

# Representing Propositional Logic Connectives With Modular Polynomials

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## Abstract

*This paper explores the relationship between  $n$ -valued propositional logic connectives and modular polynomials. Namely the representing of logic connectives using modular polynomials. The case for  $n = 2$  is explored and a method is developed for finding the coefficients of the unique polynomial that represents any given binary logic connective. Examples are then given for using the modular polynomial representations of connectives to determine the validity of propositional arguments. A similar procedure is shown for when  $n = 3$  and an evaluation of the axioms of Łukasiewicz's 3-valued logic is given using modular polynomials. The general case is explored to determine for which values of  $n$  the representation holds. It is then shown that  $\text{mod } n$  polynomial functions are sufficient for representing any  $n$ -valued logic connective if and only if  $n$  is prime.*

## 1 Introduction

Mathematics and logic have a deep-rooted relationship with one another. Gottfried Wilhelm Leibniz explored this relationship in the 17<sup>th</sup> century [9][4]. George Boole published his *Laws of Thought* in the 19<sup>th</sup> century which further cemented the relationship between mathematics and logic [3][4].

This paper will focus on exploring the relationship between  $n$ -valued propositional logic and base  $n$  modular arithmetic. More specifically it will be concerned with polynomial functions in  $\mathbb{Z}_n$  that represent  $n$ -valued propositional logic connectives. The ultimate aim of this paper is to show that there exists a unique modular polynomial function for representing any  $n$ -valued propositional logic connective when  $n$  is prime.

The following abbreviations will be used for discussing propositional logics: 'TRUE', 'FALSE', and 'INDIFFERENT' will be represented by  $T$ ,  $F$ , and  $I$  respectively. Common operators will be as follows: 'AND'- $\wedge$ , 'XOR'- $\text{xor}$ , 'NOT'- $\neg$ , 'OR'- $\vee$ , 'IF...THEN'- $\rightarrow$ , and 'IFF'- $\leftrightarrow$ .

## 2 General Propositional Logic Structure

Logical systems are composed of a syntactic element and a semantic element. For this paper, the syntax for an  $n$ -valued propositional logic will employ lower case letters with or without subscripts as variables, symbols for connectives of all arities, commas, and parentheses. Let  $\Sigma_n$  be the set composed of these elements for a given  $n$ -valued propositional logic. A finite string of symbols from  $\Sigma_n$  is allowable, or a well-formed formula (wff), if and only if it conforms to the following rules :

1. All variables are wffs.
2. If  $f$  is a symbol for a  $k$ -ary connective and  $w_1, w_2, \dots, w_k$  are wffs then  $f(w_1, w_2, \dots, w_k)$  is a wff.

Let  $\mathcal{L}_n$  be the set of all allowable strings from  $\Sigma_n$ .

Let  $V_n$  be a set of size  $n$ .  $V_n$  is the set of truth values of the logic. Further  $V_n$  contains two distinguished symbols,  $T$  and  $F$ , that represent truth and falsehood respectively.

The semantics of the logic assign meaning to the symbols for connectives. This meaning is determined based on how the connective behaves under certain valuations of the propositional variables. A valuation is a map,  $\nu$ , from the set of propositional variables to  $V_n$ . Thus the semantics is defined by specifying how to extend each valuation  $\nu$  so it is a map from all of  $\mathcal{L}_n$  to  $V_n$ . This is done by interpreting each  $k$ -ary connective,  $f$ , as a function,  $\hat{f} : V_n^k \rightarrow V_n$  such that:

$$\nu(f(p_1, \dots, p_k)) = \hat{f}(\nu(p_1), \dots, \nu(p_k))$$

This extends each valuation  $\nu$  from all of  $\mathcal{L}_n$  to  $V_n$ [1][5][11].

### 3 2-Valued Structures

The structure for 2-valued propositional logic will employ  $\Sigma_2$ ,  $\mathcal{L}_2$ , and  $V_2 = \{F, T\}$ . For example, the valuations of the binary operators,  $(p \wedge q) \in \mathcal{L}_2$  and  $(p \text{ xor } q) \in \mathcal{L}_2$  are presented in tabular form:

$p$	$q$	$p \wedge q$	$p \text{ xor } q$
$T$	$T$	$T$	$F$
$T$	$F$	$F$	$T$
$F$	$T$	$F$	$T$
$F$	$F$	$F$	$F$

The structure for base 2 modular arithmetic is as follows:  $\{\{0, 1\}, +, \times, 0, 1\}$ . The tables for  $+$  and  $\times$  are:

$+$	$0$	$1$	$\times$	$0$	$1$
$0$	$0$	$1$	$0$	$0$	$0$
$1$	$1$	$0$	$1$	$0$	$1$

Throughout the paper juxtaposition may be used in place of  $\times$ .

### 4 Representation of 2-Valued Propositional Logic Connectives Using Mod 2 Polynomials

**Lemma 4.1** (Fermat).  $p^n \equiv p \pmod n$  when  $n$  is prime. [8] □

**Theorem 4.2.** For any prime  $n$  and any polynomial  $P \in \mathbb{Z}_n[x_1, x_2, \dots, x_k]$ , there is a unique polynomial  $\hat{P}$  such that:

1. For all  $m_1, \dots, m_k \in \mathbb{Z}_n$ ,  $P(m_1, \dots, m_k) = \hat{P}(m_1, \dots, m_k)$
2. The degree of any  $x_i$  in any monomial in  $\hat{P}$  is smaller than  $n$ .

*Proof.* Let  $P$  be an arbitrary polynomial in  $\mathbb{Z}_n[x_1, x_2, \dots, x_k]$  where  $n$  is prime. Let  $m_i$  represent the degree of the variable  $x_i$  with  $1 \leq i \leq k$ . If  $0 < m_i < n$  then there is nothing to prove. If  $m_i > n$  then by lemma 4.1

$$x_i^{m_i} = x_i^{m_i - n} x_i^n \equiv x_i^{m_i - n + 1} \pmod n.$$

Now if  $0 < m_i - n + 1 < n$  then set  $m_i - n + 1$  as the value of  $m_i$  and we are done, else repeat this  $j$  times such that  $0 < m_i - j(n - 1) < n$  at which point set  $m_i - j(n - 1)$  as the value of  $m_i$ . Performing this reduction for all monomials in  $P$  yields  $\hat{P}$  such that satisfies (1) and (2).

Let  $P$  be an arbitrary polynomial in  $\mathbb{Z}_n[x_1, x_2, \dots, x_k]$  where  $n$  is prime. Note that  $\mathbb{Z}_n[x_1, x_2, \dots, x_k]$  is an integral domain, since  $\mathbb{Z}_n$  is a field when  $n$  is prime.[10] Call polynomials that satisfy 1 and 2 above reduced. Let  $P_1$  and  $P_2$  be reduced polynomials of  $P$ . So  $P(m_1, \dots, m_k) = P_1(m_1, \dots, m_k) = P_2(m_1, \dots, m_k)$  for all  $m_1, \dots, m_k$  and  $P_1 - P_2(m_1, \dots, m_k) = 0$ . Note that  $P_1 - P_2$  is itself reduced. Assume for contradiction that  $P_1 - P_2$  is not identically zero. Since  $P_1 - P_2(m_1, \dots, m_k) = 0$  for all  $m_1, \dots, m_k$ , all terms of the form  $m_i - a$ ,  $1 \leq i \leq k$  and  $a \in \mathbb{Z}_n$ , are factors of  $P_1 - P_2$ . This, however, leads to a contradiction since all of the following are factors of  $P_1 - P_2$ :  $m_i(m_i - 1)(m_i - 2) \dots (m_i - (n - 1))$ . Yet this would mean that the degree of  $m_i$  in  $P_1 - P_2$  will be at least  $n$ , but  $P_1 - P_2$  was reduced so the degree of any  $x_i$  in any monomial of  $P_1 - P_2$  is less than  $n$ . This proves that  $P_1 - P_2$  is identically zero and since  $\mathbb{Z}_n[x_1, \dots, x_k]$  is an integral domain there are no zero-divisors so that  $P_1$  is identical with  $P_2$ . Thus any polynomial in  $\mathbb{Z}_n[x_1, \dots, x_k]$  has a unique reduced form. □

Consider polynomials in  $\mathbb{Z}_2[p, q]$ . These polynomials have the reduced form  $c_1 pq + c_2 p + c_3 q + c_4$  by Theorem 4.2. Call this general reduced form  $P_2(p, q)$ . Accounting for the four possible value combinations of  $p$  and  $q$  and substituting these values into the polynomial  $P_2$  yields four linear equations:

$$\begin{aligned}
 P_2(1, 1) &= c_1 + c_2 + c_3 + c_4 \\
 P_2(1, 0) &= c_2 + c_4 \\
 P_2(0, 1) &= c_3 + c_4 \\
 P_2(0, 0) &= c_4.
 \end{aligned} \tag{1}$$

Setting up these equations into a linear system in matrix form gives us:

$$\begin{bmatrix}
 1 & 1 & 1 & 1 \\
 0 & 1 & 0 & 1 \\
 0 & 0 & 1 & 1 \\
 0 & 0 & 0 & 1
 \end{bmatrix}.$$

Since this matrix is triangular without zeros along the diagonal it is invertible and therefore spans  $\mathbb{Z}_2^4$ , the four-dimensional vector space with elements from  $\mathbb{Z}_2$ . This tells us that the coefficients of  $P_2$  span  $\mathbb{Z}_2^4$ , so there is a polynomial representation for all the vectors of  $\mathbb{Z}_2^4$  for the given valuations of  $p$  and  $q$ .

**Theorem 4.3.** *For any binary connective in  $\mathcal{L}_2$ , there is a unique polynomial in  $\mathbb{Z}_2[p, q]$  that represents this connective.*

*Proof.* To show that there is a polynomial in  $\mathbb{Z}_2[p, q]$  that can represent any binary connective of  $\mathcal{L}_2$ , it needs to be shown that the valuation of all connectives can be represented by a fourth dimensional vector. Further it must be shown that this vector can be mapped onto a vector in  $\mathbb{Z}_2^4$ . Define  $h : V_2 \rightarrow \{0, 1\}$  such that  $h(F) = 0$  and  $h(T) = 1$ . Let  $*$   $\in \Sigma_2$  be an arbitrary binary connective. Define vector  $\vec{v}$  such that  $v_i$  is the  $i^{\text{th}}$  entry of  $\vec{v}$  and each  $v_i$  is the valuation,  $g$ , of  $*$  such that  $v_1 = g(T * T)$ ,  $v_2 = g(T * F)$ ,  $v_3 = g(F * T)$  and  $v_4 = g(F * F)$ . Taking  $h_2(\vec{v})$  gives a vector in  $\mathbb{Z}_2^4$  and from above the coefficients of  $P_2 \in \mathbb{Z}_2[p, q]$  span  $\mathbb{Z}_2^4$ , so there is a unique set of coefficients for a polynomial in  $\mathbb{Z}_2[p, q]$  that represents the connective  $*$ . Let  $P_*(p, q) \in \mathbb{Z}_2[p, q]$  be the polynomial that represents the connective  $*$ . From Theorem 4.2, the reduced form of  $P_*$  is unique. Since  $*$  was an arbitrary binary connective, there exists a unique polynomial representation for all the binary connectives of  $\mathcal{L}_2$ .  $\square$

The proof for the existence of polynomial representations for 2-valued propositional connectives of any arity is proved in the general case in section 9.

To find the representative polynomial of a binary connective we can use the linear system from above to find the coefficients of the polynomial. For example, to find the polynomial that represents the material conditional ( $\rightarrow$ ) of propositional logic we would row reduce the following:

$$\left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \sim \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right].$$

This shows that the representative polynomial for  $p \rightarrow q$  is  $h(p)h(q) + h(p) + 1$  where  $h$  is the mapping from  $V_2$  to  $\{0, 1\}$ . Here  $\sim$  symbolizes that the two matrices are row equivalent.

## 5 Evaluating 2-Valued Propositional Logic Arguments Using Base 2 Modular Arithmetic

An argument in propositional logic is a set of premises that are taken to support a given conclusion [2]. For  $p_i, q \in \mathcal{L}_2$ , an argument with  $k$  premises can be symbolized as  $(p_1 \wedge p_2 \wedge \dots \wedge p_k) \rightarrow q$ . The argument is said to be valid if and only if whenever all the premises are true the conclusion is also true. Another way to say this is that  $(p_1 \wedge p_2 \wedge \dots \wedge p_k) \rightarrow q$  is a tautology [11]. An element in  $\mathcal{L}_2$  is a tautology if it is true on all possible truth values. By Theorem 4.2 and section 4, there is a unique polynomial representation for tautologies in  $\mathcal{L}_2$ . This can be determined using the method from the end of section 4:

$$\left[ \begin{array}{cccc|c} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right] \sim \left[ \begin{array}{cccc|c} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 \end{array} \right].$$

Thus the polynomial representation of tautologies is the constant 1.

So to determine if an argument is valid using its polynomial representation, find the polynomial representation of the argument and reduce the resulting polynomial. The argument is valid if and only if the polynomial reduces to 1. Take for example *Modus Ponens*:

$$\begin{aligned} 1. & p \rightarrow q \\ 2. & p \\ \therefore & q. \end{aligned} \tag{2}$$

Using the symbolic representation, this becomes:

$$[(p \rightarrow q) \wedge p] \rightarrow q. \tag{3}$$

Finding the representative polynomial of 3 gives

$$[(p \times q + p + 1) \times p] \times q + [(p \times q + p + 1) \times p] + 1 \tag{4}$$

which simplifies via Theorem 4.2 to

$$\begin{aligned}
& p^2 \times q^2 + p^2 \times q + p \times q + p^2 \times q + p^2 + p + 1 \\
& = 4 \times p \times q + 2 \times p + 1 \\
& = 1
\end{aligned} \tag{5}$$

The simplification of equation 5 to '1' shows that the argument is a tautology and therefore *Modus Ponens* is valid. Now consider the fallacy of affirming the consequent:

$$\begin{aligned}
& 1. p \rightarrow q \\
& 2. q \\
& \therefore p.
\end{aligned} \tag{6}$$

Symbolizing this argument and finding the representative polynomial gives

$$\begin{aligned}
& [(p \times q + p + 1) \times q] \times p + (p \times q + p + 1) \times q + 1 \\
& = p^2 \times q^2 + p^2 \times q + p \times q + p \times q^2 + p \times q + q + 1 \\
& = 5 \times p \times q + q + 1 \\
& = p \times q + q + 1.
\end{aligned} \tag{7}$$

So equation 7 simplifies to  $p \times q + q + 1$ . Since this is not a tautology, affirming the consequent is invalid.

## 6 3-Valued Structures

The structure for 3-valued propositional logic will employ  $\Sigma_3$ ,  $\mathcal{L}_3$ , and  $V_3 = \{F, I, T\}$ . In  $V_3$ ,  $I$  represents indifferent or undefined.

The structure for base 3 modular arithmetic is as follows:  $\{\{0, 1, 2\}, +, \times, 0, 1\}$ . The tables for  $+$  and  $\times$  are:

+	0	1	2
0	0	1	2
1	1	2	0
2	2	0	1

×	0	1	2
0	0	0	0
1	0	1	2
2	0	2	1

## 7 Representation of 3-Valued Propositional Logic Connectives Using Mod 3 Polynomials

Polynomials in  $\mathbb{Z}_3[p, q]$  have the reduced form via Theorem 4.2:

$$\begin{aligned}
& c_1 \times p^2 \times q^2 + c_2 \times p^2 \times q + c_3 \times p^2 + c_4 \times p \times q^2 \\
& + c_5 \times p \times q + c_6 \times p + c_7 \times q^2 + c_8 \times q + c_9.
\end{aligned} \tag{8}$$

Call this polynomial  $P_3(p, q)$ . Accounting for the nine possible value combinations of  $p$  and  $q$  and substituting these values into the polynomial  $P_3$  yields nine linear equations. Using the method from section 4 to create a linear system for the coefficients of  $P_3$  yields the matrix

$$\begin{bmatrix}
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\
1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 \\
1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{bmatrix}$$

which has the inverse:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 2 & 1 & 0 & 2 & 1 & 0 & 2 & 1 & 0 \\ 0 & 0 & 2 & 0 & 0 & 2 & 0 & 0 & 2 \\ 2 & 2 & 2 & 1 & 1 & 1 & 0 & 0 & 0 \\ 1 & 2 & 0 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 2 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

So the coefficients of  $P_3$  span  $\mathbb{Z}_3^9$ . A similar proof to the one presented in Theorem 4.3 shows that for any binary connective in  $\mathcal{L}_3$ , there is a unique polynomial in  $\mathbb{Z}_3[p_q]$  that represents this connective. Further the method for finding the coefficients of these polynomials as outlined in section 4 can be extended into the  $n = 3$  case.

## 8 Łukasiewicz 3-Valued Axioms

The 3-valued propositional logic employed by Łukasiewicz used the operators, expressed in Polish notation,  $C$  and  $N$  [12]. For the notation used in this paper these operators become  $\rightarrow$  and  $\neg$  respectively. The truth tables for these operators are as follows:

$p$	$q$	$p \rightarrow q$	$\neg p$
$T$	$T$	$T$	$F$
$T$	$I$	$I$	$F$
$T$	$F$	$F$	$F$
$I$	$T$	$T$	$I$
$I$	$I$	$T$	$I$
$I$	$F$	$I$	$I$
$F$	$T$	$T$	$T$
$F$	$I$	$T$	$T$
$F$	$F$	$T$	$T$

Applying  $h : V_3 \rightarrow \{0, 1, 2\}$  to the valuations of these operators such that  $h(F) = 0$ ,  $h(I) = 1$ , and  $h(T) = 2$  and entering them into vectors gives:

$$\begin{bmatrix} 2 \\ 1 \\ 0 \\ 2 \\ 2 \\ 1 \\ 2 \\ 2 \\ 2 \end{bmatrix} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 1 \\ 1 \\ 2 \\ 2 \\ 2 \end{bmatrix}$$

Augmenting the coefficient matrix from section 7 with these vectors and row reducing gives the coefficients for the polynomial functions that represent these operators.

$$\left[ \begin{array}{cccccccc|cc} 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 0 \\ 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 \end{array} \right] \sim \left[ \begin{array}{cccccccc|cc} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 2 \end{array} \right]$$

From this  $(p \rightarrow q)$  is represented by the polynomial  $2 \times h_3(p)^2 \times h_3(q)^2 + 2 \times h_3(p)^2 \times h_3(q) + 2 \times h_3(p) \times h_3(q)^2 + h_3(p) \times h_3(q) + 2 \times h_3(p) + 2$  and  $(\neg p)$  is represented by the polynomial  $2 \times h_3(p) + h_3(p)$ .

Łukasiewicz employed four axioms for his logic:

$$\begin{aligned}
Ax1 &: p \rightarrow (q \rightarrow p) \\
Ax2 &: (p \rightarrow q) \rightarrow [(q \rightarrow r) \rightarrow (p \rightarrow r)] \\
Ax3 &: (\neg p \rightarrow \neg q) \rightarrow (q \rightarrow p) \\
Ax4 &: ((p \rightarrow \neg p) \rightarrow p) \rightarrow p
\end{aligned}$$

Being axioms all four should be tautologies. Finding the representative polynomials for these axioms and reducing the polynomials using Theorem 4.2 yields a single 2 for all four axioms(See Appendix). To verify that 2 is indeed the unique polynomial that represents tautologies row reduce the following matrix:

$$\left[ \begin{array}{cccccccc|c}
1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\
1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 2 \\
0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 2 \\
1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 2 \\
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\
0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2
\end{array} \right] \sim \left[ \begin{array}{cccccccc|c}
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1
\end{array} \right]$$

Thus 2 is the polynomial that represents tautologies which means that the axioms all simplify to tautologies as expected.

## 9 Determining which values of $n$ allow for polynomial representations of logical connectives

Let  $P_n$  represent the reduced general form of  $Z_n[p, q]$ . For base  $n$  modular arithmetic, the structure will be  $\{\{0, 1, \dots, n-1\}, +, \times, 0, 1\}$ . Let  $L_n$  map  $\mathcal{L}_n$  to  $Z_n[p, q]$  and  $h_n$  be a function that maps  $\{F, I_1, I_2, \dots, I_{n-2}, T\}$  to  $\{0, 1, 2, \dots, n-1\}$  such that  $h_n(F) = 0$ ,  $h_n(I_1) = 1$ ,  $h_n(I_2) = 2$ , ...,  $h_n(I_{n-2}) = n-2$ ,  $h_n(T) = n-1$ . To examine whether or not  $n$  is viable for a given structure, two cases where  $n$  is composite and  $n$  is prime will be examined.

The method from sections 4 and 7 will be used to set up a linear system for the coefficients of  $P_n$ . The rows will be formed by setting the values of  $p$  and  $q$  in the following way:

$$\begin{aligned}
&(p = n-1, q = n-1) \\
&(p = n-1, q = n-2) \\
&\vdots \\
&(p = n-1, q = 1) \\
&(p = n-2, q = n-1) \\
&\vdots \\
&(p = 1, q = 1) \\
&(p = n-1, q = 0) \\
&(p = n-2, q = 0) \\
&\vdots \\
&(p = 1, q = 0) \\
&(p = 0, q = n-1) \\
&\vdots \\
&(p = 0, q = 1) \\
&(p = 0, q = 0)
\end{aligned}$$

The columns will represent the following powers of  $p$  and  $q$ :

$$p^{n-1}q^{n-1} \quad p^{n-1}q^{n-2} \quad \dots \quad p^{n-1}q^1 \quad p^{n-2}q^{n-1} \quad \dots \quad p^1q^1 \quad p^{n-1}q^0 \quad \dots \quad p^1q^0 \quad p^0q^{n-1} \quad \dots \quad p^0q^0$$

Call the matrix constructed to these specifications  $C_2$ .

**Theorem 9.1** (Case when  $n$  is composite). *When  $n$  is a composite number  $Z_n[p, q]$  is insufficient to represent all the connectives of  $n$ -valued propositional logic.*

*Proof.* Consider an  $n$ -valued propositional logic. Let  $n$  be a composite number such that  $n = i \times j$  where  $i$  and  $j$  are integers. From section 9 this can be translated into a base  $n$  polynomial function with a coefficient matrix  $C_2$ . There is a row in  $C_2$  such that  $p = i$  and  $q = 0$ . Multiplying this row by  $k$  yields a row that is linearly dependent with the row made when  $p = 0$  and  $q = 0$ . These two rows are shown below.

$$(p = i, q = 0) \quad [0 \quad 0 \quad \dots \quad 0 \quad i^{n-1} \quad i^{n-2} \quad \dots \quad i^1 \quad 0 \quad \dots \quad 1]$$

Multiplying this row by  $k$  gives:

$$[0 \quad 0 \quad \dots \quad 0 \quad k]$$

When  $(p = 0, q = 0)$

$$[0 \quad 0 \quad \dots \quad 0 \quad 1]$$

Since  $C_2$  contains at least two rows that are linearly dependent  $C_2$  does not span  $\mathbb{Z}_n^{n^2}$  and therefore can not span all possible binary connectives of  $\mathcal{L}_n$ . This means that the base  $n$  polynomial is insufficient for representing all binary connectives and therefore insufficient for representing all the connectives of  $n$ -valued propositional logic.  $\square$

**Theorem 9.2** (Case when  $n$  is prime). *When  $n$  is a prime number,  $\mathbb{Z}_n[x_1, x_2, \dots, x_k]$  is sufficient to represent any  $k$ -ary connective of  $n$ -valued propositional logic.*

*Proof.* Let  $n$  be prime. This will be a proof by induction on the coefficient matrices constructed from the  $k$ -variate polynomial in  $\mathbb{Z}_n[x_1, x_2, \dots, x_k]$ . The base case will be  $k = 2$ . First, though, consider the following lemma.

**Lemma 9.3.** *Let  $C_k$  represent the coefficient matrix for the generalized polynomial  $P(x_1, \dots, x_k) \in \mathbb{Z}_n[x_1, \dots, x_k]$ . If  $C_k$  is invertible then there is a representative polynomial in  $\mathbb{Z}_n[x_1, \dots, x_k]$  for all  $k$ -ary connectives in  $\mathcal{L}_n$ .*

*Proof.* This follows from an extension of theorem 4.3. If it can be shown that the valuation for any  $k$ -ary connective can be represented by a vector in  $\mathbb{Z}_n^{(n^k)}$  then, assuming  $C_k$  is invertible such that it spans  $\mathbb{Z}_n^{(n^k)}$ , there is a representative polynomial in  $\mathbb{Z}_n[x_1, \dots, x_k]$  for all  $k$ -ary connectives. Let  $f$  be the symbol for an arbitrary  $k$ -ary connective in  $\Sigma_n$  and the vector  $\vec{v}$  be such that its entries are the valuations of  $f$  for all possible combinations of valuations for  $x_1, x_2, \dots, x_k$ . Thus  $|\vec{v}| = n^k$ . Now let  $h : V_n \rightarrow \{0, 1, 2, \dots, n-1\}$  be a one-to-one and onto function. So  $h(\vec{v}) \in \mathbb{Z}_n^{(n^k)}$ .  $\square$

From lemma 9.3, to show that a representative polynomial exists for any given  $k$ -ary connective it will be sufficient to show that  $C_k$ , the coefficient matrix of  $k$  variables, is invertible.

Set  $m = n^2$  and assume for contradiction that there exists a non-trivial vector  $\vec{x} \in \mathbb{Z}_n^m$  such that  $C_2 \cdot \vec{x} = \vec{0}$ . Let  $x_i$  denote the  $i^{\text{th}}$  element of  $\vec{x}$ . Note that  $x_m = 0$ , since the  $m^{\text{th}}$  row of  $C_2$  is

$$[0 \quad 0 \quad \dots \quad 0 \quad 1]$$

So the  $m^{\text{th}}$  row of  $C_2$  multiplied by  $\vec{x}$  equals 0 and thus  $x_m = 0$ .

Now consider the  $m - (n - 1)$  through  $m - 1$  rows which take the form:

$$\begin{bmatrix} 0 & 0 & \dots & 0 & (n-1)^{n-1} & (n-1)^{n-2} & \dots & (n-1)^1 & (n-1)^0 \\ 0 & 0 & \dots & 0 & (n-2)^{n-1} & (n-2)^{n-2} & \dots & (n-2)^1 & (n-2)^0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & 2^{n-1} & 2^{n-2} & \dots & 2^1 & 2^0 \\ 0 & 0 & \dots & 0 & 1^{n-1} & 1^{n-2} & \dots & 1^1 & 1^0 \end{bmatrix}$$

Since the first  $m - n$  columns are zero and  $x_m$  is also zero, the columns of interest are  $m - n + 1$  through  $m - 1$ . These entries form an  $(n - 1) \times (n - 1)$  matrix of the form:

$$\begin{bmatrix} (n-1)^{n-1} & (n-1)^{n-2} & \dots & (n-1)^1 \\ (n-2)^{n-1} & (n-2)^{n-2} & \dots & (n-2)^1 \\ \vdots & \vdots & \ddots & \vdots \\ 2^{n-1} & 2^{n-2} & \dots & 2^1 \\ 1^{n-1} & 1^{n-2} & \dots & 1^1 \end{bmatrix}.$$

Call this matrix  $A_2$ . Now define a matrix  $B_2$  with the form:

$$(n-1) \cdot \begin{bmatrix} (n-1)^{n-1} & (n-2)^{n-1} & \dots & 1^{n-1} \\ (n-1)^1 & (n-2)^1 & \dots & 1^1 \\ (n-1)^2 & (n-2)^2 & \dots & 1^2 \\ \vdots & \vdots & \ddots & \vdots \\ (n-1)^{n-3} & (n-2)^{n-3} & \dots & 1^{n-3} \\ (n-1)^{n-2} & (n-2)^{n-2} & \dots & 1^{n-2} \end{bmatrix}$$

**Lemma 9.4.** Let  $n$  be a prime. For  $S = \sum_{i=1}^{n-1} i^r \pmod n$ , if  $(n-1) \nmid r$  then  $S \equiv 0 \pmod n$ . If  $(n-1) \mid r$  then  $S \equiv (n-1) \pmod n$ . [6] □

When multiplying  $B_2 \cdot A_2$  each entry is  $\sum_{i=1}^{n-1} i^r \pmod n$ . If necessary  $r$  can be reduced by using lemma 4.1 until  $1 \leq r < n$ . For example, the multiplication of the 1<sup>st</sup> row of  $B_2$  with the  $(n-1)^{th}$  column of  $A_2$  would be  $\sum_{i=1}^{n-1} i^{1+1} = \sum_{i=1}^{n-1} i^2$ . The entry can be determined using lemma 9.4. Note that once  $r$  has been reduced the only case when  $(n-1) \mid r$  is when  $r = (n-1)$ . It is only the case that  $r = (n-1)$  along the diagonal of the resulting matrix. So  $B_2 \cdot A_2$  takes the form:

$$(n-1) \cdot \begin{bmatrix} (n-1) & 0 & \dots & 0 \\ 0 & (n-1) & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \dots & 0 & (n-1) \end{bmatrix} = I_{n-1}$$

where  $I_{n-1}$  is the  $(n-1) \times (n-1)$  identity matrix. This follows because  $(n-1)^2 = n^2 - 2n + 1 \equiv 1 \pmod n$ . This means that  $A_2$  is invertible and for the equation  $A_2 \cdot \vec{v} = \vec{0}$ ,  $\vec{v} = \vec{0}$ . For the case at hand this means that  $x_{m-n+1}$  through  $x_{m-1}$  are equal to 0. Similarly for rows  $m-2n+2$  through  $m-n$  in  $C_2$ , because  $A_2$  occurs in columns  $m-2n+2$  through  $m-n$ . To illustrate here is the binary coefficient matrix for  $P_3$ :

$$\begin{bmatrix} 1 & 2 & 2 & 1 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 2 & 2 & 1 & 2 & 1 & 1 & 1 \\ 1 & 2 & 1 & 2 & 1 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore  $x_{m-2n+2}$  through  $x_{m-n}$  are also all equal to 0.

This leaves only the first  $m-2n+1$  rows and columns which can be partitioned into the following form:

$$\begin{bmatrix} (n-1)^{n-1} \cdot A_2 & (n-1)^{n-2} \cdot A_2 & \dots & (n-1)^1 \cdot A_2 \\ (n-2)^{n-1} \cdot A_2 & (n-2)^{n-2} \cdot A_2 & \dots & (n-2)^1 \cdot A_2 \\ \vdots & \vdots & \ddots & \vdots \\ 2^{n-1} \cdot A_2 & 2^{n-2} \cdot A_2 & \dots & 2^1 \cdot A_2 \\ 1^{n-1} \cdot A_2 & 1^{n-2} \cdot A_2 & \dots & 1^1 \cdot A_2 \end{bmatrix}.$$

Notice that the scalar multiples of  $A_2$  are exactly the entries of  $A_2$ . Therefore it has the following inverse:

$$(n-1) \cdot \begin{bmatrix} (n-1)^{n-1} \cdot B_2 & (n-2)^{n-1} \cdot B_2 & \dots & 1^{n-1} \cdot B_2 \\ (n-1)^1 \cdot B_2 & (n-2)^1 \cdot B_2 & \dots & 1^1 \cdot B_2 \\ (n-1)^2 \cdot B_2 & (n-2)^2 \cdot B_2 & \dots & 1^2 \cdot B_2 \\ \vdots & \vdots & \ddots & \vdots \\ (n-1)^{n-3} \cdot B_2 & (n-2)^{n-3} \cdot B_2 & \dots & 1^{n-3} \cdot B_2 \\ (n-1)^{n-2} \cdot B_2 & (n-2)^{n-2} \cdot B_2 & \dots & 1^{n-2} \cdot B_2 \end{bmatrix}.$$

This follows because  $A_2 \cdot B_2 = I_{n-1}$  and the scalars ensure that all the entries except those along the diagonal are zero (from lemma 9.4). Also from lemma 9.4 the scalars down the diagonal are all  $n-1$  so that when multiplied by the



scalar on the inverse,  $B_2$ , simplifies to 1. So the resulting matrix is a partitioned diagonal matrix with  $I_{n-1}$  along the diagonal which is itself the identity matrix.

Thus,  $x_1$  through  $x_{m-2n+1}$  are all equal to 0. This means that  $\vec{x}$  is trivial and therefore  $C_2 \cdot \vec{x} = \vec{0}$  has only the trivial solution. From this it follows that  $C_2$  is invertible [7].

Since  $C_2$  is invertible when  $n$  is prime,  $\mathbb{Z}_n[p, q]$  is sufficient for representing all binary connectives in  $\mathcal{L}_n$  by lemma 9.3.

Assume that  $C_k$ , the coefficient matrix of  $k$  variables, is invertible. Consider the coefficient matrix of  $k+1$  variables,  $C_{k+1}$ . It can be constructed such that it has the following form:

$$\begin{bmatrix} (n-1)^{n-1} \cdot C_k & (n-1)^{n-2} \cdot C_k & \dots & (n-1)^1 \cdot C_k & C_k \\ (n-2)^{n-1} \cdot C_k & (n-2)^{n-2} \cdot C_k & \dots & (n-2)^1 \cdot C_k & C_k \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 2^{n-1} \cdot C_k & 2^{n-2} \cdot C_k & \dots & 2^1 \cdot C_k & C_k \\ 1^{n-1} \cdot C_k & 1^{n-2} \cdot C_k & \dots & 1^1 \cdot C_k & C_k \\ 0^{n-1} \cdot C_k & 0^{n-2} \cdot C_k & \dots & 0^1 \cdot C_k & C_k \end{bmatrix}.$$

For example take  $C_2$  and  $C_3$  when  $n = 2$ :

$$C_2 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$C_3 = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C_2 & C_2 \\ 0 & C_2 \end{bmatrix}$$

Assume for contradiction that there is a non-trivial vector  $\vec{x}$  such that  $C_{k+1} \cdot \vec{x} = \vec{0}$ . Notice that the last  $n^k$  entries of  $\vec{x}$  are zero since  $C_k$  is invertible and is the only non-zero partition for the final  $n^k$  rows of  $C_{k+1}$ . Let  $A_{k+1}$  be the first  $n^{k+1} - n^k$  rows and columns of  $C_{k+1}$ .  $A_{k+1}$  is a partitioned matrix of  $C_k$  with scalar multiples from  $A_2$  and so has the following inverse:

$$(n-1) \cdot \begin{bmatrix} (n-1)^{n-1} \cdot C_k^{-1} & (n-2)^{n-1} \cdot C_k^{-1} & \dots & 1^{n-1} \cdot C_k^{-1} \\ (n-1)^1 \cdot C_k^{-1} & (n-2)^1 \cdot C_k^{-1} & \dots & 1^1 \cdot C_k^{-1} \\ (n-1)^2 \cdot C_k^{-1} & (n-2)^2 \cdot C_k^{-1} & \dots & 1^2 \cdot C_k^{-1} \\ \vdots & \vdots & \ddots & \vdots \\ (n-1)^{n-3} \cdot C_k^{-1} & (n-2)^{n-3} \cdot C_k^{-1} & \dots & 1^{n-3} \cdot C_k^{-1} \\ (n-1)^{n-2} \cdot C_k^{-1} & (n-2)^{n-2} \cdot C_k^{-1} & \dots & 1^{n-2} \cdot C_k^{-1} \end{bmatrix}.$$

This means that  $\vec{x} = \vec{0}$  and so  $C_{k+1}$  is invertible.

Therefore  $n$ -valued propositional connectives of all arities can be represented by a polynomial in  $\mathbb{Z}_n[x_1, x_2, \dots]$  if  $n$  is prime.  $\square$

Therefore from Theorems 9.1 and 9.2, all connectives of  $\mathcal{L}_n$  can be represented by modular polynomials if and only if  $n$  is prime.

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## Appendix

```

> LoMat:=proc(integer)
> local N, LoMod, A, n, m, l, k;
> N:=integer;
> A:=Matrix(N^2);
> LoMod:=sum(sum(p^i*q^j, j=0..N-1),i=0..N-1);
> for n from 0 to N-1
> do
> for m from 1 to N
> do
> for l from 1 to N^2
> do
> k:=N-m;
> A(m+n*N,l):=subs(p=N-1-n, q=k, op(N^2+1-l,LoMod)) mod N;
> end do;
> end do;
> end do;
> return A;
> end proc;

```

$$LoMat3 := \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 \\ 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 \\ 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

$LoMat3C := \langle | \rangle (LoMat3, \langle , \rangle (2, 1, 0, 2, 2, 1, 2, 2, 2))$

$$LoMat3C := \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 2 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 2 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$C := (p, q) \mapsto 2p^2q^2 + 2p^2q + 2pq^2 + pq + 2p + 2$$

$$LoMat3N := \begin{bmatrix} 1 & 2 & 1 & 2 & 1 & 2 & 1 & 2 & 1 & 0 \\ 1 & 1 & 1 & 2 & 2 & 2 & 1 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 2 & 0 & 0 & 1 & 0 \\ 1 & 2 & 1 & 1 & 2 & 1 & 1 & 2 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 2 \end{bmatrix}$$

$$N := p \mapsto 2p + 2$$

> weakSimp:=proc(expr,list)

```

> local i, j, deg, simp, varList;
> deg:=degree(expr);
> simp:=expand(expr);
> varList:=list;
> if nops(varList)=0 then
> return "varList is empty";
> end if;
> for i from 1 to nops(varList) do
> simp:=(add(coeff(simp,varList[i],n)*varList[i]^(abs((n mod 2)-2)),n=1..deg)
> +coeff(simp,varList[i],0)) mod 3;
> end do;
> return expand(simp) mod 3;
> end proc:

```

$$Ax1 := C(p, C(q, p))$$

$$Ax2 := C(C(p, q), C(C(q, r), C(p, r)))$$

$$Ax3 := C(C(N(p), N(q)), C(q, p))$$

$$Ax4 := C(C(C(p, N(p)), p), p)$$

*weakSimp*(Ax1, [p, q])

2

*weakSimp*(Ax2, [p, q, r])

2

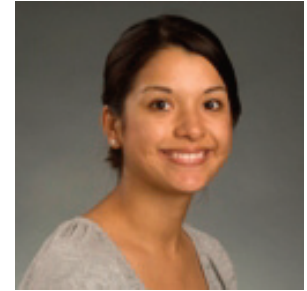
*weakSimp*(Ax3, [p, q])

2

*weakSimp*(Ax4, [p])

2

**Perceptions of Discrimination:  
An Analysis of Four National Surveys of Latinos;  
Findings from the Pew Hispanic Center/Kaiser Family  
Foundation National Survey of Latinos**



Jenny Gallegos: McNair Scholar

Dr. Ross E. Burkhart: Mentor

Political Science

**Abstract**

*Objective: Throughout the history of the United States different legal measures have resulted in efforts to deal with immigration issues, and some have resulted in adverse consequences for Latinos in particular. Massive immigration from 1850-1920 arose the historic distrust and suspicion of Anglos toward Mexicans and tended to evoke various kinds of repressive acts, excluding Mexican Americans from political participation (Garcia and de la Garza 1977). After the 9/11 attacks Latinos reported a heightened level of perceived discrimination as well as fear of deportation, even among U.S. Citizens. Methods: Using multiple-regression analysis this study analyzes four datasets of National Survey of Latinos<sup>1</sup> for years 2002, 2004, 2006, and 2007. Individual factors such as gender, citizenship, income level, and marital status were used to determine the impact on perceptions of discrimination among Latinos. Significance: Reasonable perceptions of discrimination, a lack of political trust and sporadic political participation for Latinos suggest troubling prospects for the future of race relations, the American political system, and the entire essence of democracy.*

**Introduction**

Quantitative restrictions, border patrols, work permits, economic needs tests, wage parity requirements, raids, and repatriation among others have been part of the efforts to restrict immigration (Gradstein and Shiff, 2004). The justification of these measures is first derived from labor shortages, economic hardship, the perceived threat from immigrants, or when states' legislatures are perceived to be overly responsive to minority groups (Hero and Colbert, 2001). These types of measures and other events often influence public opinion, public response, and so much as trust in the government (Michelson, 2001). In addition, an absence of socialization and familiarity with American political processes help explain why Latinos are less politically knowledgeable and involved as native born people (Nicholson, Pantoja, and Segura, 2006).

Mexicans first found themselves in a state of political vulnerability and powerlessness after the loss of land and identity in the ratification of the Treaty of Guadalupe Hidalgo (Takaki 2008). One million Mexicans immigrated to the United States between 1850 and 1920 as a result of U.S. built railroads in Mexico, the Mexican Revolution (1910-1917), and other push-pull social and economic factors (Espinosa, 2007; Ngai, 2004). In the 1920s Mexicans were enumerated as a separate race and deportation statutes were put in place. "The possibility of sweeps, detainment, interrogation, and deportation spread apprehension among Mexicans and loomed as perhaps the single greatest indicator that Mexicans did not belong" (Ngai, 2004). In 1924 the Johnson-Reed Act was the nation's first comprehensive restriction law. This act drew a new racial and ethnic map based on new categories and hierarchies of difference. "The nation was racially and spatially reimagined; immigration restriction produced the illegal alien as new legal and political subject (Ngai, 2004)." The repatriation of over 400,000 Mexicans in the early 1930s was a racial expulsion program exceeded in scale only by the Native American removals of the 19<sup>th</sup> century (Ngai, 2004).

In 1942, due to labor shortages caused by World War II, the U.S. government established the Bracero program, under which Mexico sent workers to the U.S. as temporary laborers. This program encouraged illegal entry for Mexicans economic refuge and other interests. From 1949 to 1954 over one million undocumented immigrants

<sup>1</sup> The Pew Hispanic Center bears no responsibility for the interpretations offered, or conclusions made based on analysis of the Pew Hispanic Center National Survey of Latinos.

entered the United States. Pressured to address the massive immigration rates the U.S. Immigration and Naturalization Service (INS) conducted Operation Wetback (1954). Operation Wetback was a massive enforcement effort aimed at apprehending and deporting undocumented Mexicans in the southwest. According to Commissioner General Joseph M. Swing the “alarming, ever-increasing flood tide” of undocumented migrants from Mexico constituted “an actual invasion of the U.S.” Between 1953 and 1955 801,069 immigrants were apprehended.

The Mexican American civil rights movement was at its peak from 1965-1975. Mexican Americans accepted the legitimacy and necessity of following the political process and committed themselves to developing the resources required to deal effectively in the political arena (Garcia and de la Garza, 1977). These efforts generated the establishment of grassroots organizations such as the Mexican American Legal and Education Fund and League of Latin American Citizens.

In the state of California in 1986 a proposition passed making English the official language; in 1998 Proposition 227 eliminated bilingual education in public schools; and in 1994 Proposition 187 denied social services to undocumented immigrants. In reaction to the anti-Latino atmosphere, Mexican Americans became more concerned about racism and discrimination (Michelson, 2003). Immigrants, undocumented and authorized Latinos were alarmed by the threatened enactment of House of Representatives' Bill 4437: “Border Protection, Antiterrorism, and Illegal Immigration Control of 2005.” The dynamics of the marches in protest of HR 4437 were in many ways prefigured by events that occurred in California as a response to Proposition 187 (Milkman, 2006).

The legacy of discrimination and prejudice is a major part of the context in which today’s generation of Latinos still experiences. The effects of racial exclusion and discrimination continue to influence social and political outcomes (Blank, Dabady, and Citro, 2004). Cumulative discrimination is defined as the dynamic concept that captures systematic processes occurring over time and across domains. The effects of cumulative discrimination can be transmitted through organizations and social structures of society; the ways in which discrimination effects are transmitted across domains and over generations often depend on the social organization (Blank, Dabady, and Citro, 2004).

I intend to expand the knowledge about the role that current and past discrimination may play in shaping American society. It is important to recognize three aspects of the discrimination process. They are: 1) the effects of discrimination may cumulate across generations and throughout history, 2) the effects of discrimination may cumulate over time through the course of an individual’s life across different domains, and 3) the effects of discrimination may cumulate over time through the course of an individual’s life sequentially within any one domain (Blank, Dabady, and Citro, 2004). Immigration policies with adverse consequences for racial and ethnic minorities may affect the diversity of American communities. Mistreatment of Latinos has impacted their sense of belonging in America and their ties to their national origin. The intent of this study is to sharpen the concept of discrimination and factors that affect the perception of its existence.

The social environment that surrounds Latinos makes certain aspects of their identity more significant than others according to Social Identity Theory. Social environments that foster adverse consequences for racial minorities help shape the identity of Latinos on an individual level affecting their political trust. It is acculturation that is corrosive of political trust for Latinos of Mexican descent, and trust in government impacts both government effectiveness and individual political behavior (Michelson, 2003). It is also important to recognize generational effects. For example, third-generation Mexican Americans are more acculturated according to socioeconomic status and linguistic measures and are more pessimistic about the political system than first and second generation Mexican Americans. Whether these suggestions also pertain to Latinos of other origins has not been tested.

## Literature review

While some studies suggest that larger percentages of a minority group in a given population promotes interracial contact and cultural literacy others suggest that it promotes conflict, hostility, and tension (Oliver and Wong, 2003; Dixon and Rosenbaum, 2004; Wenzel, 2006). Other literature on behavioral contact finds that contact between majority and minority populations significantly reduces prejudicial attitudes and opinions about minorities and minority based policies (Stein, Post, and Rinden, 2000). MacKuen (1981) finds evidence that direct experiences influence individual’s political concerns (as cited by Michelson, 2001). Michelson also finds that recently naturalized Mexican American voters are significantly more concerned with racism and discrimination than native-born Mexican American voters or non-naturalized Mexicans (2003); however, she does not address undocumented Mexican Americans. According to Michelson the heightened sensitivity was due to the political atmosphere of the Chicago area when the survey was taken. Events and the sequence of events often influence public opinion; trust in the government, and political behavior (Hero, 2005; Michelson, 2001). Perceptions of discrimination among Latinos motivate public opinion towards immigration and bilingual education, and collective action toward immigration

according to Gabriel Sanchez's analysis of the 1999 National Survey of Latinos. Not only does discrimination have a negative effect on health, but it is also a source of chronic stress among Hispanics (R. Cardarelli, K. Cardarelli, and Chiapa, 2007).

## Methodology

According to Blank, Dabady, and Citro, longitudinal and repeated cross-sectional data illuminate trends and changes in patterns of racially discriminatory attitudes and behaviors toward Latinos. Perceived discrimination may over/under report discrimination assessed by other methods. The guiding question of this study is to identify what social factors influence Latinos' perception of discrimination.

This research was conducted as a panel design. The data sources are the Pew Hispanic Center/Kaiser Family Foundation 2002 National Survey of Latinos, Pew Hispanic Center/Kaiser Family Foundation 2004 National Survey of Latinos: Politics and Civic Engagement, The 2006 National Survey of Latinos: The Immigration Debate, and Pew Hispanic Center 2007 National Survey of Latinos<sup>2</sup>. The 2002 survey focused on attitudes and experiences of Latinos on a wide variety of topics, 2004 focused on politics and civic participation, 2006 focused on the immigration debate, and the 2007 survey focused on illegal immigration.

For the National Survey of Latinos (NSL) in 2002 interviews were conducted by telephone for 67 days among a nationally representative sample of 2,929 adults, 18 years and older, who were selected at random. Although observations include non-Hispanics, these observations were not be used because variation cannot be measured when compared to the National Survey of Latinos for 2004, 2006, and 2007. In 2004 the sample design employed a highly stratified disproportionate Random Digit Dialing sample of the 48 contiguous states, according to the Pew Hispanic Center. The 2004 survey interviews were conducted by telephone for 48 days among a nationally representative sample of 2,288 Latino adults, 18 years and older. Unlike 2004, the 2006 survey results were weighted to better represent the distribution of adults throughout the United States. The 2006 survey and 2007 survey interviews were conducted by telephone for a one month period; the survey was drawn through Random Digit Dialing. The surveys were conducted among nationally represented samples of 2,000 Hispanics adults, 18 years and older.

I used multiple regression analysis to test which independent variables were statistically significant. The dependent variable is derived from the following question asked in each survey: "In general, do you think discrimination against Latinos is a major problem, minor problem, or not a problem in preventing Latinos from succeeding in America?" The independent variables are income, birthplace, years living in the U.S., U.S. citizenship, employment status, partisanship, marital status, educational level, and gender. By using Multiple Regression Analysis I determined which independent variables explain variation in the dependent variable. My hypotheses for this study are as follows:

## Hypotheses

1. All other things being equal, an increase in income leads to a decrease in the perception of discrimination preventing Latinos from succeeding in America.
2. All other things being equal, respondents born outside the United States will be more likely to believe perceived discrimination prevents Latinos from succeeding in America than respondents born in the U.S.
3. All other things being equal, respondents of U.S. citizenship are less likely to perceive discrimination preventing Latinos from succeeding in America than non-citizens.
4. All other things being equal, having full-time employment leads to a decreased perception of discrimination preventing Latinos from succeeding in America than respondents who did not report having a full-time job.
5. All other things being equal, respondents reporting as Republicans are less likely to perceive discrimination as preventing Latinos from succeeding in America than non-Republicans.
6. All other things being equal, respondents who are single are more likely to believe perceived discrimination prevents Latinos from succeeding in America than respondents who are not single.
7. All other things being equal, an increase in education level leads to a decrease in the belief that perceived discrimination prevents Latinos from succeeding in America.

<sup>2</sup> As titled by the Pew Hispanic Center

8. All other things being equal, women are more likely to believe perceived discrimination prevents Latinos from succeeding in America.

### Formula

$Y = a + b_1x_1 + b_2x_2 + b_3x_3 + b_4x_4 + b_5x_5 + b_6x_6 + b_7x_7 + b_8x_8 + e$   
 Y=Discrimination effect on preventing Latinos from succeeding in America  
 a=constant  
 b<sub>1</sub>= Income level  
 b<sub>2</sub>=Birthplace  
 b<sub>3</sub>=U.S. Citizenship  
 b<sub>4</sub>=Employment Status  
 b<sub>5</sub>=Partisanship  
 b<sub>6</sub>=Marital Status  
 b<sub>7</sub>=Educational Level  
 b<sub>8</sub>=Gender  
 e=Error

### Limitations

I did not address discrimination against non-Hispanics or any policies intended to alleviate discrimination. There is no data that addresses who the respondent believed committed act(s) of discrimination preventing Latinos from succeeding in America. There may be error in the term definition and concept understanding of discrimination among respondents. I did not measure discrimination, but the reports on levels of perceived discrimination against Latinos, keeping them from succeeding in America. The survey captured self-reported evidence on perceptions that are not validated. I cannot identify how much of any past outcome is due to discrimination or how much past discrimination may be affecting current outcomes.

## Findings

Figure 1. Statistical significance & null hypothesis

	2002			2004			2006			2007		
	R-sig	P-Value	T	R-sig	P-Value	T	R-sig	P-Value	T	R-sig	P-Value	T
Total Annual Income			REJ (2.646)			REJ (3.784)			REJ (2.326)			REJ (3.973)
Place of Birth		NOT S.S		NOT S.S		REJ (-5.004)		EXCLUDED		NOT S.S		NOT S.S
Citizenship	NOT S.S	NOT S.S				REJ (7.356)	NOT S.S	NOT S.S				NOT S.S
Employment Status		NOT S.S				REJ (3.441)	NOT S.S	NOT S.S				NOT S.S
Partisanship		NOT S.S						NOT S.S				NOT S.S
Marital Status		NOT S.S			NOT S.S			NOT S.S		NOT S.S		NOT S.S
Education		NOT S.S			NOT S.S				REJ (3.049)	NOT S.S		NOT S.S
Gender		NOT S.S			NOT S.S		NOT S.S	NOT S.S		NOT S.S		NOT S.S

\*p≤.05



2002	2004	2006	2007
Total Annual Income	Employment Status	Education	Total Annual Income
Citizenship	Total Annual Income	Total Annual Income	Citizenship
Partisanship	Partisanship	Marital Status	Partisanship
Marital Status	Marital Status	Partisanship	Gender
Employment Status	Education	Gender	Employment Status
Education	Gender	Citizenship	Marital Status
Gender	Citizenship	Employment Status	Gender
(Excluded P.o.B.)	(Excluded P.o.B.)	(Excluded P.o.B.)	Place of Birth

Figure 2. Beta weight rankings

Variables of the top of the column are most important; those at the bottom are least important based on beta weight.

	2002	2004	2006	2007
R	0.100	0.342	0.181	0.165
R-Sq.	0.010	0.117	0.033	0.027
Adj. R-Sq.	0.006	0.112	0.027	0.021
F	2.239	24.285	5.264	4.559
Constant	1.710	1.538	1.325	1.345

Figure 3. Statistics

As shown in Figure 1, Total Annual Income was a statistically significant in every survey. In 2004 Place of birth, Citizenship, and Employment were statistically significant predictors in addition to Total Annual Income. The independent variables explained 11.2% of the variation in the dependent variables for 2004. This was dramatically different to the surveys in 2002 (0.06%), 2006 (2.7%), and 2007 (2.1%). As shown in figure 2 the beta weight rankings for each survey were consistent with the t scores in Figure 1.

Although all the surveys were completed by the Kaiser Family Foundation/Pew Hispanic Center I analyzed each survey coding to identify disparities between survey years that might have affected the survey results from one year to another. By options, I refer to the options listed on the coding manual after each question. By coding, I refer to the methodology of each survey.

The options and coding for Total Annual Income were the same for all survey years, and in each survey year Total Annual Income was a significant factor. Place of birth and Citizenship were both significant in 2004<sup>3</sup>. The question structure, options, and coding were the same for Place of birth and Citizenship in 2004 as in the 2002 National Survey of Latinos. Yet, in 2002 neither Place of birth nor Citizenship were significant. In 2006, Education was significant and it too shared the same question structure, options, and coding as all three other surveys.

In 2007 place of birth options were listed differently than other survey years. In 2002, 2004, and 2006 U.S. was listed first and coded as 1 followed by Puerto Rico which was coded as 2. In 2007 Puerto Rico was coded 1, and the United States was coded 2.

Citizenship varied from 2002 and 2004 to 2006 and 2007. In 2002 and 2004 the question was “Now we would like to ask you about U.S. Citizenship. Are you...?” The options listed were “A U.S. Citizen, Currently applying for citizenship, Planning to apply for citizenship, and Not planning to become a citizen.” In 2006 and 2007 the question was worded differently: “Are you a citizen of the United States?” with options of “yes or no.” As previously mentioned Citizenship was only significant in 2004.

The survey question for Employment was worded the same in 2002 and 2004, but the options differed. The question structure and options were completely different in 2006 and again in 2007.

The question structure for Marital Status was the same in every survey year. The only difference was in one of the options. In 2002, 2004, and 2006 “Living with a partner” was coded as 2, whereas in 2007 “Have a partner” was coded as 2.

<sup>3</sup> By options, I refer to the options listed on the coding manual after each question. By coding, I refer to the methodology of each survey.

The predominant finding from these regression analyses is that the respondent's income level matters greatly for the perception of discrimination. This statistically significant variable is conspicuous despite coding, question wording, and option differences across the surveys. Employment and educational level are also generally important.

## **Discussion**

Differences in question structure, options, and coding did not appear to have statistically significant effects in the survey results. The results were independent of the surveys' differences. The question structure, options, and coding remained the same for the Dependent Variable and the general focus of the independent variable remained more important than the minor differences from one survey to another.

The surveys revealed interesting factors that affected the perception discrimination as it prevents Latinos from succeeding in America. With 2004 explaining variation to a much greater degree than 2002, 2006, and 2007, I was left with questioning the reasons for these results. According to Michelson (2001), extensive media attention to an issue can increase its perceived national importance, and Latinos are aware of the political world and react to changes in that environment. The survey conducted in 2002 addressed a wide variety of topics and the 2004 survey's primary focus was politics and civic participation. The 2006 National Survey of Latinos: The Immigration Debate contained new questions about the immigration debate, and followed the congressional votes on the immigration question. The Pew Hispanic Center 2007 National Survey of Latinos was conducted during a period of increased local- and state-level legislative actions, and increased enforcement measures in reaction of the illegal immigration debates. This survey included new questions regarding fears of deportation.

The general focus and new questions in each survey do not appear to influence the explained variation in each survey. The consistency of Total Annual Income as a significant factor is ever present. It is important to consider the national mood and level of media coverage, and how it may have affected Latino public opinion and their perceptions of discrimination. The 2002 survey followed the 9/11 attacks; this event followed a report of a heightened level of perceived discrimination among Latinos as well as fear of deportation, even among U.S. Citizens. The presidential election debates of 2004 may have had swayed the American national mood and ultimately the Latino reaction to perceive discrimination at a higher extent than in 2002, 2006, or 2007. Arguably the events following the 9/11 attacks such as the attention to immigration in the 2004 U.S. presidential debates may have led to the creation of House of Representatives' Bill 4437. This bill was proposed to criminalize any knowledge of undocumented persons residing in the United States without immediate reporting to INS as well as antiterrorism. The threatened enactment of Bill 4437 caused alarm throughout the United States especially in Latino communities. In 2006 over five million opponents of Bill 4437 set out to march the streets in protest.

## **Conclusion**

The general trend among Latinos and their perceptions of discrimination as a major problem in preventing Latinos from succeeding in America exists with Total Annual Income as a consistent factor. The analysis suggests that increased levels of perceived discrimination are correlated with shifts in national mood. However, the analysis also suggests that a perception of discrimination always exists in respondents with lower total annual incomes. A state's racial and ethnic composition is an important factor in shaping policy outcomes (Hero 2001). In addition to a state's composition we also have to consider political participation among racial and ethnic minorities. This is significant with the increased Latino population expected to compose 25% of the total U.S. population by 2042. Reasonable perceptions of discrimination, a lack of political trust and sporadic political participation for Latinos suggest troubling prospects for the future of race relations, the American political system, and the entire essence of democracy.

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