\mathbf{e} + \mathbf{g}(\mathbf{e}) \qquad (9)
$$

where $\mathbf{e}$, $A(\mathbf{x}_0)$, and $\mathbf{g}(\mathbf{e})$ are given in (10)-(11) at the bottom of the page.

The term $\mathbf{g}(\mathbf{e})$ consists of quadratic terms and is independent of the equilibrium point. The system is dominated by the linear term $A(\mathbf{x}_0)\mathbf{e}$ near the equilibrium point $\mathbf{x}_0$. Equation (7) which determines the equilibrium points may be rewritten as

$$
n_p M (i_{Sq0} i_{Rd0} - i_{Sd0} i_{Rq0}) - (f \omega_{R0} + \tau_{L0}) = 0 \qquad (12)
$$

$$
K \begin{bmatrix} i_{Sd0} \\ i_{Sq0} \\ i_{Rd0} \\ i_{Rq0} \end{bmatrix} = \begin{bmatrix} u_{Sd0} \\ u_{Sq0} \\ 0 \\ 0 \end{bmatrix} \qquad (13)
$$

where K is given in (14) at the bottom of the page. We selected the set-points for the voltages u_{Sd0}, u_{Sq0}, and the speed ω_{R0}, and then computed the corresponding equilibrium currents i_{Sd0}, i_{Sq0}, i_{Rd0}, and i_{Rq0} using (13). The resulting load torque τ_{L0} is determined by (12). In other words, one specifies u_{Sd0}, u_{Sq0}, and ω_{R0}, and then uses

$$
\begin{bmatrix} i_{Sd0} \\ i_{Sq0} \\ i_{Rd0} \\ i_{Rq0} \end{bmatrix} = K^{-1} \begin{bmatrix} u_{Sd0} \\ u_{Sq0} \\ 0 \\ 0 \end{bmatrix} \qquad (15)
$$

and

$$
\tau_{L0} = n_p M \left(i_{Sq0} i_{Rd0} - i_{Sd0} i_{Rq0} \right) - f \omega_{R0} \qquad (16)
$$

to obtain the currents and load torque. We further require that $\tau_{L0} \geq 0$.

$$
\mathbf{e} = \left[e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5 \right]^T
$$

$$
A(\mathbf{x}_0) \triangleq \begin{bmatrix}
-\frac{f}{J} & -\frac{n_p M}{J} i_{Rq0} & \frac{n_p M}{J} i_{Rd0} & \frac{n_p M}{J} i_{Sq0} & -\frac{n_p M}{J} i_{Sd0} \\
\frac{n_p(1-\sigma)}{\sigma} \left(\frac{L_R}{M} i_{Rq0} + i_{Sq0} \right) & -\frac{R_S}{\sigma L_S} & \frac{n_p(1-\sigma)}{\sigma} \omega_{R0} + \omega_S & \frac{R_R(1-\sigma)}{\sigma M} & \frac{n_p M}{\sigma L_S} \omega_{R0} \\
-\frac{n_p(1-\sigma)}{\sigma} \left(\frac{L_R}{M} i_{Rd0} + i_{Sd0} \right) & -\frac{n_p(1-\sigma)}{\sigma} \omega_{R0} - \omega_S & -\frac{R_S}{\sigma L_S} & -\frac{n_p M}{\sigma L_S} \omega_{R0} & \frac{R_R(1-\sigma)}{\sigma M} \\
-\frac{n_p}{\sigma} \left(\frac{M}{L_R} i_{Sq0} + i_{Rq0} \right) & \frac{R_S(1-\sigma)}{\sigma M} & -\frac{n_p M}{\sigma L_R} \omega_{R0} & -\frac{R_R}{\sigma L_R} & \omega_S - \frac{n_p}{\sigma} \omega_{R0} \\
\frac{n_p}{\sigma} \left(\frac{M}{L_R} i_{Sd0} + i_{Rd0} \right) & \frac{n_p M}{\sigma L_R} \omega_{R0} & \frac{R_S(1-\sigma)}{\sigma M} & \frac{n_p}{\sigma} \omega_{R0} - \omega_S & -\frac{R_R}{\sigma L_R}
\end{bmatrix} \qquad (10)
$$

$$
\mathbf{g}(\mathbf{e}) \triangleq \begin{bmatrix}
\frac{n_p M}{J} e_3 e_4 - \frac{n_p M}{J} e_2 e_5 \\
\frac{n_p M}{\sigma L_S} e_1 e_5 + \frac{n_p(1-\sigma)}{\sigma} e_1 e_3 \\
-\frac{n_p M}{\sigma L_S} e_1 e_4 - \frac{n_p(1-\sigma)}{\sigma} e_1 e_2 \\
-\frac{n_p M}{\sigma L_R} e_1 e_3 - \frac{n_p}{\sigma} e_1 e_5 \\
\frac{n_p M}{\sigma L_R} e_1 e_2 + \frac{n_p}{\sigma} e_1 e_4
\end{bmatrix} \qquad (11)
$$

$$
K \triangleq \begin{bmatrix}
R_S & -\omega_S L_S & 0 & -\omega_S M \\
\omega_S L_S & R_S & \omega_S M & 0 \\
0 & M \left(n_p \omega_{R0} - \omega_S \right) & R_R & L_R \left(n_p \omega_{R0} - \omega_S \right) \\
-M \left(n_p \omega_{R0} - \omega_S \right) & 0 & -L_R \left(n_p \omega_{R0} - \omega_S \right) & R_R
\end{bmatrix} \qquad (14)
$$

III. Power Balance Equation

The Lyapunov candidate function will be derived from a power balance equation that characterizes the power transfer between the input and output of the motor.

A. Power Balance Equation

First we define the magnetic field energy of the motor W_f and the mechanical energy W_J as (see [1])

$$W_f \triangleq \frac{1}{2}L_S\left(i_{Sd}^2+i_{Sq}^2\right)+\frac{1}{2}L_R\left(i_{Rd}^2+i_{Rq}^2\right) + M\begin{bmatrix} i_{Sd} & i_{Sq}\end{bmatrix}\begin{bmatrix} i_{Rd} \\ i_{Rq}\end{bmatrix} \tag{17}$$

and

$$W_J \triangleq \frac{1}{2}J\omega_R^2. \tag{18}$$

The power balance equation in terms of the stator field coordinate variables is given by

$$\frac{d}{dt}\left(W_f+W_J\right)=\begin{bmatrix} u_{Sd} & u_{Sq} & -\tau_L\end{bmatrix}\begin{bmatrix} i_{Sd} \\ i_{Sq} \\ \omega_R\end{bmatrix} - R_S i_{Sd}^2 - R_S i_{Sq}^2 - R_R i_{Rd}^2 - R_R i_{Rq}^2. \tag{19}$$

B. Error State Variables

Using (8), we now rewrite (19) in terms of the error variables taking into account the equilibrium conditions (7) as

$$\begin{aligned}\frac{d}{dt}(W_f+W_J) &= u_{Sd}e_2+u_{Sq}e_3-(fe_1^2+2fe_1\omega_{R0}+\tau_{L0}e_1)\\ &\quad - R_S(e_2^2+2e_2 i_{Sd0}) - R_S(e_3^2+2e_3 i_{Sq0})\\ &\quad - R_R(e_4^2+2e_4 i_{Rd0}) - R_R(e_5^2+2e_5 i_{Rq0})\end{aligned} \tag{20}$$

where

$$\begin{aligned}W_f &= \frac{1}{2}L_S(e_2^2+e_3^2+2e_2 i_{Sd0}+2e_3 i_{Sq0})\\ &\quad + \frac{1}{2}L_R(e_4^2+e_5^2+2e_4 i_{Rd0}+2e_5 i_{Rq0}+Me_2e_4)\\ &\quad + M(i_{Sd0}e_4+e_2 i_{Rd0}+e_3e_5+i_{Sq0}e_5+e_3 i_{Rq0})\\ &\quad + \frac{1}{2}L_S(i_{Sd0}^2+i_{Sq0}^2)+\frac{1}{2}L_R(i_{Rd0}^2+i_{Rq0}^2)\\ &\quad + M(i_{Sd0}i_{Rd0}+i_{Sq0}i_{Rq0})\\ W_J &= \frac{1}{2}J(e_1^2+2e_1\omega_{R0})+\frac{1}{2}J\omega_{R0}^2.\end{aligned}$$

IV. Lyapunov Stability of the Induction Motor

In this section, the power balance (20) is used to obtain a Lyapunov candidate function V.

Define the function $W(\mathbf{e})$ by

$$W(\mathbf{e}) \triangleq W_f + W_J - (W_f(\mathbf{0})+W_J(\mathbf{0}))$$

where

$$\begin{aligned}W_f(\mathbf{0}) &= \frac{1}{2}L_S(i_{Sd0}^2+i_{Sq0}^2)+\frac{1}{2}L_R(i_{Rd0}^2+i_{Rq0}^2)\\ &\quad + M(i_{Sd0}i_{Rd0}+i_{Sq0}i_{Rq0})\\ W_J(\mathbf{0}) &= \frac{1}{2}J\omega_{R0}^2.\end{aligned}$$

This ensures $W(\mathbf{0})=0$, however W is not assured to be positive definite. Next $W(\mathbf{e})$ is rewritten as

$$W(\mathbf{e}) = \mathbf{e}^T P\mathbf{e} + \mathbf{d}^T\mathbf{e} \tag{21}$$

where

$$P \triangleq \frac{1}{2}\begin{bmatrix} J & 0 & 0 & 0 & 0\\ 0 & L_S & 0 & M & 0\\ 0 & 0 & L_S & 0 & M\\ 0 & M & 0 & L_R & 0\\ 0 & 0 & M & 0 & L_R\end{bmatrix} \tag{22}$$

and

$$\mathbf{d} \triangleq \begin{bmatrix} J\omega_{R0}\\ L_S i_{Sd0}+Mi_{Rd0}\\ L_S i_{Sq0}+Mi_{Rq0}\\ L_R i_{Rd0}+Mi_{Sd0}\\ L_R i_{Rq0}+Mi_{Sq0}\end{bmatrix}. \tag{23}$$

The derivative of $W(\mathbf{e})$ is of course the same as the right-hand side of the power balance (20), which is now rewritten as

$$\frac{dW}{dt} = -\mathbf{e}^T Q_W\mathbf{e} - \mathbf{c}_W^T\mathbf{e} \tag{24}$$

where

$$Q_W \triangleq \begin{bmatrix} f & 0 & 0 & 0 & 0\\ 0 & R_S & 0 & 0 & 0\\ 0 & 0 & R_S & 0 & 0\\ 0 & 0 & 0 & R_R & 0\\ 0 & 0 & 0 & 0 & R_R\end{bmatrix}$$

$$\mathbf{c}_W \triangleq \begin{bmatrix} 2f\omega_{R0}+\tau_{L0}\\ 2R_S i_{Sd0}-u_{Sd}\\ 2R_S i_{Sq0}-u_{Sq}\\ 2R_R i_{Rd0}\\ 2R_R i_{Rq0}\end{bmatrix}.$$

A. Lyapunov Candidate Function and its Derivative

Next, using P as defined in (22) above, a candidate Lyapunov function V is constructed by defining

$$V \triangleq \mathbf{e}^T P\mathbf{e}. \tag{25}$$

The derivative of this Lyapunov candidate function is thus

$$\frac{dV}{dt} = -\mathbf{e}^T Q_W\mathbf{e} - \mathbf{c}_W^T\mathbf{e} - \mathbf{d}^T\dot{\mathbf{e}}.$$

Using (9) this becomes

$$\frac{dV}{dt} = -\mathbf{e}^T Q_W \mathbf{e} - \mathbf{c}_W^T \mathbf{e} - \mathbf{d}^T(\mathbf{g}(\mathbf{e}) + A(\mathbf{x}_0)\mathbf{e})$$

which can be rewritten as

$$\frac{dV}{dt} = -\mathbf{e}^T Q \mathbf{e} - \mathbf{c}^T \mathbf{e} \qquad (26)$$

where Q is given in (27) at the bottom of the page, and $\mathbf{c}^T \triangleq \mathbf{c}_W^T + \mathbf{d}^T A(\mathbf{x}_0)$ or explicitly

$$\mathbf{c} = \begin{bmatrix} f\omega_{R0} - n_p M\left(i_{Sq0} i_{Rd0} - i_{Sd0} i_{Rq0}\right) + \tau_{L0} \\ R_S i_{Sd0} - \left(L_S i_{Sq0} + M i_{Rq0}\right)\omega_S - u_{Sd0} \\ R_S i_{Sq0} + \left(L_S i_{Sd0} + M i_{Rd0}\right)\omega_S - u_{Sq0} \\ R_R i_{Rd0} + \left(M i_{Sq0} + L_R i_{Rq0}\right)\left(n_p\omega_{R0} - \omega_S\right) \\ R_R i_{Rq0} - \left(M i_{Sd0} + L_R i_{Rd0}\right)\left(n_p\omega_{R0} - \omega_S\right) \end{bmatrix}$$

where the substitution $u_{Sd} = u_{Sd0}, u_{Sq} = u_{Sq0}$ holds because in open-loop operation these input voltages are held fixed.

However, with reference to (7) one sees that the components of $\mathbf{c}$ are the equilibrium conditions and therefore $\mathbf{c} \equiv \mathbf{0}$. Hence, the Lyapunov candidate function and its derivative are, respectively

$$V \triangleq \mathbf{e}^T P \mathbf{e} \qquad (28)$$

and

$$\frac{dV}{dt} = -\mathbf{e}^T Q \mathbf{e}. \qquad (29)$$

The leading principal minors of the matrix P are

$$\pi_1 = \frac{1}{2}J > 0,\ \pi_2 = \frac{1}{4}JL_S > 0,\ \pi_3 = \frac{1}{8}JL_S^2 > 0$$
$$\pi_4 = \frac{1}{16}J\sigma L_S^2 L_R > 0,\ \pi_5 = \frac{1}{32}J\sigma^2 L_S^2 L_R^2 > 0.$$

As all of the leading principal minors of P are positive, P is positive definite [8]. Furthermore, $V = \mathbf{e}^T P \mathbf{e} \geq \lambda_{\min}(P)\,\mathbf{e}^T\mathbf{e}$ and as $\lambda_{\min}(P) > 0$ we have $V(\mathbf{e}) \to \infty$ as $\|\mathbf{e}\| \to \infty$.

B. GAS of an Unloaded Induction Motor

We first consider the special case with the machine operating at $S = 0$. This means there is no load on the machine (both $f = 0$ and $\tau_{L0} = 0$) as the machine produces no torque if the slip is zero. We first show that Q is positive semidefinite and then use LaSalle's theorem [9] to conclude global asymptotic stability. With $f = 0$ and $S = 0$ [so $i_{Rd0} = i_{Rq0} = 0$—see (42)] the matrix Q in (27) reduces to

$$\begin{bmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & R_S & 0 & 0 & -\frac{Mn_p\omega_{R0}}{2} \\ 0 & 0 & R_S & \frac{Mn_p\omega_{R0}}{2} & 0 \\ 0 & 0 & \frac{Mn_p\omega_{R0}}{2} & R_R & 0 \\ 0 & -\frac{Mn_p\omega_{R0}}{2} & 0 & 0 & R_R \end{bmatrix}.$$

We now show Q is positive semidefinite. Recall that a matrix is *positive semidefinite* if and only if *all* the principal minors are non-negative (see [8, p. 74]). The nonzero 1×1 principal minors of Q have values $R_S, R_R,$ the nonzero 2×2 principal minors of Q have values $R_S^2, R_R^2, R_S R_R,$ and $R_S R_R - (Mn_p\omega_{R0}/2)^2$, the nonzero 3×3 principal minors of Q have values $R_R(R_S R_R - (Mn_p\omega_{R0}/2)^2), R_S(R_S R_R - (Mn_p\omega_{R0}/2)^2)$, and the single nonzero 4×4 principal minor of Q has the value $\left(R_S R_R - (Mn_p\omega_{R0}/2)^2\right)^2$. Finally, the only 5×5 principal minor is zero. Consequently, Q is positive semidefinite if and only if

$$R_S R_R - \left(\frac{Mn_p\omega_{R0}}{2}\right)^2 \geq 0. \qquad (30)$$

Remark: If $R_S R_R - (Mn_p\omega_{R0}/2)^2 < 0$ then Q is neither positive semidefinite nor negative semidefinite.

LaSalle's theorem tells us that the induction motor system (9) is globally asymptotically stable in the sense of Lyapunov if (see corollary 4.2, [9, p. 129])

(a) $V(\mathbf{e}) > 0\ \forall\ \mathbf{e} \neq \mathbf{0}$, and $V(\mathbf{0}) = 0$.
(b) $dV(\mathbf{e})/dt \leq 0\ \ \forall\ \mathbf{e} \neq \mathbf{0}$.
(c) $dV(\mathbf{e})/dt \equiv 0 \Rightarrow \mathbf{e} \equiv 0$.
(d) $V(\mathbf{e}) \to \infty$ as $\|\mathbf{e}\| \to \infty$.

Conditions (a) and (d) hold for V defined as in (28). Condition (b) holds as $dV(\mathbf{e})/dt = -\mathbf{e}^T Q \mathbf{e} \leq 0$ and Q was just shown to be positive semidefinite with the assumption of condition (30). To check condition (c), suppose the derivative of the Lyapunov function V is identically zero, that is

$$\begin{aligned}\frac{d}{dt}V(\mathbf{e}) &= -\,\mathbf{e}^T(t)Q\mathbf{e}(t)\\ &= R_S e_2^2 + R_S e_3^2 + R_R e_4^2 + R_R e_5^2\\ &\quad + Mn_p\omega_{R0} e_3 e_4 - Mn_p\omega_{R0} e_5 e_2 \equiv 0. \end{aligned} \qquad (31)$$

$$Q \triangleq \begin{bmatrix} f & \frac{1}{2}n_p M i_{Rq0} & -\frac{1}{2}n_p M i_{Rd0} & \frac{1}{2}n_p L_R i_{Rq0} & -\frac{1}{2}n_p L_R i_{Rd0} \\ \frac{1}{2}n_p M i_{Rq0} & R_S & 0 & 0 & -\frac{1}{2}Mn_p\omega_{R0} \\ -\frac{1}{2}n_p M i_{Rd0} & 0 & R_S & \frac{1}{2}Mn_p\omega_{R0} & 0 \\ \frac{1}{2}n_p L_R i_{Rq0} & 0 & \frac{1}{2}Mn_p\omega_{R0} & R_R & 0 \\ -\frac{1}{2}n_p L_R i_{Rd0} & -\frac{1}{2}Mn_p\omega_{R0} & 0 & 0 & R_R \end{bmatrix} \qquad (27)$$

We begin by showing that (31) implies $e_2(t) = e_3(t) = e_4(t) = e_5(t) \equiv 0$. This in turn is accomplished by first showing that (31) implies $e_3e_4 = e_5e_2 \equiv 0$. To this end, note that

$$\left(\sqrt{R_S}e_3 \pm \sqrt{R_R}e_4\right)^2 + \left(\sqrt{R_S}e_2 \pm \sqrt{R_R}e_5\right)^2 \geq 0 \quad (32)$$

which upon expanding becomes

$$R_Se_3^2 \pm 2\sqrt{R_S}\sqrt{R_R}e_3e_4 + R_Re_4^2 + R_Se_2^2 \pm 2\sqrt{R_S}\sqrt{R_R}e_2e_5 + R_Re_5^2 \geq 0. \quad (33)$$

Subtracting (31) from (33), we have

$$\pm 2\sqrt{R_S}\sqrt{R_R}e_3e_4 - Mn_p\omega_{R0}e_3e_4 \pm 2\sqrt{R_S}\sqrt{R_R}e_2e_5 + Mn_p\omega_{R0}e_2e_5 \geq 0$$

or

$$\left(\pm 2\sqrt{R_S}\sqrt{R_R} - Mn_p\omega_{R0}\right)e_3e_4 + \left(\pm 2\sqrt{R_S}\sqrt{R_R} + Mn_p\omega_{R0}\right)e_2e_5 \geq 0. \quad (34)$$

We now require that (30) hold with *strict* inequality, that is

$$2\sqrt{R_S}\sqrt{R_R} > Mn_p\omega_{R0} = M\omega_S. \quad (35)$$

noting that $n_p\omega_{R0} = \omega_S$ as $S = 0$.

Using (35), we obtain a contradiction in (34) unless $e_3e_4 = e_2e_5 \equiv 0$. For example, if $e_3e_4 < 0$ and $e_2e_5 > 0$ then choose the $+$ sign in the first term of (34) and the $-$ sign in the second term of (34) so that the inequality (35) implies (34) is negative, which is a contradiction. Similarly, regardless of the signs e_3e_4 and e_2e_5, a contradiction will always result unless $e_3e_4 = e_2e_5 \equiv 0$. With this result, it then follows from (31) that

$$R_Se_2^2 + R_Se_3^2 + R_Re_4^2 + R_Re_5^2 \equiv 0$$

or equivalently

$$e_2(t) = e_3(t) = e_4(t) = e_5(t) \equiv 0.$$

Finally, we show $e_1(t) \equiv 0$. To do so, note that (9) still holds, that is

$$\dot{\mathbf{e}} = A(\mathbf{x}_0)\mathbf{e} + \mathbf{g}(\mathbf{e}). \quad (36)$$

With $e_2(t) = e_3(t) = e_4(t) = e_5(t) \equiv 0$ it follows that $\mathbf{g}(\mathbf{e}) = \mathbf{0}$. Also, setting $S = 0$ in (14) and solving (15) shows the steady-state rotor currents are zero. However, the steady-state stator currents are *not* zero for $S = 0$. Thus (36) reduces to

$$\begin{bmatrix} \dot{e}_1(t) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -\frac{n_p(\sigma-1)}{\sigma}i_{Sq0} \\ \frac{n_p(\sigma-1)}{\sigma}i_{Sd0} \\ -\frac{n_p}{\sigma}\frac{M}{L_R}i_{Sq0} \\ \frac{n_p}{\sigma}\frac{M}{L_R}i_{Sd0} \end{bmatrix} e_1(t) \quad (37)$$

which can only hold if $e_1(t) \equiv 0$.

Thus, with $f = 0, \tau_{L0} = 0$ (i.e., no load) and $R_SR_R - (Mn_p\omega_{R0}/2)^2 = R_SR_R - (M\omega_S/2)^2 > 0$, all the conditions of LaSalle's theorem hold and the (unloaded) induction motor system is globally asymptotically stable.

Remark: An alternative candidate Lyapunov function could be

$$V \triangleq \mathbf{e}^T P\mathbf{e}$$

where $P \triangleq \mathrm{diag}\,(J/\sigma L_S + J/\sigma L_R, 1, 1, 1, 1)$.

Also, a straightforward calculation shows that

$$\left[\left(\frac{J}{\sigma L_S} + \frac{J}{\sigma L_R}\right)e_1 \quad e_2 \quad e_3 \quad e_4 \quad e_5\right]\mathbf{g}(\mathbf{e}) \equiv 0. \quad (38)$$

Then, the derivative of the Lyapunov candidate function becomes

$$\frac{dV}{dt} = \left(A(\mathbf{x}_0)\mathbf{e} + \mathbf{g}(\mathbf{e})\right)^T P\mathbf{e} + \mathbf{e}^T P\left(A(\mathbf{x}_0)\mathbf{e} + \mathbf{g}(\mathbf{e})\right) = \mathbf{e}^T\left(A^T(\mathbf{x}_0)P + PA(\mathbf{x}_0)\right)\mathbf{e}.$$

However, straightforward calculations show that with $S = 0$ ($\Rightarrow f = 0$), the matrix $A^T(\mathbf{x}_0)P + PA(\mathbf{x}_0)$ is indefinite so this Lyapunov candidate function is not helpful.

C. GAS of an Induction Motor With Load

With V defined as in (28), the induction motor is globally asymptotically stable (GAS) in the sense of Lyapunov if (see [9, Th. 4.2, p. 124])

(a) $V(\mathbf{e}) > 0 \ \forall\, \mathbf{e} \neq \mathbf{0}$, and $V(\mathbf{0}) = 0$.
(b) $dV(\mathbf{e})/dt < 0 \ \forall\, \mathbf{e} \neq \mathbf{0}$.
(c) $V(\mathbf{e}) \to \infty$ as $\|\mathbf{e}\| \to \infty$.

We have already shown that $V = \mathbf{e}^T P\mathbf{e}$ satisfies conditions (a) and (c). We now find conditions under which Q in (27) is positive definite. Recall that a matrix is positive definite if and only if its *leading* principal minors are positive [8]. The leading principal minors of Q are

$$\begin{aligned}
\Pi_1 &= f \\
\Pi_2 &= fR_S - \frac{1}{4}n_p^2M^2i_{Rq0}^2 \\
\Pi_3 &= \frac{1}{4}R_S\left(4fR_S - n_p^2M^2\left(i_{Rd0}^2 + i_{Rq0}^2\right)\right) \\
\Pi_4 &= fR_S^2R_R + \frac{1}{4}n_p^2\left(\frac{1}{2}M^2i_{Rq0}n_p\omega_{R0} - i_{Rd0}R_SL_R\right)^2 \\
&\quad - \frac{1}{4}n_p^2R_S(fM^2\omega_{R0}^2 + (R_RM^2 + L_R^2R_S)(i_{Rd0}^2 + i_{Rq0}^2)) \\
\Pi_5 &= \frac{1}{4}\left(R_SR_R - \left(\frac{Mn_p\omega_{R0}}{2}\right)^2\right) \times (4fR_SR_R - M^2fn_p^2\omega_{R0}^2 \\
&\quad - n_p^2(R_RM^2 + L_R^2R_S)(i_{Rd0}^2 + i_{Rq0}^2)).
\end{aligned}$$

To get $\Pi_5 > 0$ we can require that the two factors of Π_5 be positive,[3] i.e.,

$$R_SR_R - \left(\frac{Mn_p\omega_{R0}}{2}\right)^2 > 0 \quad (39)$$

$$4fR_SR_R - M^2fn_p^2\omega_{R0}^2 - n_p^2\left(R_RM^2 + L_R^2R_S\right) \times \left(i_{Rd0}^2 + i_{Rq0}^2\right) > 0. \quad (40)$$

[3] We could also require both factors to be negative. However, rewriting Π_4 in terms of S and taking the limit as $S \to 0$ we have $\lim_{S\to 0}\Pi_4 = fR_S(R_SR_R - (Mn_P\omega_{R0}/2)^2)$. Hence, we could not get Q to be positive definite for small slip values with $R_SR_R - (Mn_P\omega_{R0}/2)^2 < 0$.

Clearly for (40) to be positive requires $f > 0$ which also implies $\Pi_1 > 0$. Also it is easily seen that (40) positive implies (39) is positive. Thus $\Pi_5 > 0$ if (40) is positive. Note that $\Pi_3 > 0$ implies $\Pi_2 > 0$. To consider Π_3 rewrite (40) as

$$R_R\left(4fR_S - n_p^2M^2\left(i_{Rd0}^2 + i_{Rq0}^2\right)\right) - fM^2n_p^2\omega_{R0}^2 \\ -n_p^2L_R^2R_S\left(i_{Rd0}^2 + i_{Rq0}^2\right) > 0$$

which implies $4fR_S - n_p^2M^2\left(i_{Rd0}^2 + i_{Rq0}^2\right) > 0$ so that (40) implies $\Pi_3 > 0$.

Similarly, multiplying (40) by $R_S/4 > 0$ and rearranging gives

$$fR_S^2R_R - \frac{n_p^2M^2R_S}{4}f\omega_{R0}^2 - \frac{n_p^2R_S}{4}(R_RM^2 + L_R^2R_S) \\ \times(i_{Rd0}^2 + i_{Rq0}^2) > 0 \quad (41)$$

and thus (40) implies $\Pi_4 > 0$. In summary, Q is positive definite if (40) is positive.

Using (15) with $\omega_S - n_p\omega_{R0}$ replaced by $\omega_S S$, the sum of the squares of the steady-state rotor currents can be written as

$$i_{Rd0}^2 + i_{Rq0}^2 = \frac{\omega_S^2M^2S^2V_{d0}^2}{\Delta(S)} \quad (42)$$

where $\Delta(S)$ is the determinant of the matrix K in (14) and is given by

$$\Delta(S) \triangleq \left(L_R^2R_S^2\omega_S^2 + \omega_S^4\left(L_RL_S - M^2\right)^2\right)S^2 \\ + \left(2M^2R_RR_S\omega_S^2\right)S + R_R^2R_S^2 + L_S^2R_R^2\omega_S^2. \quad (43)$$

Note that $\Delta(S) > 0$ for $S \geq 0$. Using $n_p\omega_{R0} = \omega_S(1 - S)$, (40) can be written as a function of S given by

$$4fR_SR_R - M^2f\omega_S^2(1-S)^2 - n_p^2(R_RM^2 + L_R^2R_S) \\ \times\frac{\omega_S^2M^2S^2V_{d0}^2}{\Delta(S)} > 0. \quad (44)$$

By (43), we have $\Delta(S) > 0$ for all $S \geq 0$. It then follows upon multiplying (44) through by $\Delta(S)$ that the condition for Q to be positive definite is

$$g(S) \triangleq 4fR_SR_R\Delta(S) - M^2f\omega_S^2(1-S)^2\Delta(S) \\ - n_p^2(R_RM^2 + L_R^2R_S)\omega_S^2M^2S^2V_{d0}^2 \\ > 0. \quad (45)$$

Note that

$$g(0) = 4f\left(R_SR_R - \frac{M^2\omega_S^2}{4}\right)\Delta(0) > 0$$

assuming the condition $R_SR_R - M^2\omega_S^2/4 > 0$. Thus $g(S)$ is a polynomial in S which is positive for $S = 0$. Then there exists an $S_\beta > 0$ such that $g(S_\beta) = 0$ and $g(S) > 0$ for $0 \leq S < S_\beta$. Thus we are assured of finding an interval $[0, S_\beta)$ of values of S for which Q is positive definite.

This is not quite a sufficient condition for global asymptotic stability because we are implicitly assuming the external load torque τ_{L0} given in (16) is non-negative. Using (15) to eliminate the stator and rotor currents in (16) and $n_p\omega_{R0} = \omega_S(1 - S)$, we can rewrite the condition $\tau_{L0} \geq 0$ [see (16)] as

$$\tau_{L0} = \underbrace{\omega_Sn_pM^2S\left(\frac{V_{d0}^2R_R}{\Delta(S)}\right)}_{\tau(S)} - f\frac{\omega_S}{n_p}(1-S) \geq 0. \quad (46)$$

Multiplying through by $\Delta(S)$ this becomes the polynomial condition

$$h(S) \triangleq \omega_Sn_pM^2SV_{d0}^2R_R - f\frac{\omega_S}{n_p}(1-S)\Delta(S) \geq 0. \quad (47)$$

To give some insight that there are positive values of S satisfying condition (46) [equivalently (47)], consider rewriting the steady-state torque $\tau(S)$ as

$$\tau(S) = \omega_Sn_pM^2S\left(\frac{V_{d0}^2R_R}{\Delta(S)}\right) \approx S\tau'(0)$$

which is a good approximation for small S (see Fig. 1). Then the condition (46) simplifies (approximates) to

$$\tau_{L0} \approx S\left(\tau'(0) + \frac{f\omega_S}{n_p}\right) - \frac{f\omega_S}{n_p} > 0 \quad (48)$$

or

$$S > \frac{f\omega_S}{n_p\tau'(0) + f\omega_S} \approx S_\alpha \quad (49)$$

where S_α denotes the smallest positive real root of (47).

Theorem: GAS of an Induction Motor with Load.

If

1) $R_SR_R - (Mn_p\omega_{R0}/2)^2 > 0$ and $f > 0$.
2) Assume $S_\alpha < S_\beta$ with S_α and S_β defined as above.

Then the induction motor is globally asymptotically stable for

$$S_\alpha \leq S < S_\beta.$$

Remarks: Note that with any load on the machine $i_{Rd0}^2 + i_{Rq0}^2 > 0$ so that if $f = 0$ then $\Pi_3 < 0$ and Q cannot possibly be positive definite or even positive semidefinite. Similarly, note for example that if $R_S = 0$ then Q cannot be positive definite or even positive semidefinite with $i_{Rd0}^2 + i_{Rq0}^2 > 0$.

V. Numerical Example

In this section, a numerical example is presented to demonstrate the analytical results and their application. Namely, an example is presented to illustrate the test for global asymptotic stability.

Consider a small induction motor with the following parameter values (see [1]): $M = 0.0117$ H, $L_R = 0.014$ H, $L_S = 0.014$ H, $R_S = 1.7\ \Omega$, $R_R = 3.9\ \Omega$, $f = 0.00014$ N · m/rad/s, $J = 0.00011$ kg · m^2, $n_p = 3$, $\omega_S = 2\pi \times 60$ rad/s. With the input voltages set as $u_{Sd0} = 50$ V, $u_{Sq0} = 0$ V, and the steady-state speed chosen as $\omega_{R0} = 124$ rad/s, the normalized slip is

$$S = \frac{377 - 3 \times 124}{377} = \frac{377 - 372}{377} = 0.0132.$$

First, checking condition (1) of the theorem we have

$$R_S R_R - \left(\frac{M n_p \omega_{R0}}{2}\right)^2 = 1.894 > 0.$$

Second, substituting the parameter values into (45) and (47), the numerical values of S_α and S_β can be computed. Specifically, solving for the smallest positive real root of (47) gives

$$S_\alpha = 3.93 \times 10^{-3}$$

while the other two roots of (47) are complex conjugates. Solving (45) for its smallest positive real root gives

$$S_\beta = 51.50 \times 10^{-3}$$

where the other roots of (45) consist of a complex conjugate pair and a negative real root. Thus the system is globally asymptotically stable for

$$3.93 \times 10^{-3} \leq S < 51.50 \times 10^{-3}.$$

In our example, $S = 13.2 \times 10^{-3}$ so the system is globally asymptotically stable at this slip. The corresponding equilibrium currents and external load torque set point are computed from (15) and (16) as $i_{Sd0} = +2.852$ A, $i_{Sq0} = -8.521$ A, $i_{Rd0} = -0.128$ A, $i_{Rq0} = -0.040$ A, and $\tau_{L0} = +0.025\ \mathrm{N} \cdot \mathrm{m}$.

In the no load case for this machine, that is, taking $f = 0$, $\tau_{L0} = 0$, and $\omega_{R0} = \omega_S / n_P$, it is straightforward to verify that condition (35) holds. Consequently, the machine is GAS in the no load case as well.

VI. Conclusion and Discussion

Sufficient conditions for the global asymptotic stability of an open-loop induction motor have been derived in this work. Roughly speaking, under either no load or lightly loaded conditions, it was shown that the induction motor is globally asymptotically stable if $R_S R_R - (M n_p \omega_{R0}/2)^2 > 0$. Though such behavior of a lightly loaded machine is empirically known, to the authors' knowledge this is the first analytical result showing this. In particular, as it guarantees GAS, the machine will start from rest.

With significant load on the machine, it is usually not *globally* asymptotically stable. In particular, it is well known that open loop starting of an induction machine under load is a challenge. One common method is the addition of resistance in series with the rotor (using a wound rotor machine) to shift the peak of the torque slip curve to the right so the starting torque is increased. Thus the motor parameters are *changed* to help with its starting torque. Though not the same, this is consistent with the condition $R_R R_S - (M n_p \omega_{R0}/2)^2 > 0$ given in our work, that is, increasing the value of R_R helps. The torque-slip curve has long been used to give an (imprecise) indication of the operating point stability of the induction machine. In contrast, the analysis presented here gives conditions that (precisely) ensure GAS.

The conditions for Lyapunov stability presented here appear to be rather restrictive, however this seems most likely due to the inherent dynamics of the induction machine. With the Lyapunov candidate function chosen as in (25), the system will be either GAS, that is Q in (29) is positive definite, or no conclusion on Lyapunov stability can be inferred. A question therefore presents itself as to whether another Lyapunov function exists that allows one to obtain less restrictive conditions for GAS. However, the Lyapunov function chosen here was based on energy (power) and the choice of another such function is not at all obvious to the authors.

Appendix

The linearized system matrix $A(x_0)$ given in (10) with $\mathbf{x}_0 = [\omega_{R0} \quad i_{Sd0} \quad i_{Sq0} \quad i_{Rd0} \quad i_{Rq0}]^T$ can be written as a function of the normalized slip S as $A(S)$ using (15), and $\omega_{R0} = (1 - S)\omega_S / n_p$ and setting $u_{Sd0} = V_{d0}$, $u_{Sq0} = 0$. With $f = 0$, its characteristic equation has the form

$$\det[\lambda I_5 - A(S)] = \frac{1}{\rho_5}\left(\rho_5\lambda^5 + \rho_4\lambda^4 + \rho_3\lambda^3 + \rho_2\lambda^2 + \rho_1\lambda + \rho_0\right) \tag{50}$$

where $\rho_0 \triangleq \left(M^2 n_p^2 V_{d0}^2 R_R / \left(\omega_S^2 L_R^2 \left(R_S^2 + \sigma^2 \omega_S^2 L_S^2\right)\right)\right)(S_p^2 - S^2)$.

Straightforward calculations show that $\rho_5(S) > 0$ for all $S \geq 0$. However, for $S = S_p$ the term $\rho_0(S_p) = 0$ and for $S > S_p$ the term $\rho_0(S) < 0$. Thus the linearized system is unstable for $S \geq S_p$.

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